Recall: We want to show AKE principles for tame fields.

Simone's first talk (Lemma 6.4 in [Kuh16]):
$\mathcal{C}$ elem. class of valued fields with
(CALM) every field in $\mathcal{C}$ is alg. maximal
(CRAC) if $(L,v) \in \mathcal{C}$ and K is rel. alg. closed in L s.th. Lv/Kv is alg. and vL/vK is torsion, then $(K,v) \in \mathcal{C}$ with $vL = vK$ and $Kv = Lv$
(CIMM) If $(K,v) \in \mathcal{C}$, then every hens. of an immediate function field of trdeg 1 over (K,v) is the henselization of a rational function field over K.
Then $\mathcal{C}$ has the relative emb. property.

this holds by definition in tame fields
this we have spent the last couple of weeks on and chosen to accept the remaining scraps (?)
?
Pop's Lemma (proven by Sylvy in her first talk).
Remark: the assumption "vL/vK torsion" is not actually needed for tame fields.

Def: A class $\mathcal{C}$ of valued fields has the relative embedding property if whenever $(L,v), (F,u) \in \mathcal{C}$ have a common subfield (K,v) s.th.
1. (K,v) is defectless
2. (F,u) is $|L|^+$-sat.
3. vL/vK is torsion-free & $Kv \subseteq Lv$ is separable

4. there are emb. $\rho : vL \to uF$ over vK
and $\vartheta : Lv \to Fu$ over Kv
then, there is an embedding $i : (L, v) \hookrightarrow (F, u)$
over K, inducing ρ and ϑ.

So, are we done proving (REP) for tame fields?
$\hookrightarrow$ that is of course too optimistic.

Simone left two $\boxed{\text{black boxes}}$

Lemma 5.7
Embedding Lemma 2

Lemma 6.2
Embedding Lemma 3

requires Embedding Lemma 1
& Knaf-Kuhlmann
& Lemma 2.2

TODAY!

requires

Generalized stability

Thm (Generalized stability, [Kuh 10] ~ 32 pages)
$(F|K, v)$ valued function field without transc. defect. If (K, v) is defectless, then (F, v) is defectless.
Proof: follows a similar pattern to the proof of CIMM. (add appropriate horror emoji)

Valued f.f.: fin. gen field ext. of tr.deg ≥ 1.

Without trans. defect: $\operatorname{trdeg}[F:K] = \operatorname{trdeg}[Fv : Kv] + \underbrace{\dim_{\mathbb{Q}} \mathbb{Q} \otimes (vF/vK)}_{\text{rational rank}}$

i.e. max. number of elem. of vL/vK indep. over $\mathbb{Z}$

Thm (Dimension inequality / Abhyankar-inequality, [EP, 3.4.3])
$(F|K,v)$ field ext.. Then, we have

(⋆) $\operatorname{trdeg}[F:K] \geq \operatorname{trdeg}[Fv:Kv] + \dim_{\mathbb{Q}} \mathbb{Q} \otimes (vF/vK)$

Moreover, if $F|K$ is fin. gen., and equality holds in (⋆), then $Fv|Kv$ is fin. gen. and vF/vK is a fin. gen. $\mathbb{Z}$-module.

Generalized stability is used for:

Thm (cited as 1.9 in [Kuh16], deduced from gen. stability by Knaf-Kuhlmann [KK05] as 3.4)
Take a defectless field (K,v) and a valued f.f. $(F|K,v)$ without transcendence defect.
Assume that
- $Fv|Kv$ is separable and
- $vF|vK$ is torsion-free.

Then $(F|K,v)$ is strongly inertially generated.
In fact, for every transc. basis $\mathcal{T}$ as in the def of sig. there is an element a with the required property in every hens. of F.

A valued function field $F|K$ is strongly inertially generated if there is a trans. basis
$\mathcal{T} = \{x_1, \dots, x_r, y_1, \dots, y_s\}$ of $F|K$ s.th.

a) $vF = vK(\mathcal{T}) = vK \oplus \mathbb{Z}vx_1 \oplus \dots \oplus \mathbb{Z}vx_r$

b) $\bar{y}_1, \dots, \bar{y}_s$ form a separating trans. basis of $Fv|Kv$ i.e. $Fv|Kv(\bar{y}_1, \dots, \bar{y}_s)$ is sep.

and there is some a in some henselization F^h of (F,v) s.th. $F^h = K(\mathcal{T})^h(a)$, $v(a) = 0$
and $[K(\mathcal{T})v(\bar{a}) : K(\mathcal{T})v] = [K(\mathcal{T})^h(a) : K(\mathcal{T})^h]$
and $K(\mathcal{T})v(\bar{a}) / K(\mathcal{T})v$ is separable.

We now prove Thm (Knaf-Kuhlmann)
(assuming gen. stability)

Lemma: If (K,v) is defectless, then (K^h,v^h) does not admit proper imm. alg. extensions.

Pf: (K,v) defectless $\Rightarrow$ (K^h,v^h) defectless

Sps. L/K^h immediate, finite

$$\Rightarrow \quad [L:K^h] \underset{\text{defectless}}{=} [Lv^h : K^h v^h] \cdot (v^h L : v^h K^h) = 1$$

$\Rightarrow L = K^h$ $\boxtimes$

Remark: I'm taking $\Rightarrow$ as a fact. Kuhlmann cites himself, where his other paper cites Endler for sep. defectless and doesn't give a proof for defectless.
A proof (building on Endler) is in Anscombe-Jahnke "Characterizing NIP".

Proof: Take a defectless field (K,v) and a valued f.f. $(F|K,v)$ without transcendence defect.
Assume that

- $Fv|Kv$ is separable and
- $vF|vK$ is torsion-free.

By the Abhyankar inequality, vF/vK and Fv/Kv are fin. generated.
Choose $x_1,..,x_r \in F$ s.th.

$$vF = vK \oplus \mathbb{Z}v(x_1) \oplus \dots \oplus \mathbb{Z}v(x_r)$$

where $r := \dim_{\mathbb{Q}}(\mathbb{Q} \otimes vF/vK)$
Since $Fv|Kv$ is fin. gen. and separable, it there is are $y_1,..,y_s \in F$ s.th.

$Fv / Kv(\bar{y}_1, \dots, \bar{y}_s)$ is separable, $s := \mathrm{trdeg}(Fv|Kv)$
Set $T := \{x_1, \dots, x_r, y_1, \dots, y_s\}$ and $F_0 := K(T)$.

Now choose $\alpha \in Fv$ s.th. $Fv = \underbrace{Kv(\bar{y}_1, \dots, \bar{y}_s}_{=: K_0}, \alpha)$
As α is sep.-alg. $/ F_0$, there is $a \in (F, v)^h$
s.th. $\bar{a} = \alpha$ and mipo_{a/F_0} reduces to
$\mathrm{mipo}_{\alpha/K_0}$. (Hensel's Lemma)
$\Rightarrow$ we have $v^h(a) = 0$ and
$[K(T)v(\alpha) : K(T)v] = [K(T)^h(a) : K(T)^h]$
and $K(T)v(\alpha) / K(T)v$ is separable.
Moreover, $F / F_0(a)$ is immediate.

Remains to show: $F^h = F_0^h(a)$ [In fact, we show $F^h = F_0(a)^h$.
"$\supseteq$" clear.
"$\subseteq$" WMA $F_0(a)^h \subseteq F^h$
$\Rightarrow (F^h, v^h) \supseteq (F_0(a)^h, v^h)$ is imm.

$(F_0 | K, v)$ without trans. defect
goes up alg. ext! $\Rightarrow (F_0(a) | K, v)$ without trans. defect

(K, v) defectless + generalized stability
$\Rightarrow (F_0(a), v)$ defectless

Lemma $\Rightarrow (F_0(a)^h, v^h)$ has no proper imm. ext.,
so $F_0(a)^h = F^h$. ☐
$\|$
$F_0^h(a)$