Tame valued fields reading group

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Around Lemma 4.2

Fourth session of reading group on tame fields.

We are now following:

F.-V. Kuhlmann. The algebra and model theory of tame valued fields, *J. reine angew. Math.*, 719 (2016), 1–43.
and
F. V. Kuhlmann. Elimination of ramification II: Henselian rationality, *Israel J. Math.*, 234(2019), no.2, 927–958.

Reminder:

$4.2 + 4.7 \implies 4.8$

Density lemma 4.1 (Kuhlmann–Vlahu)

If (K, v) is of rank 1 and $(K(x), v)/(K, v)$ is immediate then $K[x]$ is dense in $K(x)^h$.

Around Lemma 4.2

Aim: Lemma 4.8 (char $= p$)

Let (K, v) sep tame, rank 1, let $(K(x), v)/(K, v)$ be immediate simple transcendental, and let $E/K(x)^h$ be Galois degree p. There exists $\theta \in E$ such that $E = K(\theta)^h$.

Today: one ingredient, Lemma 4.2.

Around Lemma 4.2

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Reminder

Lemma 4.9 (mix char)

Let (K, v) be algebraically closed mixed characteristic, rank 1, let $(K(x), v)/(K, v)$ be an immediate transcendental extension, and let $E/K(x)^h$ be Galois of degree p. There exists $\eta \in E$ such that $E = K(\eta)^h$.

Around Lemma 4.2

Lemma 4.2 (char $= p$)

For every $f \in K[X]$ there exists a finite purely inseparable extension K'/K and a polynomial

$$g \in f + \wp(K'(x)^h)$$

such that

1. $z = (x - c)/d$ with $vz = 0$, $c \in K$, $d \in K^\times$,
2. $g(z) = \sum_i a_i z^i \in K'[z]$
3. for all i, ($va_i < 0$ or $a_i = 0$) and $p \mid i \implies a_i = 0$,
4. va_i (for i: $a_i \neq 0$) are distinct.

Note $K[x] = K[z]$.

Around Lemma 4.2

Proof

Formal Taylor expansion of f in the variable X around X_0:

$$f(X) = \sum_{i=0}^{n} f_i(X_0)(X - X_0)^i,$$

where the f_i are Hasse-Schmidt derivatives, purely formal. We can kill any monomial with index i divisible by p, thus: if $i = pj$ then $f_i(X_0)(X - X_0)^i$ is congruent to $f(X)$ modulo $\wp$. More precisely, modulo $\wp(K_1[X, X_0])$ where $K_1 = K(f_i(X_0)^{p^{-1}})$. This is a simple purely inseparable extension of K.

Around Lemma 4.2

Proof

Formal Taylor expansion of f in the variable X around X_0:

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Iterating this, we find K'/K finite purely inseparable such that modulo $\wp(K'[X, X_0])$ the polynomial $f(X)$ is congruent to

$$f(X_0) + \sum_j{}' \left(f_j(X_0) + \sum_{\nu}{}^{(j)} f_{jp^\nu}(X_0)^{p^{-\nu}} \right) (X - X_0)^j,$$

where $\sum_j'$ is the sum over $j \neq n$ coprime to p, and $\sum_\nu^{(j)}$ is the sum over all $\nu \geq 1$ with $jp^\nu \leq n$.

Around Lemma 4.2

Proof (cont.)

For sufficiently large $\lambda \in \mathbb{N}$

$$F_j(X_0) := \left(f_j(X_0) + \sum_{\nu}{}^{(j)} f_{jp^\nu}(X_0)^{p^{-\nu}} \right)^{p^\lambda}$$

is in $K[X_0]$.

Around Lemma 4.2

Proof (cont.)

By hypothesis, the approximation type of x over K is transcendental. Thus there exists $\alpha_0 \in v(x - K)$ such that

1. $vf(X_0)$ and
2. $vF_j(X_0)$, for $j \leq n$ coprime to p,

are constant on the 'closed' ball $\bar{B}(x; \alpha_0)$ around x of radius α_0.

Proof (cont.)

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For j coprime to p, and each $c \in \bar{B}(x; \alpha_0)$, we define

$$\beta_j := v\left(f_j(c) + \sum_{\nu} {}^{(j)} f_{jp^{\nu}}(c)^{p^{-\nu}}\right)$$

Around Lemma 4.2

Proof (cont.)

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$$\beta_j := v\left(f_j(c) + \sum_{\nu} {}^{(j)}f_{jp^{\nu}}(c)^{p^{-\nu}}\right)$$

We can choose $c \in \bar{B}(x; \alpha_0)$ such that the

1. $vf(c)$,
2. $\beta_j + j \cdot v(x - c)$, for $j \leq n$ coprime to p,

are pairwise distinct and nonzero.

Around Lemma 4.2

Proof (cont.)

Unwinding. Next we translate back: choose $d \in K$ with $vd = v(x - c)$ and $z := \frac{x-c}{d}$.

Around Lemma 4.2

Proof (cont.)

Unwinding. Next we translate back: choose $d \in K$ with $vd = v(x - c)$ and $z := \frac{x-c}{d}$. Taking $X = x$ and $X_0 = c$ (or evaluating, if you prefer), we find that K'/K is finite and purely inseparable and

$$\tilde{g}(z) = f(c) + \sum_j{}' b_j z^j,$$

where

$$b_j := d^j \left(f_j(c) + \sum_\nu{}^{(j)} f_{jp^\nu}(c)^{p^{-\nu}} \right).$$

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Proof (cont.)

Unwinding. Next we translate back: choose $d \in K$ with $vd = v(x-c)$ and $z := \frac{x-c}{d}$. Taking $X = x$ and $X_0 = c$ (or evaluating, if you prefer), we find that K'/K is finite and purely inseparable and

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where

$$b_j := d^j \left(f_j(c) + \sum_\nu {}^{(j)} f_{jp^\nu}(c)^{p^{-\nu}} \right).$$

Note $vb_j \neq 0$ and $\tilde{g}(z)$ and $f(x)$ are congruent modulo $\wp(K'[x])$.

Around Lemma 4.2

Proof (cont.)

If any $vb_i > 0$ then $b_i z^i \in \wp(K'(z)^h) = \wp(K'(x)^h)$. Thus writing $a_i = b_i$ if $vb_i < 0$, $a_i = 0$ otherwise, we have a polynomial $g(z) = \sum_{i0}^{n} a_i z^i \in K'[x]$ to which both $\tilde{g}(z)$ and $f(x)$ are congruent modulo $\wp(K'(x)^h)$. Remains to note that the values of the (nonzero) coefficients $a_i = \beta_j + jv(x - c)$ (for j coprime to p) are pairwise distinct (and different from va_0).

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Lemma 4.8 (char $= p$)

Let (K, v) sep tame, rank 1, let $(K(x), v)/(K, v)$ be immediate simple transcendental, and let $E/K(x)^h$ be Galois of degree p. There exists $\theta \in E$ such that $E = K(\theta)^h$.