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Resolution of unbounded complexes in Grothendieck categories

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Abstract

Spaltenstein showed in [Spa88] that the unbounded complex category of sheaves on a topological space has enough K-injective complexes. We extend this to the unbounded category of an arbitrary Grothendieck category. This is important for our construction of a triangulated category of equivariant motives in [Se01].

1 Introduction

If one wants for example to define sheaf cohomology one needs injective resolution of a single sheaf. In [Gro57] Grothendieck showed that in a Grothendieck category each object has an injective resolution. If one wants to define more general sheaf hypercohomology of a complex of sheaves one can use Cartan-Eilenberg resolutions for bounded below complexes or in the situation of finite cohomological dimension. In general one needs an appropriate analog of an injective resolution. That was done by Spaltenstein in [Spa88]. The right replacement is a right resolution by a so called K-injective complex. Spaltenstein showed in [Spa88] for special categories that there are enough K-injective complexes. The goal of this article is to show that for any Grothendieck category there are enough K-injective complexes. The strategy of our proof is somehow a mixture of the proof of Grothendieck cited above and the proof of Spaltenstein for the case of the category of abelian sheaf on a topological space. Recently there appeared another independent and different proof in [ATJLSS00].

The motivation for this work is an equivariant version of Voevodsky's triangulated category of motives, which the author gives in [Se01]. In this context there occur natural unbounded complexes because of a simplicial construction. Further in the equivariant situation one has no finite cohomological dimension.

In section 2 we briefly recall the most important definitions and facts we will need. After that we prove in section 3 our main result. In section 4 the connection to the derived version of the theorem of Gabriel and Popesco is established.

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2 Notations and recollections

First we want to fix some notations and recall the most important definitions and facts.

Definition 2.1. *An abelian category is called a Grothendieck category if it has a generator and it has property AB 5), i.e. it is cocomplete and the filtered colimit of exact sequences is exact.*

Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{Kom}(\mathcal{A})$ the category of complexes in $\mathcal{A}$, by $\mathcal{K}(\mathcal{A})$ the homotopy category of complexes in $\mathcal{A}$ and by $\mathcal{D}(\mathcal{A})$ the derived category of $\mathcal{A}$. $\mathcal{D}(\mathcal{A})$ exists by the properties of $\mathcal{A}$.

Definition 2.2. *A complex $I^\bullet \in \mathcal{K}(\mathcal{A})$ is called* K-injective *if for each complex $A^\bullet \in \mathcal{K}(\mathcal{A})$ the equality $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, I^\bullet) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A^\bullet, I^\bullet)$ holds.*

The following Proposition is proven in [Spa88, Prop. 1.5].

Proposition 2.3. *For a complex $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ the following conditions are equivalent:*

(i) $A^\bullet$ is K-injective.

(ii) for each acyclic complex $S^\bullet \in \mathrm{Kom}(\mathcal{A})$ the complex $\mathrm{Hom}^\bullet(S^\bullet, A^\bullet)$ is again acyclic.

(iii) the functor $\mathrm{Hom}^\bullet(\_, A^\bullet) : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(Ab)$ preserves quasi isomorphisms.

Definition 2.4. *We say that $\mathrm{Kom}(\mathcal{A})$* has enough K-injective complexes *if for each complex $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ there is a right resolution $A^\bullet \to I^\bullet$ with $I^\bullet$ K-injective.*

Definition 2.5. *An ordinal $\alpha \in Ord$ is called* regular *if each subset $X \subseteq \alpha$ with $|X| < \alpha$ is bounded by an $\alpha_0 < \alpha$.*

Remark 2.6. *For the consequences see for example .....*

3 Proof of the theorem

Let $\mathcal{A}$ be a Grothendieck category and let U be a generator of $\mathcal{A}$.

The following two lemmas are proved in [Spa88, Sec. 4].

Lemma 3.1. *Let $({S^\bullet}_j)_{j\in J}$ be a family of acyclic complexes, and let $A^\bullet \in \mathrm{Kom}(\mathcal{A})$. Then $A^\bullet$ has a right resolution $s : A^\bullet \to B^\bullet$ which is in each degree injective and for every $j \in J$ and every $w : S_j^\bullet \to A^\bullet$ the chain map $s \circ w$ is homotopic to zero.*

Lemma 3.2. *For every $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ there exists a right resolution $m : A^\bullet \to B^\bullet$, such that m is injective and for all $i \in \mathbb{Z}$ the image of $m(A^i)$ is contained in an injective subobject of B^i.*

Now we define in an appropriate way the cardinality of an object in $\mathcal{A}$.

Definition 3.3. *For an object $A \in \mathcal{A}$ we define by*

$$|A| := |\{A' \subseteq A\}|$$

the cardinality *of A.*

Remark 3.4. *Because $\mathcal{A}$ has a generator $\{A' \subseteq A\}$ is really a set.*

The following properties of the cardinality function are obvious.

Lemma 3.5. *(i) If $A' \to A$ is injective $|A'| \leq |A|$ holds.*

(ii) If $A \to^v A''$ is surjective $|A''| \leq |A|$ holds.

We have the following important

Lemma 3.6. *Let $A \in \mathcal{A}$, κ a regular ordinal with $\kappa > |A|$ and $(B_\lambda)_{\lambda<\kappa}$ an inductive system in $\mathcal{A}$ with injective morphisms. Then for any morphism*

$$v : A \longrightarrow \operatorname*{colim}_{\lambda<\kappa} B_\lambda$$

there is a $\lambda_0 < \kappa$ such that v factors over $B_{\lambda_0} \to \mathrm{colim}_{\lambda<\kappa} B_\lambda$.

Proof. Because of AB 5) of $\mathcal{A}$ the canonical map $B_{\lambda_0} \to \mathrm{colim}_{\lambda<\kappa} B_\lambda$ is injective for all $\lambda_0 < \kappa$ and with $A_\lambda := v^{-1}(B_\lambda)$ we have $\mathrm{colim}_{\lambda<\kappa} A_\lambda = A$. We have to show that there is a $\lambda_0 < \kappa$ such that $A_{\lambda_0} = A$ holds. For that we define a subset E of κ by

$$E := \{\lambda \in \kappa | \lambda = 0 \vee A_\lambda \neq \operatorname*{colim}_{\lambda'<\lambda} A_{\lambda'}\}.$$

We have $A = \mathrm{colim}_{\lambda<\kappa} A_\lambda = \mathrm{colim}_{\lambda\in E} A_\lambda$. And the map $E \to \{A' \subseteq A\}$, $E \ni \lambda \mapsto A_\lambda$ is injective. Therefore $|E| \leq |A| < \kappa$ and by regularity of κ the subset E is bounded by $\lambda_0 < \kappa$. $\square$

With that in hand we can prove the following Lemma.

Lemma 3.7. *Let $(S_j^\bullet)_{j\in J}$ be a family of acyclic complexes in $\mathcal{A}$ and $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ any complex. Then there is a right resolution $A^\bullet \to B^\bullet$ with the following properties:*

(i) for all $j \in J$ the complex $\mathrm{Hom}^\bullet(S_j^\bullet, B^\bullet)$ is acyclic,

(ii) for all $i \in \mathbb{Z}$ the object B^i is injective in $\mathcal{A}$.

Proof. For an ordinal α we define by transfinite induction an inductive system of complexes $\{B_{\alpha'}\}_{\alpha'\leq\alpha}$:

- $B_0^\bullet := A$,
- for α a limit ordinal, we define

$$B_\alpha^\bullet := \operatorname*{colim}_{\alpha'<\alpha} B_{\alpha'}^\bullet,$$

- for $\alpha = (\alpha-1)+1$ we do the following: By Lemma 3.1 and 3.2 we choose a right resolution $u_\alpha : B_{\alpha-1}^\bullet \to B_\alpha^\bullet$ with the following properties:
  (a) for all $i \in \mathbb{Z}$ the morphism $u_\alpha^i : B_{\alpha-1}^i \to B_\alpha^i$ is injective,
  (b) for all $i \in \mathbb{Z}$ the image $u_\alpha^i(B_{\alpha-1}^i)$ is contained in an injective subobject of B_α^i and
  (c) for all $j \in J$ and any morphisms $w : S_j^\bullet \to B_{\alpha-1}^\bullet$ the morphism $u_\alpha \circ w$ is homotopic to zero.

Now we choose a regular ordinal κ with

1) for all $i \in \mathbb{Z}$ and $j \in J$ the inequality $|S_j^i| < \kappa$ holds,

2) for the generator U of $\mathcal{A}$ the inequality $|U| < \kappa$ holds and

3) $|\mathbb{N}| \leq \kappa$.

Then we set $B^\bullet := B_\kappa$ and let $A^\bullet = B_0^\bullet \to B_\kappa^\bullet$ be the canonical morphism.

Obviously this is a right resolution.

By (a) and by property AB 5) of $\mathcal{A}$ the inductive system $(B_\lambda^i)_{\lambda<\kappa}$ satisfies for any $i \in \mathbb{Z}$ the condition of Lemma 3.6. Therefore by Lemma 3.6, the properties 1)-3) and the regularity of κ for any $j \in J$ any morphism $S_j^\bullet \to B^\bullet$ factories for a $\lambda < \kappa$ via $B_\lambda^\bullet \to B_{\lambda+1}^\bullet \to B_\kappa^\bullet$ By (c) it follows property (i).

By [Gro57, I,Lemma 1], to show (ii) it is enough to show the lifting property for subobjects of U. So let V be a subobject of U and let $V \to B^i$

be a morphism for some i $\in \mathbb{Z}$. We have to show that there is a continuation to U. By Lemma 4, since $|V| \leq |U| < \kappa$, the morphism $V \to B^i$ factors by Lemma 3.6 for a $\lambda < \kappa$ through $B^i_\lambda \to B^i_{\lambda+1} \to B^i_\kappa$. With (b) the claim follows. □

Our next goal is to show that any complex has a subcomplex of bounded cardinality. For that we need more properties of the cardinality function.

Lemma 3.8. *Let $A \in \mathcal{A}$. Then there is a subset $J \subseteq \mathrm{Hom}(U, A)$ with the following properties*

(i) $|J| \leq |A|$,

(ii) $\bigoplus_J U \to A$ *is surjective.*

Proof. First we choose a well ordering on the set $\mathrm{Hom}(U, A)$. So let $\mathrm{Hom}(U, A) = \{f_\nu | \nu \leq \kappa\}$ for an ordinal κ. Then we define like in the proof of Lemma 3.6 a subset of $\mathrm{Hom}(U, A)$by

$$J := \{f_\nu \in \mathrm{Hom}(U, A) | \nu = 0 \vee \mathrm{Im}(f_\nu) \not\subseteq \operatorname*{colim}_{\mu<\nu} \mathrm{Im}(f_\mu)\}.$$

We have the surjective morphism

$$\bigoplus_J U \longrightarrow A$$

and by construction the map

$$\begin{array}{ccc} J & \longrightarrow & \{A' \subset A\} \\ \nu & \longmapsto & \mathrm{colim}_{\mu \leq \nu} \mathrm{Im}(f_\mu) \end{array}$$

is injective. □

Corollary 3.9. *Let β be a cardinal number. Then the isomorphism classes of objects $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ with $|A^i| \leq \beta$ for all $i \in \mathbb{Z}$ form a set.*

Proof. By lemma 3.8 any object $B \in \mathcal{A}$ with $|B| \leq \beta$ is a quotient of $\oplus_\beta U$. And the class of subobjects of a fixed object is a set because $\mathcal{A}$ has a generator. □

Now we define by induction a sequence of cardinals:

$$\begin{array}{rccc} & a_0 & = & |\mathbb{N}| \\ \text{for } n \geq 1: & a_{2n-1} & := & |\,\mathrm{Hom}(U, \oplus_{a_{2n-2}} U)| \\ & a_{2n} & := & |\bigoplus_{a_{2n}} U| \end{array}$$

The crucial fact is the following.

Proposition 3.10. *Let $A, A'' \in \mathcal{A}$, $A \to A''$ a surjective morphism and assume for some $n \in \mathbb{N}$ that $|A''| \leq a_{2n}$ holds. Then there is a subobject $A_0 \subseteq A$ with*

(i) $A_0 \to A \to A''$ *is surjective and*

(ii) $|A_0| \leq a_{2n+2}$.

Proof. By Lemma 3.8 there is a surjective morphism $\oplus_{a_{2n}} U \to A''$. Further by the choice of a_{2n+1} there is a morphism $\oplus_{a_{2n+1}} U \to A \times_{A''} \oplus_{a_{2n}} U$, such that $\oplus_{a_{2n+1}} U \to A \times_{A''} \oplus_{a_{2n}} U \to \oplus_{a_{2n}} U$ is surjective. Now we define $A_0 \subset A$ as $\mathrm{Im}(\oplus_{a_{2n+1}} U \to A)$. Then $A_0 \to A''$ is still surjective and by Lemma 3.5 (ii) $|A_0| \leq |\oplus_{a_{2n+1}} U| = a_{2n+2}$ holds. □

Now we choose a cardinal β which is greater than each element of the sequence

$$\{a_n | n \in \mathbb{N}\}.$$

Theorem 3.11. *Any acyclic complex $0 \neq S^\bullet \in \mathrm{Kom}(\mathcal{A})$ has an acyclic subcomplex $S_0^\bullet \neq 0$ with the property that $|S_0^i| \leq \beta$ holds for all $i \in \mathbb{Z}$.*

Proof. Let $i_0 \in \mathbb{Z}$ such that $S^{i_0} \neq 0$. Now we choose a nonzero morphism $U \to S^{i_0}$.We define a subcomplex $S_0^\bullet \subseteq S^\bullet$ as follows:

- $S_0^{i_0} := \mathrm{Im}(U \to S^{i_0})$
- $S_0^{i_0+1} := \mathrm{Im}(U \to S^{i_0} \xrightarrow{d} S^{i_0+1})$
- $S_0^i = 0$ for $i > i_0 + 1$
- for $i < i_0$ we define S_0^i by induction: if S_0^{i+1} is already defined we choose by Proposition 3.10 a subobject S_0^i of $d^{-1}(\mathrm{Kern}(S_0^{i+1} \to S_0^{i+2}))$ such that $S_0^i \to \mathrm{Kern}(S_0^{i+1} \to S_0^{i+2})$ is still surjective and by induction

  $$|S_0^i| \leq a_{2(i_0 - i)}$$

  holds.

This subcomplex has the disired properties. □

Before we can prove our main result we need a last lemma.

Lemma 3.12. *Let I a well ordered set. We look at the category of those projective systems $(A_\alpha)_{\alpha \in I}$ in the category of abelian groups where all morphisms are surjective. On this category the functor $\lim_I$ is exact.*

Proof. Like in the classical theorem of Mittag-Leffler. □

Now we can prove our main result.

Theorem 3.13. *Let $\mathcal{A}$ be a Grothendieck category. Then in* $\mathrm{Kom}(\mathcal{A})$ *there are enough K-injective complexes.*

Proof. By Corollary 3.9 we choose a set of acyclic complexes $\{S_j^\bullet\}_{j\in J}$ with the property:

(i) $|S_j^i| < \beta$ holds for all $j \in J$ and all $i \in \mathbb{Z}$ and

(ii) for any acyclic complex $T^\bullet$ with property (i) there is a $j \in J$ and an isomorphism $S_j^\bullet \to T^\bullet$ in $\mathrm{Kom}(\mathcal{A})$

Now let $A^\bullet \in \mathrm{Kom}(\mathcal{A})$. We choose a right resolution $A^\bullet \to I^\bullet$ by Lemma 3.7 apply to the above family of complexes. We want to show that this $I^\bullet$ is a K-injective complex. For that let $T^\bullet$ any acyclic complex. For each ordinal α we define by transfinite induction an acyclic subcomplex $T_\alpha^\bullet \subset T^\bullet$:

- Let $T_0^\bullet$ be a nonzero acyclic subcomplex of $T^\bullet$ such that $|T_0^i| \leq \beta$ holds for all $i \in \mathbb{Z}$. It exists by Theorem 3.11.
- If α is a limit ordinal we define
$$T_\alpha^\bullet := \operatorname*{colim}_{\alpha'<\alpha} T_{\alpha'}^\bullet.$$
- If α is the successor of $\alpha-1$ we look at the projection $\pi : T^\bullet \to T^\bullet/T_{\alpha-1}^\bullet$. $T^\bullet/T_{\alpha-1}^\bullet$ is again acyclic. If $T^\bullet/T_{\alpha-1}^\bullet \neq 0$ we choose again by Theorem 3.11 a acyclic subcomplex $\tilde{T^\bullet}_\alpha \subset T^\bullet/T_{\alpha-1}^\bullet$ such that $|\tilde{T}_\alpha^i| \leq \beta$ holds for all $i \in \mathbb{Z}$. Then we define $T_\alpha^\bullet := \pi^{-1}(\tilde{T^\bullet}_\alpha)$.

Now we show by transfinite induction that $\mathrm{Hom}^\bullet(T_\alpha^\bullet, I^\bullet)$ is exact for each ordinal α.

- The complex $T_0^\bullet$ is isomorphic to a complex of $\{S_j^\bullet\}_{j\in J}$ and therefore the claim follows from the construction of $I^\bullet$.
- If $\alpha = \alpha' + 1$ we look at the exact sequence
$$0 \longrightarrow T_{\alpha'}^\bullet \longrightarrow T_\alpha^\bullet \longrightarrow \tilde{T^\bullet}_\alpha \longrightarrow 0.$$
By construction $\tilde{T^\bullet}_\alpha$ is isomorphic to a complex of $\{S_j^\bullet\}_{j\in J}$ and therefore $\mathrm{Hom}^\bullet(\tilde{T^\bullet}_\alpha, I^\bullet)$ is acyclic. The complex $\mathrm{Hom}^\bullet(T_{\alpha'}^\bullet, I^\bullet)$ is acyclic by our induction hypothesis. By the fact that I^i are injective for all $i \in \mathbb{Z}$ the claim follows.

- We want to show that for a limit ordinal α the complex $\mathrm{Hom}^\bullet(T_\alpha^\bullet, I^\bullet) = \lim_{\alpha'<\alpha} \mathrm{Hom}^\bullet(T_{\alpha'}^\bullet, I^\bullet)$ is acyclic. For ordinals $\lambda \leq \lambda'$ the morphism $T_\lambda^\bullet \to T_{\lambda'}^\bullet$ is injective and and therefore $\mathrm{Hom}^n(T_{\lambda'}^\bullet, I^\bullet) \to \mathrm{Hom}^n(T_\lambda^\bullet, I^\bullet)$ is for all $n \in \mathbb{Z}$ surjective, because of the injectivity of I^i for all $i \in \mathbb{Z}$. By Lemma 3.12 $\lim_{\alpha'<\alpha} \mathrm{Hom}^\bullet(T_{\alpha'}^\bullet, I^\bullet)$ is acyclic.

Now we choose an ordinal ψ which is greater than $\prod_{i\in\mathbb{Z}} |T^i|$. Then we have $T^\bullet = T_\psi^\bullet$ and the theorem is proven. $\square$

4 The derived version of the theorem of Gabriel Popesco

In [ATJLSS00] the authors prove first a derived version of the theorem of Gabriel-Popesco. Then the existence of enough K-injective complexes is a corollary. In the end of this article we want to show that conversely we can easily deduce the derived version of Gabriel-Popesco from theorem 3.13.

Corollary 4.1. *Let $\mathcal{A}$ be a Grothendieck category. Then there is a ring R such that $\mathcal{D}(\mathcal{A})$ is a localization of $\mathcal{D}(R)$ in the sense of Bousfield (comp. [Ne01, chap. 9]).*

Proof. The usual theorem of Gabriel-Popesco [GP] says that there is a ring R and a pair of adjoint functors

$$\begin{array}{rrccc} & a: & R-Mod & \longrightarrow & \mathcal{A} \\ \text{and} & i: & \mathcal{A} & \longrightarrow & R-Mod \end{array}$$

where a is exact and i is full, faithful and right adjoint to a. Because a is exact it induces a functor

$$a : \mathcal{D}(R) \longrightarrow \mathcal{D}(\mathcal{A})$$

and because $\mathcal{D}(\mathcal{A})$ has enough K-injective complexes the right derived functor

$$\mathbf{R}i : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(R)$$

of i exists. Furthermore i respects K-injective complexes because it is right adjoint to an exact functor. Therefore we have

$$\mathrm{Hom}_{\mathcal{D}(R)}(M^\bullet, \mathbf{R}i(A^\bullet)) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(a(M^\bullet), A^\bullet)$$

and

$$id_{\mathcal{D}(A)} = \mathbf{R}id_{\mathcal{A}} \xrightarrow{\sim} \mathbf{R}(a \circ i) \xrightarrow{\sim} a \circ \mathbf{R}i.$$

So $\mathbf{R}i$ is full and faithful and the theorem is proven. $\square$

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