Henselian Rationality continues to drag on

Working group on tame fields

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Goal

Theorem (11.1)

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Actually (and that's how we prove it) there is some $\gamma \in vK$ *such that* $K(x)^h = K(y)^h$ *for every* $y \in F$ *with* $v(x - y) \geqslant \gamma$*.*

Outline

We work in the following situation:

$$\begin{cases} (K, v) \text{ a valued field of rank 1} \\ (K(x)|K, v) \text{ immediate, } x \notin K^c \text{, and} \\ \operatorname{appr}(x, K) \text{ transcendental} \\ y \in K(x)^h \text{ transcendental over } K. \end{cases} \tag{10.1}$$

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We aim to control the degree $[K(x)^h : K(y)^h]$.

To do so, we define $h_K(x : y)$ to be $h_K(x : f)$ for any $f \in K[X]$ s.t. $v(y - f(x)) > \text{dist}(y, K)$, and prove that

$$[K(x)^h : K(y)^h] \leqslant h_K(x : y).$$

Approximative summary of Marga's talk (1/2)

Given valued fields $(L|K, v)$ and $x \in L$, define for $\alpha \in vK$

$$\mathrm{appr}(x, K)_\alpha = B_\alpha(x, L) \cap K.$$

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$$\mathrm{appr}(x, K) = \{\mathrm{appr}(x, K)_\alpha \mid \alpha \in S_{x,K}\}.$$

We say that $\mathrm{appr}(x, K)$ is immediate if $\bigcap_{\alpha \in S_{x,K}} \mathrm{appr}(x, K)_\alpha = \emptyset$.

Approximative summary of Marga's talk (2/2)

We say that $\mathrm{appr}(x, K)$ fixes the value of $f \in K[X]$ if there is $\alpha \in vK$ such that $v(f(c)) = \alpha$ for $c \nearrow x$.

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Lemma (5.2)

Take $\mathrm{appr}(x, K)$ *immediate,* $f \in K[X]$ *of degree* $\leqslant d_{x,K}$*,* $f_i = \frac{f^{(i)}}{i!}$*,* β_i *its fixed value by* $\mathrm{appr}(x, K)$*.*

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Then there is $h = h_K(x : f) \leqslant \deg(f)$ *such that for* $i \neq h$:

$$\beta_h + hv(x - c) < \beta_i + iv(x - c) \text{ for } c \nearrow x$$

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and if $\mathrm{appr}(x, K)$ *fixes the value of* f*, then*

$$v(f(x) - f(c)) > v(f(x)) = v(f(c)) \text{ for } c \nearrow x.$$

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$$x \in K^c \Leftrightarrow \text{dist}(x, K) = \infty.$$

Lemma (4.2)

Take $(L|K, v)$ and $x, x' \in L$. Assume $\text{appr}(x, K)$ *is immediate.*

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Lemma (6.2 (not proven; not needed?...))

If $\operatorname{appr}(x, K)$ *is immediate and transcendental, then* $(K(x)|K, v)$ *is immediate and transcendental.*

The hunt for f

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Under assumptions 10.1, there is $f \in K[X]$ *such that* $v(y - f(x)) \geqslant \operatorname{dist}(y, K)$.

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$$v\left(\frac{1}{g(x)} - \frac{\sum_{i=0}^{j-1} h(x)^i}{c}\right) = v\left(\frac{h(x)^j}{c(1 - h(x))}\right) > \alpha.$$

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   Assume not. Then K is dense in $K(y)^h$. Let g be the minimal polynomial of x over $K(y)^h$. We can find a polynomial $\widetilde{g}$ with coefficient close enough to g, and by continuity of roots, $\widetilde{g}$ has a root $\widetilde{x}$ such that $v(x - \widetilde{x}) \geqslant \mathrm{dist}(x, K)$. By 4.2 we have $\mathrm{appr}(x, K) = \mathrm{appr}(\widetilde{x}, K)$, but $\widetilde{x}$ is algebraic and thus $\mathrm{appr}(\widetilde{x}, K)$ is algebraic by 6.2 (or 5.5?).

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2. $y \notin K^c$.
3. f exists.
   Indeed, since $y \notin K^c$, $\operatorname{dist}(y, K) < \infty$, and since $y \in K[x]^c$, we can find some $f \in K[X]$ arbitrary close to y.

□

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$$v(g(x) - g(c)) =$$

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h_K (disambiguation)

We define $h_K(x : y) = h_K(x : f)$ and $\beta_K(x : y) = \beta_K(x : f)$, where f is a polynomial as in the previous lemma.

Lemma (10.3)

$h_K(x : y)$ and $\beta_K(x : y)$ do not depend on the choice of f.

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Lemma (10.4)

Under assumptions 10.1 and with f as above, there is $z \in \widetilde{K(y)}$ such that

$$v(x - z) \geqslant \frac{v(y - f(x)) - \beta_K(x : y)}{h_k(x : y)}$$

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$$[K(y, z)^h : K(y)^h] \leqslant h_K(x : y).$$

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We will not do the proof today.

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By Krasner's lemma, $x \in K(y)^h(z)$. Now $[K(x, y)^h : K(y)^h] \leqslant [K(y, z)^h : K(y)^h] \leqslant h_K(x : y)$ by the choice of z, but since $K(x)^h = K(x, y)^h$, we conclude. □

i'm lost. what are we doing?

Lemma (10.6)

Under assumptions 10.1, given $z \in K(y)^h$ *transcendental over* K*, 10.1 also holds for* y *and* z *in lieu of* x *and* y*, and:*

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   Now $h_K(x : y) = h_K(x : f)$, $h_K(y : z) = h_K(y : g)$, and $h_K(x : z) = h_K(x : g \circ f)$ since $v(z - g(f(x)) = v(z - g(y) + g(y) - g(f(x))) \geqslant \operatorname{dist}(z, K)$.

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And that should be all!