The relative approximation degree I

Freitag, 8. Dezember 2023 12:01

Where are we?

We want to prove (CIMM) / Henselian rationality

Proof structure

1. rank 1 + separably closed ← We are still here
2. rank 1
3. finite rank
4. arbitrary rank

→ Franzi outlined the proof two weeks ago

Black boxes:

- Galois-degree-p-extensions ⟶ Lemma 4.8 (Artin-Schreier-extension / equi-p, Sylvy last week)

  ⟶ Lemma 4.9 (Kummer extensions / mixed char, ???)

- TODAY: (+ prob. more) Kuhlmann-Vlahu, Theorem 11.1

  (K,v) valued field, rank 1,

  $(F|K,v)$ immediate function field, $\mathrm{tr.deg}(F|K)=1$

  Suppose there is $x \in F^h \setminus K^c$ with transcendental approximation type over K such that $F^h = K(x)^h$

  Then there is already some $y \in F$ such that $F^h = K(y)^h$ such that $F^h = K(y)^h$

§ Approximation Types

Definition

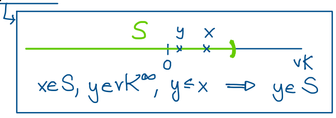
- $B_\alpha(c,K) = \{a \in K : v(a-c) \geq \alpha\}$ "closed" ball in (K,v) of radius $\alpha \in vK^\infty = vK \cup \{\infty\}$
- An approximation type over (K,v) is a collection

$$\underline{A} = \{B_\alpha(c_\alpha, K) : \alpha \in S\}$$

where $S \subseteq vK^\infty$ is an initial segment

and the balls $B_\alpha(c_\alpha, K)$ are

linearly ordered by inclusion → i.e. no two balls are disjoint!



- S is called support of $\underline{A}$,

  write $S = \mathrm{supp}\ \underline{A}$

- For $\alpha \in vK^\infty$,

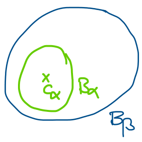
$$\underline{A}_\alpha := \begin{cases} B_\alpha(c_\alpha, K) & \text{if } \alpha \in \mathrm{supp}\ \underline{A} \\ \emptyset & \text{otherwise} \end{cases}$$

Remark

$\underline{A}$ is determined by $(\underline{A}_\alpha)_{\alpha \in T}$ where $T \subseteq_{\text{cofinal}} \mathrm{supp}\ \underline{A}$

(Because for $\beta < \alpha \in \mathrm{supp}\ \underline{A}$,

$\underline{A}_\beta = B_\beta(c_\beta, K) = B_\beta(c_\alpha, K)$)



Definition

$(L|K, v)$, $x \in L$. Define

$$\mathrm{appr}(x,K)_\alpha := \{c \in K : v(x-c) \geq \alpha\} = B_\alpha(x, L) \cap K$$

And $\mathrm{appr}(x,K) := \{\mathrm{appr}(x,K)_\alpha : \alpha \in vK^\infty, \mathrm{appr}(x,K)_\alpha \neq \emptyset\}$,

the approximation type of x over K.

§ Immediate Approximation Types

Definition

$\underline{A}$ approximation type over (K,v).

- $(L|K,v)$, $x \in L$. Say x realizes $\underline{A}$ (in (L,v)) $:\Leftrightarrow$ $\underline{A} = \operatorname{appr}(x,K)$
- $\underline{A}$ is trivial $:\Longleftrightarrow$ $\underline{A}$ is realized by some $c \in K$

  $\Longleftrightarrow$ $\underline{A}_\infty \neq \emptyset$ $\Longleftrightarrow$ $\underline{A}_\infty = \{c\}$ $\Longleftrightarrow$ $\operatorname{supp} \underline{A} = vK^\infty$
- $\underline{A}$ is immediate $:\Leftrightarrow$ $\bigcap \underline{A} := \bigcap_{\alpha \in \operatorname{supp} \underline{A}} \underline{A}_\alpha = \emptyset$

Remark (immediate $\Longrightarrow$ non-trivial)

$\underline{A}$ trivial $\Longrightarrow$ $\bigcap \underline{A} - \underline{A}_\infty \neq \emptyset$ $\Longrightarrow$ $\underline{A}$ not immediate

Fact / Kaplansky [KV, Proposition 6.6]

Every immediate approximation type is realized in some immediate simple valued field extension

Lemma [KV, Lemma 4.1]

$(L|K,v)$, $x \in L$

(a) $\operatorname{appr}(x,K)$ is immediate $\Longleftrightarrow$ $v(x-K)$ has no maximal element

(c) $\operatorname{appr}(x,K)$ is immediate $\Longrightarrow$ $\operatorname{supp} \operatorname{appr}(x,K) = v(x-K)$

Proof: (a) "$\Rightarrow$": $\underline{A} = \operatorname{appr}(x,K)$ immediate : $\bigcap_{\alpha \in \operatorname{supp} \underline{A}} \underline{A}_\alpha = \emptyset$, $c \in K$ arbitrary

$\Longrightarrow$ $\exists\, \alpha \in \operatorname{supp} \underline{A}$: $c \notin \underline{A}_\alpha = B_\alpha(x,L) \cap K$ $\Longrightarrow$ $v(x-c) > \alpha$

Let $c' \in \underline{A}_\alpha = B_\alpha(x,L) \cap K \neq \emptyset$ $\Longrightarrow$ $v(x-c') \geq \alpha > v(x-c)$

So $v(x-K)$ has no maximal element

"$\Leftarrow$": $\operatorname{appr}(x,K)$ imm. $\Longrightarrow$ $x \in L \setminus K$ (non-trivial)

Suppose $v(x-K)$ has no max : $\forall c \in K\ \exists c' \in K$: $v(x-c') > v(x-c)$

To show: for every $c \in K$ ex. $\alpha \in \operatorname{supp} \operatorname{appr}(x,K)$, s.th. $c \notin \operatorname{appr}(x,K)_\alpha$

Fix $c \in K$. There is c', c'' s.th. $v(x-c'') > v(x-c') > v(x-c)$

$\overset{\Delta\text{-ineq.}}{\Longrightarrow}$ $v(c'-c) = v((x-c)-(x-c')) = v(x-c) < v(x-c') = v(c''-c')$

$\Longrightarrow$ $v(c''-c') \in v(x-K)$, $c \notin \operatorname{appr}(x,K)_{v(c''-c')} \ni c'$

(c) "$\subseteq$": $\alpha \in \operatorname{supp} \operatorname{appr}(x,K)$ $\Longrightarrow$ $\operatorname{appr}(x,K)_\alpha \neq \emptyset$ $\longrightarrow$ $\exists c \in K$: $v(x-c) \geq \alpha$

- $v(x-c) = \alpha$ $\Longrightarrow$ $\alpha \in v(x-K)$
- $v(x-c) > \alpha$: Let $d \in K$ with $v(d) = \alpha$, then

  $v(x-(c+d)) = v(d) = \alpha \in v(x-K)$

"$\supseteq$": $c \in K$. Since $v(x-K)$ has no maximal element, there is $c' \in K$ s.th.

$v(x-c') > v(x-c)$

$\Longrightarrow$ $v(c'-c) = v((x-c)-(x-c')) = v(x-c) \in vK$

$v(x-c) \geq v(x-c)$ $\Longrightarrow$ $c \in \operatorname{appr}(x,K)_{v(x-c)} \neq \emptyset$ $\Longrightarrow$ $v(x-c) \in \operatorname{supp} \operatorname{appr}(x,K)$

§ Polynomials

Definition

$\varphi(X)$ formula. $\underline{A}$ approximation type, $(L|K,v)$, $x \in L$,
one free variable from K

$\gamma(X)$ term in vK.

- Write $\varphi(c)$ for $c \nearrow \underline{A}$ if
  there is $\alpha \in \operatorname{supp} \underline{A}$ s.th. $\varphi(c)$ holds for all $c \in \underline{A}_\alpha$
- Write "$c \nearrow x$" for "$c \nearrow \underline{A}$" if $\underline{A} = \operatorname{appr}(x, K)$
- Write $\gamma(c)$ increases for $c \nearrow x$ if
  there is $\alpha \in \operatorname{supp} \underline{A}$ s.th. for all $c' \in \underline{A}_\alpha \setminus \{x\}$:
  not necessary if $\underline{A}$ is non-trivial
  $$\gamma(c) > \gamma(c') \quad \text{for } c \nearrow x$$
  where $\underline{A} = \operatorname{appr}(x, K)$.

Definition

$\underline{A}$ approximation type over (K,v)

- $f \in K[X]$:
  $\underline{A}$ fixes the value of f $:\Longleftrightarrow$ $\exists \alpha \in vK$: $v(f(c)) = \alpha$ for $c \nearrow \underline{A}$
  Then: this α is the fixed value $v(f(c))$ for $c \nearrow \underline{A}$
- $\underline{A}$ is a transcendental approximation type
  $:\Longleftrightarrow$ $\underline{A}$ fixes the value of every $f \in K[X]$
  Otherwise: $\underline{A}$ is an algebraic approximation type
  - an associated minimal polynomial for $\underline{A}$ is a monic polynomial of minimal degree whose value is not fixed by $\underline{A}$
  - the degree of $\underline{A}$ is its associated minimal polynomial
    transcendental approximation type: degree $d = \infty$

Remark

- An associated minimal polynomial for $\underline{A}$ is irreducible
- If $\operatorname{supp} \underline{A} \subsetneq vK$, then the associated minimal polynomial is not unique

Lemma [KV, Lemma 5.2; Kaplansky, Lemma 4]

Γ ordered abelian group, $\alpha_1, \dots, \alpha_m \in \Gamma$, $\Upsilon \subseteq \Gamma$ without max element.
$t_1, \dots, t_m \in \mathbb{Z}$, distinct

Then: ex. $\beta \in \Upsilon$, permutation $\sigma: \{1,\dots,m\} \to \{1,\dots,m\}$ s.th. f.a. $\gamma \in \Upsilon$, $\gamma \geq \beta$
$$\alpha_{\sigma(1)} + t_{\sigma(1)}\gamma > \alpha_{\sigma(2)} + t_{\sigma(2)}\gamma > \dots > \alpha_{\sigma(m)} + t_{\sigma(m)}\gamma$$

Lemma [KV, Lemma 5.2]

$\underline{A} = \operatorname{appr}(x,K)$ imm. appr. type of degree d over (K,v).
$f \in K[X]$, $\deg(f) \leq d$, $f = \sum c_i X^i$

$f_i := \sum_{j=i}^{n} \binom{j}{n} c_j X^{j-i}$ formal i-th derivative of f $\quad \frac{f^{(n)}}{n!}$

$\beta_i :=$ fixed value $v(f_i(c))$ for $c \nearrow x$ $\quad (\deg(f_i) < \deg(f) \leq d)$

Then there is $h \leq \deg f$ such that

(54) $\quad \beta_h + h \cdot v(x-c) < \beta_i + i \cdot v(x-c)$ for $i \neq h$, $i = 1, \dots, \deg(f)$
for $c \nearrow x$

Hence,

$$(5.5) \qquad v(f(x)-f(c)) = \beta_h + h\cdot v(x-c) \quad \text{for } c \nearrow x$$

Consequently,

$$\begin{cases} v(f(x)-f(c)) > v(f(x)) = v(f(c)) & \text{for } c\nearrow x, \ \text{if } \underline{A} \text{ fixes the value of } f \\ v(f(x)) > v(f(c)) = \beta_h + h\cdot v(x-c) & \text{for } c\nearrow x, \ \text{if } \underline{A} \text{ doesn't fix the value of } f \end{cases}$$

Proof: Set $n = \deg(f)$.

Taylor expansion

$$(*) \qquad f(x) - f_0(c) = f_1(c)(x-c) + \dots + f_n(c)(x-c)^n$$

$$v(f_i(c)(x-c)^i) = \underbrace{\beta_i}_{\alpha_i} + \underbrace{i}_{t_i}\, \underbrace{v(x-c)}_{\gamma} \quad \text{for } c\nearrow x$$

$\underline{A}$ imm. appr. type $\overset{\text{Lemma 4.1.}}{\Longrightarrow}$ supp $\underline{A}$ has no maximal element

Lemma 5.1. with $\gamma = \operatorname{supp} \underline{A} \subseteq vK$, $\alpha_i = \beta_i$, $t_i = i$:

There is $h < \deg(f)$, s.th. (5.4):

$$\beta_h + h\cdot v(x-c) < \beta_i + i\cdot v(x-c) \quad \text{for } c\nearrow x,\ i\neq h$$

This implies (5.5) using $(*)$ and ultrametric Δ-ineq.

- if $\underline{A}$ fixes the value of f, then assume for contradiction

  $v(f(x)) \neq v(f(c))$ for $c\nearrow x$ $\Longrightarrow$ $\underbrace{v(f(x)-f(c))}_{\text{LHS of (5.5)}} = \min\{v(f(x)), v(f(c))\}$

  $\hookrightarrow$ has fixed value for $c\nearrow x$

  $\lightning$ but $\underbrace{\beta_h + h\cdot v(x-c)}_{\text{RHS of (5.5)}}$ increases for $c\nearrow x$

  $\Longrightarrow$ ✓

- if $\underline{A}$ doesn't fix the value of f, then $v(f(c)) \neq v(f(x))$ for $x\nearrow c$

  $\Longrightarrow \underbrace{v(f(x)-f(c))}_{\overset{(5.5)}{=} \beta_h + h\cdot v(x-c)} = \min\{\underbrace{v(f(x))}_{\text{constant}}, \underbrace{v(f(c))}\}$ $\nwarrow$ the minimum $\Longrightarrow$ ✓

  increases for $x\nearrow c$

Lemma 5.2. only for polynomials of degree $\leq d$. Now:

Lemma 5.4.

Take an immediate algebraic approximation type $\underline{A} - \operatorname{appr}(x, K)$ over (K, v) and an associated minimal polynomial $f \in K[X]$ for $\underline{A}$.

Further, take an arbitrary polynomial $g \in K[X]$ and write

$$g = c_k f^k + \dots + c_1 f + c_0$$

where $c_i \in K[X]$ with $\deg(c_i) < \deg(f)$.

Then there is $1 \leq m \leq k$, $\beta \in vK$ such that with h from 5.2.

$$v(g(c) - c_0(c)) = v(c_m(c)) + m\cdot v(f(c)) = \beta + m\cdot h\cdot v(x-c) \quad \text{for } c\nearrow x$$

Consequently,

$$\begin{cases} v(g(x)) = v(g(c)) = v(c_0(c)) = v(c_0(x)) < v(g(c) - c_0(c)) \quad \text{for } c\nearrow x \\ \qquad\qquad \text{if } \underline{A} \text{ fixes the value of } g \\ v(g(x)) > v(g(c)) = \beta + m\cdot h\cdot v(x-c) \quad \text{for } c\nearrow x \\ \qquad\qquad \text{if } \underline{A} \text{ doesn't fix the value of } g \end{cases}$$

Proof: Not in this talk. $\square$

§ Relative Approximation Degree

$\underline{A}$ imm. appr. type over (K,v), $\underline{A}$ - appr(x,K)

Definition

Let $f \in K[X]$, $\deg(f) \leq \deg \underline{A}$. The integer h from Lemma 5.2. is called the relative approximation degree of f in x over K and is denoted by $h_K(x:f)$.

Remark

By Lemma 5.2,

$$1 \leq h_K(x:f) \leq \deg(f)$$

and $\quad v(f(x)-f(c)) = \beta_{h_K(x:f)} + h_K(x:f) \cdot v(x-c) \quad$ for $c \nearrow x$

Observation

$g \in K[X]$, arbitrary degree.

There are unique $\beta \in vK$, $k \in \mathbb{Z}_{>0}$ s.th.

(#) $\quad v(g(x)-g(c)) = \beta + k \cdot v(x-c)$

(This is because $\underline{A}$ is immediate and so $v(x-c)$ takes infinitely many values for $c \nearrow x$)

Definition

$g \in K[X]$, arbitrary degree.

The integer k from (#) is called relative approximation degree of $g(x)$ in x, denoted by $h_K(x:g)$.

The value β from (#) is called relative approximation constant of $g(x)$ in x, denoted by $\beta_K(x:g)$.

§ Outlook – the proof of 11.1

Situation:

$$(10.1) \begin{cases} (K,v) \text{ a valued field of rank } 1 \\ (K(x)\,|\,K,\ v) \text{ an immediate extension s.th. } x \notin K^c \\ \text{appr}(x,K) \text{ is transcendental} \\ y \in K(x)^h \text{ is transcendental over } K \end{cases}$$

Q: What can we say about $[K(x)^h : K(y)^h]$?

(Want: $[K(x)^h : K(y)^h] = 1$)

Idea: Define relative approximation degree of x over y:

Find polynomial $f \in K[X]$ s.th.

$$v(y - f(x)) \geq \text{dist}(y, K)$$

smallest initial segment of $\text{div}(vK)$ containing supp appr(y,K)

Lemma 10.2: In situation (10.1), such an f exists

Definition

$h_K(x:y) := h_K(x:f)$, the relative appr. degree of y in x

$\beta_K(x:y) := \beta_K(x:f)$.

where f is as in Lemma 10.2.

Lemma 10.3. $h_K(x:y)$ and $\beta_K(x:y)$ are well-defined
i.e. do not depend on f

Theorem 10.7: In situation (10.1),
$$[K(x)^h : K(y)^h] \le h_K(x:y).$$
Needs many ingredients

Kuhlmann–Vlahu, Theorem 11.1

(K,v) valued field, rank 1,

$(F|K,v)$ immediate function field, $\operatorname{tr.deg}(F|K)=1$

Suppose there is $x \in F^h \setminus K^c$ with transcendental approximation type over K such that $F^h = K(x)^h$

Then there is already some $y \in F$ such that $F^h = K(y)^h$

Proof idea of 11.1: Find $y \in F$ such that $h_K(x:y) = 1$

Then $[K(x)^h : K(y)^h] \le h_K(x:y) = 1$ by Thm 10.7.

So $K(x)^h = K(y)^h$.

Need dist & stuff from [KV, Chapter 10]