Where are we?

Aim: (Dive deeper into) CIMM: (k,v) tame and $(F,v)/(k,v)$ immediate function field with $\mathrm{tr.deg}_k(F)=1$, then $F^h = E^h$ for some $k \subseteq E$ rational function field.

Today: More on Kuhlmann-Vlahu: Let (k,v) be a rank 1 valued field and let $(F|k,v)$ be an immediate extension with $F|k$ of tr. deg. 1, F a function field. Suppose there is some $x \in F^h \setminus k^c$ with transcendental approximation type $/k$ such that $F^h = k(x)^h$.

Then there is some $y \in F$ such that $F^h = k(y)^h$. In fact, there is some $\gamma \in vk$ such that $k(x)^h = k(y)^h$ for every $y \in F$ with $v(x-y) \geq \gamma$.

Proof:

1) $x \notin k^c \Rightarrow \exists \gamma \in vk: \ \gamma > \mathrm{dist}(x,k)$.

2) $x \in F^h$ and $F|k$ is immediate $\Rightarrow vk = vF \Rightarrow (F,v)$ has rank 1 too $\Rightarrow F^h \subseteq F^c \ni x$.

3) F is dense in F^c, so there is some $y \in F$ with $v(x-y) \geq \gamma$.

4) If $v(x-y) > \mathrm{dist}(x,k)$, then $[k(x)^h : k(y)^h] \leq h_k(x:y)$. (Theorem 10.7)

5) $h_K(x:y) = h_K(x:x) = 1$ (Lemma 10.3).

1) Distances & approximations (Omarga's session)

Recall

✓ $\mathcal{B}_\alpha(c, K) = \{a \in K : v(a-c) \geq \alpha\}$ is the "closed" ball in K of radius $\alpha \in vK^\infty$ and "center" $c \in K$.

✓ $S \subseteq vK^\infty$ initial segment, $\{c_\alpha : \alpha \in S\} \subseteq K$, $A = \{\mathcal{B}_\alpha(c_\alpha, K) : \alpha \in S\}$ a chain. We say that A is an approximation type over (K, v).

✓ $\alpha \in S \Rightarrow A_\alpha := \mathcal{B}_\alpha(c_\alpha, K)$; $\alpha \notin S \Rightarrow A_\alpha := \emptyset$.

✓ $S =: \operatorname{supp} A$.

✓ Let $(L|K, v)$ be an extension, $x \in L$, $\alpha \in vK^\infty$. Define $\operatorname{appr}(x, K)_\alpha = \{c \in K : v(x-c) \geq \alpha\}$
$= \mathcal{B}_\alpha(x, L) \cap K$.

✓ $\operatorname{appr}(x, K)_\alpha$ is either $\emptyset$ or a closed ball of radius α.

$c \in K \cap \mathcal{B}_\alpha(x, L)$

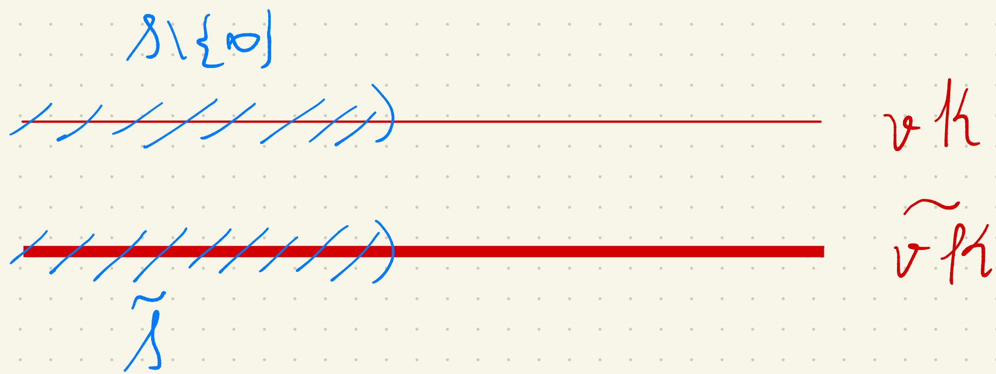
✓ The set $\{\alpha \in vK^\infty : \operatorname{appr}(x, K)_\alpha \neq \emptyset\}$ is an initial segment of vK^∞.

$\operatorname{appr}(x, K)_\alpha = \mathcal{B}_\alpha(c, K$

✓ Define $\operatorname{appr}(x,K) := \{ \operatorname{appr}(x,K)_\alpha : \alpha \in vK^\infty,\ \operatorname{appr}(x,K)_\alpha \neq \emptyset \}$. This is the approximation type of x over (K, v).

✓ $\mathcal{S} = \operatorname{supp}(\operatorname{appr}(x,K)) \Longrightarrow \mathcal{S} \cap vK = \mathcal{S} \setminus \{\infty\}$ is a (lower) cut in vK because $\mathcal{S}$ is an initial segment.

✓ This cut induces a cut in $\widetilde{vK}$ (the divisible hull of vK) with lower cut $=$ smallest initial segment of $\widetilde{vK}$ containing $\mathcal{S}\setminus\{\infty\}$.



$$\gamma \in \widetilde{\mathcal{S}} \iff \exists\ \delta \in \mathcal{S} : \gamma \leq \delta$$

✓ We call $\mathcal{S}$ the distance from x to $(\mathfrak{h}, v)$ and denote it by $\mathrm{dist}(x, \mathfrak{h})$

Question

Let ε be a bounded element in an el. extension of $(\mathbb{R}, <)$ be such that $\mathcal{S} \setminus \{\infty\} = \varepsilon^-$. Then $\mathrm{dist}(x, \mathfrak{h}) = \{\gamma \in \widetilde{v\mathfrak{h}} : \gamma < \varepsilon\}$. Well, not quite.

✓ We write $\mathrm{dist}(x, \mathfrak{h}) = \infty$ if $\mathrm{dist}(x, \mathfrak{h}) = \widetilde{v\mathfrak{h}}$, and $\mathrm{dist}(x, \mathfrak{h}) < \infty$ otherwise.

Observation:

$x \in \mathfrak{h}^c \Leftrightarrow \mathrm{dist}(x, \mathfrak{h}) = \infty$.

Proof

We want to see $x \in \mathfrak{h}^c \Leftrightarrow \mathrm{supp}(\mathrm{appr}(x, \mathfrak{h})) = v\mathfrak{h} \Leftrightarrow \mathrm{dist}(x, \mathfrak{h}) = \widetilde{v\mathfrak{h}}$:

$(\Leftarrow)$

Let $\gamma \in v\mathfrak{h}$

$\exists\, c_\gamma \in \mathfrak{h} : c_\gamma \in \mathcal{B}_\gamma(x, L)$

$\mathrm{supp}(\mathrm{appr}(x, K)) = vK$

$\gamma \in vK$

$\exists c_\gamma : c_\gamma \in \mathcal{D}_\gamma(x, L)$ for arbitrary

$(c_\gamma)_\gamma \rightsquigarrow x$ is ps.c. α γ

$\exists \nu_0 \in vK : \forall \nu \geq \nu_0 : v(c_\nu - c_{\nu_0}) > \gamma$

$$v(x - c_\nu) \geq \nu \geq \nu_0$$

$$v(c_{\nu_0} - c_\nu) = v(c_{\nu_0} - x + x - c_\nu)$$

$$> \gamma$$

$$\mathrm{supp}(\mathrm{appr}(x, P_h)) = v_h \iff \mathrm{dist}(x, P_h) = \tilde{v}_h$$

$(\Longrightarrow)$ ✓

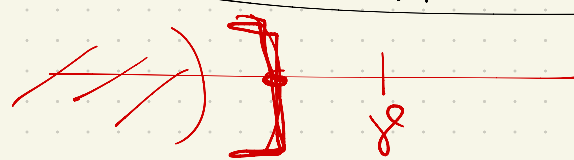
$(\Longleftarrow)$ $\subseteq$ always

$$\gamma \in v_h \subseteq \tilde{v}_h = \mathrm{dist}(x, P_h)$$

$$\nexists\, \gamma \in \mathrm{supp}(\mathrm{app}(x, P_h))$$

$$(-\infty, \gamma] \subseteq \mathrm{supp}(\mathrm{app}(x, P_h))$$

$\tilde{v}_h$



Corollary

$x \notin k^c \Rightarrow \operatorname{dist}(x, k) < \infty$, i.e., $\exists\, \gamma \in vk$: $\operatorname{supp}(\operatorname{appr}(x, k)) < \gamma$.

Towards 10.7:

Assume :
$$\begin{cases} (k, v) \text{ valued field of rank } 1 \\ (k(x)|k, v) \text{ immediate extension with } x \notin k^c \\ \operatorname{appr}(x, k) \text{ transcendental} \\ y \in k(x)^h \text{ transcendental} / k. \end{cases} \qquad (\star)$$

Observation:

Let $y \in F$ satisfy $v(x-y) \geq \gamma > \operatorname{dist}(x, k)$. Then $y \in k(x)^h$ is transcendental $/k$, so $\star$ holds.

Indeed,

$(k(x)|k, v)$ immediate $\Longrightarrow$ $\operatorname{appr}(x, k)$ immediate (i.e., $\bigcap \operatorname{appr}(x, k) = \emptyset$) (Lemma 4.1.b)

$v(x-k)$

$\Longrightarrow \operatorname{appr}(x, k) = \operatorname{appr}(y, k) \iff v(x-y) \geq \operatorname{dist}(x, k)$ (Lemma 4.2)

$\Longrightarrow \operatorname{appr}(y, k)$ is transcendental & immediate

$\Longrightarrow y$ is transc $/k$ (Corollary 6.2)

✓ (Omarga) $\mathrm{appr}(x, K)$ is immediate $\Leftrightarrow$ $v(x - K)$ has no maximal element.

$(K(x)|K)$ immediate, $c \in K \Rightarrow v(x-c) \in v(K(x)) = vK \Rightarrow \exists d \in K : v(x-c) = v(d)$

$\Rightarrow v\left(\frac{x-c}{d}\right) = 0 \Rightarrow \mathrm{res}_{K(x)}\left(\frac{x-c}{d}\right) \neq 0$ & $\exists y \in \mathcal{O}_K : \mathrm{res}_K(y) = \mathrm{res}_{K(x)}\left(\frac{x-c}{d}\right) \Rightarrow$

$\Rightarrow \mathrm{res}_{K(x)}(y) = \mathrm{res}_{K(x)}\left(\frac{x-c}{d}\right) \Rightarrow v\left(y - \frac{x-c}{d}\right) > 0 \Rightarrow v(dy - x - c) > v(d) = v(x-c)$, and

$dy - c \in K$, so $v(x - K)$ has no maximal element.

✓ Lemma 4.2: $\mathrm{appr}(x, K) = \mathrm{appr}(y, K) \iff v(y-x) \geq \mathrm{dist}(x, K)$ whenever $\mathrm{appr}(x, K)$ is immediate

$h_k(x:y)$:

<u>Assume</u> : $\begin{cases} (k, v) \text{ valued field of rank } 1 \\ (k(x)|k, v) \text{ immediate extension with } x \notin k^c \\ \operatorname{appr}(x, k) \text{ transcendental} \\ y \in k(x)^h \text{ transcendental } / k. \end{cases}$

<u>Lemma 10.2</u> (Amarga)

Under these conditions we have that $y \in k[x]^c \setminus k^c$ and there exists a polynomial $f \in k[X]$ such that

$$v(y - f(x)) \geq \operatorname{dist}(y, k)$$

We define $h_k(x:y) := h_k(x:f) = h \leq \deg f$ where

$$\beta_h + h \cdot v(x-c) < \beta_i + i\, v(x-c)$$

for $c \nearrow x$, β_i the fixed value $v(f_i(c))$, $i \neq h$, $i \in \{1, \dots, \deg f\}$,

$\deg f \leq d = \deg(\operatorname{appr}(x, k))$, $\operatorname{appr}(x, k)$ immediate.

Made with Goodnotes

Theorem 10.7

Assume:
$$\begin{cases} (k, v) \text{ valued field of rank } 1 \\ (k(x)|k, v) \text{ immediate extension with } x \notin k^c \\ \operatorname{appr}(x, k) \text{ transcendental} \\ y \in k(x)^h \text{ transcendental} / k . \end{cases}$$

Then $[k(x)^h : k(y)^h] \leq h_k(x : y)$.

Proof:

$$\begin{array}{c} k(x)^h \\ | \\ k(y)^h \end{array}$$

can be decomposed as

$$\begin{array}{cl} k(x)^h & \\ | & \leftarrow \text{inseparable, nec. of deg } p^n = [k(x)^h : k(y)^h]_i . \\ L & \\ | & \leftarrow \text{maximal separable subextension} \\ k(y)^h & \end{array}$$

x^{p^n} is separable over $k(y)^h$

$x^{p^n} \in L$

$k(x^{p^n})^h \subseteq L^h = L$ (because L is rel. sep. closed in the henselian $k(x)^h$).

$k(x)^h = k(x^{p^n})^h(x)$ ✓

$$p^n \geq [k(x)^h : k(x^{p^n})^h] = [k(x)^h : L][L : k(x^{p^n})^h] = p^n [L : k(x^{p^n})^h]$$

$k(x)^h = k(x^{p^n})^h(x)$

and $[k(x)^h : k(x^{p^n})^h(x)] \leq p^n$

because $X^{p^n} - x^{p^n} \in k(x^{p^n})^h[X]$

is satisfied by x.

$$\implies [L : k(x^{p^n})^h] \leq 1$$

$$\implies L = k(x^{p^n})^h$$

(Lemma 10.6) If $y \in k(x)^h$ is transc $/k$ and $z \in k(y)^h$ is trans $/k$, then $z \in k(x)^h$, $h(y:z)$ is defined and $h(x:z) = h(x:y)\,h(y:z)$.

$$\implies h(x:y) = h(x:x^{p^n})\,h(x^{p^n}:y)$$

$$= p^n \; h(x^{p^n}:y)$$

Moreover,

Lemma 10.5 (Reduction to the separable case)

If $k(x)^h | k(y)^h$ is separable, then $[k(x)^h : k(y)^h] \leq h_k(x:y)$

So $[k(x)^h : k(y)^h] = [k(x)^h : k(x^{p^n})^h][k(x^{p^n})^h : k(y)^h]$

$$= p^n \cdot [k(x^{p^n})^h : k(y)^h]$$

$$\leq p^n \cdot h(x^{p^n} : y)$$

$$= h(x:y)$$

We used Corollary 9.2:

Under some other (fulfilled?) hypotheses, if (k, v) is henselian, $x \in k^{alg}$, $d = [k(x) : k]$ and $f = \mu_{x,k}$, then

$$d = h(x:f) = p^t$$

TO DO !!!!