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Master Thesis

Towards $\mathsf{MM}^{*,++}$ in $\mathbb{P}_{\max}$ extensions

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Abstract

We study a strengthening of MM^{++} which is called $\mathsf{MM}^{*,++}$ and is introduced by Schindler [29], in $\mathbb{P}_{\max}$ extensions. We study a general framework to force $\mathsf{MM}^{*,++}$ by $\mathbb{P}_{\max}$ forcing, and we force $\mathsf{MM}^{*,++}_{\mathfrak{c}}$, which is stronger than both $\mathsf{MM}^{++}(\mathfrak{c})$ and BMM^{++}, by $\mathbb{P}_{\max}$ forcing. Furthermore, we give a candidate for the ground model using the result by Gappo and Sargsyan [11], and the generalized derived model construction given by Larson, Sargsyan, and Wilson [18].

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"Nunc est bibendum, nunc pede libero pulsanda tellus"

Horace, *Odes* I.37.

Chapter 1

Introduction

Cantor proved $\mathbb{R}$ is uncountable and conjectured that every uncountable subset of $\mathbb{R}$ has the same size as $\mathbb{R}$, i.e., $2^{\aleph_0} = \aleph_1$. The conjecture is known as *Cantor's Continuum Hypothesis* (CH). Gödel showed that CH holds in his *constructible universe* L and proved CH is consistent with the standard axiom system ZFC. He also showed that L is the minimal transitive model of ZFC which contains all the ordinals, i.e., the minimal *inner model*. Cohen discovered *forcing* and showed that CH does not follow from ZFC. Originating from their great work, modern set theory includes the following branches.

Forcing: While Cohen's initial motivation was to produce a counterexample of CH, forcing was developed by Solovay and has been used to produce various generic objects and to give consistency results.

Inner Model Theory: Starting with Gödel's L, it studies canonical models of ZFC with large cardinals[1]. Later, the canonicity was formulated by Jensen [14] as *fine structure*.

Descriptive Set Theory and Determinacy: It started with the work of Cantor and Bendixon and studies nice properties of definable sets of reals. *Determinacy* is one of the most important notions of descriptive set theory and provides a rich structure for sets of reals.

Currently, many branches of set theory are flourishing, and the undecidability of many propositions, including CH, is known. One view is that ZFC as a standard axiom system is insufficient to decide such propositions, especially CH. Then, what would be the correct extension for ZFC? What should the universe V look like? For example, is the axiom $V = L$, stating that the universe V is the minimal inner model L, correct? If we take the position that the universe

[1] Large cardinals give us transcendental objects for the analysis of strong propositions

should be as rich and maximal as possible[2], which the author calls *maximalism*, the axiom $V = L$ should be rejected. For example, Scott [32] proved that L cannot have measurable cardinals [3] Then, could *large cardinal axioms*, stating the existence of large cardinals, be the candidate? Unfortunately, Levy and Solovay [21] showed that by and large, large cardinals are preserved in any small forcings, and it implies they do not decide CH. However, note that this does not negate large cardinal axioms. This suggests just another possibility.

Besides large cardinal axioms, *forcing axioms* are well-known "maximality principles". Forcing axioms state that the universe is closed under taking generic objects by forcings in a fixed class of forcings Γ. In general, let κ be an infinite cardinal and let Γ be a class of forcings. Then $\mathsf{FA}_\kappa(\Gamma)$ says that whenever $\mathbb{P} \in \Gamma$ and $\{A_i \mid i < \omega_1\}$ is a family of maximal antichains in $\mathbb{P}$ such that A_i has size at most κ for each $i < \omega_1$, then there is a filter G in $\mathbb{P}$ such that $G \cap A_i \neq \emptyset$ for all $i < \omega_1$. $\mathsf{FA}(\Gamma)$ states that $\mathsf{FA}_\kappa(\Gamma)$ holds true for all κ. For example, if Γ is the class of all c.c.c., proper, and stationary set preserving forcings, then each $\mathsf{FA}(\Gamma)$ are known as the "Martin's Axiom" (MA), the "Proper Forcing Axiom" (PFA), and the "Martin's Maximum" (MM).

As the study of forcing progressed, stronger forcing axioms were proved to be consistent from large cardinals. For example, Baumgartner and Shelah forced PFA from $\mathsf{ZFC}+$ "there is a supercompact cardinal", and Foreman, Magidor, and Shelah [10] forced MM from the same assumption. Furthermore, it has become clear that forcing axioms imply various(!) propositions, including $\neg\mathsf{CH}$, in fact, $2^{\aleph_0} = \aleph_2$. For example, see [8], [10], and [40]. Moreover, recently, Asperó and Schindler [1] proved that the stronger version of MM, called MM^{++}, implies another well-known maximality principle the $\mathbb{P}_{\max}$ axiom $(*)$. Forcing axioms, especially MM (or even MM^{++}), seem reasonable candidates for additional axioms from the perspective of maximalism.

However, there is one itchy thing. We do not know the exact consistency strength of MM (and PFA). It has been a longstanding important open problem of set theory. Here is the conjecture one expects:

Conjecture. The following are equiconsistent.

(1). $\mathsf{ZFC} + \mathsf{MM}$

(2). $\mathsf{ZFC} + \mathsf{PFA}$

(3). $\mathsf{ZFC}+$ "there is a supercompact cardinal"

Besides iterated forcing, forcing over models of determinacy is another approach to understanding the relationships between forcing axioms and large cardinals.

[2] The author takes this position.
[3] It is one of the large cardinals.

Currently, methods that force full PFA or MM need the Laver function, which requires supercompact cardinals. One advantage of it is that we can start from determinacy models. Thanks to the derived model construction and inner model theory, we have access to such models with significantly lower consistency strength than supercompact cardinals.

The program obtaining fragments of forcing axioms, especially MM, by forcing over models of determinacy started with the work of Steel and Wesep [39]. They obtained the consistency of saturation of NS_{ω_1} forcing over models of $\mathsf{AD}_{\mathbb{R}}+$ "Θ is regular". Woodin [41, Definition 4.33, Theorem 9.39, Theorem 10.99] introduced the the partial order $\mathbb{P}_{\max}$ and forced a restricted version of MM called $\mathsf{MM}(\mathfrak{c})$ over models of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular", and the "Bounded Martin's Maximum" (BMM) over models of AD + "V is closed under the $M_1^\sharp$ operator". Zoble [43] studied a strengthening of BMM which is called BMM^s and states that if a Σ_1 fact about ω_2^V holds in a stationary set preserving extension then it holds in V for a stationary set of ordinals less than ω_2^V, and showed it does not hold in Woodin's BMM model. Larson and Sargsyan [20] forced failures of the square over Chang models by $\mathbb{P}_{\max}$ forcing. Note that in the context of $\mathbb{P}_{\max}$ extensions, we naturally get the ++ version and all of the above results can be replaced with the ++ version. Schindler [29, Definition 2.10] introduced $\mathsf{MM}^{*,++}$ as a strengthening of MM^{++}. $\mathsf{MM}^{*,++}$ is defined by replacing the "$\varphi(\mathcal{M})$ may be forced to hold in stationary set preserving forcing extensions of V" part of MM^{++} in the formulations of [4] and [6] with "$\varphi(\mathcal{M})$ is honestly consistent". $\mathsf{MM}^{*,++}$ is a natural formulation in the context of $\mathbb{P}_{\max}$ extensions, and the ultimate goal of the program seems to be to get models of $\mathsf{MM}^{*,++}$ using $\mathbb{P}_{\max}$ forcing.

Our main result is along the program and gives a sufficient condition for a ground model to force $\mathsf{MM}_{\mathfrak{c}}^{*,++}$. $\mathsf{MM}_{\mathfrak{c}}^{*,++}$ is a global fragment of $\mathsf{MM}^{*,++}$ stronger than both BMM^{++} and $\mathsf{MM}^{++}(\mathfrak{c})$, which are obtained by Woodin using $\mathbb{P}_{\max}$.

We let Γ^∞ be the set of universally Baire sets of reals. Here is our main result.

Theorem 1.0.1. *Assume that*

(1). $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$ + "Θ *is regular"* + *"Every set of reals is universally Baire";*

(2). $\Gamma^\infty(=\mathcal{P}(\mathbb{R}))$ *is productive;*

(3). $V^{\mathbb{P}_{\max}} \models \omega_2\text{–}\mathsf{DC}$ *and* $V^{\mathbb{P}_{\max}*\mathrm{Add}(\omega_3,1)} \models \mathsf{ZFC}$*; and*

(4). for every $A \in \Gamma^\infty$ *there is some* (N,Σ) *which strongly captures* A *and the function* $F\colon X \mapsto M_2^{\sharp,\Sigma}(X)$*, where* $X \in \mathrm{HC}$ *is self-well-ordered and* $N \in L_1(X)$*, is well-defined, total, and strongly universally Baire in the codes.*

Then

$$V^{\mathbb{P}_{\max} * \mathrm{Add}(\omega_3,1)} \models \mathsf{MM}^{*,++}_{\mathfrak{c}}.$$

Since the $*$-version implies the original version. We get the following.

Corollary 1.0.2. *Assume the same assumptions as Theorem 1.0.1. Then*

$$V^{\mathbb{P}_{\max} * \mathrm{Add}(\omega_3,1)} \models \mathsf{MM}^{++}_{\mathfrak{c}}.$$

It remains open how to get the ground model for Theorem 1.0.1. However, in chapter 5, by combining the results in [11, Lemma 3.4] and [18, Main Theorem], we will construct a model that is a strong candidate for that as follows.

Theorem 1.0.3. *Suppose that V is self-iterable. Let λ be an inaccessible cardinal which is a limit of Woodin cardinals and a limit of strong cardinals, and let $G \subset \mathrm{Col}(\omega, <\lambda)$ be a V-generic filter. Let F_{uB} be as defined in [18, Definition 4.4]. Then*

(1). $L^{F_{\mathrm{uB}}}(\mathbb{R}^*_G, \mathrm{Hom}^*_G) \models \mathsf{AD}_{\mathbb{R}}$+"$\Theta$ *is regular*"+ "*Every set of reals is universally Baire*",

(2). $L^{F_{\mathrm{uB}}}(\mathbb{R}^*_G, \mathrm{Hom}^*_G) \cap \mathcal{P}(\mathbb{R}^*_G) = \mathrm{Hom}^*_G$.

Furthermore, if $H \subset \mathbb{P}_{\max}$ is $L^{F_{\mathrm{uB}}}(\mathbb{R}^*_G, \mathrm{Hom}^*_G)$-generic, then

(3). $L^{F_{\mathrm{uB}}}(\mathbb{R}^*_G, \mathrm{Hom}^*_G)[H] \models \omega_2\text{-}\mathsf{DC}$.

Chapter 2 lists the preliminaries needed for the proof of the main theorem. In Chapter 3, we discuss the general framework for forcing $\mathsf{MM}^{*,++}$ using $\mathbb{P}_{\max}$ and give the BMM model. Chapter 4 is the proof of the main theorem, and Chapter 5 is about partial results.

Notation.

- Let $\mathbb{R} = \omega^\omega$ as usual.
- For $\alpha < \omega_1$, let WO_α be the set of reals coding α. Let $\mathrm{WO} = \bigcup_{\alpha<\omega_1} \mathrm{WO}_\alpha$. For more details, see [27, 4A].
- Let ON be the class of all the ordinals.
- Let $\mathcal{L}_{\dot\in}$ be the language of set theory, and let $\mathcal{L}_{\dot\in, \dot I_{\mathsf{NS}}}$ be the language of set theory augmented by a predicate $\dot I_{\mathsf{NS}}$ for NS_{ω_1}. In transitive models $\mathcal{M}$ of ZFC^- + "ω_1 exists", $\dot\in$ is always to be interpreted by $\in\restriction\mathcal{M}$, $\dot I_{\mathsf{NS}}$ is always to be interpreted by NS_{ω_1} in the sense of $\mathcal{M}$.
- For a forcing $\mathbb{P}$ and a $\mathbb{P}$-name τ, we denote τ^G by the interpretation of τ by $\mathbb{P}$-generic G.

Chapter 2

Preliminaries

This chapter lists the tools and theorems used in this paper briefly.

2.1 Forcing axioms

In this section, we introduce bounded forcing axioms and their characterization.

Definition 2.1.1. Let Γ be a class of forcings, i.e., complete Boolean algebras, and let κ be an uncountable cardinal.

$\mathsf{FA}_\kappa(\Gamma)$, or FA_κ for forcings in Γ, denotes the statement that whenever $\mathbb{P} \in \Gamma$ and $\{A_i \mid i < \omega_1\}$ is a family of maximal antichains in $\mathbb{P}$ such that A_i has size at most κ for each $i < \omega_1$, then there is a filter G in $\mathbb{P}$ such that $G \cap A_i \neq \emptyset$ for all $i < \omega_1$.

$\mathsf{FA}_\kappa^{++}(\Gamma)$, or FA_κ^{++} for forcings in Γ, denotes the statement that whenever $\mathbb{P} \in \Gamma$, $\{A_i \mid i < \omega_1\}$ is a family of maximal antichains in $\mathbb{P}$, and $\{\tau_i \mid i < \omega_1\}$ is a family of terms for stationary subsets of ω_1 such that A_i has size at most κ for each $i < \omega_1$, then there is a filter G in $\mathbb{P}$ such that $G \cap A_i \neq \emptyset$ for all $i < \omega_1$ and

$$\tau_i^G = \{\alpha < \omega_1 \mid \exists p \in G(p \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{\tau}_i)\}$$

is stationary for all $i < \omega_1$.

We have BMM is $\mathsf{FA}_{\aleph_1}$ for stationary set preserving forcings, $\mathsf{MM}_{\mathfrak{c}}$ is $\mathsf{FA}_{\mathfrak{c}}$ for stationary set preserving forcings, and MM is FA_κ for stationary set preserving forcings and all κ. The same goes for the ++ version.

Definition 2.1.2. Let $\mathcal{M} = (M, \in, \vec{R})$ be a transitive structure such that $\vec{R} = (R_i : i < \omega_1)$ is a list of $\aleph_1$ relations on M, and let φ be a Σ_1 formula. Let $\Psi(\mathcal{M}, \varphi)$ be the statement that there is some transitive structure $\bar{\mathcal{M}}$ of size $\aleph_1$, some elementary $\pi : \bar{\mathcal{M}} = (\bar{M}, \in, (\bar{R}_i : i < \omega_1)) \to \mathcal{M}$, and $\varphi(\bar{\mathcal{M}})$ holds true.

Honest consistency is motivated by the following characterization of $\mathsf{FA}_\kappa^{++}(\Gamma)$.

Lemma 2.1.1 ([4, Theorem 5], [6, Theorem 1.3])**.** *Let Γ be a class of forcings. The following are equivalent.*

(1). $\mathsf{FA}_\kappa^{++}(\Gamma)$*.*

(2). For all $\mathbb{P} \in \Gamma$, for all transitive structures $\mathcal{M}$ of size at most κ, and for all Σ_1 formulae φ in $\mathcal{L}_{\dot{\in}, \dot{I}_{\mathsf{NS}}}$,

$$V^{\mathbb{P}} \models \varphi(\mathcal{M}) \Longrightarrow V \models \Psi(\mathcal{M}, \varphi).$$

Proof. We only prove that (2) implies (1). For more details, see [4, Theorem 5] and [6, Theorem 1.3].

Let $\mathbb{P}$ be stationary set preserving complete Boolean algebra, let $(A_i : i < \omega_1)$ be a family of maximal antichains in $\mathbb{P}$ such that each A_i has size at most κ, and let $(\tau_i : i < \omega_1)$ be a faimily of names for stationary subsets of ω_1. Let

$$B_i = \{(\alpha, ||\check{\alpha} \in \dot{\tau}_i||) \mid \alpha < \omega_1\}$$

for $i < \omega_1$. Let θ be sufficiently large, and let

$$\sigma : \mathcal{M} = (M, \in, \bar{\mathbb{P}}, (\bar{A}_i : i < \omega_1), (\bar{B}_i : i < \omega_1)) \to (H_\theta, \in, \mathbb{P}, (A_i : i < \omega_1), (B_i : i < \omega_1)),$$

where M is transitive and of size κ, and

$$(\omega_1 + 1) \cup \{\mathbb{P}\} \cup \{\tau_i \mid i < \omega_1\} \cup \bigcup_{i<\omega_1} A_i \cup \bigcup_{i<\omega_1} B_i \subset \operatorname{ran}(\sigma).$$

Let $\varphi(\mathcal{M})$ say

$$\begin{aligned}
\varphi(\mathcal{M}) \equiv & \exists \tilde{G}\, \exists (S_i \mid i < \omega_1) \\
& [\bar{G} \text{ is a filter in } \bar{\mathbb{P}} \\
& \wedge \forall i < \omega_1 (\bar{A}_i \cap \bar{G} \neq \emptyset) \\
& \wedge \forall i < \omega_1 (S_i \subset \omega_1 \wedge S_i \notin \dot{I}_{\mathsf{NS}}) \\
& \wedge \forall i < \omega_1 \forall \alpha \in S_i \exists p \in \bar{G} \\
& ((\alpha, p) \in \bar{B}_i)].
\end{aligned}$$

Let $G \subset \mathbb{P}$ be V-generic. Then

$$V[G] \models \varphi(\mathcal{M})$$

as witnessed by the σ-preimage of G and $(\tau_i^G \colon i < \omega_1)$.

By our hypothesis, let $\pi \colon \bar{\mathcal{M}} \to \mathcal{M}$ be elementary and such that $\varphi(\bar{\mathcal{M}})$ holds. Let $\bar{G}$ and $(S_i \colon i < \omega_1)$ witness that $\varphi(\bar{\mathcal{M}})$ holds true. Let G be the filter in $\mathbb{P}$ generated by $(\sigma \circ \pi)''\bar{G}$. Then we have that

$$G \cap A_i \neq \emptyset,$$

since $\bar{G} \cap \bar{A}_i \neq \emptyset$. Moreover, since $S_i \subset \tau_i^{\bar{G}}$ for each $i < \omega$, $\tau_i^{\bar{G}}$ is stationary for each $i < \omega_1$. Hence G is the desired filter. Therefore, $\mathsf{FA}_\kappa^{++}(\Gamma)$ holds. $\square$

2.2 The axiom of determinacy and $\mathbb{P}_{\max}$

In this section, we list the facts about descriptive set theory, determinacy and $\mathbb{P}_{\max}$. For more details, see [19] and [41].

Definition 2.2.1. Let X be nonempty

- For $A \subset X^\omega$, *the game* G_A consists of two players, who take turns playing an element a_i of X ω many times. The first player wins if the sequence $(a_i \colon i \in \omega)$ is an element of A, otherwise, the second player wins.
- For $A \subset X^\omega$, we say A is *determined* if one of the two players has a winning strategy for the game G_A.
- *The Axiom of Determinacy* AD is AD_ω.
- The ordinal Θ is the supremum of ordinals which are surjective images of $\mathbb{R}$.

In the AC context, clearly $\Theta = (2^{\aleph_0})^+$. However, in the AD context, Θ is quite large.

Theorem 2.2.1 (Moschovakis)**.** *Assume* AD*. Then* Θ *is a limit cardinal.*

Proof. See [27, 7D.8 and 7D.19]. $\square$

Many of the important theorems of descriptive set theory are derived from tree representations, also known as Suslin representations, of sets of reals.

Definition 2.2.2. Let α be an ordinal. We say a set of reals is *α-Suslin* if there exists a tree T on $\omega \times \alpha$ such that

$$A = p[T] = \{x \in \mathbb{R} \mid \exists f((x, f) \in [T])\}.$$

We say a set of reals A is *Suslin* if A is α-Suslin for some α.

Example.

- Every $\mathbf{\Sigma}^1_1$ set is ω-Suslin.
- Every $\mathbf{\Sigma}^1_2$ set is ω_1-Suslin.
- (Martin, Steel, [23]) Assume $\mathsf{AD}^{L(\mathbb{R})}$. Then every $(\mathbf{\Sigma}^2_1)^{L(\mathbb{R})}$ set is Suslin in $L(\mathbb{R})$. On the other hand, there is a $(\Pi^2_1)^{L(\mathbb{R})}$ of reals which is not Suslin in $L(\mathbb{R})$.

$\mathsf{AD}_{\mathbb{R}}$ is closely related to Suslin sets.

Theorem 2.2.2 (Martin, Woodin)**.** *If* AD *holds and every set of reals is Suslin, then* $\mathsf{AD}_{\mathbb{R}}$ *holds.*

Theorem 2.2.3 (Woodin)**.** *If* $\mathsf{AD}_{\mathbb{R}}$ *holds, then all sets of reals are Suslin.*

Hence under AD, $\mathsf{AD}_{\mathbb{R}}$ is equivalent to the statement that every set of reals is Suslin.

Next is about $\mathbb{P}_{\max}$.

Definition 2.2.3. Let M be a countable transitive model of a sufficiently large fragment of ZFC, and let I be an ideal on ω_1^M which is normal in M. Let $\alpha \leq \omega_1$. We say $\langle M_\xi, G_\xi, j_{\xi,\zeta} \colon \xi \leq \zeta \leq \alpha\rangle$ is *an iteration of* (M, I) *of length* α if the following holds:

(1). $M_0 = M$, and each G_ξ is an M_ξ-generic for $(\mathcal{P}(\omega_1)/j_{0,\xi}(I))^{M_\xi}$;

(2). $j_{\xi,\xi+1}$ is the generic ultrapower embedding induced by G_α;

(3). $j_{\xi,\xi} = \mathrm{id}$, $j_{\xi,\zeta} = j_{\xi+1,\zeta} \circ j_{\xi,\xi+1}$; and

(4). for each limit ordinal $\eta \leq \alpha$, M_η is the direct limit of the system $\langle M_\xi, j_{\xi,\zeta} \colon \xi \leq \zeta < \eta\rangle$, and $j_{\xi,\eta}$ is the direct limit embedding for each $\xi < \eta$.

The models M_α are called *iterates* of (M, I). We say (M, I) is *iterable* if every iterate of (M, I) is well-founded. Note that the fact that (M, I) is iterable is Π^1_2-statement in the code.

Definition 2.2.4 (Woodin)**.** The partial order $\mathbb{P}_{\max}$ consists of all pairs $(M, \in, I, a)$ such that

(1). M is a countable transitive model of a sufficiently large fragment of $\mathsf{ZFC}+\mathsf{MA}_{\aleph_1}$;

(2). I is a normal ideal on ω_1 in M, and $a \in \mathcal{P}(\omega_1)^M$;

(3). (M, I) is iterable; and

(4). there is an $x \in \mathbb{R}^M$ such that $\omega_1^M = \omega_1^{L[a,x]}$.

$(M, \in, I, a) <_{\mathbb{P}_{\max}} (N, \in, J, b)$ if $N \in \mathrm{HC}^M$ and there is an iteration $j\colon (N, J) \to (N', J')$ in M such that $j(b) = a$ and $I \cap N' = J'$.

We say $(M, \in, I)$ is a $\mathbb{P}_{\max}$ *precondition* if there is $a \subset \omega_1^M$ such that $(M, \in, I, a) \in \mathbb{P}_{\max}$.

Notation. For a $\mathbb{P}_{\max}$ condition $p = (M, \in, I, a) \in \mathbb{P}_{\max}$, we often identify p with its universe M for notational simplicity.

Note that, by $\mathsf{MA}_{\aleph_1}$, more precisely almost disjoint coding, iteration is uniquely determined by an image of the augmented subset of ω_1.

Definition 2.2.5. Let G be $\mathbb{P}_{\max}$-generic.

- $A_G := \bigcup\{a \mid \exists (M, I)\, [(M, \in, I, a) \in G]\}$.
- We let $\mathcal{P}(\omega_1)_G$ be the set of all B such that there is a $(M, \in, I, a) \in G$ and $b \in \mathcal{P}(\omega_1)^M$ such that $j(b) = B$ where j is the unique generic iteration of (M, I) of length ω_1 sending a to A_G.
- We define I_G to be the set of all B such that there is a $(M, \in, I, a) \in G$ and $b \in I$ such that $j(b) = B$ where j is the unique generic iteration of (M, I) of length ω_1 sending a to A_G.

Note that A_G is well-defined, since letting G be $\mathbb{P}_{\max}$-generic, then for $(M, \in, I, a), (N, \in, J, b) \in G$, we have that

$$a \cap \gamma = b \cap \gamma$$

where $\gamma = \min\{\omega_1^M, \omega_1^N\}$.

Lemma 2.2.4. *If $x^\dagger$ exists for all $x \in \mathbb{R}$, then we have that*

(1). $\mathbb{P}_{\max} \neq \emptyset$, and

(2). $\mathbb{P}_{\max}$ is σ-closed and homogeneous.

In the $\mathbb{P}_{\max}$ analysis, it is important to see for each $A \subset \mathbb{R}$ there are densely many conditions that keep track of a name of A. It comes from Suslin representations.

Definition 2.2.6. Let A be a set of reals. We say a precondition $(M, \in, I)$ is *A-iterable* if

(1). $A \cap M \in M$, and

(2). $j(A \cap M) = A \cap M'$ whenever $j \colon (M, I) \to (M', I')$ is an iteration of (M, I).

Determinacy ensures that there are densely many A-iterable conditions for each A. Here, AD^+ is a technical variant of AD.

Lemma 2.2.5 (Woodin)**.** *Assume* AD^+*. Let* A *be a set of reals. Then there are densely many conditions* $p = (M, \in, I, a) \in \mathbb{P}_{\max}$ *such that*

(1). $(\mathrm{HC}^M, \in, A \cap M) \prec (\mathrm{HC}, \in, A)$*,*

(2). $(M, \in, I)$ *is* A*-iterable, and*

(3). if $j \colon (M, I) \to (M', I')$ *is any iteration of* (M, I)*, then*

$$(\mathrm{HC}^{M'}, \in, A \cap M') \prec (\mathrm{HC}, \in, A).$$

Proof. The key fact is the following.

Fact 1 (Woodin, [35, Theorem 7.1], [38])**.** Assume AD^+. Then

(1). The pointclass Σ^2_1 has the scale property, and

(2). Every lightface Σ^2_1 collection of sets of reals has a lightface Δ^2_1 member.

Now, suppose otherwise. Let $p_0 = (M_0, \in, I_0, a_0) \in \mathbb{P}_{\max}$, and let $A \subset \mathbb{R}$ be a conterexample to the statement of the theorem. Then we may assume that A is Suslin and co-Sudlin by Fact 1. Let T and U be trees projecting A and its complement respectively.

Since $\Delta^2_1(p_0)$ is closed under complements, projections, and countable unions, so there exist a $\Delta^2_1(p_0)$ set $B \subset \mathbb{R} \times \mathbb{R}$ such that whenever $F \colon \mathbb{R} \to \mathbb{R}$ uniformizes B and N is a transitive model of ZF closed under F, then

$$(\mathrm{HC}^N, \in, A \cap N) \prec (\mathrm{HC}, \in, A).$$

Again, by Fact 1, let $F \colon \mathbb{R} \to \mathbb{R}$ be $\Delta^2_1(p_0)$ and uniformize B. Let V and W be trees projecting F and its complement respectively.

The next key fact is the following.

Fact 2 ([17, Theorem 5.4])**.** Assume AD. Let S be a set of ordinals. Then there exists a real x such that for all reals y with $x \in L[S, y]$,

$$\mathrm{HOD}_S^{L[S,y]} \models \mathsf{ZFC} + \omega_2^{L[S,y]} \text{ is a Woodin cardinal.}$$

By Fact 2, let M be a transitive proper class model of ZFC such that

(1). $p_0, T, U, V, W \in M$, and

(2). there exists a countable ordinal δ such that $M \models$ "δ is a Woodin cardinal".

Let $\kappa < \lambda < \delta$ be such that κ is measurable in M and λ is inaccessible in M. Let $g_0 \subset \mathrm{Col}(\omega, <\kappa)$ be M-generic, and let g_1 be $M[g_0]$-generic for the standard c.c.c. poset to force MA. Then in $M[g_0]$, an ideal dual to a fixed normal measure on κ in M generates a precipitous ideal in $M[g_0]$. Since c.c.c. forcings preserve precipitous ideals[1], it also generates a precipitous ideal in $M[g_0, g_1]$.

Let I be a precipitous ideal in $M[g_0, g_1]$. Then $(M_\lambda[g_0, g_1], \in, I)$ is iterable.

Since $T \in M$, $A \cap M_\lambda[g_0, g_1] \in M_\lambda[g_0, g_1]$. Since $M_\lambda[g_0, g_1]$ is closed under F, we have that

$$(\mathrm{HC}^{M_\lambda[g_0,g_1]}, \in, A \cap M_\lambda[g_0, g_1]) \prec (\mathrm{HC}, \in, A).$$

Fix an iteration $j\colon (M_\lambda[g_0, g_1], I) \to (M', I')$. Let $j^*\colon (M[g_0, g_1], I) \to (M^*, I^*)$ be the lift of j. Then we have that

$$p[T] = p[j^*(T)], \text{ and } p[U] = p[j^*(U)],$$

and similarly,

$$p[V] = p[j^*(V)], \text{ and } p[W] = p[j^*(W)].$$

Hence $A \cap M' \in M'$, and M^* is closed under F. So we have that

$$(\mathrm{HC}^{M'}, \in, A \cap M') \prec (\mathrm{HC}, \in, A).$$

This shows that A is not a counterexample to the statement of the theorem. This is a contradiction! $\square$

The next theorem lists what is possible in the $\mathbb{P}_{\max}$ extension in general:

Theorem 2.2.6 (Woodin)**.** *Assume AD^+. Let $G \subset \mathbb{P}_{\max}$ be a V-generic filter. Then in $V[G]$ the following hold:*

- $\mathcal{P}(\omega_1)_G = \mathcal{P}(\omega_1)$,
- $I_G = \mathsf{NS}_{\omega_1}$,
- NS_1 *is saturated,*
- $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$,

[1] See [22].

- $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[G]$, *and*

  moreover if we assume Θ *is regular in* V, *then*

- $\Theta^V = \omega_3$.

Proof. See [19, 5.1 Theorem] and [41, Theorem 4.49]. □

2.3 Universally Baire property

The notion of universally Baireness is introduced by Feng, Magidor, and Woodin [9]. We use the following definition of universally Baireness because we will work in choiceless models.

Definition 2.3.1. Let T and U be trees on $\omega^k \times \mathrm{ON}$, and let Z be a set. We say the pair (T, U) is Z*-absolutely complementing* if we have

$$V^{\mathrm{Col}(\omega,Z)} \models p[T] = \mathbb{R}^k \setminus p[U].$$

We say a set of reals A is Z*-universally Baire* if there is a Z-absolutely complementing pair (T, U) of trees such that $A = p[T] = \mathbb{R} \setminus p[U]$ in V. We call such a pair (T, U) Z*-absolutely complementing pair of trees for* A. We say a set of reals A is *universally Baire* if A is Z-universally Baire for any set Z.

In the AC context, being universally Baire is equivalent to being $<$ ON-universally Baire, i.e., κ-universally Baire for all $\kappa \in \mathrm{ON}$. However, in the absence of AC, the author does not know that they are still equivalent. Since the definition is enough powerful for our purpose, we use it.

Notation. Let Z be a set. Let A be a Z-universally Baire, and let (T, U) be a Z-absolutely complementing pair of trees for A. Let G be $\mathrm{Col}(\omega, Z)$-generic. Then we denote the canonical expansion of A to $V[G]$ by

$$A^G = p[T]^{V[G]},$$

or if G is clear from the context, we denote it by A^*. [2]

Note that the canonical expansion does not depend on the choice of an absolutely complementing pair of trees. And notice that since trees T and U witnessing that a set of reals A is Z-universally Baire is well-orderable, we do not need DC when we use the absoluteness of well-foundedness for T and U.

We shall use projective generic absoluteness with names for sets of reals in the proof of Theorem 1.0.1. For that, we need tree representations compatible with projections.

[2]For example, when we work in $V^{\mathrm{Col}(\omega,Z)}$, we use A^*.

Definition 2.3.2. Let $\Gamma \subset \bigcup_{1\leq k<\omega} \mathcal{P}(\mathbb{R}^k)$ be a pointclass of universally Baire sets of reals. We say that Γ is *productive* if

(1). Γ is closed under taking complements and projections, and

(2). for all $k < \omega$ and all $D \in \Gamma \cap \mathcal{P}(\mathbb{R}^{k+2})$, if the trees T and U on $\omega^{k+2} \times \mathrm{ON}$ witness that D is universally Baire and if

$$\tilde{T} = \{(s \restriction (k+1), (s(k+1), t)) \mid (s,t) \in T\},$$

then there is a tree $\tilde{U}$ on $\omega^{k+1} \times \mathrm{ON}$ such that for all sets Z,

$$V^{\mathrm{Col}(\omega,Z)} \models p[\tilde{T}] = \mathbb{R}^{k+1} \setminus p[\tilde{U}].$$

Lemma 2.3.1 is shown by an induction on the complexity of formulae.

Lemma 2.3.1. *Let Γ be productive, and let $A \in \Gamma$. Then any projective statement about A is absolute between V and any forcing extension of V.*

Note that Γ^∞ is productive under enough large cardinals.

Theorem 2.3.2 (Woodin)**.** *Assume that there is a proper class of Woodin cardinals. Then Γ^∞ is productive.*

For the main ingredients of Theorem 2.3.2, see [24], [25], [35], and [42].

2.4 $\mathsf{MM}^{*,++}$

In this section, we shall define $\mathsf{MM}^{*,++}$. The notations are mainly the same as [29]. We say $x \in \mathbb{R}$ *codes a transitive set* if

$$E_x = \{(n,m) \mid x(\langle n,m\rangle) = 0\}$$

is an extensional and well-founded relation on ω. Let us write WF for the set of reals coding a transitive set. Note that WF is the Π^1_1-complete set of reals[3]. If $x \in \mathrm{WF}$, then let π_x be the transitive collapse of the structure (ω, E_x) and let $\mathrm{decode}(x) = \pi_x(0)$. If $x, y \in \mathrm{WF}$, then we shall write $x \simeq y$ to express that $\mathrm{decode}(x) = \mathrm{decode}(y)$.

We say a function $f \colon \mathbb{R} \to \mathbb{R}$ is *universally Baire* if the graph of f is a universally Baire subset of $\mathbb{R}^2$.

Definition 2.4.1. We say a function $F \colon \mathrm{HC} \to \mathrm{HC}$ is *strongly universally Baire in the codes* if there is a universally Baire function $f \colon \mathbb{R} \to \mathbb{R}$ such that

[3] See [27, 4A].

(1). if $z \in \mathrm{HC}$ and $x \in \mathrm{WF}$ with $z = \mathrm{decode}(x)$, then $f(x) \in \mathrm{WF}$ and $F(z) = \mathrm{decode}(f(x))$;

let (T, U) witness that f is universally Baire with $f = p[T]$[4]. Then for all posets $\mathbb{P}$,

(2). $V^{\mathbb{P}} \models$ "$p[T]$ is a function from $\mathbb{R}$ to $\mathbb{R}$";

(3). $V^{\mathbb{P}} \models \forall\{x, x', y, y'\} \subset \mathbb{R}[(x, y), (x', y') \in p[T] \wedge x \simeq x' \longrightarrow y \simeq y']$.

Let $F\colon \mathrm{HC} \to \mathrm{HC}$ be strongly universally Baire in the codes as witnessed by f (and a pair (T, U) of trees). Let $\mathbb{P}$ be a poset, and let $g \subset \mathbb{P}$ be V-generic. Then F canonically extends a total map

$$F^{\mathbb{P},g}\colon V[g] \to V[g]$$

as follows. Let $X \in V[g]$. Let θ be any sufficiently large cardinal, let $H \subset \mathrm{Col}(\omega, \theta)$ be $V[g]$-generic, and let $x \in \mathbb{R} \cap V[g][H]$ be such that $X = \mathrm{decode}(x)$. Let $y \in \mathbb{R} \cap V[g][H]$ be such that $(x, y) \in p[T]$. We set $F^{\mathbb{P},g}(X) = \mathrm{decode}(y)$. Then $F^{\mathbb{P},g}$ is well-defined and does not depend on the choice of f, (T, U), θ, and H by the standard argument.

Mouse operators, like $x \mapsto x^\sharp$ and $x \mapsto M_n^\sharp(x)$, are examples of such strongly universally Baire functions F.

Definition 2.4.2. Let $F\colon \mathrm{HC} \to \mathrm{HC}$ be strongly universally Baire. Let $\theta \in \mathrm{ON}$, let $g \subset \mathrm{Col}(\omega, \theta)$ be V-generic, and let $\mathfrak{A} \in V[g]$ be transitive. We say $\mathfrak{A}$ is F-closed if

(1). $\mathfrak{A}$ is closed under $F^{\mathrm{Col}(\omega,\theta),g}$, and

(2). $F^{\mathrm{Col}(\omega,\theta),g} \restriction X \in \mathfrak{A}$ for every $X \in \mathfrak{A}$.

Mouse operators, like $x \mapsto x^\sharp$ and $x \mapsto M_n^\sharp(x)$, are examples of such F.

Definition 2.4.3 (Schindler, [29, Definition 2.8])**.** Let φ be a formula in the language $\mathcal{L}_{\dot{\in}, \dot{I}_{\mathsf{NS}}}$, and let $M \in V$. Let $\theta = \aleph_1 + |\mathrm{TC}(\{M\})|$. We say $\varphi(M)$ is *honestly consistent* if for every $F\colon \mathrm{HC} \to \mathrm{HC}$ which is strongly universally Baire in the codes, if $g \subset \mathrm{Col}(\omega, 2^\theta)$ is V-generic, then in $V[g]$ there is a transitive model $\mathfrak{A}$ such that

(1). $\mathfrak{A}$ is F-closed,

(2). $\mathfrak{A} \models \mathsf{ZFC}^-$,

(3). $(H_{\theta^+})^V \in \mathfrak{A}$,

[4]By amalgamating them, one can find a single class-sized witness for universally Baireness.

(4). $\dot{I}^{\mathfrak{A}}_{\mathsf{NS}} \cap V = \mathsf{NS}^V_{\omega_1}$, and

(5). $\mathfrak{A} \models \varphi(M)$.

Definition 2.4.4 (Schindler, [29, Definition 2.10])**.** Let κ be an infinite cardinal.

- *Martin's Maximum*$^{*,++}_{\kappa}$, abbreviated by $\mathsf{MM}^{*,++}_{\kappa}$, is the statement that whenever $\mathcal{M} = (M, \in, \vec{R})$ is a models, where M is transitive with $|M| \leq \kappa$ and $\vec{R}$ is a list of $\aleph_1$ relations on M and whenever φ is a Σ_1 formula in the language $\mathcal{L}_{\dot{\in}, \dot{I}_{\mathsf{NS}}}$ such that $\varphi(\mathcal{M})$ is honestly consistent, then $\Psi(\mathcal{M}, \varphi)$ holds true in V.
- $\mathsf{MM}^{*,++}$ is the statement that $\mathsf{MM}^{*,++}_{\kappa}$ holds for all κ.
- $\mathsf{BMM}^{*,++}$ is $\mathsf{MM}^{*,++}_{\aleph_1}$.

Asperó and Schindler [2, Definition 2.3 and Theorem 3.1] introduced a weaker notion, called 1-*honestly consistency*, and proved that MM is Σ_2-complete.

The same argument as in Lemma 2.1.1 shows that:

Theorem 2.4.1 (Schindler, [29, Theorem 2.11])**.** $\mathsf{MM}^{*,++}_{\kappa}$ *implies* $\mathsf{MM}^{++}_{\kappa}$*.*

2.5 Inner model theory

In this section, we shall introduce preliminaries for Theorem 2.5.5, which is proved by Schindler [29, Theorem 3.14].

Regarding fine structure theory and theory of mice, see [26], [31], and [34].

First, we introduce good properties for iteration strategies.

Definition 2.5.1 (Branch condensation, [28, Definition 2.14])**.** Let Σ be an iteration strategy for M. Then Σ has *branch condensation* if for any two iteration trees $\mathcal{T}$ and $\mathcal{U}$ on M and any branch c of $\mathcal{U}$ if

(1). $\mathcal{T}$ and $\mathcal{U}$ are according to Σ,

(2). $\mathrm{lh}(\mathcal{U})$ is limit and $\mathrm{lh}(\mathcal{T}) = \gamma + 1$,

(3). for some $\pi \colon \mathcal{M}^{\mathcal{U}}_c \to_{\Sigma_1} \mathcal{M}^{\mathcal{T}}_{\gamma}$, $\pi^{\mathcal{T}}_{0,\gamma} = \pi \circ \pi^{\mathcal{U}}_c$,

then $c = \Sigma(\mathcal{U})$.

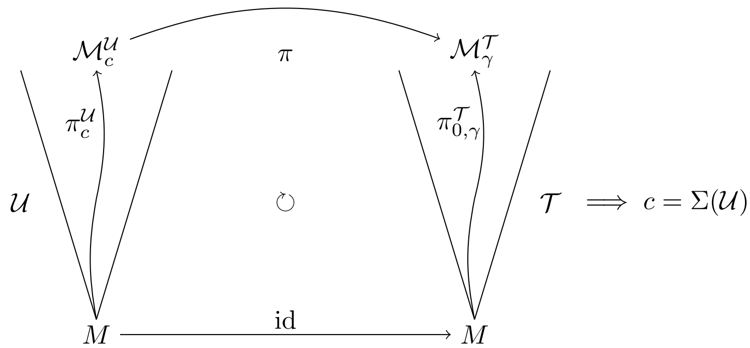

Figure: branch condensation

For the definition of the hull of a normal tree and the hull of a stack, see [28, Definition 1.29 and 1.30]

Definition 2.5.2 (Hull condensation, [28, Definition 1.31])**.** Let Σ be an iteration strategy for M. We say Σ has *hull condensation* if for any two stacks $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ on M if $\vec{\mathcal{T}}$ is according to Σ and $(M,\vec{\mathcal{U}})$ is a hull of $(M,\vec{\mathcal{T}})$, then $\vec{\mathcal{U}}$ is according to Σ.

Let Σ be an iteration strategy for M. Then we let

$I(M,\Sigma) = \{(\vec{\mathcal{T}}, N) \mid \vec{\mathcal{T}}$ is on M according to Σ with last model N and $\pi^{\vec{\mathcal{T}}}$ exists $\}$

Definition 2.5.3 ([28, Definition 2.35])**.** Let Σ be an iteration strategy for M.

(1). (**Weakly positional**) Σ is *weakly positional* is whenever $(\vec{\mathcal{T}}, N) \in I(M,\Sigma)$ and $(\vec{\mathcal{U}}, N) \in I(M,\Sigma)$, then $\Sigma_{N,\vec{\mathcal{T}}} = \Sigma_{N,\vec{\mathcal{U}}}$.

(2). (**Positional**) Σ is *positional* if whenever $(\vec{\mathcal{T}}, N) \in I(M,\Sigma)$, $\Sigma_{N,\vec{\mathcal{T}}}$ is weakly positional.

(3). (**Weakly pullback consistent**) Σ is *weakly pullback consistent* if $\Sigma^{\vec{\mathcal{T}}} = \Sigma$ whenever $\vec{\mathcal{T}}$ is a stack according to Σ and $\pi^{\vec{\mathcal{T}}}$ exists.

(4). (**Pullback consistent**) Σ is *pullback consistent* if whenever $(\vec{\mathcal{T}}, N) \in I(M,\Sigma)$, $\Sigma_{N,\vec{\mathcal{T}}}$ is weakly pullback consistent.

(5). (**Weakly commuting**) Σ is *weakly commuting* if whenever $(\vec{\mathcal{T}}, N) \in I(M,\Sigma)$ and $(\vec{\mathcal{U}}, N) \in I(M,\Sigma)$, then $\pi^{\vec{\mathcal{T}}} = \pi^{\vec{\mathcal{U}}}$.

(6). (**Commuting**) Σ is *commuting* if whenever $(\vec{\mathcal{T}}, N) \in I(M,\Sigma)$, $\Sigma_{N,\vec{\mathcal{T}}}$ is weakly commuting.

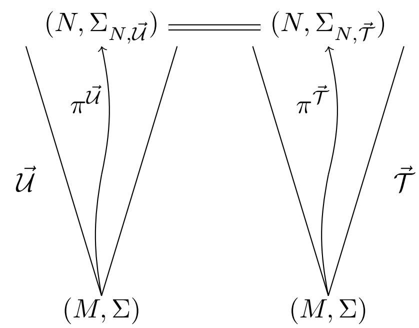

Figure: weak positionality

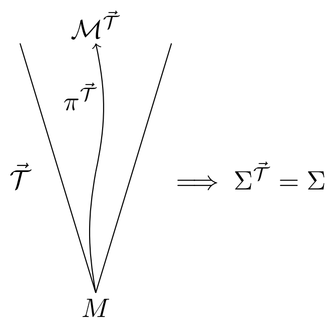

Figure: weak pullback consistency

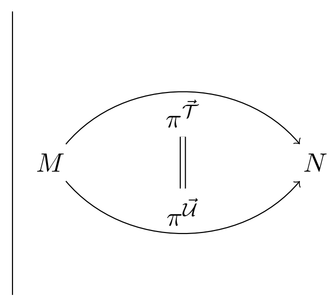

Figure: weak commutativity

The relationship between these properties is as follows.

Lemma 2.5.1 ([28, Proposition 2.36])**.** *Let Σ be an iteration strategy for M. Then the following holds.*

(1). If Σ has hull condensation, then it is pullback consistent.

(2). If Σ is positional and pullback consistent, then it is commuting.

The following theorem is due to Woodin.

Theorem 2.5.2 (Woodin, [34, Theorem 7.14])**.** *Let Σ be am $(\omega_1, \omega_1 + 1)$-iteration strategy for M, and suppose that $\kappa < \delta$ are countable ordinals such that*

$$M \models \mathsf{ZF}^- + \delta \text{ is Woodin },$$

then there is a $\mathbb{Q} \subset V_\delta^M$ such that

(1). $M \models \mathbb{Q}$ is a δ-c.c. complete Boolean algebra, and

(2). for any real x, there is a countable iteration tree $\mathcal{T}$ on M according to Σ based on the window (κ, δ) with the last model $\mathcal{M}_\alpha^{\mathcal{T}}$ such that $i_{0,\alpha}^{\mathcal{T}}$ exists and x is $i_{0,\alpha}^{\mathcal{T}}(\mathbb{Q})$-generic over $\mathcal{M}_\alpha^{\mathcal{T}}$.

We call the poset $\mathbb{Q}$ Woodin's extender algebra of M in the window (κ, δ) and denote it by $\mathbb{B}^M_{(\kappa,\delta)}$. In the case that $\kappa = 0$, we denote it by $\mathbb{B}^M_\delta$

The process of iterating M to make a real generic is called genericity iteration.

The following concept is due to Woodin.

Definition 2.5.4 (Woodin, term capturing)**.** Let $A \subset \mathbb{R}$. Let M be a countable premouse, let $\delta \in M$, and assume that $M \models \mathsf{ZFC}^- +$ "δ is a Woodin cardinal". Let Σ be an $(\omega_1, \omega_1 + 1)$-iteration strategy for M. Let $\tau \in M^{\mathrm{Col}(\omega,\delta)}$.

Then we say $(M, \delta, \tau, \Sigma)$ *captures* A if the following hold:

(1). Σ satisfies hull condensation, and branch condensation, and is positional;

(2). if $\mathcal{T}$ is an iteration tree on M of successor length $\theta + 1 < \omega_1$ which is according to Σ such that the main branch $[0, \theta]_{\mathcal{T}}$, if

$$\pi^{\mathcal{T}}_{0,\theta} \colon M \to \mathcal{M}^{\mathcal{T}}_\theta$$

is the iteration map, and if $g \in V$ is $\mathrm{Col}(\omega, \pi^{\mathcal{T}}_{0,\theta}(\delta))$-generic over $\mathcal{M}^{\mathcal{T}}_\theta$, then

$$\pi^{\mathcal{T}}_{0,\theta}(\tau)^g = A \cap \mathcal{M}^{\mathcal{T}}_\theta[g].$$

We say that (M, Σ) captures A if there is $\tau, \delta \in M$ such that $(M, \delta, \tau, \Sigma)$ captures A.

Term capturing brings out the Suslin representations of A and its complement. Let us describe the construction. Let $(M, \delta, \tau, \Sigma)$ capture A.

We let

$$I(M, \Sigma, \aleph_1) = \{(\vec{\mathcal{T}}, N) \mid \vec{\mathcal{T}} \text{ is on } M \text{ of length} < \omega_1 \text{ according to } \Sigma \\ \text{with last model } N \text{ and } \pi^{\vec{\mathcal{T}}} \text{ exists } \}$$

Then by Theorem 2.5.2, we have that

$$A = \bigcup \{\pi^{\vec{\mathcal{T}}}(\tau)^g \mid \exists N[(\vec{\mathcal{T}}, N) \in I(M, \Sigma, \aleph_1) \text{ and } g \text{ is } \mathrm{Col}(\omega, \pi^{\vec{\mathcal{T}}}(\delta))\text{-generic over } N]\}$$

Now we let

$$\mathcal{F} = \{N \mid \text{ there is } \vec{\mathcal{T}} \text{ such that } (N, \vec{\mathcal{T}}) \in I(M, \Sigma, \aleph_1)\},$$

and for N, N' in $\mathcal{F}$, we let

$N \prec^* N' \iff N'$ is a non-dropping iterate of N according to the tale strategy Σ_N.

Note that Σ_N is well-defined since by the positionality of Σ, Σ_N is uniquely determined only by the final model N.

For $N, N' \in \mathcal{F}$ with $N \prec^* N'$, we let

$$\pi_{N,N'} := \text{ the unique iteration map from } N \text{ to } N'.$$

Note that $\pi_{N,N'}$ is well-defined, since Σ is commuting by Lemma 2.5.1. By the comparison lemma, $\prec^*$ is directed. Moreover, $\prec^*$ is countably directed again by the comparison lemma. Hence the direct limit is well-founded.

Therefore, we let

$$M_\infty^{<\omega_1}, (\pi_{N,\infty} \colon N \in \mathcal{F})$$

be the direct limit of the directed system

$$(\mathcal{F}, \prec^*, (\pi_{N,N'} \colon N \prec^* N' \wedge N, N' \in \mathcal{F})).$$

We define tree representations for A and its complement as follows.

We let $x \in p[T]$ if there is an iteration tree $\mathcal{T}$ on M of length $\theta + 1 < \omega_1$ such that

(1). $\mathcal{T}$ has no drops at all,

(2). there is a system $(\psi_i \colon i \leq \theta)$ of elementary embeddings such that $\psi_0 = \pi_{M,\infty}$ and for all $i \leq_{\mathcal{T}} j \leq \theta$, $\psi_i \colon \mathcal{M}_i^{\mathcal{T}} \to M_\infty^{<\omega_1}$ and $\psi_j \circ \pi_{i,j}^{\mathcal{T}} = \psi_i$, and

(3). there is some g which is $\mathrm{Col}(\omega, \pi_{0,\theta}^{\mathcal{T}}(\delta))$-generic over $\mathcal{M}_\theta^{\mathcal{T}}$ such that $x \in \pi_{0,\theta}^{\mathcal{T}}(\tau)^g$.

We let $x \in p[U]$ be defined in exactly the same way except for that in (3), we replace "$x \in \pi_{0,\theta}^{\mathcal{T}}(\tau)^g$" by "$x \notin \pi_{0,\theta}^{\mathcal{T}}(\tau)^g$". Then we have the following.

Lemma 2.5.3. *Let (T, U) be as above. Then*

$$A = p[T] \text{ and } \mathbb{R} \setminus A = p[U].$$

Note that the fact that Σ has branch condensation is used here to organize trees T and U.

Again suppose that $(M, \delta, \tau, \Sigma)$ captures A. Let $\kappa \geq \aleph_1$ be a cardinal. Also, suppose that there is a $(\kappa^+, \kappa^+ + 1)$-iteration strategy $\tilde{\Sigma}$ for M extending Σ and such that $\tilde{\Sigma}$ has hull condensation, and branch condensation, and is positional. Note that such $\tilde{\Sigma}$ is unique if it exists. Then we let

$$I(M, \tilde{\Sigma}, \kappa^+) = \{(\vec{\mathcal{T}}, N) \mid \vec{\mathcal{T}} \text{ is on } M \text{ of length} < \kappa^+ \text{ according to } \tilde{\Sigma} \\ \text{with last model } N \text{ and } \pi^{\vec{\mathcal{T}}} \text{ exists } \}$$

We may define the directed system

$$(\mathcal{F}^{<\kappa^+}, \prec^*, (\pi_{N,N'} \colon N \prec^* N' \wedge N, N' \in \mathcal{F}^{<\kappa^+})),$$

and the direct limit

$$M_\infty^{<\kappa^+}, (\pi_{N,\infty} \colon N \in \mathcal{F}^{<\kappa^+})$$

in the same way by replacing $I(M, \Sigma, \aleph_1)$ with $I(M, \tilde{\Sigma}, \kappa^+)$.

Also, we may defined (T^κ, U^κ) in the same way as (T, U) defined. Our claim is the following.

Lemma 2.5.4 ([29, pp. 12]). *Let (T^κ, U^κ) as above. Then (T^κ, U^κ) witnesses that A is κ-universally Baire.*

Proof. Let $g \subset \mathrm{Col}(\omega, \kappa)$ be V-generic. It is enough to see that

$$V[g] \models p[T^\kappa] \cup p[U^\kappa] = \mathbb{R}.$$

Let $x \in V[g] \cap \mathbb{R}$. Let $\tau^g = x$. Inside of V, we shall construct an iteration tree $\mathcal{T}$ on M of length $\leq \kappa^+ + 1$ such that $\mathcal{T}$ has no drops at all and is according to Σ by generically genericity iteration.

Having defined $\mathcal{T} \restriction i+1$, if there is some $E_\nu^{\mathcal{M}_i^\mathcal{T}} \neq \emptyset$ which is total on $\mathcal{M}_i^\mathcal{T}$ and such that for some $p \in \mathrm{Col}(\omega, \kappa)$,

$$p \Vdash^{V}_{\mathrm{Col}(\omega,\kappa)} \dot{\tau} \text{ violates the axiom of } \mathbb{B}^{\mathcal{M}_i^\mathcal{T}}_{\pi^\mathcal{T}_{0,i}(\delta)} \text{ associated with } E_\nu^{\mathcal{M}_i^\mathcal{T}}, \tag{2.1}$$

then pick $E_\nu^{\mathcal{M}_i^\mathcal{T}}$ for the least such ν as the next extender. If there is no such ν, then stop the construction. This defines an iteration tree $\mathcal{T}$ on M.

We claim that $\mathrm{lh}(\mathcal{T}) < \kappa^+$. Suppose that $\mathrm{lh}(\mathcal{T}) = \kappa^+ + 1$. Then since $[0, \kappa^+)_\mathcal{T}$ is a club subset of κ^+ and $|\mathrm{Col}(\omega, \kappa)| = \kappa$, there is $p \in \mathrm{Col}(\omega, \kappa)$ and a stationary set $S \subset [0, \kappa^+)_\mathcal{T}$ such that (2.1) is true for all $i \in S$.

Let $g' \subset \mathrm{Col}(\omega, \kappa)$ be V-generic with $p \in g'$. Then S is also stationary in $V[g']$, since $\mathrm{Col}(\omega, \kappa)$ is small. Therefore, the usual Skolem hull argument in [34, Theorem 7.14] gives us $i \in S$ such that $\tau^{g'}$ satisfies the axiom of $\mathbb{B}^{\mathcal{M}_i^\mathcal{T}}_{\pi^\mathcal{T}_{0,i}(\delta)}$ associated with $E_{\nu_i}^{\mathcal{M}_i^\mathcal{T}}$. This is a contradiction!

Let $\theta = \mathrm{lh}(\mathcal{T}) < \kappa^+$, and let $\psi_i = \pi_{\mathcal{M}_i^{\mathcal{T}},\infty}$. Then we have that x is $\mathbb{B}^{\mathcal{M}_\theta^{\mathcal{T}}}_{\pi^{\mathcal{T}}_{0,\theta}(\delta)}$-geneirc over $\mathcal{M}_\theta^{\mathcal{T}}$. Let $g \subset \mathrm{Col}(\omega, \pi^{\mathcal{T}}_{0,\theta}(\delta))$ be $\mathcal{M}_\theta^{\mathcal{T}}$-generic such that $x \in \mathcal{M}_\theta^{\mathcal{T}}[g]$. Then $\mathcal{T}$, $(\psi_i : i < \theta)$, and g witness that

$$x \in p[T^\kappa] \cup p[U^\kappa].$$

This finishes the proof. $\square$

Definition 2.5.5. Let $A \subset \mathbb{R}$. We say (M, Σ) *strongly captures* A if (M, Σ) captures A and for all $\kappa \geq \aleph_1$, there is some $(\kappa^+, \kappa^+ + 1)$-iteration strategy $\tilde{\Sigma}$ for M extending Σ and such that $\tilde{\Sigma}$ has hull condensation, and branch condensation, and is positional.

Note that lemma 2.5.4 shows that if A is strongly captured by some (N, Σ), then A is $<$ ON-universally Baire.

We will make use of Σ-mice to produce A-iterable conditions for $A \in \Gamma^\infty$. For more details for Σ-mice, see [30].

Definition 2.5.6 ([29, pp. 13])**.** Let M be a premouse, and let Σ be an iteration strategy for M. Let X be a self-well-ordered transitive set with $M \in L_1(X)$. We say $\mathcal{N}$ is a Σ-*premouse* over X if $\mathcal{N}$ is a J-model of the form $J_\alpha[\vec{E}, \vec{S}, X]$ satisfying the following;

(1). $\vec{E}$ codes a sequence of extenders satisfying the usual axioms for fine extender sequences. (See [26], [30], and [34].)

(2). $\vec{S}$ codes a partial iteration strategy for M as follows:

let $\gamma < \gamma + \delta \leq \alpha$ be such that $J_\gamma[\vec{E}, \vec{S}, X] \models \mathsf{ZFC}^-$ and γ is the largest cardinal of $J_{\gamma+\delta}[\vec{E}, \vec{S}, X]$. Suppose that $\mathcal{T} \in J_\gamma[\vec{E}, \vec{S}, X]$ is $J_\gamma[\vec{E}, \vec{S}, X]$-least such that $\mathcal{T}$ is an iteration tree on M of limit length, $\mathcal{T}$ is according to $\vec{S} \restriction \gamma$, but $(\vec{S} \restriction \gamma)(\mathcal{T})$ is undefined, Suppose also that $\delta = \mathrm{lh}(\mathcal{T})$, and δ does not have measurable cofinality in $J_{\gamma+\delta}[\vec{E}, \vec{S}, X]$. Then $\Sigma(\mathcal{T})$ is defined, and $\vec{S}(\gamma + \delta)$ is an amenable code for $(\mathcal{T}, \Sigma(\mathcal{T}))$.

Let $\mathcal{N}$ be a Σ-premouse over X, and let Γ be an iteration strategy for $\mathcal{N}$. Then we say Γ *moves* Σ *correctly* if every iterate $\mathcal{N}'$ of $\mathcal{N}$ according to Γ is again a Σ-premouse over X.

We say $\mathcal{N}$ is a Σ-*mouse over* X if for every sufficiently elementary $\pi : \bar{\mathcal{N}} \to \mathcal{N}$ with $\bar{\mathcal{N}}$ being countable and transitive, there is some iteration strategy Γ for $\bar{\mathcal{N}}$ which witnesses $\bar{\mathcal{N}}$ is $\omega_1 + 1$-iterable and which moves Σ correctly.

The next is the Σ-mouse version of $M_n^\sharp$.

Definition 2.5.7. Let M be a countable premouse, and let Σ be an iteration strategy for N. Let X be a self-well-ordered transitive set with $N \in L_1(X)$, and let $n \in \omega$. Then we denote by

$$M_n^{\sharp,\Sigma}(X)$$

the unique Σ-mouse over X which is sound above X, not n-small above X, and such that every proper initial segment is n-small above X, if it exists.

The next theorem is crucial for Theorem 1.0.1.

Theorem 2.5.5 (Schindler, [29, Theorem 3.14]). *Let $A \in \Gamma^\infty$, and suppose that $(M, \delta, \tau, \Sigma)$ captures A. Let $X \in \mathrm{HC}$, and suppose that*

$$N = M_2^{\sharp,\Sigma}(N, X)$$

exists. Let δ_0 be the bottom Woodin cardinal of N, let $g_0 \in V$ be $(\mathrm{Col}(\omega_1, <\delta_0))^N$-generic over N, and let $g_1 \in V$ be $\mathbb{Q}$-generic over $N[g_0]$, where $\mathbb{Q} \in N[g_0]$ is the standard c.c.c. forcing for Martin's Axiom. Let κ be the critical point of the top extender of N. Then

$$p = ((N||\kappa)[g_0, g_1], \in, (\mathsf{NS}_{\omega_1})^{(N||\kappa)[g_0,g_1]})$$

is an A-iterable $\mathbb{P}_{\max}$ precondition.

Chapter 3

A strategy to force $\mathsf{MM}^{*,++}$ and the BMM model

In this chapter, we shall describe a general framework to force $\mathsf{MM}^{*,++}$ using $\mathbb{P}_{\max}$ and give Woodin's BMM model as a specific example.

Our ultimate goal is to find a determinacy model V such that $V[G] \models \mathsf{MM}^{*,++}$ where $G \subset \mathbb{P}_{\max}$ is V-generic. We shall give a sufficient condition for that and, for now, we will not argue forcing AC. Let $\mathcal{M} = \dot{\tau}^G \in V[G]$ be a transitive structure with $\aleph_1$ relations and let φ be a Σ_1 formula in $\mathcal{L}_{\dot{\in}, \dot{I}_{\mathsf{NS}}}$. Suppose that $\varphi(\mathcal{M})$ is honestly consistent in $V[G]$, and suppose that $p_0 \in G$ forces the fact.

We say a pair $(p, g, \bar{\tau})$ is *good* for p_0, τ, and φ if the following holds:

(1). $p \in \mathbb{P}_{\max}$, $g, \bar{\tau} \in p$, and $p_0 \in g$;

(2). $\bar{\tau}$ is a $\mathbb{P}_{\max}$-name;

(3). $g \subset \mathbb{P}_{\max}$ is a filter, $p <_{\mathbb{P}_{\max}} q$ for all $q \in g$;

(4). $p \models \varphi(\bar{\tau}^g)$;

(5). p forces that there is an elementary embedding $\pi_p \colon \bar{\tau}^g \to \tau$ such that $\pi_p \restriction \omega_1^p = \mathrm{id}$, and for any $j \colon p \to p'$ which is an iteration of p of length $\leq \omega_1$ induced by a $\mathbb{P}_{\max}$-generic $\dot{G}$, there is an elementary embedding $\pi_{p'} \colon j(\bar{\tau})^{j(g)} \to \dot{\tau}$ such that $\pi_{p'} \restriction \omega_1^{p'} = \mathrm{id}$ and $\pi_p = j \circ \pi_{p'}$.

Condition (5) is a generalization of A-iterability for a set of reals A. *The density of good conditions* is the statement that for every p_0, τ, and φ as above, there is a good pair $(p, g, \bar{\tau})$ for p_0, τ, and φ.

The density of good conditions enables us to force $\mathsf{MM}^{*,++}$.

Theorem 3.0.1. *Assume that $V \models \mathsf{AD}^+$, and assume that the density of good conditions holds. Then*

$$V^{\mathbb{P}_{\max}} \models \mathsf{MM}^{*,++}.$$

Proof. Suppose not. Let (p_0, τ, φ) be such that:

(1). $p_0 \in \mathbb{P}_{\max}$, τ is a $\mathbb{P}_{\max}$-name, and φ is a Σ_1 formula in $\mathcal{L}_{\dot{\in},\dot{I}_{\mathsf{NS}}}$;

(2). $p_0 \Vdash_{\mathbb{P}_{\max}}$ "τ is a transitive structure with $\aleph_1$ relations";

(3). $p_0 \Vdash_{\mathbb{P}_{\max}}$ "$\varphi(\tau)$ is honestly consistent"; and

(4). $p_0 \Vdash_{\mathbb{P}_{\max}} \neg\Psi(\tau, \varphi)$.

Then by our hypothesis, let $(p, g, \bar{\tau})$ be a good pair for p_0, τ, and φ.

Let $G \subset \mathbb{P}_{\max}$ be V-generic with $p \in G$, and let $j \colon p \to p^*$ be the iteration of p of length ω_1 induced by G. Let $\tau^* = j(\bar{\tau})$, and let $g^* = j(g)$. Then by the goodness of $(p, g, \bar{\tau})$, there is an elementary embedding $\pi_G \colon \tau^{*,g^*} \to \tau^G$. And since $p \models \varphi(\bar{\tau}^g)$, we have that

$$V[G] \models \varphi(\tau^{*,g^*}).$$

Hence $\pi_G \colon \tau^{*,g^*} \to \tau^G$ witnesses that $\Psi(\tau^G, \varphi)$ holds in $V[G]$. This is a contradiction! Therefore, $V[G] \models \mathsf{MM}^{*,++}$. □

If we restrict the size of $\mathcal{M}$ is at most $\aleph_2$, then A-iterability for a set of reals $A \in V$ is enough to show the density of good conditions. However, if we allow the size of $\mathcal{M}$ to be $\aleph_3$, we may need densely many $\mathcal{A}$-iterable conditions for a set of sets of reals $\mathcal{A}$ to show the density of good condition. AD_2 is one candidate for a base theory for finding such iterable conditions. Since this paper deals with only structures of size at most $\aleph_2$, AD^+ is sufficient for our purpose.

In the rest of this chapter, as a concrete example, we shall give Woodin's BMM model and show $\mathsf{BMM}^{*,++}$ holds there. The proof essentially shows the density of good conditions.

Theorem 3.0.2 (Woodin, [41, Theorem 10.99])**.** *Let N be the minimal model containing $\mathbb{R}$ and closed under the $M_1^\sharp$ operation, and assume AD. Then*

$$N^{\mathbb{P}_{\max}} \models \mathsf{ZFC} + \mathsf{BMM}^{*,++}.$$

Proof. Note that
$$N^{\mathbb{P}_{\max}} \models \mathsf{ZFC}$$
by the minimality of N. Suppose that
$$N^{\mathbb{P}_{\max}} \models \neg\mathsf{BMM}^{*,++}.$$
Let us assume that

(1). $p_0 \in \mathbb{P}_{\max}$, $M \in p_0$;

(2). φ is a Σ_1 formula in $\mathcal{L}_{\dot{\in},\dot{I}_{\mathsf{NS}}}$;

(3). $p_0 \models$ "M is a transitive structure of size $\aleph_1$ with $\aleph_1$-many predicates";

(4). $p_0 \Vdash^N_{\mathbb{P}_{\max}}$ "$\varphi(j(\check{M}))$ is honestly consistent, where $j\colon p_0 \to p'$ is the iteration of p_0 of length ω_1 induced by a $\mathbb{P}_{\max}$ generic $\dot{G}$" ; and

(5). $p_0 \Vdash^N_{\mathbb{P}_{\max}} \neg\varphi(j(\check{M}))$, where j is same as above.

Let $G \subset \mathbb{P}_{\max}$ be N-generic with $p_0 \in G$. Let $g \subset \mathrm{Col}(\omega, 2^{2^{\omega_1}})$ be $N[G]$-generic. Let $j\colon p_0 \to p'$ be the iteration of p_0 of length ω_1 induced by G. We claim that in $N[G][g]$, there is $p \in \mathbb{P}_{\max}$ such that $p <_{\mathbb{P}_{\max}} p_0$ as witnessed by j, and $p \models j(M)$.

Let $\mathfrak{A} \in N[G][g]$ be an $M_1^\sharp$-closed witness to the fact that $\varphi(j(M))$ is honestly consistent. Let $X \in \mathfrak{A}$ be transitive and such that

(1). $(\mathcal{P}(\omega_1) \cap \mathfrak{A}) \cup \{(\mathsf{NS}_{\omega_1})^{\mathfrak{A}}\} \in X$, and

(2). $X \models \varphi(j(M))$.

Let $M = M_1^\sharp(X)$. Let g be M-generic making NS_{ω_1} presaturated and forcing $\mathsf{MA}_{\aleph_1}$. Let
$$p = (M[g], \in, \mathsf{NS}_{\omega_1}^{M[g]}, A_G).$$
Since we have
$$\mathsf{NS}_{\omega_1}^{M[g]} \cap N[G] = \mathsf{NS}_{\omega_1}^{N[G]},$$
$p <_{\mathbb{P}_{\max}} p_0$ as witnessed by j, and by construction, $p \models \varphi(j(M))$. Hence p is the desired condition.

Now, let Y be a countable elementary submodel of a large enough fragment of $N[G]$ containing everything relevant. Let P be the transitive collapse of Y. Let $g' \in N[G]$ be P-generic for $\mathrm{Col}(\omega, 2^{2^{\omega_1^P}})$. Then in $P[g']$, there is a condition $p <_{\mathbb{P}_{\max}} p_0$ such that
$$p \models \varphi(j(M))$$

where $j\colon p_0 \to p'$ witnesses that $p <_{\mathbb{P}_{\max}} p_0$. Since $P[g']$ is closed under sharps, p is iterable also in $N[G]$. Moreover, since $\mathbb{P}_{\max}$ is σ-closed, $p \in N$. Finally,

$$p \Vdash^{N}_{\mathbb{P}_{\max}} \varphi(j(M))$$

where j is the generic iteration map on p induced by a $\mathbb{P}_{\max}$-generic. This is a contradiction! Therefore, $N^{\mathbb{P}_{\max}} \models \mathsf{BMM}^{*,++}$. $\square$

Chapter 4

Proof of the main theorem

We prove the main theorem.

Theorem 4.0.1. *Assume that*

(1). $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$ + "Θ *is regular*" + "*Every set of reals is universally Baire*";

(2). $\Gamma^\infty (= \mathcal{P}(\mathbb{R}))$ *is productive;*

(3). $V^{\mathbb{P}_{\max}} \models \omega_2\text{–}\mathsf{DC}$ *and* $V^{\mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1)} \models \mathsf{ZFC}$*; and*

(4). *for every* $A \in \Gamma^\infty$ *there is some* (N, Σ) *which strongly captures* A *and the function* $F \colon X \mapsto M_2^{\sharp, \Sigma}(X)$*, where* $X \in \mathrm{HC}$ *is self-well-ordered and* $N \in L_1(X)$*, is well-defined, total, and strongly universally Baire in the codes.*

Then

$$V^{\mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1)} \models \mathsf{MM}^{*,++}_{\mathfrak{c}}.$$

Proof. Let us fix a standard coding of $\mathbb{P}_{\max}$-conditions into reals and identify a $\mathbb{P}_{\max}$-condition and its code. For a filter $h \subset \mathbb{P}_{\max}$, we define

$$\mathbb{R}_h = \bigcup_{p \in h} \mathbb{R} \cap p.$$

For V-generic $G \subset \mathbb{P}_{\max}$ and $V[G]$-generic $H \subset \mathrm{Add}(\omega_3, 1)$, we note that

$$\mathbb{R}_G = \mathbb{R} \cap V = \mathbb{R} \cap V[G] = \mathbb{R} \cap V[G][H],$$

since $\mathbb{P}_{\max}$ is σ-closed and $\mathrm{Add}(\omega_3, 1)$ is ω_3-closed. Also, note that

$$\mathcal{P}(\mathbb{R}) \cap V[G] = \mathcal{P}(\mathbb{R}) \cap V[G][H],$$

since $\mathrm{Add}(\omega_3, 1)$ is ω_3-closed and $2^{\aleph_0} = \aleph_2$ in $V[G]$. Now suppose that

$$V^{\mathbb{P}_{\max} * \mathrm{Add}(\omega_3,1)} \models \neg \mathsf{MM}^{*,++}_{\mathfrak{c}}.$$

Assume that we have the following.

(1). φ is a Σ_1 formula;

(2). $\dot{\mathcal{M}} = (\dot{M}, \in, \dot{\vec{R}})$ is a $\mathbb{P}_{\max}$-name for a transitive structure, and $\dot{f} \in V^{\mathbb{P}_{\max}}$;

(3). $(p, \dot{s}) \in \mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1)$;

(4). $p \Vdash_{\mathbb{P}_{\max}} \dot{f} \colon \mathbb{R} \to \dot{M}$ is bijective; and

(5). $(p, \dot{s}) \Vdash_{\mathbb{P}_{\max} * \mathrm{Add}(\omega_3,1)} \varphi(\dot{\mathcal{M}})$ is honestly consistent; and

(6). $(p, \dot{s}) \Vdash_{\mathbb{P}_{\max} * \mathrm{Add}(\omega_3,1)} \neg \Psi(\dot{\mathcal{M}}, \varphi)$.

We shall derive a contradiction by finding $q <_{\mathbb{P}_{\max}} p$ such that

$$(q, \dot{s}) \Vdash_{\mathbb{P}_{\max} * \mathrm{Add}(\omega_3,1)} \Psi(\dot{\mathcal{M}}, \varphi).$$

Let

$$B_{=} = \{(p, x, y) \in \mathbb{R}^3 \mid p \in \mathbb{P}_{\max} \wedge p \Vdash_{\mathbb{P}_{\max}} \dot{f}(x) = \dot{f}(y)\},$$

let

$$B_{\in} = \{(p, x, y) \in \mathbb{R}^3 \mid p \in \mathbb{P}_{\max} \wedge p \Vdash_{\mathbb{P}_{\max}} \dot{f}(x) \in \dot{f}(y)\},$$

and let

$$B_{\vec{R}} = \{(p, x, y) \in \mathbb{R}^3 \mid p \in \mathbb{P}_{\max} \wedge x \in \mathrm{WO} \wedge p \Vdash_{\mathbb{P}_{\max}} \dot{f}(y) \in R_{|x|_{\mathrm{WO}}}\}.$$

For each formula and for each $\vec{z} \in \mathbb{R}^{<\omega}$, let $E_{\psi,\vec{z}}$ be a set of conditions $p \in \mathbb{P}_{\max}$ such that

(1). p decides the sentence "$\dot{\mathcal{M}} \models \exists x \psi(x, \dot{f}(\check{\vec{z}}))$", and

(2). if $p \Vdash_{\mathbb{P}_{\max}} \dot{\mathcal{M}} \models \exists x \psi(x, \dot{f}(\check{\vec{z}}))$, then there is $x \in p$ such that

$$p \Vdash_{\mathbb{P}_{\max}} \dot{\mathcal{M}} \models \psi(\dot{f}(\check{x}), \dot{f}(\check{\vec{z}})).$$

Note that $E_{\psi,\vec{z}}$ is also a dense subset of $\mathbb{P}_{\max}$. Let

$$E = \{(p, \ulcorner \psi \urcorner, \vec{z}) \mid p \in E_{\psi,\vec{z}}\}.$$

Suppose a pair (q, h) satisfies:

(1). $q = (N, \in, J, a) \in \mathbb{P}_{\max}$, $h \in q$ is a filter in $\mathbb{P}_{\max}$;

(2). $(N, \in) \models \mathsf{ZFC}^-$

(3). $q <_{\mathbb{P}_{\max}} p$ for all $p \in h$; and

(4). q is $(B_= \oplus B_\in \oplus B_{\vec{R}} \oplus E)$-iterable.

We define a structure $(\mathbb{R}_h/{\sim}, \tilde{\in}, \tilde{R})$ as follows. For $x, y \in \mathbb{R}_h$ and for $\alpha < \omega_1^q$, we define

$$x{\sim}y \iff \exists r \in h\, (r, x, y) \in B_=,$$

and

$$x \tilde{\in} y \iff \exists r \in h\, (r, x, y) \in B_\in,$$

and

$$\tilde{R}_\alpha(x) \iff \exists r \in G\, \exists w \in \mathrm{WO}_\alpha\, (r, w, x) \in B_{\vec{R}}.$$

Since q is $(B_= \oplus B_\in \oplus B_{\vec{R}} \oplus E)$-iterable and h is a filter, the quotient $(\mathbb{R}_h/{\sim}, \tilde{\in}, \tilde{R})$ is well-defined. Note that the relation $\tilde{\in}$ is extensional and well-founded since h is a filter and $\mathbb{P}_{\max}$ is σ-closed. Then we let

$$\sigma_h \colon \mathcal{M}_h := (M_h, \in, \vec{R}_h) \simeq (\mathbb{R}_h/{\sim}, \tilde{\in}, \tilde{R})$$

be the uncollapsing map.

Now we say a pair (q, h) is 2-*good*[1] if (q, h) satisfies:

(1). $q = (N, \in, J, a) \in \mathbb{P}_{\max}$, $h \in q$ is a filter in $\mathbb{P}_{\max}$;

(2). $(N, \in) \models \mathsf{ZFC}^-$

(3). $q <_{\mathbb{P}_{\max}} p$ for all $p \in h$;

(4). q is $(B_= \oplus B_\in \oplus B_{\vec{R}} \oplus E)$-iterable;

(5). $q \models \varphi(\mathcal{M}_h)$; and

(6). for every formula ψ and for every $\vec{z} \in (\mathbb{R}_h)^{<\omega}$, $E_{\psi, \vec{z}} \cap h \neq \emptyset$.

Note that "being a 2-good pair" is Σ^1_2 statement in $B_=$, $B_\in$, $B_{\vec{R}}$, and E. We shall find a 2-good pair in the generic extension of V by $\mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1) * \mathrm{Col}(\omega, 2^{\omega_2})$, and using Lemma 2.3.1, we shall find a 2-good pair in V.

Claim. Let $G \subset \mathbb{P}_{\max}$ be V-generic with $p \in G$, let $H \subset \mathrm{Add}(\omega_3, 1)$ be $V[G]$-generic with $\dot{s}_G \in H$, and let $g \subset \mathrm{Col}(\omega, 2^{\omega_2})$ be $V[G][H]$-generic. Then, in $V[G][H][g]$ there is $q \in \mathbb{P}_{\max}$ such that (q, G) is 2-good.

Proof of Claim. By our hypothesis, let $(N, \delta, \tau, \Sigma)$ be such that

[1] The 2 in "2-good" represents the 2 in $\aleph_2$.

(1). $(N, \delta, \tau, \Sigma)$ strongly captures $(B_= \oplus B_\in \oplus B_{\vec{R}} \oplus E)$, and

(2). the function $F\colon X \mapsto M_2^{\sharp,\Sigma}(X)$, where $X \in \mathrm{HC}$ is self-well-ordered and $N \in L_1(X)$, is well-defined, total, and strongly universally Baire in the codes.

Let $\mathfrak{A} \in V[G][H][g]$ be an F-closed witness to the fact that $\varphi(\dot{\mathcal{M}}^G)$ is honestly consistent. Let $X \in \mathfrak{A}$ be transitive and such that

(1). $(\mathcal{P}(\omega_1) \cap \mathfrak{A}) \cup \{(\mathsf{NS}_{\omega_1})^{\mathfrak{A}}, G, \dot{\mathcal{M}}^G\} \in X$, and

(2). $X \models \varphi(\dot{\mathcal{M}}^G)$.

Let $M = M_2^{\sharp,\Sigma}(X)$, and let δ_0, g_0, g_1, $\mathbb{Q}$, and κ be as in the statement of Theorem 2.5.5. Then by Theorem 2.5.5, inside $V[G][H][g]$ we have

$$q = ((M||\kappa)[g_0, g_1], \in, (\mathsf{NS}_{\omega_1})^{(M||\kappa)[g_0,g_1]}, A_G)$$

is such that

(1). $q \in \mathbb{P}_{\max}$ and $G \in q$;

(2). $((M||\kappa)[g_0, g_1], \in) \models \mathsf{ZFC}$;

(3). $q <_{\mathbb{P}_{\max}} r$ for all $r \in G$; and

(4). q is $(B_= \oplus B_\in \oplus B_{\vec{R}} \oplus E)^*$-iterable.

Note that $\mathcal{M}_G = \dot{\mathcal{M}}^G$ where $\mathcal{M}_G$ is the transitive collapse of $(\mathbb{R}_G/{\sim}, \tilde{\in}, \tilde{R})$. Hence we have

$$q \models \varphi(\mathcal{M}_G).$$

Also, we have

$$E_{\psi,\vec{z}} \cap G \neq \emptyset$$

for every formula ψ and for every $\vec{z} \in (\mathbb{R}_G)^{<\omega}$ by the genericity of G. Therefore, (q, G) is a 2-good pair. **(Q.E.D. Claim.)**

Therefore, by Lemma 2.3.1, there is a 2-good pair (q, h) in V with $q <_{\mathbb{P}_{\max}} p$. We claim that

$$(q, \dot{s}) \Vdash_{\mathbb{P}_{\max} * \mathrm{Add}(\omega_3,1)} \Psi(\dot{\mathcal{M}}, \varphi).$$

Let $G \subset \mathbb{P}_{\max}$ be V-generic with $q \in G$, let $H \subset \mathrm{Add}(\omega_3, 1)$ be $V[G]$-generic with $\dot{s}^G \in H$. Let $j\colon q \to q^*$ is the generic iteration of length ω_1 induced by G, and let $h^* = j(h)$.

Since $q \models \varphi(\mathcal{M}_h)$, we have $q^* \models \varphi(\mathcal{M}_{h^*})$. By the upward absoluteness of Σ_1 formulae, $\varphi(\mathcal{M}_{h^*})$ holds in $V[G]$.

We define $\pi\colon \mathcal{M}_{h^*} \to \dot{\mathcal{M}}^G$ as follows. For $x \in \mathcal{M}_{h^*}$, letting $y \in \mathbb{R}_{h^*}$ be such that $\sigma_{h^*}(x) = [y]_\sim$ where $\sigma_{h^*}\colon \mathcal{M}_{h^*} \simeq (\mathbb{R}_{h^*}/{\sim}, \tilde{\in}, \tilde{R})$ is uncollapsing and $[y]_\sim$ is the $\sim$-equivalence class of y, we define $\pi(x) = \dot{f}^G(y)$. π is well-defined, since $h^* \subset G$ and q is $(B_= \oplus B_\in \oplus B_{\vec{R}} \oplus E)$-iterable. We shall show that π is elementary by Tarski–Vaught test.

Let $\vec{z} \in (\mathbb{R}_{h^*})^{<\omega}$ and let ψ be a formula, and suppose that

$$\dot{\mathcal{M}}_G \models \exists x \psi(x, \dot{f}^G(\vec{z})).$$

Since (q, h) is good and j is elementary,

$$E_{\psi,\vec{z}} \cap h^* \neq \emptyset.$$

Let $r \in E_{\psi,\vec{z}} \cap h^*$. Then there is $x \in r$ such that

$$r \Vdash_{\mathbb{P}_{\max}} \psi(\dot{f}(x), \dot{f}(\vec{z})).$$

Hence we have $\dot{f}^G(x)$ is in the image of π and $\dot{\mathcal{M}}^G \models \psi(\dot{f}^G(x), \dot{f}^G(\vec{z}))$, since $r \in h^* \subset G$. This shows $\pi\colon \mathcal{M}_{h^*} \to \dot{\mathcal{M}}^G$ is elementary. Therefore, we showed that $\pi\colon \mathcal{M}_{h^*} \to \dot{\mathcal{M}}^G$ witnesses that $\Psi(\dot{\mathcal{M}}^G, \varphi)$ holds in $V[G][H]$. This is a contradiction! This finishes the proof. $\square$

Theorem 2.4.1 gives us the following corollary.

Corollary 4.0.2. *Assume the same assumptions as Theorem 4.0.1. Then*

$$V^{\mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1)} \models \mathsf{MM}^{++}_{\mathfrak{c}}.$$

Chapter 5

Partial results

In this chapter, we provide partial results related to thr ground model for Theorem 1.0.1. Theorem 1.0.1 uses the following assumptions:

(1). $\mathsf{ZF}+\mathsf{AD}^{+}+\mathsf{AD}_{\mathbb{R}}+$ "Θ is regular" + "Every set of reals is universally Baire",

(2). Γ^{∞} is productive,

(3). $V^{\mathbb{P}_{\max}} \models \omega_2\text{−}\mathsf{DC}$ and $V^{\mathbb{P}_{\max} * \mathrm{Add}(\omega_3,1)} \models \mathsf{ZFC}$; and

(4). for every $A \in \Gamma^{\infty}$ there is some (N, Σ) which strongly captures A and the function $F\colon X \mapsto M_2^{\sharp,\Sigma}(X)$, where $X \in \mathrm{HC}$ is self-well-ordered and $N \in L_1(X)$, is well-defined, total, and strongly universally Baire in the codes.

The author expects that an appropriate generalization of the derived model constitution gives us models satisfying the assumptions of Theorem 1.0.1. In the rest of this chapter, we construct a candidate for the ground model.

We discuss how to construct a model satisfying (1), (2), and (3). First, we recall notations from [35].

Definition 5.0.1. Let λ be a limit of Woodin cardinals and $G \subset \mathrm{Col}(\omega, <\lambda)$ be a V-generic filter. Define:

- $\mathbb{R}^*_G = \bigcup_{\alpha<\lambda} \mathbb{R}^{V[G\restriction\alpha]}$,
- Hom^*_G is the pointclass of sets $\bigcup_{\alpha<\lambda} A^{G\restriction\alpha}$ for $<\lambda$-universally Baire sets of reals A appearing in $V[G\restriction\alpha]$ where $\alpha < \lambda$.

We call $L(\mathbb{R}^*_G, \mathrm{Hom}^*_G)$ *a derived model at* λ.

If G is clear from the context, then we omit the subscript G.

Theorem 5.0.1 (Derived model theorem, Woodin, [35])**.** *Let λ be a limit of Woodin cardinals, let $G \subset \mathrm{Col}(\omega, <\lambda)$ be a V-generic filter, and let $L(\mathbb{R}^*, \mathrm{Hom}^*)$ be a derived model at λ. Then*

(1). $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}^+$*, and*

(2). $\mathrm{Hom}^* = \{A \subset \mathbb{R}^* \mid A \text{ is Suslin, co-Suslin in } L(\mathbb{R}^*, \mathrm{Hom}^*)\}$.

Moreover, if λ is a limit of $<\lambda$-strong cardinals, then

(1). $\mathcal{P}(\mathbb{R}^*) \cap L(\mathbb{R}^*, \mathrm{Hom}^*) = \mathrm{Hom}^*$*, and*

(2). $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}_{\mathbb{R}}$.

The paper [18] constructs a generalized derived model that satisfies $\mathsf{ZF} + \mathsf{AD}^+ +$ "Every set of reals is universally Baire".

Definition 5.0.2 ([18, Definition 4.4])**.** The predicate F_{uB} consists of all quadruples (A, Z, p, x) such that

- A is universally Baire, Z is a set, $p \in \mathrm{Col}(\omega, Z)$, x is a $\mathrm{Col}(\omega, Z)$-name for a real, and
- $p \Vdash_{\mathrm{Col}(\omega,Z)} x \in A^{\dot{G}}$.

Notation. In the rest of this paper, the predicate F_{uB} always denotes the predicate defined in Definition 5.0.2.

The paper [18, Main Theorem] shows that the predicate F_{uB} has enough information to get universally Baire representations.

Theorem 5.0.2 ([18, Main Theorem])**.** *Let λ be a limit of Woodin cardinals and a limit of strong cardinals, let $G \subset \mathrm{Col}(\omega, <\lambda)$ be a V-generic filter, and define the model*

$$\mathcal{M} = (L^{F_{\mathrm{uB}}}(\mathbb{R}^*, \mathrm{Hom}^*))^{V(\mathbb{R}^*)}.$$

Then

$$\mathcal{M} \cap \mathcal{P}(\mathbb{R}^*) = \mathrm{Hom}^*,$$

and

$$\mathcal{M} \models \mathsf{AD}^+ + \text{“Every set of reals is universally Baire”}.$$

Note that the model $\mathcal{M}$ also satisfies $\mathsf{AD}_{\mathbb{R}}$.

[18] shows the following.

Lemma 5.0.3. *Let λ, G, $\mathcal{M}$ be as in Theorem 5.0.2. Then*

$$\mathcal{M} \models \text{“}\mathrm{Hom}^* \text{ is productive”}.$$

Proof. See [18, Lemma 6.1]. $\square$

The same argument in [11, Lemma 3.4] gives us the regularity of Θ. For the definition of being self-iterable, see [11, Section 2]. The author would like to thank Takehiko Gappo for his help in proving the following theorem.

Theorem 5.0.4. *Suppose that V is self-iterable. Let λ be an inaccessible cardinal which is a limit of Woodin cardinals and a limit of strong cardinals, and let $G \subset \mathrm{Col}(\omega, <\lambda)$ be a V-generic filter. Then*

(1). $L^{F_{\mathrm{uB}}}(\mathbb{R}^*_G, \mathrm{Hom}^*_G) \models \mathsf{AD}_{\mathbb{R}} +$“$\Theta$ *is regular*”$+$ “*Every set of reals is universally Baire*”,

(2). $L^{F_{\mathrm{uB}}}(\mathbb{R}^*_G, \mathrm{Hom}^*_G) \cap \mathcal{P}(\mathbb{R}^*_G) = \mathrm{Hom}^*_G$.
Furthermore, if $H \subset \mathbb{P}_{\max}$ is $L^{F_{\mathrm{uB}}}(\mathbb{R}^*_G, \mathrm{Hom}^*_G)$-generic, then

(3). $L^{F_{\mathrm{uB}}}(\mathbb{R}^*_G, \mathrm{Hom}^*_G)[H] \models \omega_2\text{−}\mathsf{DC}$.

Proof of (1) and (2). This proof is by [11, Lemma 3.4], except for Claim 1.

By Theorem 5.0.2,

$$L^{F_{\mathrm{uB}}}(\mathbb{R}^*_G, \mathrm{Hom}^*_G) \models \mathsf{AD}_{\mathbb{R}} + \text{ “Every set of reals is universally Baire”},$$

and

$$L^{F_{\mathrm{uB}}}(\mathbb{R}^*_G, \mathrm{Hom}^*_G) \cap \mathcal{P}(\mathbb{R}^*_G) = \mathrm{Hom}^*_G\,.$$

Let $x \in \mathbb{R}^*_G$, and $\delta < \lambda$ be a Woodin cardinal. Let $\mathcal{T}$ be a normal iteration tree on V with last model W making x generic for the extender algebra at δ, and let $\mathcal{U}$ be an iteration tree on W of length λ based on the window (δ, λ) resulting from $\mathbb{R}^*_G$-genericity iteration. Let $\bar{W}$ be the well-founded direct limit along the unique cofinal branch through $\mathcal{S} = \mathcal{T}^\frown\mathcal{U}$. Since λ is inaccessible, $\pi^{\mathcal{S}}(\lambda) = \lambda$. By construction, we pick a $(\bar{W}, \mathrm{Col}(\omega, <\lambda))$-generic $H \in V[G]$ such that $\mathbb{R}^*_G = \mathbb{R}^*_H$. Let $\mathbb{R}^* = \mathbb{R}^*_G = \mathbb{R}^*_H$. Note that $\bar{W}[H] \subset V[G]$. [11, Lemma 3.4] shows

$$L(\mathbb{R}^*, \mathrm{Hom}^*)^{V(\mathbb{R}^*)} = L(\mathbb{R}^*, \mathrm{Hom}^*)^{\bar{W}(\mathbb{R}^*)}.$$

For the same argument in [11, Lemma 3.4] to hold, it is enough to show:

Claim 1.
$$L^{F_{\mathrm{uB}}}(\mathbb{R}^*, \mathrm{Hom}^*)^{V(\mathbb{R}^*)} = L^{F_{\mathrm{uB}}}(\mathbb{R}^*, \mathrm{Hom}^*)^{\bar{W}(\mathbb{R}^*)}.$$

Proof. Let $Z \in \bar{W}(\mathbb{R}^*)$ be a set and let $A \in (\mathrm{Hom}^*)^{V(\mathbb{R}^*)} = (\mathrm{Hom}^*)^{\bar{W}(\mathbb{R}^*)}$. Let $K \subset \mathrm{Col}(\omega, Z)$ be $V(\mathbb{R}^*)$-generic. Then we have that

$$(A^K)^{V(\mathbb{R}^*)[K]} \cap \bar{W}(\mathbb{R}^*)[K] = (A^K)^{\bar{W}(\mathbb{R}^*)[K]}.$$

Hence we have that

$$p \Vdash^{V(\mathbb{R}^*)}_{\mathrm{Col}(\omega,Z)} \dot{x} \in A^{\dot{K}} \iff p \Vdash^{\bar{W}(\mathbb{R}^*)}_{\mathrm{Col}(\omega,Z)} \dot{x} \in A^{\dot{K}}$$

where $p \in \mathrm{Col}(\omega, Z)$ and $\dot{x} \in \bar{W}(\mathbb{R}^*)$ is a $\mathrm{Col}(\omega, Z)$-name for a real.

Therefore, we have that for all $X \in \bar{W}(\mathbb{R}^*)$,

$$X \cap (F_{\mathrm{uB}})^{V(\mathbb{R}^*)} = X \cap (F_{\mathrm{uB}})^{\bar{W}(\mathbb{R}^*)}.$$

where $\dot{K}$ is the canonical $\mathrm{Col}(\omega, Z)$-name for a generic.

It follows that

$$(J^{F_{\mathrm{ub}}}_\alpha(\mathbb{R}^*, \mathrm{Hom}^*))^{V(\mathbb{R}^*)} = (J^{F_{\mathrm{ub}}}_\alpha(\mathbb{R}^*, \mathrm{Hom}^*))^{\bar{W}(\mathbb{R}^*)}$$

for all α, by induction on the level of J-hierarchy. **(Q.E.D. Claim 1.)**

This finishes the proof of (1) and (2). □

The proof of (3) is by [41, Theorem 9.35]. Here we shall give a more detailed proof.

Proof of (3). For each $A \in \Gamma$, we denote $w(A)$ by the Wadge rank of A. Let $F = F_{\mathrm{uB}}$, let $\Gamma = \mathrm{Hom}^*_G$, and let $\mathbb{R} = \mathbb{R}^*_G$. Note that we have

$$L^F(\Gamma, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = \Gamma.$$

Let $H \subset \mathbb{P}_{\max}$ be $L^F(\Gamma, \mathbb{R})$-generic. Suppose that $R = \tau^H \in L^F(\Gamma, \mathbb{R})[H]$ is a binary relation. Fix an ordinal α such that $\tau \in J^F_\alpha(\Gamma, \mathbb{R})$ and $J^F_\alpha(\Gamma, \mathbb{R})$ satisfies a sufficiently large fragment of ZF. For each $\gamma < \Theta$, let

$$Z_\gamma = \mathrm{Hull}^{J^F_\alpha(\Gamma,\mathbb{R})}_{\Sigma_1}(\{\tau\} \cup \Gamma_{<\gamma})^{1}$$

where $\Gamma_{<\gamma}$ is the set of reals $A \in \Gamma$ with $w(A) < \gamma$. Since

$$L^F(\Gamma, \mathbb{R}) \models \mathsf{AD}_\mathbb{R} + \text{“}\Theta \text{ is regular”},$$

[1] Σ_1-Skolem hull of $\{\tau\} \cup \Gamma_{<\gamma}$ in $J^F_\alpha(\Gamma, \mathbb{R})$

there exists $\gamma < \Theta$ such that

$$Z_\gamma \cap \Theta = \gamma$$

and such that in $L^F(\Gamma, \mathbb{R})$, $\mathrm{cf}(\gamma) = \omega_2$.

We claim that

$$Z_\gamma \prec_{\Sigma_1} J^F_\alpha(\Gamma, \mathbb{R}).$$

To prove this, we shall use the fine structure of $L^F(\Gamma, \mathbb{R})$. For more details, see [37, §1]. The same argument as [37, Lemma 1.10] shows $J^F_\alpha(\Gamma, \mathbb{R})$ satisfies Σ_1 selection, i.e., whenever $R \subset J^F_\alpha(\Gamma, \mathbb{R}) \times [\omega\alpha]^{<\omega}$ is $\Sigma_1(J^F_\alpha(\Gamma, \mathbb{R}), \{p\})$ for some $p \in J^F_\alpha(\Gamma, \mathbb{R})$, there is a $\Sigma_1(J^F_\alpha(\Gamma, \mathbb{R}), \{p\})$ set $S \subset R$ such that

$$\forall a(\exists F\, R(a, F) \implies \exists! F\, S(a, F)).$$

The standard argument of fine structure theory gives us uniformly $\Sigma_1(J^F_\alpha(\Gamma, \mathbb{R}))$ maps f_α such that

$$f_\alpha \colon [\omega\alpha]^{<\omega} \times \Gamma \times \mathbb{R} \twoheadrightarrow J^F_\alpha(\Gamma, \mathbb{R}).$$

Hence we obtain a Σ_1-Skolem function of $J^F_\alpha(\Gamma, \mathbb{R})$

$$h \colon J^F_\alpha(\Gamma, \mathbb{R}) \times \Gamma \times \mathbb{R} \to J^F_\alpha(\Gamma, \mathbb{R})$$

such that

(1). h is a partial $\Sigma_1(J^F_\alpha(\Gamma, \mathbb{R}))$ map, and

(2). whenever S is $\Sigma_1(J^F_\alpha(\Gamma, \mathbb{R}), \{a\})$ and $S \neq \emptyset$, then

$$\exists A \in \Gamma\, \exists x \in \mathbb{R}(h(a, A, x) \in S),$$

as follows. Let $\langle \varphi_i \colon i < \omega \rangle$ be a recursive enumeration of the Σ_1 formulae of two free variables. Let

$$R(a, A, x, F) \iff J^F_\alpha(\Gamma, \mathbb{R}) \models \varphi_{x(0)}[a, f_\alpha(F, A, \lambda i.x(i+1))].$$

Since R is $\Sigma_1(J^F_\alpha(\Gamma, \mathbb{R}))$, by Σ_1 selection, we have a partial $\Sigma_1(J^F_\alpha(\Gamma, \mathbb{R}))$ map g uniformizing R. Let

$$h(a, A, x) = f_\alpha(g(a, A, x), \lambda i.x(i+1)).$$

h is the desired Skolem function.

Now let $a \in Z_\gamma$, and let $\varphi(v_0, v_1)$ be a Σ_0 formula. Suppose that

$$J^F_\alpha(\Gamma, \mathbb{R}) \models \exists x\, \varphi(x, a).$$

Then there exist $x \in \mathbb{R}$ and $A \in \Gamma$ such that

$$J^F_\alpha(\Gamma, \mathbb{R}) \models \varphi(h(a, A, x), a).$$

It is enough to see that there is such an A in $\Gamma_{<\gamma}$. Suppose not. Let $\eta < \Theta$ be the minimal ordinal such that there is such an A with $w(A) = \eta$. Then $\eta \in Z_\gamma$, since η is defined by

$$\begin{aligned}\exists\beta < \omega\alpha[&\exists x \in S^F_\beta(\Gamma, \mathbb{R})\, \varphi(x, a)\wedge\\ &\forall\gamma < \beta\forall x \in S^F_\gamma(\Gamma, \mathbb{R})\, \neg\varphi(x, a)\wedge\\ &\exists A \in \Gamma\exists x \in \mathbb{R}\exists y \in S^F_\beta(\Gamma, \mathbb{R})(y = h_\beta(a, A, x) \wedge w(A) = \eta \wedge \varphi(y, a))\wedge\\ &\forall\zeta < \eta\forall A \in \Gamma\forall x \in \mathbb{R}\forall y \in S^F_\beta(\Gamma, \mathbb{R})(w(A) = \xi \wedge y = h_\beta(a, Ax) \longrightarrow \neg\varphi(y, a))],\end{aligned}$$

which is Σ_1. However, since $Z_\gamma \cap \Theta = \gamma$ and $\eta \in [\gamma, \Theta)$, this is a contradiction!!

We have proved

$$Z_\gamma \prec_{\Sigma_1} J^F_\alpha(\Gamma, \mathbb{R}).$$

Hence we have

$$Z_\gamma[G] \prec_{\Sigma_1} J^F_\alpha(\Gamma, \mathbb{R})[G],$$

since $\mathbb{P}_{\max} \subset \mathrm{HC}$. Let N be the transitive collapse of Z_γ. Let τ_N be the image of τ under collapsing, and let $R_N = \tau^G_N$. Then $N[G]$ is the transitive collapse of $Z_\gamma[G]$ and R_N is the image of R under collapsing. Since N can be coded into a set of reals, there exists $A \in \Gamma$ such that $N \in L^F(A, \mathbb{R})$.

Note that $\gamma = \Theta^N = \omega_3^{N[G]}$. We claim that

$$L^F(A, \mathbb{R})[G] \models \mathrm{cf}(\gamma) \geq \omega_2.$$

Suppose not. Let $f\colon \omega_1 \to \gamma$ be cofinal and such that $f \in L^F(A, \mathbb{R})[G]$. Then there is a function $g \in L^F(A, \mathbb{R})$ such that

$$\{\alpha < \omega_1 \mid g(\alpha) = f(\alpha)\}$$

is stationary[2]. We have $g \in L^F(\Gamma, \mathbb{R})$. Then g is bounded in γ, since

$$L^F(\Gamma, \mathbb{R}) \models \mathrm{cf}(\gamma) = \omega_2.$$

This is a contradiction! Hence $L^F(A, \mathbb{R})[G] \models \mathrm{cf}(\gamma) \geq \omega_2$, it follows that

$$N[G]^{\omega_1} \subset N[G]$$

in $L^F(A, \mathbb{R})[G]$.

Let

$$\pi\colon N[G] \simeq Z_\gamma[G] \prec_{\Sigma_1} J^F_\alpha(\Gamma, \mathbb{R})[G]$$

be uncollapsing. Then $\mathrm{crit}(\pi) = \gamma = \Theta^N = \omega_3^{N[G]}$.

[2]See [41, Theorem 3.42].

Now since $L^F(A,\mathbb{R})[G] \models \mathsf{ZFC}$, we have either for some $\beta \leq \omega_1$ there is an increasing sequence $(a_\alpha \colon \alpha < \beta)$ of elements of R_N with no upper bound or there is an increasing sequence $(a_\alpha \colon \alpha < \omega_2)$.

In the first case, $(a_\alpha \colon \alpha < \beta) \in N[G]$ and $\pi((a_\alpha \colon \alpha < \beta))$ is an increasing sequence of elements of R with no upper bound. In the second case, $(\pi(a_\alpha) \colon \alpha < \omega_2)$ is an ω_2 increasing sequence of elements of R.

This finishes the proof of (3). $\square$

Here is our conjecture:

Conjecture 1. the model constructed in Theorem 5.0.4 is a ground model for Theorem 1.0.1.

Closing

We have studied fragments of MM below $\aleph_3$. The natural next step is $\mathsf{MM}_{\aleph_3}$. To accomplish this, it seems that we need an iterability notion of $\mathbb{P}_{\max}$ for a set of sets of reals. Here is the next problem:

Problem 1. Define appropriate $\mathcal{A}$-iterability of $\mathbb{P}_{\max}$-conditions for a set of sets of reals $\mathcal{A}$, and prove the density of $\mathcal{A}$-iterable conditions.

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Declaration

Hereby I declare that I wrote this master's thesis independently and that I have not used sources other than the specified and made quotations.