The last bit of light before the long tunnel of generalized stability

Montag, 18. März 2024 18:15

Last week (2 weeks ago), Franzi:

ingredients for Embedding Lemma II:

- Embedding Lemma I
- Knaf-Kuhlmann ✓, (Franzi) requires Generalized Stability
- Lemma 2.2.

TODAY! YAY!

Aim for today: Lemma 5.6 (Embedding Lemma I)

Let $(F|K, v)$ be a SIG function field and let (K^*, v^*) be a henselian extension of (K, v).
Let vF/vK be torsion-free and Fv/Kv separable.
Assume there are embeddings

$$\rho: vF \hookrightarrow_{vK} v^*K^*$$

$$\sigma: Fv \hookrightarrow_{Kv} K^*v^*$$

then there exists an embedding

$$\iota: (F, v) \hookrightarrow_K (K^*, v^*)$$

inducing ρ and σ, i.e.

$$v^*(\iota(a)) = \rho(v(a)) \quad \& \quad res_{v^*}(\iota(a)) = \sigma(res_v(a))$$

DEFINITION of SIG (strongly inertially generated)

$(F|K, v)$ valued function field is called SIG if there is a transcendence basis

$$\mathcal{T} = \{x_1, \dots, x_r, y_1, \dots, y_s\}$$

such that:

- $vF = vK \oplus v(x_1)\mathbb{Z} \oplus \dots \oplus v(x_r)\mathbb{Z}$ AND
- $\{res_v(y_1), \dots, res_v(y_s)\}$ is a separating transcendence basis of $Fv|Kv$, i.e. a transcendence basis such that $Fv|Kv(res_v(y_1), \dots, res_v(y_s))$ is separable

and there is $a \in F^h$ such that

- $F^h = K(\mathcal{T})^h(a)$
- $v(a) = 0$
- $res_v(a)$ is separable over $K(\mathcal{T})v$ and $\deg(res_v(a) \mid K(\mathcal{T})v) = \deg(a \mid K(\mathcal{T}))$ $(f = n)$

DETOUR - LEMMA 2.2. – "Gauß extension on fire"

Recall some facts –

- $(F|K, v)$ algebraic extension $\Rightarrow$ vF/vK torsion-group (ALG-val)
  $Fv|Kv$ alg. extension (ALG-res)
- (Gauß extension) (K, v) valued field, t transcendental $/K$
  There is exactly one extension of v to $K(t)$ with

$v(t)=0$ & $\mathrm{res}_v(t)$ transc over Kv.

We have $vK(t)=vK$ and $K(t)v = Kv(\mathrm{res}_v(t))$. (TR-res)

- (K,v) valued field, t transcendental $/K$,
  γ rat. indep. over vK. (TR-val)
  There is exactly one extension of v to $K(t)$ with $v(t)=\gamma$. We have $vK(t) = vK \oplus \gamma\mathbb{Z}$, $K(t)v = Kv$.

Lemma 2.2.

$(F|K, v)$ extension of valued fields

$x_i \in F$, $i \in I$ s.th. $v(x_i)$, $i \in I$ are rat. indep. over vK

$y_j \in F$, $j \in J$ s.th. $\mathrm{res}_v(y_j)$, $j \in J$ are alg. indep. over Kv

$\Longrightarrow$ $\mathcal{T} = \{x_i : i \in I\} \cup \{y_j : j \in J\}$ are alg. independent over K

We have: $vK(\mathcal{T}) = vK \oplus \bigoplus_{i\in I} v(x_i)\mathbb{Z}$

$K(\mathcal{T})v = Kv(\mathrm{res}_v(y_j) : j \in J)$

valuation v on $K(\mathcal{T})$ is uniquely determined by:

- restriction to K
- $v(x_i)$, $i \in I$
- the fact that $\mathrm{res}_v(y_j)$, $j \in J$ are alg. indep. over Kv

residue map res_v on $K(\mathcal{T})$ is uniquely determined by

- restriction to K
- $\mathrm{res}_v(y_j)$, $j \in J$
- the fact that $v(x_i)$, $i \in I$ are rat. indep. over vK

Moreover, if

$$f = \sum_{\underline{\alpha} \in \mathbb{N}_0^{I \cup J}} c_{\underline{\alpha}} \, \underline{x}^{\underline{\alpha}_I} \underline{y}^{\underline{\alpha}_J}$$

$\underline{\alpha} = (\underline{\alpha}_I, \underline{\alpha}_J)$, $\underline{y}^{\underline{\alpha}_J} = \prod_{j\in J} y_j^{\alpha_j}$, $\underline{x}^{\underline{\alpha}_I} = \prod_{i \in I} x_i^{\alpha_i}$

then

$$v(f) = \min_{\underline{\alpha} \in \mathbb{N}_0^{I\cup J}} v(c_{\underline{\alpha}}) + \sum_{i \in I} \alpha_i \cdot v(x_i)$$

WHY?

Proof:

① $v(x_i)$, $i \in I$ rat. indep over vK & $\mathrm{res}_v(y_j)$, $j \in J$ alg. indep. over Kv $\Big\} \Longrightarrow \mathcal{T}$ alg. indep over K

Take $(x_1, \dots, x_r, y_1, \dots, y_s)$ finite subtuple of $\mathcal{T}$.

We prove it is alg. indep. over K by induction on $r+s$

- if $r+s=0$: nothing to do, assume $r+s>0$
- We show that $(x_1, \dots, x_r, y_1, \dots, y_s)$ is alg. indep. over K and $K(x_1, \dots, x_r, y_1, \dots, y_s)$ has value group

  $vK \oplus v(x_1)\mathbb{Z} \oplus \dots \oplus v(y_1)\mathbb{Z}$

  and residue field $Kv(\mathrm{res}_v(y_1), \dots, \mathrm{res}_v(y_s))$

  We assume this is true for subtuples of $\mathcal{T}$ of length $r+s-1$. $\Big\}$ (IH)

(a) if $s>0$, we know by (IH) that $(x_1, \dots, x_r, y_1, \dots, y_{s-1})$ is alg. indep. over K

Thus it is enough to show that

y_s is transcendental over $K(x_1,\dots,x_r,y_1,\dots,y_{s-1})$

We know by assumption that

$(\mathrm{res}(y_1),\dots,\mathrm{res}(y_s))$ is alg. indep. over Kv,

so in particular, $\mathrm{res}(y_s)$ is transcendental

over $\underbrace{Kv(\mathrm{res}(y_1),\dots,\mathrm{res}(y_{s-1}))}$

$\subset$ residue field of $K(x_1,\dots,x_r,y_1,\dots,y_{s-1})$ by (IH)

By (ALG-res), y_s cannot be alg. over $K(x_1,\dots,x_r,y_1,\dots,y_{s-1})$

By (TR-res), we have uniqueness and

$$K(x_1,\dots,x_r,y_1,\dots,y_s)v = (K(x_1,\dots,x_r,y_1,\dots,y_{s-1})v)(\mathrm{res}_v(y_{s-1}))$$
$$= Kv(\mathrm{res}(y_1),\dots,\mathrm{res}_v(y_s))$$

$$vK(x_1,\dots,x_r,y_1,\dots,y_s) = v(K(x_1,\dots,x_r,y_1,\dots,y_{s-1}))$$
$$= vK \oplus v(x_1)\mathbb{Z} \oplus \dots \oplus v(x_r)\mathbb{Z}$$

(b) if $s=0$, then $r>0$, we know by (IH) that

$(x_1,\dots,x_{r-1})$ is alg. indep. over K

Thus it is enough to show that

x_r is transcendental over $K(x_1,\dots,x_{r-1})$

Analogous, we (ALG-val) & (TR-val)

About the WHY? part

We are given. $f = \sum_{\underline{\alpha}\in \mathbb{N}_0^{[\omega]},\ \underline{\alpha}=(\alpha_I,\alpha_J)} c_{\underline{\alpha}}\, \underline{x}^{\alpha_I}\, \underline{y}^{\alpha_J}$, where $\underline{y}^{\alpha_J} = \prod_{j\in J} y_j^{\alpha_j}$ and $\underline{x}^{\alpha_I} = \prod_{i\in I} x_i^{\alpha_i}$

Calculate

$$v(f) \overset{?}{=} \min_{\underline{\alpha}\in\mathbb{N}_0^{[\omega]}} v\left(c_{\underline{\alpha}}\, \underline{x}^{\alpha_I}\, \underline{y}^{\alpha_J}\right)$$

have valuation 0

?

$$= \min_{\underline{\alpha}\in\mathbb{N}_0^{[\omega]}} v(c_{\underline{\alpha}}) + \sum_{i\in I} \alpha_i \cdot v(x_i)$$

$\square$

NOW. Embedding Lemma I

Lemma 5.6 (Embedding Lemma I)

Let $(F|K, v)$ be a SIG function field and let (K^*, v^*) be a henselian extension of (K,v)

Let vF/vK be torsion-free and Fv/Kv separable

Assume there are embeddings

$$\rho: vF \hookrightarrow_{vK} v^*K^*$$
$$\sigma: Fv \hookrightarrow_{Kv} K^*v^*$$

then there exists an embedding

$$\iota: (F,v) \hookrightarrow_K (K^*,v^*)$$

inducing ρ and σ, i.e.

$$v^*(\iota(a)) = \rho(v(a)) \quad \& \quad \mathrm{res}_{v^*}(\iota(a)) = \sigma(\mathrm{res}_v(a))$$

Proof.

DEFINITION of SIG (strongly inertially generated)

$(F|K,v)$ valued function field is called SIG if there is a transcendence basis

$$\mathcal{T} = \{x_1,\dots,x_r,y_1,\dots,y_s\}$$

such that:

- $vF = vK \oplus v(x_1)\mathbb{Z} \oplus \ldots \oplus v(x_r)\mathbb{Z}$ AND
- $\{\mathrm{res}_v(y_1), \ldots, \mathrm{res}_v(y_s)\}$ is a separating transcendence basis of $Fv|Kv$, i.e. a transcendence basis such that $Fv|Kv(\mathrm{res}_v(y_1), \ldots, \mathrm{res}_v(y_s))$ is separable

and there is $a \in F^h$ such that

- $F^h = K(\mathcal{T})^h(a)$
- $v(a) = 0$
- $\mathrm{res}_v(a)$ is separable over $K(\mathcal{T})v$ and
  $\deg(\mathrm{res}_v(a) \mid K(\mathcal{T})v) = \deg(a \mid K(\mathcal{T}))$

STRATEGY: ① First construct embedding for $K(\mathcal{T})$
② Then extend to F

① embedding for $K(\mathcal{T})$

Find $x_1', \ldots, x_r' \in K^*$ s.th. $v^*(x_i') = \rho(v(x_i))$ for $i = 1, \ldots, r$

$\Longrightarrow$ $v^*(x_i')$ are rat. indep. over vK

Find $y_1', \ldots, y_s' \in K^*$ s.th. $\mathrm{res}_{v^*}(y_j') = \sigma(\mathrm{res}_v(y_j))$ for $j = 1, \ldots, s$

$\Longrightarrow$ $\mathrm{res}_{v^*}(y_j')$ are alg. indep. over Kv

$\underset{\text{Lemma 2.2}}{\Longrightarrow}$ $\mathcal{T}' := \{x_1', \ldots, x_r', y_1', \ldots, y_s'\}$ is alg. indep. over K

Get Isomorphism

$$K(\mathcal{T}) \xrightarrow{\ \cong\ } K(\mathcal{T}') \subseteq K^*$$
$$x_i \longmapsto x_i'$$
$$y_j \longmapsto y_j'$$

To show: ι respects ρ and σ

Let $f = \sum_{\underline{\alpha} \in \mathbb{N}_0^r} \sum_{\underline{\beta} \in \mathbb{N}_0^s} c_{\underline{\alpha}, \underline{\beta}} \, \underline{x}^{\underline{\alpha}} \underline{y}^{\underline{\beta}}$

For ρ: use suspicious formula from Lemma 2.2.

$$v^*(\iota(f)) = v^*\left(\sum_{\underline{\alpha}} \sum_{\underline{\beta}} c_{\underline{\alpha}, \underline{\beta}} \, \underline{x}'^{\underline{\alpha}} \underline{y}'^{\underline{\beta}}\right)$$

$$\overset{\text{Lemma 2.2}}{=} \min_{(\underline{\alpha}, \underline{\beta})} \left(v^*(c_{\underline{\alpha}, \underline{\beta}}) + \sum_{i=1}^{r} \alpha_i \cdot v^*(x_i')\right)$$

$$= \min_{(\underline{\alpha}, \underline{\beta})} \left(v(c_{\underline{\alpha}, \underline{\beta}}) + \sum_{i=1}^{r} \alpha_i \cdot \rho(v(x_i))\right)$$

$$= \rho\left(\min_{(\underline{\alpha}, \underline{\beta})} \left(v(c_{\underline{\alpha}, \underline{\beta}}) + \sum_{i=1}^{r} \alpha_i \cdot v(x_i)\right)\right)$$

$$\overset{\text{Lemma 2.2}}{=} \rho(v(f))$$

For σ: use that res_{v^*}, res_v are homomorphisms

② Extend to F

Write $F_0 = K(\mathcal{T})$

Goal: Embedding $(F^h, v^h) \hookrightarrow_K (K^*, v^*)$

$\rightsquigarrow$ can then take restriction to F

By the universal property of the Henselization, we can extend $\iota: F_0 \hookrightarrow K^*$

$$F_0^h \dashrightarrow K^*$$

to an embedding $\qquad F_0^h \hookrightarrow K^*$

$$\begin{tikzcd} K \cup F \supseteq F_0 \arrow[dr, no head] \arrow[dr, hook, bend left, "\iota"] & \\ & K \end{tikzcd}$$

Henselization is immediate $\Rightarrow$ ι induces ρ & σ

From now on: identify F_0^h with $\iota(F_0^h)$

By SIG we know $\quad F^h = F_0^h(a)$

Let $\bar{f}$ be the minimal polynomial of $\mathrm{res}_v(a) \in F^h v$ over $F_0^h v$, note $\bar{f}$ is separable by SIG

Let $f \in \mathcal{O}_{F_0^h}[X]$ be a monic lift of $\bar{f}$.

$\underset{\text{Lemma}}{\overset{\text{Hensel's}}{\Longrightarrow}}$ there is unique $a' \in \mathcal{O}_{K^*}$ s.th.

$$f(a') = 0 \quad \text{and} \quad \mathrm{res}_{v^*}(a') = \sigma(\mathrm{res}_v(a))$$

$\rightsquigarrow$ Isomorphism

$$\iota : F_0^h(a) \xrightarrow{\cong} F_0^h(a')$$

compatible with the valuation (F_0^h henselian!)

$$\begin{tikzcd} F^h = F_0^h(a) \arrow[d, no head, "n"] \\ F_0^h \end{tikzcd}$$

$r = 1$ (henselian)
$f = n$ (SIG)
$e = 1$ (fundamental inequality)

$\Rightarrow$ $vF^h = vF_0^h$

$\Rightarrow$ ι respects ρ

<u>To show: ι respects σ</u>

write $\quad \mathrm{res}_v(a) = \alpha$

$\rightsquigarrow$ $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are lin. indep. over $F_0^h v$

$\Rightarrow$ $1, a, a^2, \ldots, a^{n-1}$ form a standard valuation basis of $F^h | F_0^h$, that is,

$$v\Big(\sum_i c_i a^i\Big) = \min_i v(c_i)$$

Every $b \in F_0^h(a)$ can be expressed as

$$b = g(a) \quad \text{for some } g \in F_0^h[X], \ \deg(g) < n$$

If $v(b) = 0$, then $g \in \mathcal{O}_{F_0^h}[X] \rightsquigarrow \bar{g} \in F_0^h v[X]$

$$\mathrm{res}_{v^*}(\iota(b)) = \mathrm{res}_{v^*}(\iota(g(a))) = \mathrm{res}_{v^*}(g(a')) = \bar{g}(\mathrm{res}_{v^*}(a'))$$

$$= \bar{g}(\sigma(\mathrm{res}_v(a))) = \sigma(\bar{g}(\mathrm{res}_v(a))) = \sigma(\mathrm{res}_v(g(a))) = \sigma(\mathrm{res}_v(b)).$$