Henselian rationality

Aim (CIMM): (K,v) tame and $(F,v)/(K,v)$ immediate function field with $\mathrm{trdeg}_K F = 1$, then $F^h = E^h$ for some $K \subseteq E$ rational function field.

Proof structure:

5.2 - rank 1 + sep-alg. closed ← TODAY [well, not quite!]

5.3 - rank 1

5.6 - finite rank

5.7 - arbitrary rank

Proposition 5.2: Every immediate sep. function field $(F|K,v)$ of transcendence degree 1 over a sep.-alg. closed field K of rank 1 is henselian rational.

Pf: Note: v not trivial on K as o/w

$vK = vF = \{0\} \Rightarrow v$ trivial on F

$K = Kv = Fv = F \Rightarrow F = K$ ↯

K SCF $\Rightarrow$ vK divisible and Kv alg. closed

Take $x \in F$ s.th. $F/K(x)$ is separable (exists, because F/K is separable!)

As $(K(x)|K,v)$ is immediate

$\Rightarrow K(x)v = Kv \models ACF$, $vK(x) = vK \equiv \mathbb{Q}$.

If $F \not\subseteq K(x)^h$, then

$F \cdot K(x)^h / K(x)$ is separable, and a tower of Galois ext. of degree p

(By Ostrowski: $F \cdot K(x)^h \supseteq K(x)^h$ finite

$\Rightarrow [F \cdot K(x)^h : K(x)^h] = p^n$

p-groups are solvable $\Rightarrow$ get tower of degree-p ext.)

$\rightsquigarrow K(x)^h \overset{\deg p}{\subseteq} M_1 \overset{p}{\subseteq} \dots \overset{p}{\subseteq} M_n = F \cdot K(x)^h \subseteq F^h$

Lemma 4.8: (K,v) sep. tame, $\mathrm{char}(K) = p > 0$, rk 1. If $(K(x)|K,v)$ is immediate, $E/K(x)^h$ Galois, of degree p, then there is $\vartheta \in E$ s.th. $E = K(\vartheta)^h$

Lemma 4.9: (K,v) alg. closed, $\mathrm{char}(K) = 0$, rk 1. If $(K(x)|K,v)$ is immediate, $E/K(x)^h$ Galois of degree $p = \mathrm{char}(Kv) > 0$, then there is $\eta \in E$ s.th. $E = K(\eta)^h$

PROOFS LATER

$\Rightarrow$ $M_1 = K(\vartheta_1)^h$, $M_2 = (K(\vartheta_1)^h)(\vartheta_2)^h$
$= K(\vartheta_1, \vartheta_2)^h = ?$

Induction $\Rightarrow$ $F \cdot K(x)^h = K(y)^h$ for some $y \in F^h$
$\Rightarrow F \subseteq K(y)^h$

1. If $y \in K^c$ ← completion of (K,v)
(K,v) rank-1 $\overset{\text{Hensel's Lemma}}{\Rightarrow}$ (K^c, v^c) henselian
wlog $\Rightarrow$ $K(y) \subseteq K^c \Rightarrow F \subseteq K^c$

Theorem 2.3: (K,v) hens., $(F|K,v)$ separable func. field. If $F \subseteq K^c$, then $(F|K,v)$ is henselian rational, more precisely, $F \subset K(T)^h$ for every separating trans. base T of $F|K$.

Pf.: T sep. tr. base, i.e. $F/K(T)$ sep. alg.
$\Rightarrow$ $F \cdot K(T)^h \mid K(T)^h$ is a sep.-alg. subext. of $K^c / K(T)^h$.
$K(T)^h \subseteq K^c$ rel. sep. alg. closed (as $K(T)^h$ is henselian)
$\Rightarrow$ $F \cdot K(T)^h = K(T)^h$. $\boxed{/}_{23}$

$\Rightarrow$ $F \subseteq K(x)^h$ $\Rightarrow$ $F^h = K(x)^h$.

(2) If $y \notin K^c$

Kuhlmann-Vlahu, Theorem 11.1:
(K,v) valued, rank 1, $(F|K)$ immediate f.f., $\operatorname{trdeg}_K F = 1$. Suppose $F^h = K(x)^h$ for some $x \in F^h \setminus K^c$ of transcendental approx. type.
Then, there is $y \in F$ s.th. $F^h = K(y)^h$.

PROOF: Needs an entire session

$\Rightarrow$ $F^h = K(y)^h$ for an appropriate $y \in F$.

modulo 4.8, 4.9. + [KV] $\boxed{/}$ 5.2

We say that the approximation type of x over K is transcendental if for every polynomial $h(Y) \in K[Y]$ there is some $\alpha \in v(x-K)$ s.th. for all $c \in K$ with $v(x-c) \geq \alpha$ the value $v(h(c))$ is fixed.

§ Galois extensions of degree p of $K(x)^h$

Next stop:

Lemma 4.8: (K,v) sep. tame, $\mathrm{char}(K) = p > 0$, rk 1.
If $(K(x)|K,v)$ is immediate, $E/K(x)^h$ Galois of degree p, then there is $\vartheta \in E$ s.th. $E = K(\vartheta)^h$.

&

Lemma 4.9: (K,v) alg. closed, $\mathrm{char}(K) = 0$, rk 1.
If $(K(x)|K,v)$ is immediate, $E/K(x)^h$ Galois of degree $p = \mathrm{char}(Kv) > 0$, then there is $\eta \in E$ s.th. $E = K(\eta)^h$

Setup: $(K(x)|K,v)$ immediate & transc.
We want to understand $E/K(x)^h$ Galois with $[E : K(x)^h] = p = \mathrm{char}(Kv) > 0$.

1. $\mathrm{char}(K) = p > 0$.

$\Rightarrow$ $E/K(x)^h$ is Artin-Schreier ext., i.e. generated by $\vartheta \in E$ s.th.
$$\vartheta^p - \vartheta = a \in K(x)^h.$$

Define $\wp(X) = X^p - X$ (additive poly)
$\Rightarrow$ can replace a by any
be $a + \wp(K(x)^h)$
without changing the ext.

Hensel's Lemma $\Rightarrow$ $\mathcal{M}_{K(x)^h} \subseteq \wp(K(x)^h)$

as $X^p - X$ has a simple root in Kv.
$\Rightarrow$ can replace a by any
$b \in a + \mathfrak{m}_{K(x)^h}$
without changing the ext.

② char$(K) = 0 \neq p =$ char(Kv)

If $\zeta_p \in K \Rightarrow E/K(x)^h$ is Kummer,
i.e. generated by $\eta \in E$ s.th.
$\eta^p = a \in K(x)^h$

prim. pth root of unity

can replace a by any $b \in a \cdot (K(x)^h)^p$

Q: What else can we say about a?

Lemma 4.1: If (K,v) has rk-1 and
$(K(x)|K,v)$ is immediate, then $K[X]$
is dense in $K(x)^h$.

Pf: Any rk-1 field is dense in its henselization $\Rightarrow$ STS $K[X]$ dense in $K(x)$
Take $f(x) \in K[X]$, $\gamma \in vK$
NTS: ex. $g(x) \in K[X]$ s.th.
$v(g(x) - 1/f(x)) > \gamma$

$K(x)|K$ immediate $\Rightarrow$ we may
choose $c \in K$ s.th. $v(c) = v(f(x))$ and
$\text{res}(f(x)/c) = 1$

$$\Rightarrow v\left(1 - \frac{f(x)}{c}\right) > 0$$

$rk = 1 \Rightarrow$ ex. $j \in \mathbb{N}$ s.th.
$$j \cdot v(1 - f(x)/c) > \gamma + v(c)$$

for $h = 1 - f(x)/c \in K[X]$, we get

$$v\left(1/f(x) - \underbrace{c^{-1}\sum_{i=0}^{j-1} h^i}_{\in K[X]}\right) = v\Big(\frac{1}{c(\underbrace{1-h}_{f(x)/c})} - c^{-1}\sum_{i=0}^{j-1} h^i\Big)$$
$$\overset{\text{geom. series}}{=} v(c^{-1}h^j) > \gamma \quad \boxtimes$$

① Assume (K,v) has rk 1, $\mathrm{char}(K) = p > 0$.
By 4.1 for every $a \in K(x)^h$ there is $f(x) \in K[X]$ s.th. $a - f(x) \in \mathfrak{m}_{K(x)^h}$.

$\overset{\text{wlog}}{\Rightarrow}$ $E = K(x)^h(\eta)$, with $\eta^p - \eta = f(X) \in K[X]$

② Assume (K,v) has rk 1, $\mathrm{char}(K) = 0$
If K is closed under pth roots:
a/K is immediate
$\Rightarrow v(a) \in vK \Rightarrow$ ex. $d_1 \in K$ s.th. $v(d_1^p a) = 0$
& $\mathrm{res}(d_1^p \cdot a) \in Kv \Rightarrow$ ex. $d_2 \in K$ s.th. $\mathrm{res}(d_1^p d_2^p a) = 1$
For $d = d_1 \cdot d_2 \in K$, we get
$v((d\eta)^p - 1) > 0$ & $d\eta$ generates $E/K(x)^h$ with $(d\eta)^p = 1 + a' \in K(x)^h$ with $v(a') > 0$.

By 4.1 $\Rightarrow$ ex. $f(x) \in K[X]$ s.th.

$$v(f(X) - a') > \frac{p}{p-1} v(p)$$
$$\Rightarrow v(f(X)) > 0 \quad \& \quad 1 + f(X) \in 1 + \mathfrak{m}_{K(X)}$$

Claim (Lemma 3.1 a)

Any root of $X^p - (1 + f(X))$ generates the same ext. as α_n

Pf: We show two subclaims:

(A) (F, v) hens., $\zeta_p \in F$, $\mathrm{char}(F) = 0$

Then $v(b) > \frac{p}{p-1} v(p) \Rightarrow 1 + b \in (F^\times)^p$

$\leadsto 1 + f(X) - a' \in ((K(X)^h)^\times)^p$

It's a fun calculation with HL to see $\zeta_p \in F \Rightarrow C \in F$.

Pf: Consider $X^p - (1+b)$ and take $\rightarrow C \in \overline{\mathbb{Q}}$, $C^{p-1} = -p$

Setting $X = CY + 1$, we obtain

$$f(Y) = (CY + 1)^p - (1+b)$$

Note: $C^p = -pC$

$$= C^p \left(Y^p + g(Y) - Y - \frac{b}{C^p} \right)$$

with $g(Y) = \sum_{i=2}^{p-1} \binom{p}{i} C^{i-p} Y^i$

If $v(b) > p/_{p-1} v(p) = v(C^p)$

$\Rightarrow \mathrm{res}(1/C^p \cdot f) = Y^p - Y$ splits over Kv

Hensel $\Rightarrow f(Y)$ splits over K

$\Rightarrow X^p - (1+b)$ has a root. $\boxtimes$ (A)

(B) (F, v) hens., $\zeta_p \in F$, $1+b, 1+c \in 1 + \mathfrak{m}_v$.

Then $1 + b \in (1 + b + c) \cdot (K^\times)^p$

if $v(c) > p/_{p-1} v(p)$.

$\leadsto 1 + f(X) \in (1 + \cancel{f(X)} \underbrace{- \cancel{f(X)} + a'}_{v(\ldots) > p/_{p-1} v(p)})((K(X)^h)^\times)^p$,

so Claim follows.

Pf: $1 + b \in (1 + b + c)(K^\times)^p$

$$\Leftrightarrow \frac{1+b+c}{1+b} = 1 + \frac{c}{1+b} \in (K^\times)^p$$

This follows from (A)

[as $v(b) > 0$, $v(c/1+b) = v(c)$] $\boxtimes$ (B)

Claim $\Rightarrow$ wma $E = K(X)^n(\eta)$

with $\eta^p = 1 + f(X) \in K(X)$

1-unit.

Next: zoom in on $f(X)$.

ENOUGH IS ENOUGH! ☺ tbc...