Invariant metrics of positive scalar curvature on S^1-manifolds

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Outline

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1. Introduction
2. The case where M^{S^1} has codimension two
3. The case where codim $M^{S^1} \geq 4$

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Scalar curvature

- Let (M, g) be a Riemannian manifold.
- The scalar curvature of M is a function $scal : M \to \mathbb{R}$
- For small $r > 0$ and $x \in M$ we have :

$$\mathrm{vol}(B_r(x)) = \mathrm{vol}_{euclid}(B_r(0))(1 - \frac{scal(x)}{6(n+2)} r^2 + O(r^4))$$

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A basic question

Question

Assume that a compact connected Lie group G acts effectively on a closed connected manifold M.
Does there exist an G-invariant metric of positive scalar curvature on M?

$G = \{1\}$

Theorem (Gromov-Lawson 1980)

Assume that $\pi_1(M) = 0$, $\dim M \geq 5$ and M does not admit a spin-structure.
Then M admits a metric of positive scalar curvature.

psc-metrics and Spin-structures

If M is spin and admits a metric of positive scalar curvature, then

- the Dirac-operator D on M is invertible (Lichnerowicz 1963).
- Hence its index vanishes.
- ind $D = \widehat{A}(M)$ is an invariant of the spin-bordism type of M (Atiyah-Singer 1968).

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$G = \{1\}$

Theorem (Stolz 1992)

Assume that $\pi_1(M) = 0$, $\dim M \geq 5$ and M admits a spin structure.
Then M admits a metric of positive scalar curvature if and only if $\alpha(M) = 0$.

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Proof.

1. If M is constructed from N by a surgery of codimension at least three and N admits a metric of positive scalar curvature, then the same holds for M. (Gromov-Lawson, Schoen-Yau)
2. Hence, M admits a metric of positive scalar curvature, if and only if its class in a certain bordism group can be represented by a manifold with such a metric.
3. Find all bordism classes which can be represented by such manifolds.

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Non-abelian groups

Theorem (Lawson-Yau 1974)

If G is non-abelian,
then there is always a G-invariant metric of positive scalar curvature on M.

- Therefore in the following we assume that $G = T$ is a torus or $G = S^1$

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Abelian groups

Theorem (Bérard Bergery 1983)

Assume that a torus T acts freely on M.
Then M admits an invariant metric of positive scalar curvature if and only if M/T admits a metric of positive scalar curvature.

Examples

- $\exists$ manifolds which admit a non-trivial S^1-action but no metric of positive scalar curvature:
  - Exotic spheres with $\alpha(\Sigma) \neq 0$ (Bredon, Schultz, Joseph 1967-1981)
- $\exists$ S^1-manifolds which admit metrics of positive scalar curvature but no invariant such metric:
  - simply connected S^1-bundles over $K3$-surfaces (Bérard Bergery).

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First main theorem

Theorem (2013)

Let M be a connected $(G \times S^1)$-manifold such that codim $M^{S^1} = 2$.
Then M admits a $(G \times S^1)$-invariant metric of positive scalar curvature.

Corollary

Every torus manifold admits an invariant metric of positive scalar curvature.

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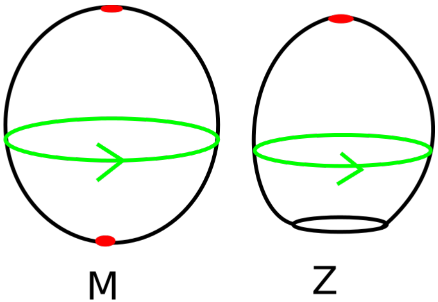
Corollary

Every torus manifold admits an invariant metric of positive scalar curvature.

The proof of the Theorem

- Let $Z = M - N(F, M)$, where $F \subset M^{S^1}$ component with codim $F = 2$.
- $\exists$ a $(G \times S^1)$-handle decomposition of Z without handles of codimension zero



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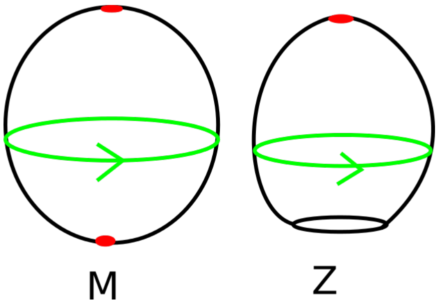

- $Z \times D^2$ is $(G \times S^1 \times S^1)$-manifold.
- $\exists$ a $(G \times S^1 \times S^1)$-handle decomposition of $Z \times D^2$ without handles of codimension < 3.
- $\partial(Z \times D^2) = SN(F, M) \times D^2 \cup Z \times S^1$ admits invariant metric of positive scalar curvature
- diag$(S^1 \times S^1)$ acts freely on $\partial(Z \times D^2)$ with orbit space M.

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Some more corollaries

Corollary

Let M be an effective S^1-manifold $\dim M \geq 5$.

- *Assume that the principal orbits in M are null-homotopic.*
- *If $\tilde{M}$ is spin, assume that the lifted S^1-action on $\tilde{M}$ is of odd type.*

Then M admits a non-invariant metric of positive scalar curvature.

Corollary (Ono 1991)

Let M be a spin manifold with an effective S^1-action of odd type, then $\alpha(M) = 0$.

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A related result of M. Bendersky

Theorem (Bendersky, Ochanine, Ono 1990-1992)

Let M be a spin manifold with effective S^1-action of odd type, then the Ochanine-genus of M vanishes.

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Proof of Corollary 2

Corollary (Ono 1991)

Let M be a spin manifold with an effective S^1-action of odd type, then $\alpha(M) = 0$.

- A neighborhood of a principal orbit in M is equivariantly diffeomorphic to $S^1 \times \mathbb{R}^{n-1}$.
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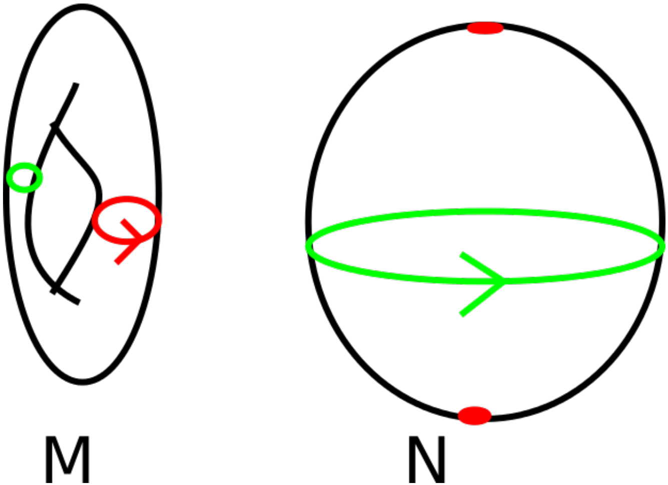
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- If M is spin and S^1-action on M of odd type, then N is spin.

Proof of Corollary 1

Corollary

Let M be an effective S^1-manifold $\dim M \geq 5$.

- *Assume that the principal orbits in M are null-homotopic.*
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Then M admits a non-invariant metric of positive scalar curvature.

- First construct N as in the proof of the previous corollary.
- If principal orbits are null-homotopic, then $N \cong M \# S^2 \times S^{n-2}$.
- So by surgery on S^2 we can recover M.

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Obstructions to positive scalar curvature and to S^1-actions

Corollary

Let M be a manifold with $\dim M \geq 5$, $\chi(M) \neq 0$ and non-spin universal covering.
If M does not admit a metric of positive scalar curvature then there is no non-trivial S^1-action on M.

- The only known obstruction to a metric of positive scalar curvature on a manifold as above comes from the minimal hypersurface method of Schoen and Yau (1979).
- This gives obstructions for manifolds of dimensions $n \leq 8$.
- Without using scalar curvature we can prove that there is a similar obstruction to non-trivial S^1-actions.
- This works in all dimensions.

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In this part assume that $\pi_1(M_{max}) = 0$, codim $M^{S^1} \geq 4$ and that the action satisfies the following condition:

Condition C

- *For all subgroups $H \subset S^1$, $N(M^H, M)$ is a S^1-equivariant complex vector bundle.*
- *For $H \subset K \subset S^1$, there is an isomorphism of S^1-equivariant complex vector bundles*

$$N(M^K, M) \cong N(M^K, M^H) \oplus N(M^H, M)|_{M^K}.$$

This condition is always satisfied if no isotropy group of a point in M has even order.

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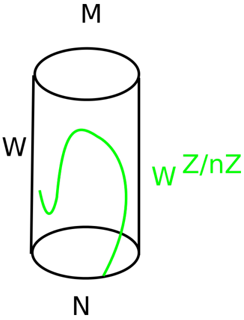
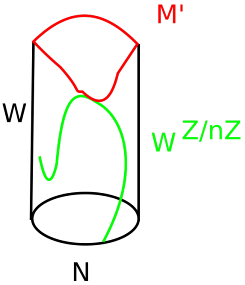
Some notations

- Let $\Omega^{C,SO,S^1}_{\geq 4,n}$ the bordism group of oriented n-manifolds as above
- Let $\Omega^{C,Spin,S^1}_{\geq 4,n}$ the bordism group of n-Spin-manifolds as above

- We want to prove a bordism principle for these actions.
- Here singular strata of codimension two in the bordisms cause some problems.



- This has been dealt with essentially by Hanke (2008).



normally symmetric metrics

A invariant metric g is called *normally symmetric in codimension two* if

- For each component $F \subset M^H$ with codim $F = 2$,
  - $\exists$ a invariant neighborhood U of F in M
  - and an S^1-action on U which
    - has $U^{S^1} = F$
    - commutes with the original S^1-action and
    - leaves g invariant.
- If codim $M_{(\mathbb{Z}_2)} > 2$, then any metric g can be deformed to a normally symmetric metric.

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The bordism principle

Theorem

If $\dim M \geq 6$ *and* M_{max} *is not spin, then* M *admits a normally symmetric metric of positive scalar curvature if and only if its class in* $\Omega^{C,SO,S^1}_{\geq 4,n}$ *can be represented by a manifold which admits such a metric.*

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Existence results

Theorem (2015)

If dim $M \geq 6$ *and*

- M_{max} *is not spin, or*
- *M is spin and the* S^1*-action of odd type,*

then there is an $\ell \in \mathbb{N}$ *such that the equivariant connected sum of* 2^ℓ *copies of M admits an invariant normally symmetric metric of positive scalar curvature.*

- In the first case ℓ can be taken to be 1.
- If the action is semi-free, ℓ can be taken to be 1.

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Existence results II

Theorem (2015)

If $\dim M \geq 6$, M is spin and the S^1-action of even type, then $\hat{A}_{S^1}(M/S^1) = 0$ if and only if there is an $\ell \in \mathbb{N}$ such that the equivariant connected sum of 2^ℓ copies of M admits an invariant normally symmetric metric of positive scalar curvature.

- $\hat{A}_{S^1}(M/S^1)$ is a $\mathbb{Z}[\frac{1}{2}]$-valued equivariant bordism invariant of M.
- For free actions it is the $\hat{A}$-genus of the orbit space.
- For semi-free actions it was defined by Lott (2000).

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A corollary

Corollary (Atiyah-Hirzebruch 1970)

Let M be a spin-manifold with $\dim M \geq 6$ *which admits a non-trivial* S^1*-action which satisfies Condition C.*
Then $\widehat{A}(M) = 0$.

- We may assume that $\dim M = 4k$.
- Since $\widehat{A}_{S^1}(M/S^1) \neq 0$ implies $\dim M = 4k+1$, $2^\ell M$ is equivariantly spin-bordant to an S^1-manifold N with an invariant metric of positive scalar curvature.
- Hence, $2^\ell \widehat{A}(M) = \widehat{A}(N) = 0 \;\Rightarrow \widehat{A}(M) = 0$

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Let M be a spin-manifold with $\dim M \geq 6$ *which admits a non-trivial* S^1*-action which satisfies Condition C.*
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For simply connected S^1-manifolds M with dim $M \geq 6$ the following holds:

- If codim $M^{S^1} = 2$, then there is always a invariant psc-metric on M.
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