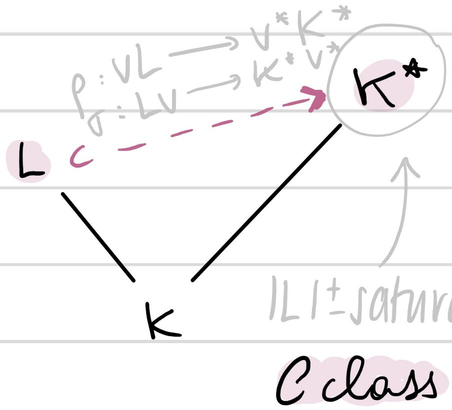
Def.^{n} If $\mathcal{C}$ is a class of valued fields, we say that $\mathcal{C}$ has the relative embedding property (REP) if whenever $(L,v), (K^*,v^*) \in \mathcal{C}$ have a common subfield (K,v) s.t.

1. (K,v) is defectless,
2. (K^*,v^*) is $|L|^+$-saturated,
3. vL/vK is torsion-free & $Kv \subset Lv$ is separable,
4. there are embeddings $\rho: vL \to v^*K^*$ over vK
   $\sigma: Lv \to K^*v^*$ over Kv

then there is an embedding $i: (L,v) \to (K^*,v^*)$ over K, inducing ρ and σ.

Def.^{n} A class $\mathcal{C}$ of valued fields satisfies:



- (CALM) if all $K \in \mathcal{C}$ are algebraically maximal,
- (CRAC) if whenever $L \in \mathcal{C}$ and $K \subset L$ is rel. alg. closed with $Kv \subset Lv$ algebraic & $vK \subset vL$ torsion, then $K \in \mathcal{C}$ and $Kv = Lv$, $vK = vL$,
- (CIMM) if $K \in \mathcal{C}$ and there is an immediate function field F with $\text{trdeg}_K F = 1$, then $F^h = E^h$ for some $K \subset E$ rational function field.

$\mathcal{C} = \{\text{tame fields}\}$ satisfies CALM, CRAC & CIMM

if $\mathcal{C}$ satisfies CALM, CRAC & CIMM
$\Rightarrow \mathcal{C}$ has REP

this first

Lemma 6.4. if C is an elementary class of valued fields that satisfies (CALM), (CRAC) & (CIMM), then C has the REP.

Proof. we are given

- $(L,v), (K^*,v^*) \in C$,
- a common subfield (K,v) which is defectless,
- vL/vK is torsion-free, $Kv \subset Lv$ is separable,
- $\rho: vL \hookrightarrow v^*K^*$ over vK & $\sigma: Lv \hookrightarrow K^*v^*$ over Kv.

① take $T = \{x_i, y_j \mid i \in I, j \in J\} \subset L$ s.that
- $\{v(x_i) : i \in I\} \subset vL$ is a maximal set of rationally independent over vK elements,
- $\{res(y_j) : j \in J\} \subset Lv$ is a transcendence basis of $Kv \subset Lv$. Adjoin T to K: then, if $K' = K(T)^{alg} \cap L$,

vL/vK' has torsion

&

$Lv/K'v$ is algebraic.

$\Rightarrow$ we can use (CRAC): $(K',v) \in C$ with $Lv = K'v$ and $vL = vK'$.

② T is a standard valuation basis for K'/K, hence the extension is without transcendence defect. We can then use "Embedding Lemma II" (Lemma 5.7). We get an embedding $i: (K',v) \hookrightarrow (K^*,v^*)$ over K. Identify K' & $i(K')$.

③ By compactness, it is now enough to embed f.g. subextensions of L/K'. Say $K' \subset F \subset L$ has transcend. basis $\{t_1, \ldots t_n\}$ & define

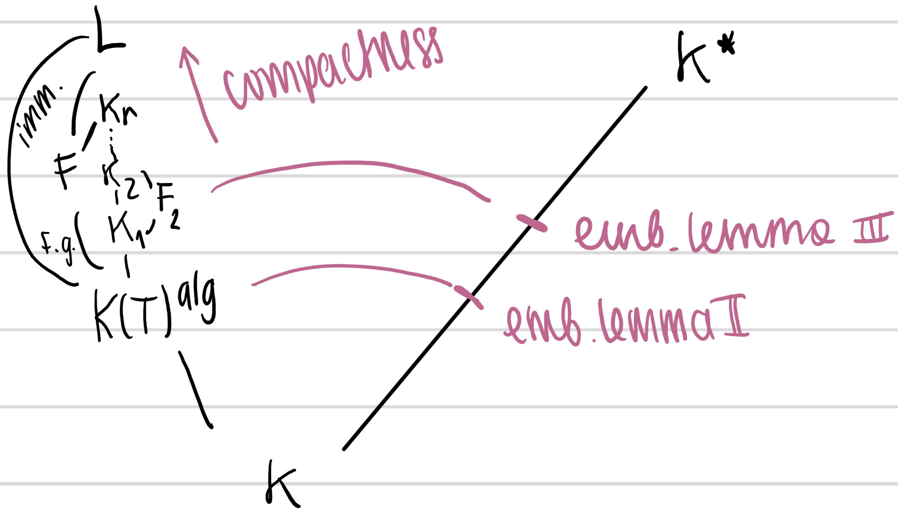
$$K_0 = K' \subset \underbrace{K'(t_1)^{alg} \cap L}_{K_1} \subset \underbrace{K'(t_1,t_2)^{alg} \cap L}_{K_2} \subset \ldots \subset \underbrace{K'(t_1, \ldots t_n)^{alg} \cap L}_{K_n}.$$

Note: $F \subset K_n$. By (CRAC), each $K_i \in \mathcal{C}$.
By induction on n, suppose we have embedded K_i. Then $K_i \subset K_{i+1}$ is immediate of tr.deg. 1. It is then again enough to find embeddings for f.g. subextensions

$$K_i \subset F_{i+1} \subset K_{i+1}.$$

But now F_{i+1} is an immediate function field of tr. deg. 1, by (CIMM) $F_{i+1}^h = K_i(x_{i+1})^h$.

④ as K_i is algebraically maximal (CALM), x_{i+1} is the limit of a pC sequence of tr. type (Theorem 2.24).
$\Rightarrow$ use "Embedding Lemma III" (Lemma 6.2). $\square$



BLACKBOXES:
- Lemma 5.7
- Lemma 6.2
- (Theorem 2.24)

TAME FIELDS

Axioms: say $\mathrm{Char}(Kv) = p > 0$

(VGD$_p$) $\forall x \exists y \, (v(xy^p) = 0 \vee x = 0)$.

(RFD$_p$) $\forall x \exists y \, (v(x) = 0 \to v(xy^p - 1) > 0)$.

(HENS) v is henselian.

(MAXP) $\forall x_0, x_1, \dots x_n \exists y \forall z \, \left(v\left(\sum_{i=0}^{n} x_i y^i\right) \geq v\left(\sum_{i=0}^{n} x_i z^i\right)\right)$. $(\forall n)$

$\hookrightarrow$ (K, v) tame $\iff$ $(K, v) \models$ these axioms.

Now, $\mathcal{C}$ class of models of these axioms.

- (Theorem 2.25) (HENS + MAXP) $\Rightarrow$ (CALM).
- (Lemma 3.7) $\mathcal{C}$ satisfies (CRAC), Pop's Lemma
- (Theorem 1.10) $\mathcal{C}$ satisfies (CIMM). henselian rat.

Getting started on the blackboxes

Lemma 6.2. $K \subset K(x)$ immediate, x transcendental$/K$.
If there is a pseudo-Cauchy sequence of transcendental type with x as pseudolimit, then $K(x)^h$ embeds into any $|K|^+$-saturated henselian ext^n. of K.

λ some limit cardinal,
$(a_\rho)_\rho \subset K$ is pseudo-Cauchy if
there is $\rho_0 < \lambda$ s.t., for all $\rho' > \rho'' > \rho''' \geq \rho_0$,
$v(a_{\rho'} - a_{\rho''}) > v(a_{\rho''} - a_{\rho'''})$.

$\Downarrow$

think of $(a_\rho)_\rho$ as the sequence $(B_{\gamma_\rho}(a_\rho))_{\rho<\lambda}$

$\left\{ \text{for all } \rho \geq \rho_0, \ \gamma_\rho := v(a_\rho - a_{\rho+1}) = v(a_\rho - a_{\rho'}), \ \rho' > \rho \right.$

$\hookrightarrow$ A pseudolimit is an element of $\bigcap_{\rho<\lambda} B_{\gamma_\rho}(a_\rho)$,
i.e. $a \in K$ s.t. $v(a - a_\rho) = \gamma_\rho \ \forall \rho < \lambda$.
Write $a_\rho \Rightarrow a$.

Algebraic type - if $\exists p(X) \in K[X]$ s.t. $v(p(a_\rho))_\rho \subset VK$ is strictly increasing. Otherwise transcendental type.

Proof. take $(a_\rho)_\rho \subset K$ with $a_\rho \Rightarrow a$, tr. type.

$\rightsquigarrow$ the type $\pi(x) = \{ v(x - a_\rho) = \gamma_\rho : \rho < \lambda \}$ is fin. sat.

$\Rightarrow$ realize it in some $K^* \succcurlyeq K$, $|K|^+$-sat., find $x^* \models \pi$.

Then $K(x) \cong K(x^*)$: as fields it is clear, then if $f \in K[X]$,

$v(f(x))$ = eventual value of $v(f(a_\rho))_\rho$

$= v(f(x^*))$.

Hence $(K(x), v) \cong (K(x^*), v) \Rightarrow (K(x), v)^h \cong (K(x^*), v)^h$. $\square$