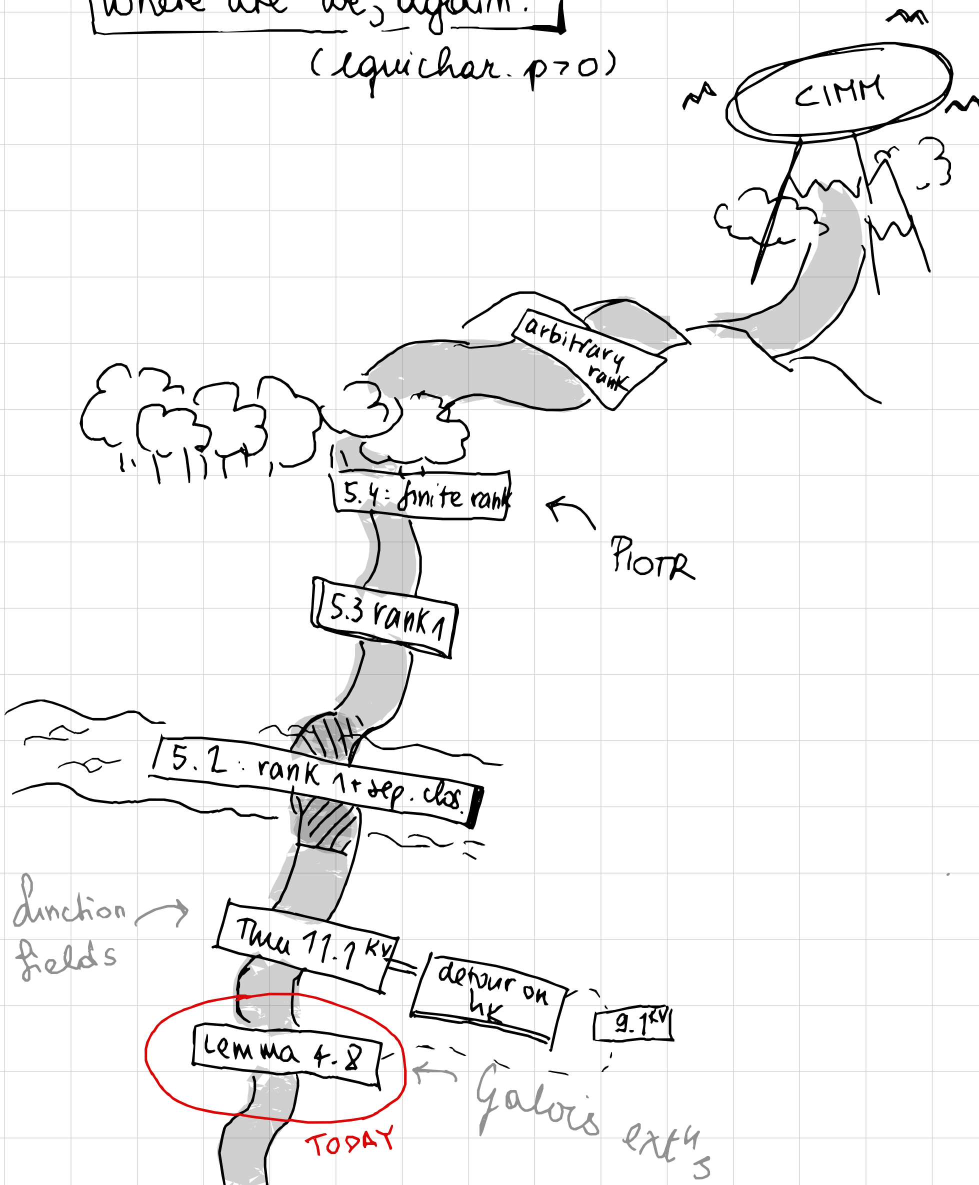
THE CHICKEN & THE DRAGON

Tying up some loose ends from ~~last week~~. two weeks ago

$\boxed{\text{Where are we, again?}}$

(equichar. $p>0$)



~~Lemma~~ Proposition 4.8: (K,v) separably tame of $\mathrm{char}(K)=p>0$ and rank 1, $K(x)$ immediate & transcendental, $K(x)^h \subset E$ Galois of degree p.
Then $\exists \theta \in E$ such that $E = K(\theta)^h$.

Proof structure.

- we can find $\theta \in E$ s.t.

$$E = K(x)^h(\theta), \quad \theta^p - \theta = f(X) \in K[x].$$

(using that here $K[x] \subset K(X)^h$ is dense, 10.1 from KV).

- then, $\mathrm{appr}(x,K)$ is transcendental. (Lemma 4.6)
- we can assume $\theta^p - \theta = g(Z)$, where

  i. $Z = \frac{X-c}{d}$, $v(Z)=0$, $c \in K$, $d \in K^\times$,

  ii. if $g(Z) = a_n Z^n + \ldots + a_1 Z + a_0 \in K[Z]$, then $\forall i>0$ either $a_i = 0$ or $v(a_i)<0$, and whenever $p \mid i$, $a_i = 0$,

  iii. $(v(a_i))_{i>0}$, if non-zero, are distinct.

  (Lemma 4.2)

- as $g(Z) \in K[\theta]$, $K(g(Z))^h \subset K(\theta)^h$,
- then $K(Z)^h = K(g(Z))^h$ (Lemma 4.7), so

$$K(X)^h = K(Z)^h = K(g(Z))^h \subset K(\theta)^h.$$

- we deduce

$$E = K(x)^h(\theta) = K(x,\theta)^h$$
$$= K(\theta)^h. \qquad \square$$

Lemma 10.1 (KV).

K rank 1, $K(x)$ immediate, then $K[x] \underset{\text{dense}}{\subset} K(x)^h$.

<u>Proof.</u> Enough to show $K[x] \underset{\text{dense}}{\subset} K(x)$.

For this, we show:

$$\forall f(x) \in K[x],\ \forall \alpha \in vK,\ \exists g(x) \in K[x] \text{ s.that}$$
$$v\left(g(x) - \frac{1}{f(x)}\right) > \alpha.$$

As $K \subset K(x)$ is immediate, there is $c \in K$ with

$$v(c) = v(f(x))$$
$$\& \quad v(c - f(x)) > v(c),$$

so $v\left(1 - \frac{f(x)}{c}\right) > 0$.

By rank 1 of vK, $\exists j \geq 0$ s.t.

$$j \cdot v\Big(\underbrace{1 - \frac{f(x)}{c}}_{h(x)}\Big) > \alpha + v(c).$$

Thus we can compute (i.e., I won't)

$$v\left(\frac{1}{f(x)} - \frac{1}{c} \cdot \sum_{i=0}^{j-1} h(x)^i\right) > \alpha. \qquad \square$$

Lemma 4.6.
(K,v) separably-algebraically maximal,
$K \subset K(x)$ immediate transcendental
$\Rightarrow$ $\mathrm{appr}(x,K)$ is transcendental.

This follows from the same result for pseudo-Cauchy sequences, using that $K \subset K^{perf}$ is dense.

Lemma 4.7.
$(K,v) \subset (K(z),v)$ immediate, $v(z)=0$, $\mathrm{appr}(z,K)$ transcendental. If
$$f(X) = a_n X^n + \ldots + a_1 X + a_0 \in K[X]$$
is such that, for some $i_0 \in \{1, \ldots n\}$, $p \nmid i_0$ and $\min_{i=1\ldots n} \{v(a_i)\} = v(a_{i_0})$ distinct from all others, then
$$K(z)^h = K(f(z)^h).$$

For this, we need two things:
Theorem 9.1 & Corollary 7.7.

Theorem 9.1. (KV)
Assume $\mathbb{A} = \mathrm{appr}(x, K)$ is an immediate approximation type over (K, v), of degree $\underline{d}$; let $f(x) \in K[X] \setminus K$ be of degree $\leq \underline{d}$, $\underline{h} = h_K(x:f)$, [and β_i = the fixed value of $v(f_i(c))$ for $c \nearrow x$.] not sure why this is useful
Then, $[K(x)^h : K(f(x))^h] \leq h_K(x:f)$.

Corollary 7.7. (KV) (follows from Lemma 7.6).
Assume $v(x) = 0$ and $e \geqslant 1$. Suppose that all nonzero coefficients c_i of f have different values and that whenever $p^e \mid i$, $c_i = 0$.
Then $h_K(x:f) < p^e$.

$$\Downarrow$$

$$[K(x)^h : K(f(x))^h] < p$$

$$\Rightarrow [K(x)^h : K(f(x))^h] = 1. \qquad \blacksquare \ (4.7)$$

Proof (of Theorem 9.1).

$\mathbb{A} = \operatorname{appr}(x, K)$ immediate appr. type over (K, v)

$\underline{d}$ = degree of $\mathbb{A}$, $f \in K[x] - K$, $\underline{h} = h_K(x : f)$.

- expand $f(X) = \sum_{i=0}^{n} f_i(c)(X-c)^i$, for $c \in K$.
- use Lemma 5.2 to get

$$\beta_{\underline{h}} + \underline{h} \cdot v(x-c) < \beta_i + i \cdot v(x-c)$$

whenever $i \neq \underline{h}$, $1 \leq i \leq \deg(f)$, $c \nearrow x$.

- choose such a c & $d \in K$ s.t.

$$v(d) = -v(x-c).$$

- set $x_0 = d \cdot (x-c)$ so $K(x) = K(x_0)$ &

$$v(f_i(c)\, d^{-i}) > v(f_{\underline{h}}(c)\, d^{-\underline{h}}), \quad i \neq \underline{h}, \; 1 \leq i \leq \deg(f)$$

& (also from Lemma 5.2)

$$v(f(x) - f(c)) = v(f_{\underline{h}}(c))\, d^{-\underline{h}}.$$

- moreover,

$$\operatorname{res}_v\left(\frac{d^{\underline{h}}}{f_{\underline{h}}(c)} \cdot \left(f(c) - f(x)\right)\right) = \operatorname{res}_v(x_0)^{\underline{h}}.$$

- set $\tilde{f}(Z) = \sum_{i=0}^{\deg(f)} f_i(c)\, d^{-i} Z^i$ and consider

$$F(Z) = \frac{d^{\underline{h}}}{f_{\underline{h}}(c)} \left(\tilde{f}(Z) - \tilde{f}(x_0)\right)$$

over $K(\tilde{f}(x_0)) = K(f(x))$, of which x_0 is a zero.

- compute:

$$\mathrm{res}_v(F(Z)) = Z^{\underline{h}} - (\mathrm{res}_v(x_0))^{\underline{h}}$$
$$= (Z - \mathrm{res}_v(x_0))^{\underline{h}}$$

because $\underline{h}$ is a power of p (Prop. 7.4).

- henselianity:

$$F(Z) = G(Z)\,H(Z)$$

over $K(f(x))^h$ with

$$\mathrm{res}_v(G(Z)) = Z^{\underline{h}} - (\mathrm{res}_v(x_0))^{\underline{h}}$$

& $\deg(G(Z)) = \deg(\mathrm{res}_v(G(Z))) = \underline{h}$.

- since $\mathrm{res}_v(H(Z)) = 1$, then $G(x_0) = 0$,

so

$$[K(x)^h : K(f(x))^h]$$
$$= [K(x_0)^h : K(f(x))^h]$$
$$\leq \underline{h}.$$

$\blacksquare$