Preprintreihe

SFB 478 – Geometrische Strukturen in der Mathematik

Heft 173

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ISSN 1435-1188

Preprintreihe

des SFB 478 – Geometrische Strukturen in der Mathematik

des Mathematischen Instituts
der Westfälischen Wilhelms-Universität Münster

Heft 173

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Juli 2001

ISSN 1435-1188

Diese Arbeit ist im Sonderforschungsbereich 478, Geometrische Strukturen in der Mathematik, Münster, entstanden und wurde auf seine Veranlassung unter Verwendung der ihm von der Deutschen Forschungsgemeinschaft zur Verfügung gestellten Mittel gedruckt.

A triangulated category of equivariant motives

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August 2, 2001

Abstract

We construct, for a discrete group, an equivariant version of Voevodsky's triangulated category of mixed motives. Afterwards we compare this with known constructions from algebraic geometry and topology.

Contents

1 Introduction

In this article we construct for a discrete group G, an equivariant version ${}^{G}DM(k)$ of Voevodsky's triangulated category of mixed motives together with two covariant functors

$${}^{\mathrm{G}}\mathrm{M} : {}^{G}Sch/k \longrightarrow {}^{G}DM(k)$$

and

$${}^{\mathrm{G}}\mathrm{M}^{c} : {}^{G}Sch/k^{prop} \longrightarrow {}^{G}DM(k).$$

Our way is of course only one possibility and is the analog of the Borel construction from topology. Using [Se01a] or [ATJLSS00] we can calculate the Hom-groups of ${}^{G}DM(k)$ via derived functors. Therewith many properties of ${}^{G}DM(k)$, for example Mayer-Vietoris exact triangle and homotopy invariance, follow easily from the same properties of $DM^{eff}(k)$. For most of this we need only the resolution for the ground field k, and no equivariant resolution. Afterwards, in special situations, we compare the result with Borel cohomology in topology and with the equivariant Chow-groups of Edidin and Graham.

This article is organized as follows. In section 2 we collect the necessary properties from equivariant cohomology. In section 3 we construct ${}^{G}DM(k)$ and prove good properties for it. In section 4 we discuss briefly some supplements to the construction. In the last section we give the connections to the Borel cohomology from topology and to the Chow groups of Edidin and Graham.

I would like to thank C. Deninger, A. Huber–Klawitter, B. Köck, M. Joachim H. Reich, F. Bars, W. Büchner, S. Feil, S. Gille, N. Naumann, G. Standfest, T. Wenger and M. Kisin for helpful discussions and comments.

2 Equivariant geometry

In this section we want to recall the needed facts from equivariant geometry. We restrict ourself to our specific situation, even if the statements are true more generally.

2.1 Equivariant cohomology

First we fix some notation. Let k be a field and for a k-scheme X let Sm/X be the category of smooth schemes of finite type over X. We fix a discrete group G and we denote by ${}^{G}Sm/k$ the category of G-schemes which are smooth and of finite type over k . Let $\mathcal{T}$ be the etale, Nisnevich or Zariski topology on Sm/X. Let $\mathcal{C}$ be the category of sets or the category of A-modules for a ring A. Then we denote by $Shv_{\mathcal{T}}^{\mathcal{C}}(Sm/X)$ the category of sheaves on Sm/X with respect to the topology $\mathcal{T}$ and with values in $\mathcal{C}$. The category $Shv_{\mathcal{T}}^{Sets}(Sm/k)$ is a topos and by [SGA72a, IV.1.2] the category of G-objects in $Shv_{\mathcal{T}}^{\mathcal{C}}(Sm/k)$ which we denote by B_G is again a topos. Now an $X \in {}^{G}Sm/k$ can be viewed as an object in B_G and so it makes sense to look at the localization B_G/X.

Definition 2.1. *For $X \in {}^{G}Sm/k$ we call*

$${}^{G}Shv_{\mathcal{T}}^{Sets}(Sm/X) := B_G/X$$

the category of G-sheaves on X. Further we define as usual

$$ {}^{G}Shv_{\mathcal{T}}^{Ab}(Sm/X) := (B_G/X)_{Ab} $$

as the abelian objects in B_G.

There is also a more direct and more standard definition of G-sheaves. We now explain the connection to it.

Remark 2.2. *By [SGA72a, III.5.4] we have an equivalence of categories*

$$ \Psi : \; Shv_{\mathcal{T}}^{\mathcal{C}}(Sm/X)/X \xrightarrow{\sim} Shv_{\mathcal{T}}^{\mathcal{C}}(Sm/X). $$

For G-sheaves this means the following. If $X \in {}^{G}Sm/k$ then each $g \in G$ induces a functor

$$ g^* : \; Shv_{\mathcal{T}}^{\mathcal{C}}(Sm/X) \longrightarrow Shv_{\mathcal{T}}^{\mathcal{C}}(Sm/X) $$

which is given by

$$ (g^*F)(U \longrightarrow X) = F({}^{g}(U \longrightarrow X)) $$

where ${}^{g}(U \to X) := (U \to X \xrightarrow{g} X)$. Now a G-sheaf on Sm/X is a sheaf $F \in Shv_{\mathcal{T}}^{\mathcal{C}}(Sm/X)$ together with a map

$$ \phi_g : g^*F \longrightarrow F $$

for each $g \in G$ such that for all $g, h \in G$ and $(U \to X) \in Sm/X$ the diagram

$$ \begin{tikzcd} F({}^{gh}(U \to X)) \arrow[rr, "\phi_{gh}(U \to X)"] \arrow[d, equal] & & F(U \to X) \\ F({}^{g}({}^{h}(U \to X))) \arrow[rr, "\phi_g({}^{h}(U \to X))"] & & F({}^{h}(U \to X)). \arrow[u, "\phi_h(U \to X)"'] \end{tikzcd} $$

is commutative.

Remark 2.3. *If G act trivial on X we have*

$$ {}^{G}Shv_{\mathcal{T}}^{\mathcal{C}}(Sm/X) = \; Shv_{\mathcal{T}}^{{}^{G}\mathcal{C}}(Sm/X) $$

where ${}^{G}\mathcal{C}$ is the category of G-objects in $\mathcal{C}$.

Now B_G is, as mentioned above, a topos and therefore $(B_G)_{Ab}$ is a Grothendieck category. So by [Se01a] or [ATJLSS00] the unbounded derived category has enough K-injective complexes. Therefore we have the derived functor

$$ \mathbf{R}\mathrm{Hom}_{(B_G)_{Ab}} : \mathcal{D}((B_G)_{Ab})^{op} \times \mathcal{D}((B_G)_{Ab}) \longrightarrow \mathcal{D}(Ab) $$

which is in each variable a functor of triangulated categories. For $X \in {}^{G}Sm/k$ we denote by ${}^{G}\mathbb{Z}(X)$ the free abelian sheaf generated by X viewed as an object in B_G. For $A^\bullet \in \mathcal{D}((B_G)_{Ab})$ we write

$$ H^i((B_G)_{Ab}; X, A^\bullet) := H^i(G; X, A^\bullet) := h^i(\mathbf{R}\mathrm{Hom}_{(B_G)_{Ab}}({}^{G}\mathbb{Z}(X), A^\bullet) $$

and call this equivariant cohomology of X with values in $A^\bullet$. Now we come to a factorization of this which will allow us to carry over many results from the non equivariant to the equivariant case.

For an object $A \in (B_G)_{Ab}$ we look at the functor

$$\mathrm{Hom}\ _{Shv^{e}_{Ab}(Sm/k)}(A, \_) : (B_G)_{Ab} \longrightarrow \mathbb{Z}[G] - \mathrm{Mod}$$

where for two objects $N, M \in (B_G)_{Ab}$ the abelian group $\mathrm{Hom}_{T_{Ab}}(M, N)$ gets via

$$(g \cdot f)(m) := gf(g^{-1}m)$$

for all $g \in G$ the G-module structure. Together with the invariant functor

$$\mathrm{Hom}_{\mathbb{Z}[G]-\mathrm{Mod}}(\mathbb{Z}, \_) : \mathbb{Z}[G] - \mathrm{Mod} \longrightarrow Ab$$

we have the factorization

$$\mathrm{Hom}_{(B_G)_{Ab}}(A, \_) = \mathrm{Hom}_{\mathbb{Z}[G]-\mathrm{Mod}}(\mathbb{Z}, \_) \circ \mathrm{Hom}_{T_{Ab}}(A, \_).$$

To get such an derived factorization we first observe that for $X \in {}^{G}Sm/k$ the functor

$$\begin{array}{ccc} \mathbb{Z}[G] - \mathrm{Mod} & \longrightarrow & (B_G)_{Ab} \\ M & \longmapsto & M \otimes {}^{G}\mathbb{Z}(X) \end{array}$$

is exact and left adjoint to $\mathrm{Hom}_{\mathcal{T}_{Ab}}({}^{G}\mathbb{Z}(X), \_)$. So by standard results from homological algebra we get the derived factorization

$$\mathbf{R}\mathrm{Hom}_{(B_G)_{Ab}}({}^{G}\mathbb{Z}(X), \_) = \mathbf{R}\mathrm{Hom}_{\mathbb{Z}[G]-\mathrm{Mod}}(\mathbb{Z}, \_) \circ \mathbf{R}\mathrm{Hom}_{T_{Ab}}({}^{G}\mathbb{Z}(X), \_). \quad (1)$$

2.2 Equivariant Hilbert 90

In this section we prove an equivariant version of Hilbert 90.

First we look at line bundles in a ringed topos. So let (E, A) be a commutative ringed topos, $e = e_E$ the final object in E and ${}_AE$ the category of A-modules in E.

Definition 2.4. *A* line bundle *in ${}_AE$ is an A-module L, such that there is a covering $\{U_i \to e\}_{i \in I}$ of e with the property that for all $i \in I$ the A_{U_i}-module L_{U_i} is isomorphic to A_{U_i}.*

Now as in [Del77, Prop.2.3] one sees:

Theorem 2.5. *The map*

$$A^* : U \in E \longmapsto Isom_{A_{|U}}(A_{|U}, A_{|U})$$

gives a natural bijection between the isomorphism classes of line bundles in (E, A) and the isomorphism classes of A^*-torseurs

Now in [Gir71, rem. 3.5.4] the following theorem is proven.

Theorem 2.6. *There is a natural isomorphism between the group of isomorphism classes of line bundles in (E, A) and $H^1(E; e, A^*)$.*

Now we look at a topos T, a discrete group G and a G-object X. Let B_G be the topos of G-objects in T, E the topos B_G/X and A a commutative ring object in E. We look at the functor

$$\mathrm{ver} : E = B_G/X \longrightarrow T/X$$

which simple forgets the G-structure. Then $\mathrm{ver}(A)$ is a commutative ring object in T/X.

Lemma 2.7. *An A-module L is a line bundle if and only if the* $\mathrm{ver}(A)$*-module* $\mathrm{ver}(L)$ *is a line bundle.*

Proof. The only if is clear. For the other direction we first observe that L is trivial on the covering $(U \to X) \in B_G/X$ if and only if $\mathrm{ver}(L)$ is trivial on the covering $\mathrm{ver}(U \to X) \in T/X$. Now if $\{U_i \xrightarrow{\pi_i} X\}$ is a covering of X in T where $\mathrm{ver}(L)$ trivializes then $\mathrm{ver}(L)$ trivializes also on the equivariant covering $\{G \times U_i \to X,\ (g,u) \mapsto g\pi_i(u)\}$. $\square$

Now we come back to our main situation. So let $\mathcal{T} = Shv_{\mathcal{T}}^{Sets}(Sm/k)$, $\mathcal{T} = et, Nis$ or Zar and A be the G-sheaf $\mathcal{O}_X$. In the non equivariant situation we have

$$H^1_{et}(X, \mathcal{O}^*_X) = H^1_{Nis}(X, \mathcal{O}^*_X) = H^1_{Zar}(X, \mathcal{O}^*_X).$$

By a spectral sequence argument (compare [Gro68, §2]) it follows

$$H^1_{et}(G, X; \mathcal{O}^*_X) = H^1_{Nis}(G, X; \mathcal{O}^*_X) = H^1_{Zar}(G, X; \mathcal{O}^*_X)$$

for a G-scheme X.

In [MFK94] for a $X \in {}^GSm/k$ the equivariant Picardgroup $\mathrm{Pic}^G(X)$ is defined as follows. One looks at isomorphism classes of invertible $\mathcal{O}_X$-Module together with a G-structure as an $\mathcal{O}_X$-module. For details see [MFK94, 1.3]. By the above comparison, theorem remark 2.2, theorem 2.6 and lemma 2.7 we get the following theorem.

Theorem 2.8 (Equivariant Hilbert 90). *There is, natural in X, an isomorphism*

$$Pic^G(X) \xrightarrow{\sim} H^1(G; X, \mathcal{O}^*_X).$$

Furthermore from the construction it follows that the diagram

$$\begin{tikzcd} Pic^G(X) \arrow[r, "\sim"] \arrow[d] & H^1(G; X, \mathcal{O}^*_X) \arrow[d] \\ Pic(X) \arrow[r, "\sim"] & H^1(X, \mathcal{O}^*_X) \end{tikzcd}$$

is commutative. Here the vertical arrow are induced by the forgetful functor.

Remark 2.9. *Equivariant cohomology is a mixture of sheaf cohomology and group cohomology. If we look at coefficients with trivial action we can express equivariant sheaf cohomology in terms of usual cohomology. So let $B^\bullet \in$*

$\mathcal{K}((C_{G}^{\sim})_{Ab})$ be a complex with trivial G action and X a G-object. Let $EG \times G$ be the simplicial G-scheme

$$\underbrace{G \times \cdots \times G}_{n\text{-}times} \times X$$

with the G-action

$$g(h_1, h_2, \ldots, h_n, x) = (h_1 g^{-1}, h_2 g^{-1}, \ldots, h_n g^{-1}, gx).$$

Further let $EG \times_G X := EG \times X/G$ and for an object B let $\mathbb{Z}(B)$ and $\mathbb{Z}^{ps}(B)$ be the free abelian sheaf resp. presheaf generated by B. If B is a simplicial object then take the associated chain complex. Then one observes that

$$\mathbb{Z}^{ps}(EG \times X) \longrightarrow \mathbb{Z}^{ps}(X)$$

a projective resolution. From that one can deduce that

$$H^i(G; X, B^\bullet) = h^i(\mathbf{R}\mathrm{Hom}_{C_{Ab}^{\sim}}(\mathbb{Z}(EG \times_G X), B^\bullet)))$$

holds. For more details see [Se01b, sec. 2.3].

3 Constructions and properties

In this section the actual construction is carried out. In the first part we construct the triangulated tensor category ${}^{G}DM^{eff}(k)$ and a covariant functor

$${}^{\mathrm{G}}\mathrm{M} : {}^{G}Sm/k \longrightarrow {}^{G}DM^{eff}(k).$$

In the second part we show the main properties of ${}^{G}DM^{eff}(k)$. In the third part we extend the functor ${}^{\mathrm{G}}\mathrm{M}$ to the category of all scheme of finite type over k. In the last part we establish a connection to higher Chow-groups.

3.1 Main construction

First we want to recall some notations from [Voe00]. We refer to op. cit. for the details. We denote by Sm/k the category of schemes which are smooth and of finite type over a fixed base field k. For $X, Y \in Sm/k$ we have the group $Cor(X, Y)$ of finite correspondences and it is possible to take these as the new morphisms in the category Sm/k and so to get the new category $SmCor(k)$ of smooth correspondences. The graph induces a functor

$$\begin{array}{rccc} [\_] : & Sm/k & \longrightarrow & SmCor(k) \\ & X & \longmapsto & [X]. \end{array}$$

and with

$$[X] \oplus [Y] = [X \amalg Y]$$

$SmCor(k)$ becomes an additive category.

A presheaf with transfers is just a contravariant functor $F : SmCor(k) \to Ab$. We denote by $PreShv(SmCor(k))$ the category of presheaves with transfers. If we restrict a presheaf with transfers so Sm/k it makes sense to ask, whether it is a sheaf with respect to a topology $\mathcal{T}$ on Sm/k. So denote by

$Shv_{\mathcal{T}}(SmCor(k))$ the category of sheaves with transfers. Now if $X \in Sm/k$ we denote by $\mathbb{Z}_{tr}(X)$ the presheaf with transfers with is represented by X. For $\mathcal{T} = Zar, et$ or Nis this is already a sheaf. So we get a covariant functor

$$\mathbb{Z}_{tr} : Sm/k \longrightarrow Shv_{\mathcal{T}}(SmCor(k)).$$

For $\mathcal{T} = Nis, et$ the forgetful functor from sheaves with transfers to presheaves with transfers has a right adjoint. This is just sheafification. For details see again [Voe00]. Therefore sheaves with transfers form an abelian category and furthermore a Grothendieck category.

Now we want to equip everything with an group action. For this let G be a discrete group. A G-presheaf with transfers on Sm/k is just an G-object in $PreShv(SmCor(k))$. Correspondingly a G-sheaf with transfers is a G-object in $Shv_{\mathcal{T}}(SmCor(k))$. We denote these categories by ${}^{G}PreShv(SmCor(k))$ and ${}^{G}Shv_{\mathcal{T}}(SmCor(k))$. From the above it is clear that ${}^{G}Shv_{\mathcal{T}}(SmCor(k))$ is a Grothendieck category for $\mathcal{T} = Nis, et$. Therefore again the results of [ATJLSS00] or [Se01a] are applicable. That means that the category of unbounded complex has enough K-injective complexes. If $X \in {}^{G}Sm/k$ then $\mathbb{Z}_{tr}(X)$ has a natural G-structure and therefore we denote it by ${}^{G}\mathbb{Z}_{tr}(X) \in {}^{G}Shv_{Nis}(SmCor(k))$. This gives us a covariant functor

$${}^{G}\mathbb{Z}_{tr} : {}^{G}Sm/k \longrightarrow {}^{G}Shv_{Nis}(SmCor(k)).$$

As in 2 one sees easily that we have for $X \in {}^{G}Sm/k$ the factorization

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \_) \circ \mathbf{R}\mathrm{Hom}_{Shv_{Nis}(SmCor(k))}({}^{G}\mathbb{Z}_{tr}(X), \_)& \\ = \mathbf{R}\mathrm{Hom}_{{}^{G}Shv_{Nis}(SmCor(k))}&({}^{G}\mathbb{Z}_{tr}(Z), \_).\end{aligned}$$

where $\mathbf{R}\mathrm{Hom}_{Shv_{Nis}(SmCor(k))}({}^{G}\mathbb{Z}_{tr}(X), _)$is the total derived functor of

$$\mathrm{Hom}_{Shv_{Nis}(SmCor(k))}({}^{G}\mathbb{Z}_{tr}(X), \_) : {}^{G}Shv_{Nis}(SmCor(k)) \longrightarrow \mathbb{Z}[G] - Mod,$$

with the G-structure

$$(g \cdot f)(a) := gf(g^{-1}a)$$

for $f \in \mathrm{Hom}_{Shv_{Nis}(SmCor(k))}({}^{G}\mathbb{Z}_{tr}(X), B)$ and $g \in G$.

There are the four exact forget functors

$$Shv_{\mathcal{T}}(SmCor(k)) \longrightarrow Shv_{\mathcal{T}}(Sm/k),$$
$${}^{G}Shv_{\mathcal{T}}(SmCor(k)) \longrightarrow {}^{G}Shv_{\mathcal{T}}(Sm/k),$$

$$\mathrm{Ver} : {}^{G}Shv_{\mathcal{T}}(Sm/k) \longrightarrow Shv_{\mathcal{T}}(Sm/k) \text{ and}$$
$$\mathrm{Ver} : {}^{G}Shv_{\mathcal{T}}(SmCor(k)) \longrightarrow Shv_{\mathcal{T}}(SmCor(k)).$$

The following easy proposition will be quite useful.

Proposition 3.1. *Let $A^\bullet, B^\bullet \in \mathcal{D}({}^{G}Shv_{Nis}(SmCor(k)))$ and $f : A^\bullet \to B^\bullet$ a morphism in $\mathcal{D}({}^{G}Shv_{Nis}(SmCor(k)))$.Then f is an isomorphism if and only if the morphism $\mathrm{Ver}(f) : \mathrm{Ver}(A^\bullet) \to \mathrm{Ver}(B^\bullet)$ is an isomorphism in the category $\mathcal{D}(Shv_{Nis}(SmCor(k)))$. In particular $A^\bullet \in \mathcal{D}({}^{G}Shv_{Nis}(SmCor(k)))$ is acyclic if and only if $\mathrm{Ver}(A^\bullet) \in \mathcal{D}(Shv_{Nis}(SmCor(k)))$ is acyclic.*

Proof. The morphism f is an isomorphism if and only if the cone $C(f)$ is zero in $\mathcal{D}({}^G Shv_{Nis}(SmCor(k)))$, i.e. $C(f)$ is exact. Further $\mathrm{Ver}(C(f)) \cong C(\mathrm{Ver}(f))$ holds and $C(f)$ is exact if and only if $\mathrm{Ver}(C(f))$ is exact. Now $\mathrm{Ver}(f)$ is an isomorphism in $\mathcal{D}(Shv_{Nis}(SmCor(k)))$ if and only if $C(\mathrm{Ver}(F)) = 0$ in $\mathcal{D}(Shv_{Nis}(SmCor(k)))$. □

Later we will need the following

Lemma 3.2. *The induced functors*

$$\mathrm{Ver} : \mathcal{K}({}^G Shv_{\mathcal{T}}(Sm/k)) \longrightarrow \mathcal{K}(Shv_{\mathcal{T}}(Sm/k))$$

and

$$\mathrm{Ver} : \mathcal{K}({}^G Shv_{\mathcal{T}}(SmCor(k))) \longrightarrow \mathcal{K}(Shv_{\mathcal{T}}(SmCor(k)))$$

respect K-injective complexes.

Proof. Ver is right adjoint to an exact functor. □

Now we can prove

Theorem 3.3. *Let $X \in {}^G Sm/k$ and $A^\bullet \in \mathrm{Kom}^-({}^G Shv_{Nis}(SmCor(k)))$. Then one has*

$$\mathrm{Hom}_{\mathcal{D}({}^G Shv_{Nis}(SmCor(k)))}({}^G\mathbb{Z}_{tr}(X), A^\bullet) = \mathrm{Hom}_{\mathcal{D}({}^G Shv_{Nis}(Sm/k))}({}^G\mathbb{Z}(X), A^\bullet).$$

Proof. Both sides can be written via derived functors, namely

$$\mathrm{Hom}_{\mathcal{D}({}^G Shv_{Nis}(SmCor(k)))}({}^G\mathbb{Z}_{tr}(X), A^\bullet) = h^0(\mathbf{R}\mathrm{Hom}_{{}^G Shv_{Nis}(SmCor(k))}({}^G\mathbb{Z}_{tr}(X), \_)(A^\bullet))$$

and

$$\mathrm{Hom}_{\mathcal{D}({}^G Shv_{Nis}(Sm/k))}({}^G\mathbb{Z}(X), A^\bullet) = h^0(\mathbf{R}\mathrm{Hom}_{{}^G Shv_{Nis}(Sm/k)}({}^G\mathbb{Z}(X), \_)(A^\bullet)).$$

We have already factored the right hand sides through $\mathbf{R}\mathrm{Hom}_{\mathbb{Z}[G]-Mod}(\mathbb{Z}, \_)$, so we get the commutative diagram

$$\begin{tikzcd}
\mathcal{D}^-({}^G Shv_{Nis}(SmCor(k))) \arrow[d] \arrow[dr] & \\
\mathcal{D}(\mathbb{Z}[G]-Mod) \arrow[r] & \mathcal{D}(Ab) \\
\mathcal{D}({}^G Shv_{Nis}(Sm/k)). \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Therefore we have a natural morphism

$$\phi : \mathbf{R}\mathrm{Hom}_{Shv_{Nis}(Sm/k)}({}^G\mathbb{Z}(X), \_) \longrightarrow \mathbf{R}\mathrm{Hom}_{Shv_{Nis}(SmCor(k))}({}^G\mathbb{Z}_{tr}(X), \_).$$

Now it is obviously enough to show that $\mathrm{Ver}(\phi)$ is a quasi isomorphism. For that we look at the diagram

$$\begin{tikzcd}
\mathcal{D}^-({}^G Shv_{Nis}(SmCor(k))) \arrow[r, "\mathrm{Ver}"] \arrow[d, "\mathbf{R}\mathrm{Hom}_{Shv_{Nis}(SmCor(k))}({}^G\mathbb{Z}_{tr}(X),_)"'] & \mathcal{D}^-(Shv_{Nis}(SmCor(k))) \arrow[d, "\mathbf{R}\mathrm{Hom}_{Shv_{Nis}(SmCor(k))}(\mathbb{Z}_{tr}(X),_)"'] \\
\mathcal{D}(\mathbb{Z}[G]-Mod) \arrow[r, "\mathrm{Ver}"] & \mathcal{D}(Ab) \\
\mathcal{D}^-({}^G Shv_{Nis}(Sm/k)) \arrow[r, "\mathrm{Ver}"] \arrow[u, "\mathbf{R}\mathrm{Hom}_{Shv_{Nis}(Sm/k)}({}^G\mathbb{Z}(X),_)"] & \mathcal{D}^-(Shv_{Nis}(Sm/k)). \arrow[u, "\mathbf{R}\mathrm{Hom}_{Shv_{Nis}(Sm/k)}(\mathbb{Z}(X),_)"']
\end{tikzcd}$$

By lemma 3.2 the diagram is commutative and with the non-equivariant result [Voe00, 3.1.9] the claim follows. □

Definition 3.4. *A G-presheaf (or a sheaf) F with transfers is called* homotopy invariant, *if for all $X \in Sm/k$ the projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism $F(X) \to F(X \times \mathbb{A}^1)$.*

As in the non-equivariant situation the full sub-category of homotopy invariant G-sheaves with transfers forms an abelian sub-category and the inclusion is exact. Therefore we have

Corollary 3.5. *${}^G DM^{eff}(k) \subset \mathcal{D}^-({}^G Shv_{Nis}(SmCor(k)))$ is a triangulated sub-category.*

As usual we denote by $\Delta^\bullet$ in Sm/k the standard cosimplicial object, and for a G-presheaf F and a $U \in Sm/k$ we let $C_n(F)(U) = F(U \times \Delta^n)$. Then $C_*(F)$ is a simplicial G-presheaf with transfers. The associated chain complex is also denoted by $C_*(F)$ and we define $C^*(F)$ by $C^i(F) := C_{-i}(F)$, to write everything in cohomological terms. Because we have $C^*(\mathrm{Ver}(F)) = \mathrm{Ver}(C^*(F))$ and $h^i(\mathrm{Ver}(C^*(F))) = \mathrm{Ver}(h^i(C^*(F)))$ for an $F \in {}^G Shv_{Nis}(SmCor(k))$, we get, from the analogous non-equivariant result, the equivariant functor of motives

$$\begin{array}{rccc} {}^G\mathrm{M}: & {}^G Sm/k & \longrightarrow & {}^G DM^{eff}(k) \\ & X & \longmapsto & C^*({}^G\mathbb{Z}_{tr}(X)). \end{array}$$

As in the non-equivariant case one has the

Proposition 3.6. *Let $A^\bullet \in \mathrm{Kom}^-({}^G Shv_{Nis}(SmCor(k)))$ be a complex with homotopy-invariant cohomology. Then the natural embedding $A^\bullet \to C^*(A^\bullet)$ is a quasi-isomorphism.*

Therefore C^* induces a functor

$$C^* : \mathcal{K}^-({}^G Shv_{Nis}(SmCor(k))) \longrightarrow {}^G DM^{eff}(k).$$

Furthermore from the non-equivariant result and 3.1 we get

$$C^* : \mathcal{D}^-({}^G Shv_{Nis}(SmCor(k))) \longrightarrow {}^G DM^{eff}(k).$$

Analogously to the non-equivariant result (see [SV00]) one can prove the following

Theorem 3.7. *The functor*

$$C^* : \mathcal{D}^-({}^G Shv_{Nis}(SmCor(k))) \longrightarrow {}^G DM^{eff}(k)$$

is left adjoint to the canonical inclusion

$${}^G DM^{eff}(k) \longrightarrow \mathcal{D}^-({}^G Shv_{Nis}(SmCor(k)))$$

and is essentially surjective. Therefore the category ${}^G DM^{eff}(k)$ is the localization of $\mathcal{D}^-({}^G Shv_{Nis}(SmCor(k)))$ with respect to the thick sub-category of all complexes $A^\bullet \in \mathcal{D}^-({}^G Shv_{Nis}(SmCor(k)))$ with the property $C^*(A^\bullet) = 0$ in $\mathcal{D}^-({}^G Shv_{Nis}(SmCor(k)))$.

The functor Ver induces a natural morphism

$$\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^G\mathrm{M}(X), A^\bullet) \longrightarrow \mathrm{Hom}_{DM^{eff}(k)}(M(\mathrm{Ver}(X)), \mathrm{Ver}(A^\bullet)).$$

Now we can formulate the main result of this section.

Theorem 3.8. *For $X \in {}^G Sm/k$, $A^\bullet \in {}^G DM^{eff}(k)$ and with the notation of section 2, there is for all $i \in \mathbb{Z}$ a natural isomorphism*

$$H^i(G, X; A^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^G\mathrm{M}(X), A^\bullet[i]).$$

Furthermore the diagram

$$\begin{tikzcd} H^i(G, X; A^\bullet) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^G\mathrm{M}(X), A^\bullet[i]) \arrow[d] \\ H^i(\mathrm{Ver}(X), \mathrm{Ver}(A^\bullet)) \arrow[r, "\sim"] & \mathrm{Hom}_{DM(k)}(\mathrm{Ver}({}^G\mathrm{M}(X)), \mathrm{Ver}(A^\bullet)[i]) \end{tikzcd}$$

commutes.

Proof. By theorem 3.7 for all $i \in \mathbb{Z}$

$$\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^G\mathrm{M}(X), A^\bullet[i]) = \mathrm{Hom}_{\mathcal{D}^-({}^G Shv_{Nis}(SmCor(k)))}({}^G\mathbb{Z}_{tr}(X), A^\bullet[i])$$

holds and by theorem 3.3

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{D}^-({}^G Shv_{Nis}(SmCor(k)))}({}^G\mathbb{Z}_{tr}(X), A^\bullet[i]) \\ &\qquad = \mathrm{Hom}_{\mathcal{D}^-({}^G Shv_{Nis}(Sm/k))}({}^G\mathbb{Z}(X), A^\bullet[i]) \\ &\qquad\qquad = H^i(G, X; A^\bullet). \end{aligned}$$

The commutativity is clear by construction. □

Next we want to define a tensor product in ${}^G DM^{eff}(k)$. First we define as in [SV00], a tensor product in $\mathrm{Kom}^-({}^G Shv_{Nis}(SmCor(k)))$ such that for all $X, Y \in {}^G Sm/k$

$${}^G\mathbb{Z}_{tr}(X) \otimes {}^G\mathbb{Z}_{tr}(Y) = {}^G\mathbb{Z}_{tr}(X \times Y) \tag{2}$$

holds. We call directs sums of G-sheaves with transfers of the form ${}^G\mathbb{Z}_{tr}(X)$ with $X \in {}^G Sm/k$ free. For free G-sheaves with transfers we can define with equation 2 the tensor product and with the usual constructions also for bounded above complexes of free G-sheaves with transfers. Now for all sheaves with tranfers $\mathcal{F} \in {}^G Shv_{Nis}(SmCor(k))$ and $X \in Sm/k$

$$\mathcal{F}(X) = \mathrm{Hom}_{{}^G Shv_{Nis}(SmCor(k))}({}^G\mathbb{Z}_{tr}(X \times G), \mathcal{F})$$

holds, where $X \times G$ has via the second component the G-structure. Therefore we have a canonical surjection

$$\bigoplus_{X \in Sm/k} {}^G\mathbb{Z}_{tr}(X \times G) \otimes \mathcal{F}(X) \longrightarrow \mathcal{F}$$

and so we can build successive a free resolution $\chi^\bullet(A^\bullet) \to \mathcal{F}$ of $\mathcal{F}$. Doing this in the usual manner also with complexes we can define for $A^\bullet, B^\bullet \in \mathrm{Kom}^-({}^G Shv_{Nis}(SmCor(k)))$ by

$$A^\bullet \otimes^L B^\bullet := \chi^\bullet(A^\bullet) \otimes \chi^\bullet(B^\bullet)$$

a tensor product. Then as in [SV00] and [Voe00] is done for the non-equivariant case, one sees that this induces a tensor product on ${}^G DM^{eff}(k)$ and that ${}^G DM^{eff}(k)$ becomes a triangulated tensor category. Furthermore the functor

$$\mathrm{Ver} : {}^G DM^{eff}(k) \longrightarrow DM^{eff}(k)$$

respects the tensor structures.

Now we define the Tate objects $\mathbb{Z}(n)$ for all $n \in \mathbb{N}$. We define $\mathbb{Z} := \mathbb{Z}(0) := {}^{\mathrm{G}}\mathrm{M}(Spec(k))$ and obviously this is the neutral element with respect to the tensor product in ${}^G DM^{eff}(k)$. Let $\mathbb{Z}(1)$ be the complex in ${}^G DM^{eff}(k)$ which has only in degree zero the object $\mathcal{O}^*$ with trivial G-action. That works because of $\Gamma(X, \mathcal{O}^*_X) = \Gamma(X \times \mathbb{A}^1, \mathcal{O}^*_{X \times \mathbb{A}^1})$. Further we set $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$ and for an object $M \in {}^G DM^{eff}(k)$ and $n \in \mathbb{N}$

$$M(n) := M \otimes \mathbb{Z}(n).$$

We have obviously $\mathrm{Ver}(\mathbb{Z}(1)) = \mathbb{Z}(1)$ and therefore

$$\mathrm{Ver}({}^{\mathrm{G}}\mathrm{M}(X)(n)) = M(\mathrm{Ver}(X))(n).$$

3.2 First properties

As a corollary of the equivariant Hilbert 90 we have

Proposition 3.9. *Let $X \in {}^G Sm/k$. Then there is a natural isomorphism*

$$\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(X), \mathbb{Z}(1)[2]) \xrightarrow{\sim} Pic^G(X).$$

Furthermore the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Z), \mathbb{Z}(1)[2]) \arrow[r] \arrow[d] & Pic^G(X) \arrow[d] \\
\mathrm{Hom}_{DM(k)}(M(\mathrm{Ver}(X)), \mathbb{Z}(1)[2]) \arrow[r] & Pic(X)
\end{tikzcd}$$

commutes.

Proof. This follows from theorem 3.8 and theorem 2.8. ☐

If $\mathcal{E}$ is a locally free G-sheaf on a $X \in {}^G Sm/k$ then $\mathbb{P}(\mathcal{E})$ is again a G-scheme and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is a G-line bundle on it. For detail we see for example [Köc98].

Theorem 3.10. *Let $\mathcal{E}$ be a locally free G-sheaf on a G-scheme $X \in {}^G Sm/k$. Then there is a natural isomorphism*

$$ {}^{\mathrm{G}}\mathrm{M}(\mathbb{P}(\mathcal{E})) \xrightarrow{\sim} {}^{\mathrm{G}}\mathrm{M}(X) \oplus {}^{\mathrm{G}}\mathrm{M}(X)(1)[2] \oplus \cdots {}^{\mathrm{G}}\mathrm{M}(X)(n)[2n]. $$

Proof. The canonical bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ defines by 3.9 a morphism

$$ {}^{\mathrm{G}}\mathrm{M}(\mathbb{P}(\mathcal{E})) \longrightarrow \mathbb{Z}(1)[2]. $$

For $X, Y \in {}^G Sm/k$ the tensor product induces morphisms

$$ \begin{aligned} &\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(X), \mathbb{Z}(n)[i]) \otimes \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y), \mathbb{Z}(m)[j]) \\ &\qquad \longrightarrow \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(X \times Y), \mathbb{Z}(n+m)[i+j])) \end{aligned} \tag{3} $$

and together with the diagonal $\Delta : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E})$ we get a morphism

$$ {}^{\mathrm{G}}\mathrm{M}(\mathbb{P}(\mathcal{E})) \longrightarrow \mathbb{Z}(1)[2] \oplus \cdots \oplus \mathbb{Z}(n)[2n], $$

and finally, using the natural morphism $\mathbb{P}(\mathcal{E}) \to X$ the morphism

$$ \phi\ {}^{\mathrm{G}}\mathrm{M}(\mathbb{P}(\mathcal{E})) \xrightarrow{\sim} {}^{\mathrm{G}}\mathrm{M}(X) \oplus {}^{\mathrm{G}}\mathrm{M}(X)(1)[2] \oplus \cdots {}^{\mathrm{G}}\mathrm{M}(X)(n)[2n]. $$

By the commutative diagram of proposition3.9 and by construction, $\mathrm{Ver}(\phi)$ is just the analogous non-equivariant morphism. Because by [Voe00, 3.5.1] this is an isomorphism by proposition 3.1 ϕ is also an isomorphism. ☐

Remark 3.11. *This theorem might be quite surprising because ${}^{\mathrm{G}}\mathrm{M}(\mathbb{P}(\mathcal{E}))$ does not depend on the action of $\mathcal{E}$. But it is consistent with with the decomposition of the equivariant etale cohomology of the projective bundle in [Gro68, §2].*

Proposition 3.12 (Mayer-Vietoris sequence). *Let $X \in {}^G Sm/k$ and $U \cup V = X$ an equivariant Zariski-covering. Then there is an exact triangle*

$$ {}^{\mathrm{G}}\mathrm{M}(U \cap V) \to {}^{\mathrm{G}}\mathrm{M}(U) \oplus {}^{\mathrm{G}}\mathrm{M}(V) \to {}^{\mathrm{G}}\mathrm{M}(X) \to {}^{\mathrm{G}}\mathrm{M}(U \cap V)[1]. $$

Proof. The sequence

$$ 0 \to {}^G\mathbb{Z}_{tr}(U \cap V) \to {}^G\mathbb{Z}_{tr}(U) \oplus {}^G\mathbb{Z}_{tr}(V) \to {}^G\mathbb{Z}_{tr}(X) \to 0 $$

is exact because it is exact if one forgets the G-structure. Now one applys $C^*(\_)$ and the result follows. ☐

Using the same argument the following two theorems follow.

Proposition 3.13 (Homotopy invariance). *Let $\mathcal{E}$ a locally free G-sheaf on a G-scheme $X \in {}^G Sm/k$. Then $Y = \mathcal{S}pecS^*(\mathcal{E}^{\#})) \in {}^G Sm/k$ and the canonical morphism $p : Y \to X$ is equivariant and induces an isomorphism*

$$ {}^{\mathrm{G}}\mathrm{M}(p) : {}^{\mathrm{G}}\mathrm{M}(Y) \xrightarrow{\sim} {}^{\mathrm{G}}\mathrm{M}(X). $$

Theorem 3.14 (Exact triangle for blow-ups). *Let $X, Y \in {}^G Sm/k$ and $Y \hookrightarrow X$ an equivariant closed immersion. Then the blow-up $p : B_Y(X) \to X$ of X in Y is again equivariant and there is a natural exact triangle*

$$ {}^{\mathrm{G}}\mathrm{M}(p^{-1}(Y)) \to {}^{\mathrm{G}}\mathrm{M}(Y) \oplus {}^{\mathrm{G}}\mathrm{M}(B_Y(X)) \to {}^{\mathrm{G}}\mathrm{M}(X) \to {}^{\mathrm{G}}\mathrm{M}(p^{-1}(Y))[1]. $$

3.3 Motives of schemes of finite type

As in [Voe00] we define covariant functors

$$^{G}\mathbb{Z}_{tr}(\_) : {}^{G}Sch/k \longrightarrow {}^{G}Shv_{Nis}(SmCor(k))$$

and

$$^{G}\mathbb{Z}^{c}_{tr}(\_) : {}^{G}Sch/k^{prop} \longrightarrow {}^{G}Shv_{Nis}(SmCor(k)).$$

where we denote by $^{G}Sch/k^{prop}$ the category of schemes of finite type over k with proper morphisms and with

$$^{G}\mathbb{Z}^{c}_{tr}(X)(U) := \mathbb{Z}\left[\left\{ Z \hookrightarrow X \times U \,\middle|\, \begin{array}{c} Z \text{ is an integral sub-scheme of } X \times U, \\ \text{which is quasi finite and dominant over} \\ \text{a component of } U \end{array} \right\}\right].$$

Again using

$$C^{*} : {}^{G}Shv_{Nis}(SmCor(k)) \longrightarrow {}^{G}DM^{eff}(k)$$

we get functors

$$^{\mathrm{G}}\mathrm{M} : {}^{G}Sch/k \longrightarrow {}^{G}DM^{eff}(k)$$

and

$$^{\mathrm{G}}\mathrm{M}^{\mathrm{c}} : {}^{G}Sch/k^{prop} \longrightarrow {}^{G}DM^{eff}(k).$$

As in proposition 3.12 together with [Voe00, 4.1.1] one proves

Proposition 3.15 (Mayer-Vietoris sequence). *Let $X \in {}^{G}Sch/k$ and $U \cup V = X$ an equivariant Zariski-covering. Then there is a natural exact triangle*

$$^{\mathrm{G}}\mathrm{M}(U \cap V) \to {}^{\mathrm{G}}\mathrm{M}(U) \oplus {}^{\mathrm{G}}\mathrm{M}(V) \to {}^{\mathrm{G}}\mathrm{M}(X) \to {}^{\mathrm{G}}\mathrm{M}(U \cap V)[1].$$

From now on we need resolution of singularities of the ground field k. But here we still do not need an equivariant resolution.

Proposition 3.16. *Let k be a field which admits resolution of singularities, $X \in {}^{G}Sch/k$ and $Z \hookrightarrow X$ a closed G-equivariant sub-scheme. Then there is a natural exact triangle*

$$^{\mathrm{G}}\mathrm{M}^{\mathrm{c}}(Z) \longrightarrow {}^{\mathrm{G}}\mathrm{M}^{\mathrm{c}}(X) \longrightarrow {}^{\mathrm{G}}\mathrm{M}^{\mathrm{c}}(X - Z) \longrightarrow {}^{\mathrm{G}}\mathrm{M}^{\mathrm{c}}(Z)[1].$$

Proof. The sequence

$$0 \longrightarrow {}^{G}\mathbb{Z}^{c}_{tr}(Z) \longrightarrow {}^{G}\mathbb{Z}^{c}_{tr}(X) \longrightarrow {}^{G}\mathbb{Z}^{c}_{tr}(X - Z) \longrightarrow 0$$

is exact because

$$0 \longrightarrow \mathbb{Z}^{c}_{tr}(Z) \longrightarrow \mathbb{Z}^{c}_{tr}(X) \longrightarrow \mathbb{Z}^{c}_{tr}(X - Z) \longrightarrow 0$$

is exact. The proposition follows by applying C^{*}. □

Proposition 3.17. *Let k be a field which admits resolution of singularities and $X, Y \in {}^{G}Sch/k$. Then there are natural isomorphisms*

$$^{\mathrm{G}}\mathrm{M}(X \times Y) \xrightarrow{\sim} {}^{\mathrm{G}}\mathrm{M}(X) \otimes {}^{\mathrm{G}}\mathrm{M}(Y)$$

and

$$^{\mathrm{G}}\mathrm{M}^{\mathrm{c}}(X \times Y) \xrightarrow{\sim} {}^{\mathrm{G}}\mathrm{M}^{\mathrm{c}}(X) \otimes {}^{\mathrm{G}}\mathrm{M}^{\mathrm{c}}(Y).$$

Proof. This follows from the analogous non equivariant statement [Voe00, 4.1.7] and by proposition 3.1 and the compatibility of Ver and the tensor product. $\square$

Now theorem 3.10 generalized with [Voe00, 4.1.11] to

Proposition 3.18. *Let k be a field which admits resolution of singularities, $X \in {}^G Sch/k$ a G-scheme and $\mathcal{E}$ a locally free G-sheaf on a G-scheme $X \in {}^G Sch/k$. Then there is a natural isomorphism*

$$^{\mathrm{G}}\mathrm{M}(\mathbb{P}(\mathcal{E})) \xrightarrow{\sim} {}^{\mathrm{G}}\mathrm{M}(X) \oplus {}^{\mathrm{G}}\mathrm{M}(X)(1)[2] \oplus \cdots {}^{\mathrm{G}}\mathrm{M}(X)(n)[2n].$$

As a corollary of proposition 3.16 and proposition 3.17 we have

Corollary 3.19. *Let k be a field which admits resolution of singularities, $X \in {}^G Sch/k$ and $\mathbb{A}^1$ with an arbitrary G-structure. Then*

$$^{\mathrm{G}}\mathrm{M}(X \times \mathbb{A}^1) = {}^{\mathrm{G}}\mathrm{M}(X)$$

holds. Furthermore for $\mathbb{A}^1$ with trivial G-structure

$$^{\mathrm{G}}\mathrm{M}^{\mathrm{c}}(X \times \mathbb{A}^1) = {}^{\mathrm{G}}\mathrm{M}^{\mathrm{c}}(X)(1)[2]$$

holds. In particular we have ${}^{\mathrm{G}}\mathrm{M}^{\mathrm{c}}(\mathbb{A}^n) = \mathbb{Z}(n)[2n]$ where G acts trivially on $\mathbb{A}^n$.

As a generalization of theorem 3.8 one sees using [Voe00, 4.1.10]

Theorem 3.20. *Let k be a field which admits resolution of singularities, $X \in {}^G Sch/k$, $A^\bullet \in {}^G DM^{eff}(k)$ and $(A^\bullet)^\sim_{cdh}$ the sheafification of $A^\bullet$ with respect to the cdh-topology. Then there is for each $i \in \mathbb{Z}$ a canonical isomorphism*

$$H^i_{cdh}(G, X; (A^\bullet)^\sim_{cdh}) \xrightarrow{\sim} \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(X), A^\bullet).$$

In particular for $X \in {}^G Sm/k$ and $i \in \mathbb{Z}$ there is a natural isomorphism

$$H^i_{Nis}(G, X; A^\bullet) \xrightarrow{\sim} H^i_{cdh}(G, X; (A^\bullet)^\sim_{cdh}).$$

3.4 Connection to Chow groups

First we want to recall the following sheaf with transfers

$$z_{equi}(X, r) : SmCor(k) \longrightarrow Ab,$$

which is for $U \in Sm/k$ defined by

$$z_{equi}(X,r)(U) := \mathbb{Z}\left[\left\{ Z \hookrightarrow X \times U \,\middle|\, \begin{array}{c} Z \text{ is integer sub-scheme of } X \times U, \\ \text{which is equidimensional of relative} \\ \text{dimension r over } U \end{array} \right\}\right].$$

For details see [Voe00]. If $X \in {}^G Sch/k$ then $z_{equi}(X, r) \in {}^G Shv_{Nis}(SmCor(k))$ and $z_{equi}(X, 0) = {}^G\mathbb{Z}^c_{tr}(X)$. These sheaves with transfers are covariant with respect to proper morphisms and contravariant with respect to flat equidimensional morphisms. Furthermore we have for a $V \in {}^G Sm/k$ and $X \in {}^G Sch/k$ the G-sheaf with transfers

$$z_{equi}(V, X, r) = z_{equi}(X, r)(V \times \_).$$

Here for an $U \in Sm/k$ we have

$$z_{equi}(V, X, r)(U) = \mathrm{Hom}_{Shv_{Nis}(SmCor(k))}({}^G\mathbb{Z}_{tr}(V \times U), z_{equi}(X, r))$$

and the G-structure is defined by $(gf)(s) = gf(g^{-1}s)$ for all $g \in G$, $f \in z_{equi}(V, X, r)(U)$ and a section s. The G-structure of $V \times U$ is defined via the first factor. We collect in the following theorem the results of the bivariant cycle theory of Friedlander and Voevodsky.

Theorem 3.21. *Let k be a field which admits resolution of singularities, $X, Y \in {}^G Sch/k$, $i \in \mathbb{Z}$ and $r \in \mathbb{N}$.*

(i) Let $U \in {}^G Sm/k$ be equidimensional of dimension n. Then there is a natural isomorphism

$$\begin{aligned}&\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y \times U)[i], C^*(z_{equi}(X, r)))\\ &\quad\xrightarrow{\sim} \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y)[i], C^*(z_{equi}(U \times X, r+n))).\end{aligned} \tag{4}$$

(ii) The projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism

$$\begin{aligned}&\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y)[i], C^*(z_{equi}(X, r)))\\ &\quad\xrightarrow{\sim} \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y)[i], C^*(z_{equi}(X \times \mathbb{A}^1, r+1))).\end{aligned} \tag{5}$$

(iii) Let $p : X \times \mathbb{P}^1 \to X$ and $i : X \to X \times \mathbb{P}^1$ be the projection and the embedding. Then the morphism

$$i_* \oplus p^* : z_{equi}(X, r+1) \oplus z_{equi}(X, r) \to z_{equi}(X \times \mathbb{P}^1, r+1)$$

induced an isomorphism

$$\begin{aligned}&\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y)[i], C^*(z_{equi}(X, r+1)))\oplus\\ &\qquad \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y)[i], C^*(z_{equi}(X, r)))\\ &\quad\xrightarrow{\sim} \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y)[i], C^*(z_{equi}(X \times \mathbb{P}^1, r+1))).\end{aligned}$$

(iv) There is a natural isomorphism

$$\begin{aligned}&\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y \times \mathbb{P}^1)[i], C^*(z_{equi}(X, r))) \xrightarrow{\sim}\\ &\qquad \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y)[i], C^*(z_{equi}(X, r)))\oplus\\ &\qquad\qquad \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y)[i], C^*(z_{equi}(X, r+1))).\end{aligned}$$

Here the projection

$$\begin{aligned}&\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y \times \mathbb{P}^1)[i], C^*(z_{equi}(X, r))) \to\\ &\qquad \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y)[i], C^*(z_{equi}(X, r)))\end{aligned}$$

is induces by the embedding $Y \to Y \times \mathbb{P}^1$ and the embedding

$$\begin{aligned}&\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y)[i], C^*(z_{equi}(X,r))) \to \\ &\qquad \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y \times \mathbb{P}^1)[i], C^*(z_{equi}(X,r)))\end{aligned}$$

is induced by the projection $Y \times \mathbb{P}^1 \to Y$.

Proof. We show how (i) follows from the non-equivariant result. The claims (ii) and (iii) one shows in the same way and (iv) follows from (i) and (iii).

By theorem 3.20 we have

$$\begin{aligned}&\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y \times U)[i], C^*(z_{equi}(X,r))) \\ &\qquad = h^i(\mathbf{R}\mathrm{Hom}_{{}^G Shv_{cdh}(\mathrm{Sm}/k)}({}^G\mathbb{Z}(Y \times U), (C^*(z_{equi}(X,r)))^{\sim}_{cdh})),\end{aligned}$$

and there is a canonical morphism

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}_{{}^G Shv_{cdh}(\mathrm{Sm}/k)}({}^G\mathbb{Z}(Y), (C^*(z_{equi}(U,X,r)))^{\sim}_{cdh}) \\ &\qquad \longrightarrow \mathbf{R}\mathrm{Hom}_{{}^G Shv_{cdh}(\mathrm{Sm}/k)}({}^G\mathbb{Z}(Y \times U), (C^*(z_{equi}(X,r)))^{\sim}_{cdh}).\end{aligned}$$

Without a G this is an isomorphism by [FV00, 8.2]. So also here for the same reason

$$(C^*(z_{equi}(U,X,r)))^{\sim}_{cdh} \xrightarrow{\sim} (C^*(z_{equi}(U \times X, r+n)))^{\sim}_{cdh}$$

is an isomorphism. So (i) follows. □

We come to important corollaries.

Corollary 3.22. *Let k be a field which admits resolution of singularities, $X, Y \in {}^G Sm/k$ proper with X of pure dimension n. Then we have*

$$\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(X), {}^{\mathrm{G}}\mathrm{M}(Y)) = H^0(G, z_{equi}(X \times Y, n)(\Delta^\bullet)).$$

Proof. By the properness of Y we have

$$\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(X), {}^{\mathrm{G}}\mathrm{M}(Y)) = \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(X), C^*(z_{equi}(Y,0))).$$

So by theorem 3.21 (i)

$$\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(X), {}^{\mathrm{G}}\mathrm{M}(Y)) = \mathrm{Hom}_{{}^G DM^{eff}(k)}(\mathbb{Z}, C^*(z_{equi}(X \times Y, n)))$$

and by theorem 3.8 the claims follows. □

By theorem 3.21 (iv) we have the following, as in [Voe00, 4.2.3].

Proposition 3.23. *Let k be a field which admits resolution of singularities and $X, Y \in {}^G Sch/k$. Then there is for each $r \in \mathbb{N}$ and $i \in \mathbb{Z}$ a natural isomorphism*

$$\begin{aligned}&\mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y)(r)[2r+i], {}^{\mathrm{G}}\mathrm{M}^c(X)) \xrightarrow{\sim} \\ &\qquad \mathrm{Hom}_{{}^G DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(Y)[i], C^*(z_{equi}(X,r)))\end{aligned}$$

Now we come to the connection to Bloch's cycle complexes. We denote by $z^n_{Bl}(X,*)$ the complex of [Blo86] with calculates the higher Chow groups. To do everything in cohomological term we define

$$^k z^n_{Bl}(X,i) = z^n_{Bl}(X,-i).$$

For a G-scheme X the complex $^k z^n_{Bl}(X,*)$ is automatically a complex of $\mathbb{Z}[G]$-modules. Then we can write the Borel-Moore homology of our theory as follows.

Proposition 3.24. *Let k be a field which admits resolution of singularities and X an equidimensional quasi-projective scheme over k of dimension n. Then there is for each $i,j \in \mathbb{Z}$ a natural isomorphism*

$$H^{2i-j}(G,\ ^k z^{n-i}_{Bl}(X,*)) \xrightarrow{\sim} \begin{cases} \mathrm{Hom}_{^G DM^{eff}(k)}(\mathbb{Z}(i)[j],\ ^{\mathrm{G}}\mathrm{M}^c(X)) \ \textit{for } i \geq 0 \\ \mathrm{Hom}_{^G DM^{eff}(k)}(\mathbb{Z},\ ^{\mathrm{G}}\mathrm{M}^c(X)(-i)[-j]) \ \textit{for } i \leq 0. \end{cases}$$

Proof. First we prove it for $i \geq 0$. By proposition 3.23 and theorem 3.20 we have

$$\begin{aligned} \mathrm{Hom}_{^G DM^{eff}(k)}(\mathbb{Z}(i)[j],\ ^{\mathrm{G}}\mathrm{M}^c(X)) &= \mathrm{Hom}_{^G DM^{eff}(k)}(\mathbb{Z}[j-2i], C^*(z_{equi}(X,i))) \\ &= H^{2i-j}_{cdh}(Spec(k), G; (C^*(z_{equi}(X,i)))^{\sim}_{cdh}) \\ &= H^{2i-j}(G, C^*(z_{equi}(X,i))(Spec(k))). \end{aligned}$$

Now the natural morphism (comp. [Voe00, 4.2.9])

$$\Psi : C^*(z_{equi}(X,i))(Spec(k)) \longrightarrow\ ^k z^{n-i}_{Bl}(X,*)$$

is a morphism of complexes of $\mathbb{Z}[G]$-modules and an isomorphism because it is one in the non-equivariant case. So the claim follows. For $i \leq 0$ we identify using corollary 3.19

$$\begin{aligned} \mathrm{Hom}_{^G DM^{eff}(k)}(\mathbb{Z},\ ^{\mathrm{G}}\mathrm{M}^c(X)(-i)[-j]) &= \mathrm{Hom}_{^G DM^{eff}(k)}(\mathbb{Z}[j-2i],\ ^{\mathrm{G}}\mathrm{M}^c(X)(-i)[-2i]) \\ &= \mathrm{Hom}_{^G DM^{eff}(k)}(\mathbb{Z}[j-2i],\ ^{\mathrm{G}}\mathrm{M}^c(X \times \mathbb{A}^{-i})) \end{aligned}$$

Then one argues as in the first case and uses the homotopy invariance of the higher Chow groups. □

Now we define the category $^G DM(k)$ as the localization of $^G DM^{eff}(k)$ with respect to the functor $\_ \otimes \mathbb{Z}(1)$. That means the objects of $^G DM(k)$ are pairs (A,n) with $A \in\ ^G DM^{eff}(k)$ and $n \in \mathbb{Z}$ and morphism for $(A,n),(B,m) \in\ ^G DM(k)$ are defined by

$$\mathrm{Hom}_{^G DM(k)}((A,n),B,m)) := \lim_{k \geq -m,-n} \mathrm{Hom}_{^G DM^{eff}(k)}(A(k+n), B(k+m)).$$

$^G DM(k)$ is again a triangulated category and the following lemma shows that the tensor product continuous to */gdme*.

Lemma 3.25. *The permutation on $\mathbb{Z}(1) \otimes \mathbb{Z}(1)$ induce the identity in $^G DM^{eff}(k)$.*

Proof. Clear with the non-equivariant result [Voe00, 2.1.5]. □

Whether the functor

$$^G DM^{eff}(k) \longrightarrow\ ^G DM(k)$$

is a full embedding is not clear. For that one would need an equivariant resolution of singularities. But if one assumes only resolution of singularities for the field k one easily sees as in the non equivariant theory that for proper $X, Y \in {}^{G}Sm/k$ the morphism

$$\_ \otimes id_{\mathbb{Z}(1)} : \operatorname{Hom}_{{}^{G}DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(X)[i], {}^{\mathrm{G}}\mathrm{M}(Y)) \longrightarrow \operatorname{Hom}_{{}^{G}DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(X)(1)[i], {}^{\mathrm{G}}\mathrm{M}(Y)(1))$$

is an isomorphism.

We end this section with a definition of equivariant motivic cohomology.

Definition 3.26 (equivariant motivic cohomology). *Let $A = \mathbb{Z}, \mathbb{Z}/n$ or $\mathbb{Q}$ and $A(i) := \mathbb{Z}(i) \otimes A$ for all $i \in \mathbb{N}$. We define equivariant motivic cohomology with coefficients in A of a scheme $X \in {}^{G}Sch/k$ as*

$${}^{G}H^{i}_{\mathcal{M}}(X, A(j)) := \begin{cases} \operatorname{Hom}_{{}^{G}DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(X), A(j)[i]) \text{ for } j \geq 0 \\ \operatorname{Hom}_{{}^{G}DM^{eff}(k)}({}^{\mathrm{G}}\mathrm{M}(X)(-j)[-i], A) \text{ for } j \leq 0. \end{cases}$$

If $X \in {}^{G}Sch/k$ smooth and proper we have by the above remark

$${}^{G}H^{i}_{\mathcal{M}}(X, A(j)) = \operatorname{Hom}_{{}^{G}DM(k)}({}^{\mathrm{G}}\mathrm{M}(X), A(j)[i])$$

for all $i, j \in \mathbb{Z}$.

The properties of ${}^{G}DM^{eff}(k)$ show that motivic cohomology is homotopy-invariant, has a long Mayer-Vietoris and blow-up sequence and the projective bundle theorem holds.

4 Supplements

4.1 Coefficients

In the same way one can construct a triangulated category of equivariant motives with rational coefficients. For that one starts with the category

$${}^{G}Shv^{\mathbb{Q}}_{Nis}(SmCor(k))$$

of Nisnevich-sheaves with transfers and values in the category of $\mathbb{Q}$-vectors paces. Then again ${}^{G}DM^{eff}(k, \mathbb{Q})$ is the full sub-category of $\mathcal{D}^{-}({}^{G}Shv^{\mathbb{Q}}_{Nis}(SmCor(k)))$ of objects with homotopy-invariant cohomology. Again we have functors

$$\begin{array}{rccc} {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M} : & Sch/k & \longrightarrow & {}^{G}DM^{eff}(k, \mathbb{Q}) \\ & X & \longmapsto & C^{*}(\mathbb{Z}_{tr}(X) \otimes \mathbb{Q}) \end{array}$$

and

$$\begin{array}{rccc} {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}^{c} : & Sch/k^{prop} & \longrightarrow & {}^{G}DM^{eff}(k, \mathbb{Q}) \\ & X & \longmapsto & C^{*}(z_{equi}(X, 0) \otimes \mathbb{Q}). \end{array}$$

One can prove most properties in the same way. We only want to mention the following

Proposition 4.1. *Let k be a field which admits resolution of singularities and X an equidimensional quasi-projective scheme over k of dimension n. Then there is for all $i, j \in \mathbb{Z}$ a natural isomorphism*

$$H^{2i-j}(G, {}^{k}z^{n-i}_{Bl}(X, *) \otimes \mathbb{Q}) \xrightarrow{\sim} \begin{cases} \operatorname{Hom}_{{}^{G}DM^{eff}(k, \mathbb{Q})}(\mathbb{Q}(i)[j], {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}^{c}(X)) \text{ for } i \geq 0 \\ \operatorname{Hom}_{{}^{G}DM^{eff}(k, \mathbb{Q})}(\mathbb{Q}, {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}^{c}(X)(-i)[-j]) \text{ for } i \leq 0. \end{cases}$$

4.2 Etale version

For that we have to restrict for technical reasons to the case where the field k has finite etale cohomological dimension (comp. [Voe00, 3.3]). But then we get in the same way the category ${}^{G}DM^{eff}_{ét}(k, A)$ for $A = \mathbb{Z}$ or $\mathbb{Q}$ if we replace the Nisnevich topology by etale topology. Sheafification gives a functor

$$ {}^{G}DM^{eff}(k, A) \longrightarrow {}^{G}DM^{eff}_{ét}(k, A). $$

4.3 Restriction and transfer maps

If the group G varies one can define as usual restriction and transfer maps.

5 Comparison with other equivariant theories

In this section we want to establish connections between the triangulated category of equivariant motives we have just defined and other equivariant theories.

In the first part we show that for finite groups our category rationally contains the category of equivariant Chow motives of Laterveer as a full subcategory.

In the second part, we show that, in an appropriate situation, there is a strong connection with the equivariant Borel cohomology in topology.

5.1 Connection with equivariant Chow motives

In [Lat98] Laterveer constructs for an algebraic closed field k and an algebraic group a category of equivariant Chow motives with using the equivariant Chow groups of Edidin and Graham.

Let k be an algebraically closed field with admits resolution of singularities and G a finite discrete group.

To avoid equivariant resolution of singularities let $\mathcal{V}_G$ be the category of smooth projective varieties over k. Edidin and Graham construct in [EG98] equivariant Chow groups

$$ A^*_G : \mathcal{V}_G \longrightarrow Ab $$

and equivariant higher Chow groups

$$ CH^*_G(\_, *) : \mathcal{V}_G \longrightarrow Ab. $$

For details and properties see [EG98]. We only look at rational coefficients. The connection to our theory is given by the following.

Proposition 5.1. *(i) For all $X \in \mathcal{V}_G$ and all $i \in \mathbb{Z}$ there is a natural isomorphism*

$$ A^{dim(X)-i}_G(X) \xrightarrow{\sim} \mathrm{Hom}_{{}^{G}DM^{eff}(k,\mathbb{Q})}(\mathbb{Q}(i)[2i], {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X)). $$

(ii) For all $X \in \mathcal{V}_G$ with linear G-operation and quasi-projective quotient and all $i, j \in \mathbb{Z}$ there is a natural isomorphism

$$ CH^{dim(X)-i}_G(X, j) \xrightarrow{\sim} \mathrm{Hom}_{{}^{G}DM^{eff}(k,\mathbb{Q})}(\mathbb{Q}(i)[2i + j], {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X)). $$

Proof. By [EG98] proposition 13 for $X \in \mathcal{V}_G$

$$CH^i_G(X,0) = h_0((z^i_{Bl}(X,*) \otimes \mathbb{Q})^G)$$

and for (ii)

$$CH^i_G(X,j) = h_j((z^i_{Bl}(X,*) \otimes \mathbb{Q})^G).$$

By the finiteness of G, the invariant functor is exact on the category of $\mathbb{Q}[G]$-modules. Therefore the assertion follows from proposition 3.24 resp. proposition 4.1. $\square$

We now, just briefly recall the construction of [Lat98]. For $X, Y \in \mathcal{V}_G$, G-correspondences of degree $r \in \mathbb{Z}$ for an irreducible X are defined by

$$Corr^r_G(X,Y) := A_G^{dim X + r}(X \times Y).$$

In the usual manner one defines a product on G-correspondences. Then the category $\mathcal{M}_G$ of equivariant Chow motives is the category of triples (X,p,m) with $X \in \mathcal{V}_G$, $p \in Corr^0_G(X,X)$ with $p^2 = p$ and $m \in \mathbb{Z}$ with

$$\mathrm{Hom}_{\mathcal{M}_G}((X,p,m),(Y,q,n)) = qCorr^{n-m}_G(X,Y)p \subset Corr^{n-m}_G(X,Y)$$

as morphisms for $(X,p,m),(Y,q,n) \in \mathcal{M}_G$. The tensor product for two objects $(X,p,m),(Y,q,n) \in \mathcal{M}_G$ is defined by

$$(X,p,m) \otimes (Y,q,n) := (X \times Y, p \times q, m+n).$$

One gets a contravariant functor

$$\begin{array}{rccc} h_G: & \mathcal{V}_G & \longrightarrow & \mathcal{M}_G \\ & X & \longmapsto & (X, \mathrm{id}_X, 0), \end{array}$$

and $\mathbb{L} = (Spec(k), id, -1)$ is called the Lefschetz motive. By an easy calculation one shows

Lemma 5.2. *Let A and B be objects of a pseudo abelian category $\mathcal{A}$, p a projector on A and q a projector on B. Then*

$$\mathrm{Hom}_{\mathcal{A}}(Im(p), Im(q)) = q\,\mathrm{Hom}_{\mathcal{A}}(A,B)p$$

holds.

Further we need

Lemma 5.3. *The category ${}^G DM^{eff}(k,\mathbb{Q})$ is pseudo-abelian.*

Proof. The category $\mathcal{D}({}^G Shv^{\mathbb{Q}}_{Nis}(SmCor(k)))$ is by [Lev98, II,2.4.9 (iii)] pseudo abelian. So if $A \in {}^G DM^{eff}(k,\mathbb{Q})$ and p is a projector on A then $\ker p$ exists in $\mathcal{D}({}^G Shv^{\mathbb{Q}}_{Nis}(SmCor(k)))$ and $\ker p \oplus \ker(1-p) \to A$ is an isomorphism. Now $\mathrm{Ver}(\ker p) = \ker(\mathrm{Ver}(p))$ holds and because $DM^{eff}(k,\mathbb{Q})$ is pseudo abelian $\mathrm{Ver}(\ker p) \in DM^{eff}(k,\mathbb{Q})$ follows. Therefore $\ker p$ and $\ker(1-p)$ are in ${}^G DM^{eff}(k,\mathbb{Q})$. $\square$

The main step is now the following.

Proposition 5.4. *For all $X, Y \in \mathcal{V}_G$ and all $m, n \in \mathbb{Z}$ there is a natural isomorphism*

$$\mathrm{Hom}_{^G DM(k,\mathbb{Q})}({}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X)(m)[2m], {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(Y)(n)[2n]) \xrightarrow{\sim} A_G^{dim(Y)+n-m}(X \times Y).$$

Proof. First let $m \geq n$. Then we have

$$\begin{aligned}
&\mathrm{Hom}_{^G DM(k,\mathbb{Q})}({}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X)(m)[2m], {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(Y)(n)[2n]) \\
&\quad = \mathrm{Hom}_{^G DM^{eff}(k,\mathbb{Q})}({}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X), {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(Y)(n-m)[2n-2m]) \\
&\quad = \mathrm{Hom}_{^G DM^{eff}(k,\mathbb{Q})}({}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X), {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}^c(Y \times \mathbb{A}^{n-m})) \\
&\quad = \mathrm{Hom}_{^G DM^{eff}(k,\mathbb{Q})}({}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X), C^*(z_{equi}(Y \times \mathbb{A}^{n-m}, 0) \otimes \mathbb{Q})) \\
&\quad = \mathrm{Hom}_{^G DM^{eff}(k,\mathbb{Q})}(\mathbb{Q}, C^*(z_{equi}(X \times Y \times \mathbb{A}^{n-m}, dim(X)) \otimes \mathbb{Q})).
\end{aligned}$$

The first equality holds by the remark at the end of section 3.4, the second by corollary 3.19, the third one by definition and the last one by theorem 3.21 (i). Further it follows from proposition 3.23 that

$$\begin{aligned}
&\mathrm{Hom}_{^G DM^{eff}(k,\mathbb{Q})}(\mathbb{Q}, C^*(z_{equi}(X \times Y \times \mathbb{A}^{n-m}, dim(X)) \otimes \mathbb{Q})) = \\
&\qquad \mathrm{Hom}_{^G DM^{eff}(k,\mathbb{Q})}(\mathbb{Q}(dim(X))[2dim(X)], {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}^c(X \times Y \times \mathbb{A}^{n-m})),
\end{aligned}$$

and by proposition 5.1 and by the homotopy invariance of the equivariant Chow groups

$$\begin{aligned}
&\mathrm{Hom}_{^G DM^{eff}(k,\mathbb{Q})}(\mathbb{Q}(dim(X))[2dim(X)], {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}^c(X \times Y \times \mathbb{A}^{n-m})) = \\
&\qquad A_G^{dim(Y)+n-m}(X \times Y \times \mathbb{A}^{n-m}) = A_G^{dim(Y)+n-m}(X \times Y).
\end{aligned}$$

For $m \leq n$ we have by a similar argument (comp. [Voe00])

$$\begin{aligned}
&\mathrm{Hom}_{^G DM(k,\mathbb{Q})}({}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X)(m)[2m], {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(Y)(n)[2n]) \\
&\quad = \mathrm{Hom}_{^G DM^{eff}(k,\mathbb{Q})}({}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X)(m-n)[2m-2n], {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(Y)) \\
&\quad = \mathrm{Hom}_{^G DM^{eff}(k,\mathbb{Q})}({}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X), C^*(z_{equi}(Y, m-n) \otimes \mathbb{Q})).
\end{aligned}$$

Then continue as in the first case. □

Now we can prove the main result of this section.

Theorem 5.5. *There is a contravariant fully faithful functor*

$$F : \mathcal{M}_G \longrightarrow {}^G DM(k, \mathbb{Q}),$$

which is compatible with the tensor structures.

Proof. First we define the functor on objects. For that let $(X, p, m) \in \mathcal{M}_G$. By proposition 5.4 p also defines a projector on ${}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X)$. By lemma 5.3 we can define

$$F((X, p, m)) := Bild(p : {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X) \to {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X)) \otimes \mathbb{Q}(-m)[-2m]$$

. For $(X,p,m),(Y,q,n) \in \mathcal{M}_G$ by lemma 5.2 and proposition 5.4, we have

$$\mathrm{Hom}_{{}^G DM(k,\mathbb{Q})}(F((X,p,m)),F((Y,q,n))) =$$
$$q \otimes Id_{\mathbb{Q}(-n)[-2n]} \mathrm{Hom}({}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(X)(-m)[-2m], {}^{\mathrm{G}}_{\mathbb{Q}}\mathrm{M}(Y)(-n)[-2n]) p \otimes Id_{\mathbb{Q}(-m)[-2m]} =$$
$$qA_G^{dim(Y)+m-n}(X \times Y)p.$$

By the construction it follows that this isomorphism is compatible with composition and the tensor structure. ☐

5.2 Connection with equivariant Borel cohomology

In this part we compare for $k = \mathbb{C}$ etale equivariant motivic cohomology with coefficients in $\mathbb{Z}/n$ with equivariant Borel cohomology from topology.

For that we first review some facts from topology. Let G be a discrete group. Then there is a contractable topological space E_G with has a free G-operation. The quotient E_G/G is denoted by B_G and is called classifying space of G.

For a topological G-space X one defines for an abelian group A by

$${}^B_G H^i(X,A) := H^i_{sing}(X \times_G E_G, A)$$

the equivariant Borel cohomology where the right hand side denotes usual singular cohomology. To express this without the use of E_G we first observe that the singular chain complex $C_*(X)$ is a complex of $\mathbb{Z}[G]$-modules.

We have the following theorem which is well known in topology.

Theorem 5.6. *Let X be a topological G-space and A an abelian group. Then there is a natural isomorphism*

$${}^B_G H^i(X,A) \xrightarrow{\sim} h^i(\mathbf{R}\mathrm{Hom}_{\mathbb{Z}[G]}(C_*(X),A)).$$

For the proof we need the following lemma from [Wei94].

Lemma 5.7. *In the above situation $C_*(E_G \times X)$ is a free $\mathbb{Z}[G]$–module and there is a natural isomorphism*

$$C_*(E_G \times X) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \xrightarrow{\sim} C_*(E_G \times_G X).$$

Proof. [Wei94, Lemma 6.10.2]. ☐

Proof. By the contractability of E_G

$$C_*(E_G \times X) \longrightarrow C_*(X)$$

is a quasi isomorphism of $\mathbb{Z}[G]$–modules. Further, by the above lemma $C_*(E_G \times X)$ is a free $\mathbb{Z}[G]$–module therefore K-projective . So we have

$$\mathbf{R}\mathrm{Hom}_{\mathbb{Z}[G]}(C_*(X),A) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathbb{Z}[G]}(C_*(E_G \times X),A)$$
$$= \mathrm{Hom}_{\mathbb{Z}[G]}(C_*(E_G \times X),A) = \mathrm{Hom}_{\mathbb{Z}}(C_*(E_G \times X) \otimes_{\mathbb{Z}[G]} \mathbb{Z}, A).$$

Again by the above lemma we have

$$\mathrm{Hom}_{\mathbb{Z}}(C_*(E_G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}, A) = \mathrm{Hom}(C_*(E_G \times_G X), A)$$

and the assertion follows. ☐

Let μ_m be the sheaf of m-th roots of unity.

Theorem 5.8. *For $X \in {}^G Sch/k$ we have*

$$\mathrm{Hom}_{{}^G DM^{eff}_{\acute{e}t}(k)}({}^{\mathrm{G}}\mathrm{M}(X), \mathbb{Z}/m(j)[i]) \xrightarrow{\sim} H^i_{et}(X, G; \mu_m^{\otimes j}).$$

Proof. By [Fri97, 4.5] there is a natural non equivariant quasi isomorphism

$$\mu_m^{\otimes j} \longrightarrow \mathbb{Z}/m(j).$$

Because G acts trivially on both sides the claim follows. ☐

Further we need the following easy

Lemma 5.9. *Let $A^\bullet, B^\bullet \in \mathrm{Kom}^-(\mathbb{Z}[G] - Mod)$. Then in $\mathcal{D}(Ab)$ there is a natural isomorphism*

$$\mathbf{R}\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbf{R}\mathrm{Hom}_{\mathbb{Z}}(A^\bullet, B^\bullet)) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathbb{Z}[G]}(A^\bullet, B^\bullet).$$

Now we can prove the following

Theorem 5.10. *Let $k = \mathbb{C}$ and $X \in {}^G Sm/\mathbb{C}$. We denote by $X(\mathbb{C})$ the complex manifold associated to X. Then there is a natural isomorphism*

$$\mathrm{Hom}_{{}^G DM^{eff}_{\acute{e}t}(k)}({}^{\mathrm{G}}\mathrm{M}(X), \mathbb{Z}/m[i]) \xrightarrow{\sim} H^i_{sing}(X(\mathbb{C}) \times_G E_G, \mathbb{Z}/m).$$

Proof. Using theorem 3.8 we identify the left hand side with

$$h^i(\mathbf{R}\mathrm{Hom}_{{}^G Shv_{et}(Sm/k)}({}^G\mathbb{Z}(X), \mathbb{Z}/m)).$$

Then by the factorization 1 with

$$h^i(\mathbf{R}\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbf{R}\mathrm{Hom}_{Shv_{et}(Sm/k)}({}^G\mathbb{Z}(X), \mathbb{Z}/m)).$$

The right hand side is equal to, by theorem 5.6

$$h^i(\mathbf{R}\mathrm{Hom}(C_*(X(\mathbb{C})), \mathbb{Z}/m))$$

and then to

$$h^i(\mathbf{R}\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbf{R}\mathrm{Hom}_{\mathbb{Z}}(C_*(X(\mathbb{C})), \mathbb{Z}/m)))$$

bu lemma 5.9. Because $C_*(X(\mathbb{C}))$ is a free $\mathbb{Z}$–module this is equal to

$$h^i(\mathbf{R}\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \mathrm{Hom}_{\mathbb{Z}}(C_*(X(\mathbb{C})), \mathbb{Z}/m))).$$

Now by the non equivariant comparison theorems $\mathrm{Hom}_{\mathbb{Z}}(C_*(X(\mathbb{C})), \mathbb{Z}/m)$ and $\mathbf{R}\mathrm{Hom}_{Shv_{et}(Sm/k)}({}^G\mathbb{Z}(X), \mathbb{Z}/m)$ are isomorphic in $\mathcal{D}(\mathbb{Z}[G]-Mod)$. So the claim follows. ☐

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