The free pseudospace

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Recall: last time we introduced the L_n-theory T_n of free n-pseudospaces. $\{V_0, \dots V_n\} \cup \{E\}$

We proved T_n is consistent and complete: it is the theory of the Fraïssé limit M_n of the class $(\mathcal{K}_n, \leq)$.

FACT (2.24) | An L_n-structure $M \models T_n$ is ω-saturated iff it is $\mathcal{K}_n$-saturated.

⊛ Fix an ω-saturated model $M \models T_n$.

DEF (2,16) | A subset $A \subseteq M$ is <u>nice</u> if $A \in \mathcal{K}_n$ and for any $a, b \in A$, if there is a reduced path from a to b, then there is an equivalent one in A.

LEM (2.19) | If $A \subseteq M$ is finite and nice and $a \notin A$, then there is $B \geq A$ finite, nice with $a \in B$.

<u>Consequences:</u> if $A \subset M$ finite, then there is

$A \subseteq B$ finite and nice. [Start from $\emptyset$ and apply 2.19 with elements of A.] In particular, if $a, b \in M$ there are only finitely many reduced paths (up to equivalence) between a and b. [Apply previous fact to $\{a, b\}$.]

(*) goal for today:

T_n is n-ample but not $(n+1)$-ample.

ingredients: i.) describe algebraic closure

ii.) describe $\downarrow$

1.) algebraic closure

PROP (2.28) | a vertex $c \neq a, b$ is algebraic over a, b iff there is a reduced path from a to b that changes direction in c.

<u>Remark:</u> if there is D nice s.that $a, b \in D$ but $c \notin D$, then $c \notin acl(a,b)$. In fact, take $D' \geq D$ nice with $c \in D'$ then

$$\underbrace{D' \otimes_D \cdots \otimes_D D'}_{n\text{-times}} \in K_n \qquad \forall n$$

so there are infinitely many copies of D' (hence c) over D in M. In particular $c \notin acl(ab)$.

<u>PROOF (of 2.28):</u> ($\Leftarrow$) follows from the fact that there are only finitely many up to equivalence.

($\Rightarrow$) suppose there are no reduced paths from a to b changing direction in c.

If there are none at all, $\{a_1 0\} \not\ni C$ is nice and we are done. Otherwise pick

$$\gamma = (x_0 = a, \dots x_\ell = b)$$

so that $C \notin \gamma$, $|\gamma \cap R(C)|$ is minimal.

If $|\gamma \cap R(C)| = 0$, we are done again (2.13).

Otherwise, choose $x_i, x_{i+k} \in \gamma \cap R(C)$ with $k \in \mathbb{N}$ maximal. Proceed by induction on n.

$n=1$: ok

inductive step: note that γ must change direction in x_i, x_{i+k}, otherwise we could increase k. Let $C \in U_m$, $m \leq n$. There are three cases:

(a) $k>0$, $x_i, x_{i+k} \in R_<(C)$ or $R_>(C)$ ("on the same side"),

(b) $k=0$, $x_i = x_{i+k} \in R(C)$,

(c) $k>0$, $x_i \in R_<(C)$, $x_{i+k} \in R_>(C)$ or viceversa ("on different sides").

We only prove (a).

Suppose e.g. that $x_i, x_{i+k} \in R_<(C)$ (otherwise "dualize").

In this case, $R_<(C) \cap \gamma = (x_i, \dots x_{i+k})$ and it can't be a flag, so it must change direction at some point. Let $z = x_i \vee x_{i+k}$.

Claim: $z \neq C$.

Proof of the claim: suppose $z = C$, then we let v be the last change of direction before x_i and u be the last change of direction after x_{i+k}. Let $z_1 = v \wedge C$ and $z_2 = u \wedge C$: the path γ' obtained from γ by changing $(v, \dots u)$ with $(v, \dots z_1, \dots C, \dots z_2, \dots u)$ is a reduced path from a to b changing direction in C. $\lightning$

Assume further that $(x_i, \dots z)$ and $(z, \dots x_{i+k})$ are flags. By inductive assumption on the m-pseudospace $R_<(C)$, there is a nice set $D_1 \subseteq R_<(C)$ that contains both flags but not C. Now, since $x_{i-1}, x_{i+k+1} \in V_0 \cup \dots \cup V_{m-1}$, we apply the inductive hypothesis and find $D_2 \supseteq D_1$ nice containing x_{i-1}, x_{i+k+1} but not C. Finally, using 2.19 we find $D_3 \supseteq D_2$ containing $(x_0, \dots x_{i-2})$ and $(x_{i+k+2}, \dots x_e)$, D_3 nice. Since $C \notin R(x_j)$ for

$j = 0, \ldots i-2, i+k+2, \ldots \ell$, then $c \notin D_3$. This finishes the proof of ⓐ.

Sketch of ⓑ: suppose $k=0$, $x_i \in R_<(C)$. Let a', b' be the last and first place before and after x_i where γ changes directions. For example, $a', b' \in R_>(x_i)$. Then there is no reduced path between a' and b' changing direction in C. So we apply the hypothesis to $R_>(x_i)$ and then extend.

Sketch of ⓒ: $x_i \in R_<(C)$, $x_{i+k} \in R_>(C)$. Then $(x_i, \ldots x_{i+k})$ is a flag, hence nice. Again by cases: if $x_{i+k} \in V_\ell$, $\ell < n$ then we proceed as before. If $\ell = n$ and every reduced path from a to 0 goes through x_{i+k}, we use induction on $V_0 \cup \cdots \cup V_{n-1}$ and $(x_{i-1}, \ldots x_{i+k-1})$, then we extend to V_n; we do the same for $(x_{i+k+1}, \ldots b)$ and then take the union. Finally, if there is a path from x_{i+k-1} to x_{i+k+1} not going through x_{i+k}, we can find one in $R_<(x_{i+k})$ and use the inductive hypothesis and then extend.

$\boxed{}$

PROP. (2.27) | for any $A \subseteq M$, $acl(A) = \bigcap \{B \supseteq A \text{ nice}\}$.

COR. (2.29) | for any $A \subseteq M$, $acl(A) = dcl(A)$.

11.) Forking independence.

We introduce (without proofs) projections.

PROP. (2.31) | Suppose $A \subseteq M$ and $a \notin acl(A)$. Then there is a flag $C \subseteq acl(A)$ such that for every reduced path from a to b there is an equivalent one that enters $acl(A)$ through C, where it changes direction for the last time.

COR. (2.33) | For every type $tp(a/A)$ there is a unique flag C, minimal with the property mentioned above.

Call such C the <u>projection</u> from a to A, $C = proj(a/A)$.

<u>Remark:</u> one can prove that T_n is ω-stable by counting types through projections.

We deduce superstability from characterizing forking independence.

THM (2.35) | If $A, B, C \subseteq M$, then

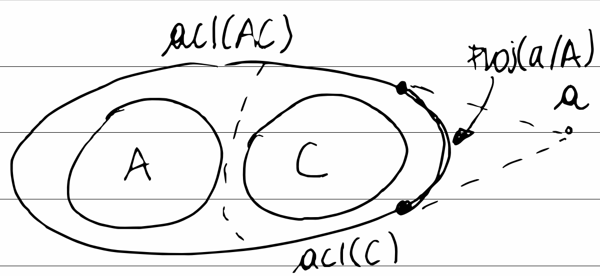
$$A \underset{C}{\perp\!\!\!\perp} B$$

iff for all $a \in acl(AC)$ and $b \in acl(BC)$, if there is a reduced path from a to b then there is an equivalent one going through $acl(C)$.

COR. | T_n is superstable.

<u>PROOF:</u> take $tp(a/A)$, then there is a finite $A_0 \subseteq A$ such that $a \underset{A_0}{\perp\!\!\!\perp} A$, namely $proj(a/A)$. ▨

COR. (2.37) | for $a \in M$, $A, C \subseteq M$ we have $a \underset{C}{\perp\!\!\!\perp} A$ iff $proj(a/AC) \subseteq acl(C)$.



iii.) ampleness.

DEF. (3.1) | a theory T is <u>n-ample</u> if, possibly after naming parameters, there are $a_0, \dots a_n$ in $M \models T$ such that

① for $i = 0, \dots n-1$ we have

$$acl(a_0, \dots a_{i-1}, a_i) \cap acl(a_0, \dots a_{i-1}, a_{i+1}) = acl(a_0, \dots a_{i-1}),$$

② $a_n \not\perp a_0$,

③ $a_0, \dots a_{i-1} \underset{a_i}{\perp} a_{i+1}, \dots a_n$ for $i = 0, \dots n-1$.

THM. (3.3) | T_n is n-ample but not $(n+1)$-ample.

<u>PROOF:</u> • <u>n-ample:</u> a maximal flag $(x_0, \dots x_n)$ in M_n is a witness for n-ampleness.

① follows from

$$acl(x_0, \dots x_i) = \bigcap \{B \text{ nice} \mid x_0, \dots x_i \in B\}$$

② any reduced path from x_i to x_j is a flag, hence $x_n, \dots x_{i+1} \underset{x_i}{\perp} x_{i-1}, \dots x_0$.

③ there is a path from a_0 to a_n.

- <u>not $(n+1)$-ample:</u>

we show something more: that if $a_0, \dots a_n$ witness n-ampleness, there are $b_i \in acl(a_i)$ such that $(b_0, \dots b_n)$ is a flag.

In particular T_n cannot be $(n+1)$-ample.

Let $A_0, \dots A_{n+1}$ witness $(n+1)$-ampleness over some set of parameters A. Then, in particular,

$$A_{n+1} \not\mathop{\smile\!\!\!\!\mid}_A A_0$$

so there are vertices $a_0 \in acl(A_0 A) - acl(A)$ and $a_{n+1} \in acl(A_{n+1}A) - acl(A)$ such that

$$a_{n+1} \not\mathop{\smile\!\!\!\!\mid}_A a_0.$$

We can choose them "minimally", i.e. so that no reduced path from a_0 to a_{n+1} contains $b \in acl(A_0 A)$ with $b \not\mathop{\smile\!\!\!\!\mid}_A a_{n+1}$ (and viceversa).

Then $a_0 \mathop{\smile\!\!\!\!\mid}_{A_n A} a_{n+1}$ and $a_0 \not\mathop{\smile\!\!\!\!\mid}_A a_{n+1}$ mean that there is a flag $C_n \subseteq acl(A_n A)$ with $C_n \not\subseteq acl(A)$ and for any reduced path from a_0 to a_{n+1} there is an equivalent one going through C_n. In particular, $a_0 \mathop{\smile\!\!\!\!\mid}_{C_n} a_{n+1}$

Choose such C_n minimal.

Let γ be a reduced path from a_0 to a_{n+1}, not going through $acl(A)$. Pick $a_n \in C_n$ so that some equivalent path goes through a_n: then $a_0 \not\perp_A a_n$.
Moreover, γ cannot change direction between a_0 and a_n: otherwise there would be $b \in acl(a_0, a_{n+1}) \cap acl(a_0, a_n)$

$$\subseteq acl(A_0 A)$$

with $b \not\perp_A a_{n+1}$. Hence, (a_0, a_n) is a flag and $a_n \notin V_0 \cup V_n$, as otherwise it would change direction in a_n.
Choose then $C_{n-1} \subseteq acl(A_{n-1}A)$, $C_{n-1} \not\subseteq acl(A)$ flag s.that $a_0 \perp_{C_{n-1}} a_n$. Proceed inductively to build $a_i \in acl(A_i A)$ such that $(a_0, a_i, \dots a_n)$ is a flag for any i. This is impossible as $a_n \notin V_0 \cup V_n$. $\boxtimes$

COR. (3.4) | If $a_0, \dots a_n$ witness that T_n is n-ample, then there are $b_i \in acl(a_i)$ such that $(b_0, \dots b_n)$ is a flag.

Remark: if T_ω is the theory of ω-graphs such that the restriction to $V_i \cup \dots \cup V_{i+j}$ is a model of T_j $\forall i,j<\omega$, then T_ω is n-ample for all $n<\omega$.