add dragons to taste

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Overview

For the past month we were working on the proof of CIMM/Henselian Rationality.

1. rank 1 and separably closed $\leftarrow$ Franzi 28.11
2. rank 1 $\leftarrow$ Marga 12.12
3. finite rank $\leftarrow$ hopefully me today
4. any rank

How we can bring back rank to 1?

Proposition (5.6 , Elimination of ramification II)

Every immediate separable function field $(F|K, v)$ of transcendence degree 1 over a separably tame field (K, v) of finite rank is henselian rational.

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Since F has finite rank and $F|K$ is an immediate extension of transcendence degree 1, there exists a decomposition $v = v_3 v_2 v_1$ such that

1. v_2 has rank 1,
2. $\operatorname{trdeg}(Fv_1|Kv_1) = 1$,
3. $\operatorname{trdeg}(Fv_2 v_1|Kv_2 v_1) = 0$.

What about the separable tame condition?

Lemma (3.14, algebra and model theory of tame valued fields)

Take a separably tame field $(K, v_3 v_2 v_1)$ of characteristic $p > 0$ and assume that v_2 is nontrivial. Then (Kv_1, v_2) is a separably tame field. If also v_1 is nontrivial, then (Kv_1, v_2) is a tame field.

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By the above Lemma, the hypothesis that (K, v) is a separably tame field yields

1. (K, v_1) and (Kv_1, v_2) are separably tame fields,
2. if v_1 is nontrivial, then (Kv_1, v_2) is a tame field,
3. since v_2 is nontrivial, $(Kv_2 v_1, v_3)$ is a tame field.

Immediate Pull

Since $(F|K, v)$ is immediate, it follows that $v_1 F = v_1 K$, $v_2(Fv_1) = v_2(Kv_1)$, and that the algebraic extension $(Fv_2 v_1|Kv_2 v_1, v_3)$ is immediate.

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Because $(Kv_2 v_1, v_3)$ is tame, the latter extension must be trivial. This yields that also $(Fv_1|Kv_1, v_2)$ is an immediate extension.

Finally finitely generated

Since $\mathrm{trdeg}(Fv_1|Kv_1) = 1$, we find $t \in F$ such that $\mathrm{trdeg}(Fv_1|K(t)v_1) = 0$. Moreover $F|K(t)$ is finite, therefore $Fv_1|Kv_1$ is finitely generated.

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If v_1 is nontrivial then from Lemma 3.14 we have (Kv_1, v_2) is a tame field and hence perfect. It follows that $Fv_1|Kv_1$ is separable.
If v_1 is trivial, then since $F|K$ is separable by assumption, also $Fv_1|Kv_1$ is separable.

Rationalize it

We have shown that $(Fv_1|Kv_1, v_2)$ is an immediate function field of transcendence degree 1, rank 1, and over separably tame field (Kv_1, v_2). We can now use henselian rationality for rank 1!

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Now we apply henselian rationality for rank 1 to find some $x \in F$ such that

$$Fv_1^{h(v_2)} = Kv_1(xv_1)^{h(v_2)}$$

Make x separating element again

Let us choose any separating element z of $F|K$ and an element $a \in K^x$ which satisfies $v_1(a) > v_1(x)$ and $v_1(az) > v_1(x)$.
Both $F|K(x,z)$ and $K(x,z)|K$ are separable extensions.

$$K(x,z,a) = K(x, x+az, x+a)$$

We have $v_1 x = v_1(x+az) = v_1(x+a)$ and $xv_1 = (x+az)v_1 = (x+a)v_1$.
At least one of the elements $x, x+az, x+a$ must be a separating element for the separable extension $K(x, x+az, x+a)|K$.
Therefore $F|K(x)$ is separable and algebraic for suitable $x \in F$.

$F^{h(v)} = K(x)^{h(v)}$ for trivial v_1

Because $(Fv_1|Kv_1, v_2)$ is immediate we have that
$F^{h(v_2 v_1)} v_2 v_1 = F v_2 v_1 = K v_2 v_1$ and $K(x)^{h(v_2 v_1)} v_2 v_1 = K(x) v_2 v_1 = K v_2 v_1$.

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Since $(K(x)v_2 v_1, v_3)$ is a tame field, it is henselian, therefore $(F^{h(v_2 v_1)}, v)$ also is henselian and $F^{h(v_2 v_1)} = F^{h(v)}$.
Similarly we have $K(x)^{h(v2v1)} = K(x)^{h(v)}$.

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$$K(x)^{h(v)} v_1 \subseteq F^{h(v)} v_1 = F v_1^{h(v_2)} = K v_1 (x v_1)^{h(v_2)} \subseteq K(x)^{h(v)} v_1$$

Because equality holds everywhere, we have that $K(x)^{h(v)} = F^{h(v)}$ for trivial v_1.

$F^{h(v)} = K(x)^{h(v)}$

Since $(F^{h(v)}|K, v)$ is immediate we have that $v_1 F^{h(v)} = v_1 K$. Together with $K(x)^{h(v)} v_1 = F^{h(v)} v_1$, we have that $(F^{h(v)}|K(x)^{h(v)}, v_1)$ is immediate.

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We have already noted that (K, v_1) is separably tame, therefore we can use **Generalized Stability Theorem** to find that $(K(x), v_1)$ and $(K(x)^{h(v)}, v_1)$ are separably defectless fields.

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Since the extension $F|K(x)$ is finite and separable, the same is true for the extension $(F^{h(v)}|K(x)^{h(v)}), v_1)$. As this extension is also immediate and $(K(x)^{h(v)}, v_1)$ is a henselian separably defectless field, it follows that the extension must be trivial, as desired.

new boxed dragon

Theorem (1.1, elimination of ramification I)

Let $(F|K, v)$ be a valued function field without transcendence defect. If (K, v) is a defectless field, then (F, v) is a defectless field. The same holds for "inseparably defectless" in the place of "defectless". If vK is cofinal in vF, then it also holds for "separably defectless" in the place of "defectless".