OUTLINE

0. some topology in o-minimal structures
1. cells and cell decompositions
2. the proof of the theorem
3. some consequences

0. Some topology

From an o-minimal structure $(M, <, \dots)$ you can recover a topology on M^n:

* order topology on M,
* product topology on M^n, $n > 1$.

This is a very natural topology, but it is often not enough for our purposes.

Example: consider an ultrapower ${}^*\overline{\mathbb{R}}$ via a non principal ultrafilter.

Theorem: the set of infinitesimals in ${}^*\overline{\mathbb{R}}$ is a clopen in the order topology.

In particular, ${}^*\overline{\mathbb{R}}$ is not connected in the order topology, which is a bit awkward.

Def. a definable set $X \subseteq M^n$ is **definably connected** if there are no non-empty, open, definable, disjoint $X_1, X_2 \subseteq X$ such that $X_1 \cup X_2 = X$.

facts: (w/ proofs)

1. definable continuous images of definably connected subspaces are definably connected.

2. the only definably connected subsets of M are of the form

$$[a,b] \quad (a,b)$$
$$[a,b) \quad (a,b]$$

for $a,b \in M \cup \{\pm\infty\}$.

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§1. CELLS

theorem. a structure is o-minimal if and only if for every non-empty definable $A \subsetneq M$ we have $0 < |\partial A| < \infty$. (no proof)

In particular, if $\partial A = \{a_1, \dots a_k\}$ then for every $i \in \{1, \dots k\}$ we have either

$(a_i, a_{i+1}) \subseteq A$ or $(a_i, a_{i+1}) \cap A = \emptyset$.

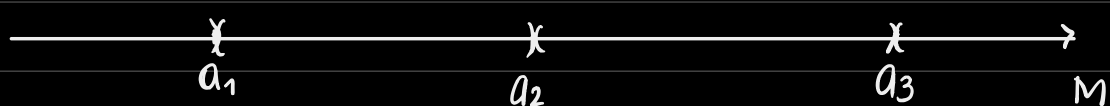
Q: can we mimick this in higher dimensions?

DESIDERATA:

1. a notion of basic building block, what we shall call a cell, which is "simple" enough.
2. a way to write definable sets as the union of finitely many cells.

<u>Easy cases, 1</u>: dimension 1, $A = M$
(this is just the theorem above)

Basic building blocks: points and intervals



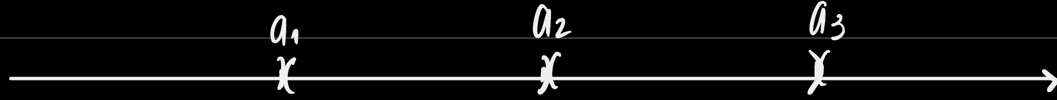
<u>points</u>: a_1, a_2, a_3 dimension 0

<u>intervals</u>: $(-\infty, a_1)$, (a_1, a_2), (a_2, a_3), $(a_3, +\infty)$ dimension 1

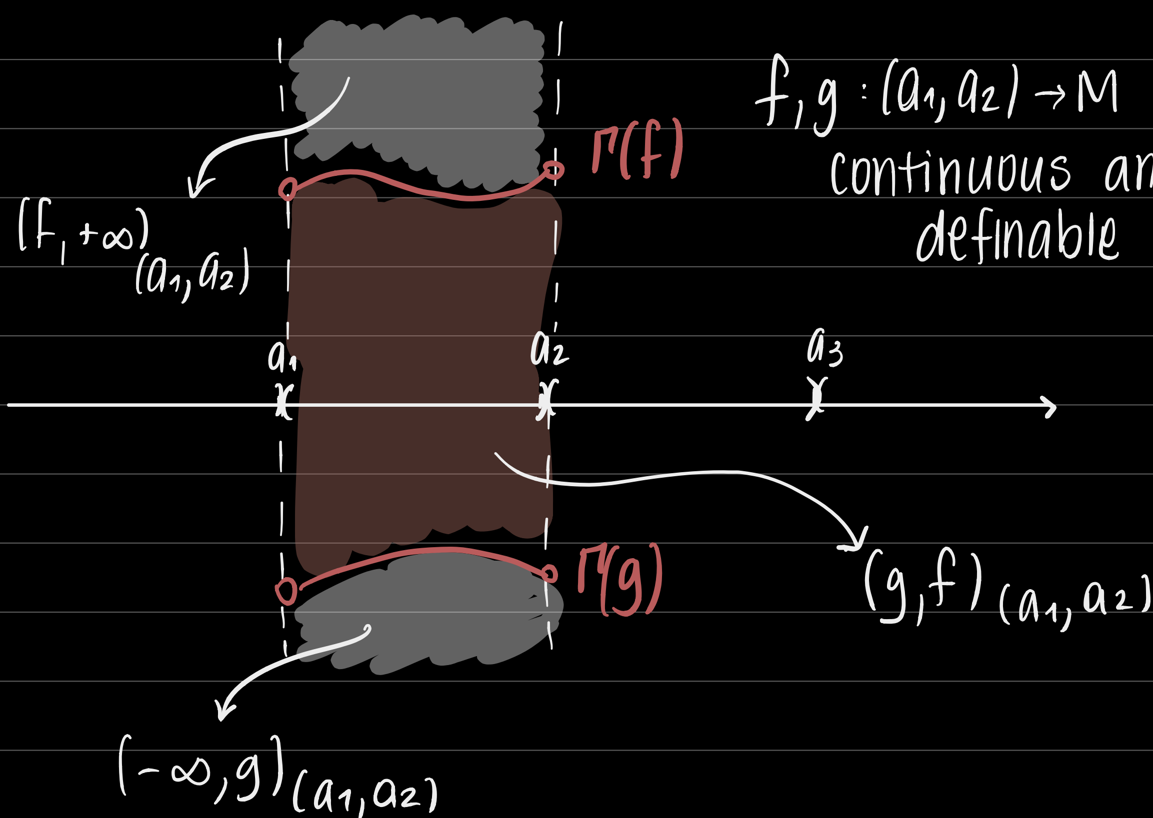
Easy cases, 2: dimension 2, $A = M^2$.

Basic building blocks:

- start with a decomposition of M:



- to build a decomposition of M^2 we work on each "strip" above a cell of M:



$f, g : (a_1, a_2) \to M$ continuous and definable

Def. An $(i_1, \dots i_n)$-cell of M^n is defined by induction on n, where $(i_1, \dots i_n) \in 2^n$.

- A 0-cell is a point $p \in M$,
- A 1-cell is an open interval $(a,b) \subseteq M$.

Given a $(i_1, \dots i_n)$-cell $A \subseteq M^n$,

- a $(i_1, \dots i_n, 0)$-cell is of the form
$$\Gamma(f) = \{(x, f(x)) \in M^{n+1} : x \in A\}$$
where
$$f \in C(A) = \{g : A \to M : \text{continuous \& definable}\}.$$

- a $(i_1, \dots i_n, 1)$-cell is of the form
$$(f,g)_A = \{(x,y) \in M^{n+1} : f(x) < y < g(x),\ x \in A\}$$
where $f,g \in C(A) \cup \{\pm\infty\}$.

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facts. (w/ proofs)

1. every cell is locally closed and definably connected.

2. every cell (of positive dimension) is isomorphic to an open one via coordinate projection.

Remark: we call "open" an $(1, 1, \dots 1)$-cell.

Def. inductively on n, a cell decomposition of M^n is a finite partition of M^n into cells $\{C_1, \dots C_e\}$ such that

$$\{\pi(C_1), \dots \pi(C_e)\}$$

is a cell decomposition of M^{n-1}.

Remark: under this definition, cell decompositions in dimension n are built from cell decompositions in dimension $n-1$ the way we built one of M^2 from one of M.

Recall: our goal was replicating the phenomenon seen in dimension 1, in higher dimensions.

Def. a cell decomposition of M^n is adapted to a definable set $A \subseteq M^n$ if for every cell C, either $C \subseteq A$ or $A \cap C = \emptyset$.

This mimicks what happens in dimension 1.
The existence of adapted cell decompositions would mean

o-minimality $\Rightarrow$ o-minimality in higher dimensions.

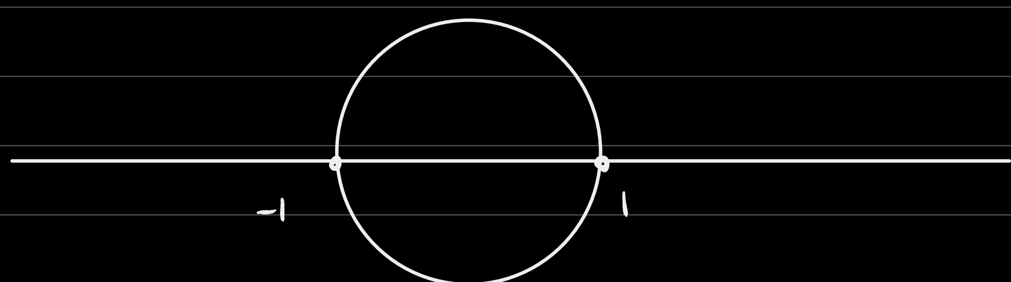
Q: how do we find an adapted cell decomposition of a definable set?

⋛ A TOY EXAMPLE ⋚

In $\overline{\mathbb{R}}$ consider

$$S =$$

Let's find a cell decomposition of $\mathbb{R}^2$ adapted to S. First, we find a cell decomposition of $\mathbb{R}$.



$(-\infty, -1)$ $\quad -1 \quad$ $(-1, +1)$ $\quad +1 \quad$ $(+1, +\infty)$

Then we move to $\mathbb{R}^2$, working on each strip: above intervals like $(-\infty,-1)$ or $(+1,+\infty)$ we only need cells of the form

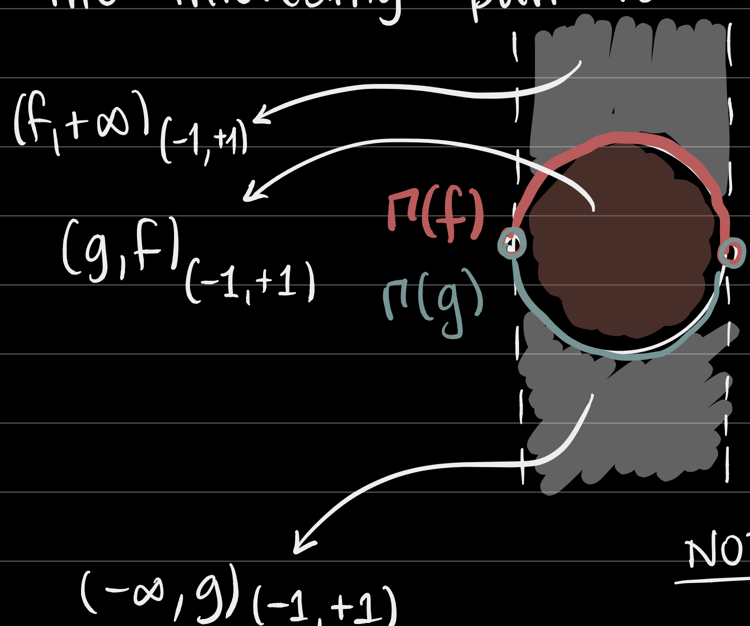
$$(-\infty,+\infty)_{(-\infty,-1)} \quad \text{or} \quad (-\infty,+\infty)_{(+1,+\infty)}.$$

Above points like $+1$ we can find a decomposition like

$$(-\infty,0)_{+1} \qquad \Gamma(0) \qquad (0,+\infty)_{+1}$$

where we see 0 as the constant function on $+1$.

The interesting part is:



$$f:(-1,1)\longrightarrow\mathbb{R}$$
$$x\longmapsto\sqrt{1-x^2}$$

$$g:(-1,1)\longrightarrow\mathbb{R}$$
$$x\longmapsto-\sqrt{1-x^2}$$

NOTE: f,g are continuous and definable.

Putting all the cells together we are done. $\boxtimes$

<u>NOTE:</u> this cell decomposition is also adapted to
$$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}.$$

<u>General strategy:</u> parametrize the boundary of the definable set with definable, continuous functions. Use them to build the decomposition.

§2: THE THEOREM

(UF) for all $n \geq 2$ and definable $A \subseteq M^n$, if A is finite over M^{n-1} it is also uniformly finite over M^{n-1}.

(CD) for all $n \geq 1$ and definable $A_1, \dots A_\ell$ in M^n, there is a cell decomposition of M^n adapted to $A_1, \dots A_\ell$.

(PC) for all $n \geq 1$ and definable

$$f : A \subseteq M^n \longrightarrow M,$$

there is a cell decomposition of M^n adapted to A s.t. if $C \subseteq A$ is a cell, $f|_C$ is continuous.

Remarks: (CD_1) is just o-minimality.

(PC_1) is the monotonicity theorem.

(UF_2) was quoted in the last talk. (We will not prove this case.)

§ 3. THE PROOF

Assume (CD_{n-1}), (UF_{n-1}), (PC_{n-1}).

(UF_n) if $A \subseteq M^n$ is finite over M^{n-1}, then it is uniformly finite over M^{n-1}.

Def. say a box $B \subseteq M^{n-1}$ is A-good if for every $(x,y) \in A$ such that $x \in B$ there are an interval $I \subseteq M$ around y and a continuous definable

$$f : B \longrightarrow M$$

such that $(B \times I) \cap A = \Gamma(f)$.

Say a point is A-good if there is an A-good box around it.
The set of A-good points is definable.

fact (1): if B is A-good, then there are continuous, definable functions
$$f_1 < \dots < f_k : B \longrightarrow M$$
such that
$$(B \times M) \cap A = \Gamma(f_1) \cup \dots \cup \Gamma(f_k).$$

proof: pick any $x \in B$. Let $A_x = \{y_1, \dots y_k\}$:
since x is A-good there are continuous and definable functions $f_i : B \to M$ and intervals $I_i \subseteq M$ around y_i such that $(B \times I_i) \cap A = \Gamma(f_i)$.
We claim $(B \times M) \cap A = \Gamma(f_1) \cup \dots \cup \Gamma(f_k)$.
Suppose not: $(\bar{x}, y) \in (B \times M) \cap A - (\Gamma(f_1) \cup \dots \cup \Gamma(f_k))$
so that $y \neq f_i(\bar{x})$ for every $i = 1, \dots k$.
Since B is A-good, there is a function
$$f : B \longrightarrow M$$
continuous and definable such that
$$(B \times I) \cap A = \Gamma(f)$$
for some interval $y \in I \subseteq M$. Consider $f(x)$: we must have $f(x) = f_i(x)$ for some i. By continuity of f, there is a $U \subseteq B$ with $f(U) \subseteq I_i$, hence $f|_U = f_i|_U$. This shows $\{z \in B : f(z) = f_i(z)\}$ is open; the same applies for

$\{z \in B : f(z) \gtrless f_i(z)\}$. By definable connectedness, $f(z) = f_i(z)$ for all $z \in B$. In particular, $f(\bar{x}) = f_i(\bar{x})$: a contradiction.

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fact (2): suppose Y is definably connected and each of its points is A-good, then there are definable continuous functions

$$f_1 < \dots < f_k : Y \longrightarrow M$$

such that

$$(Y \times M) \cap A = \Gamma(f_1) \cup \dots \cup \Gamma(f_k).$$

<u>proof (sketch)</u>: Consider

$$\{z \in Y : |A_z| = |A_x|\}$$

for some fixed $x \in Y$. By definable connectedness it is the whole Y, hence

$$|A_z| = |A_x|$$

for each $z \in Y$. Cover Y in boxes, use <u>fact 1</u> and then glue the functions together.

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fact (3): every open cell contains an A-good point.

<u>proof</u>: without loss of generality work with a box $B = B' \times (a, b)$.

We use the following lemma:

lemma: if $A \subseteq M^2$ is definable and finite over M, there are $a_1 < \cdots < a_k$ and, for each $j = 1, \ldots k-1$, definable continuous
$$f_{j1} < \cdots < f_{ji_j} : (a_j, a_{j+1}) \longrightarrow M$$
such that
$$((a_j, a_{j+1}) \times M) \cap A = \Gamma(f_{j1}) \cup \cdots \cup \Gamma(f_{ji_j}).$$

We apply it to
$$A(x) = \{(y, z) \in (a, b) \times M : (x, y, z) \in A\}$$
for $x \in B'$. Since for $y \in M$
$$A(x)_y = A_{(x,y)}$$
it is finite over M, hence there are only finitely many $A(x)$-"bad" points. In particular
$$\mathrm{Bad}(A) = \{(x, z) \in B : z \text{ is not } A(x)\text{-good}\}$$
has empty interior.

We now use (CD_{n-1}) on B and $\mathrm{Bad}(A)$. Since no open cell can be contained in $\mathrm{Bad}(A)$, we can restrict B to an open cell and assume $\mathrm{Bad}(A) = \emptyset$.

Now, if $x \in B'$ every $z \in (a, b)$ is $A(x)$-good, so by fact (2) the fiber $A(x)_r$ is finite and hence so is $A(x,r)$, uniformly in r. Say $|A(x,r)| = K(x)$.

Consider $A^{z} = \{(x,y) \in M^{n-1} : (x,z,y) \in A\}$, which is definable and finite over M^{n-2} (because $A^{z}_{x} = A(x,z)$) and so, by (UFn-1),

$$|A^{z}_{x}| \leq N \quad \text{for every } x \in B'.$$

This shows $K(x) \leq N$ for $x \in B'$.

Consider $B_i = \{x \in B : |A_x| = i\}$ for $i \leq N$. For every $x \in B_i$, suppose $A_x = \{y_1, \dots y_i\}$ and define $f_{ij}(x) = y_j$. We get $f_{ij} : B_i \to M$ which are definable.

Use (PCn-1) and (CDn-1) to find a common refinement of the original cell decomposition: since B is open, it contains an open cell and, by taking the decomposition adapted to the B_is, we can assume it is contained in some B_i. All points of that open cell are A-good.

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Finally, use (CDn-1) to find a cell decomposition of M^{n-1} adapted to the set of A-good points.

Consider its cells:

- if C is open, all of its points are A-good so there is $N_C \geq 0$ such that $|A_x| \leq N_C$ for all $x \in C$.

- if C is not open, it is isomorphic to an open cell $C' \subseteq M^d$: we can use this isomorphism to find a similar bound $N_C \geq 0$.

Since the cell decomposition is finite, we have for all $x \in M^{n-1}$,

$$|A_x| \leq \max_C N_C.$$

$\square$

(CDn) let $A_1, \dots A_\ell \subseteq M^n$ be definable: there is a cell decomposition of M^n adapted to $A_1, \dots A_\ell$.

Def. if $A \subseteq M^n$ is definable, then
$$\partial_n A = \{(x,y) \in M^n : y \in \partial A_x\}.$$

<u>Remark:</u> for every $x \in M^{n-1}$,
$$(\partial_n A)_x = \partial A_x$$
so $\partial_n A$ is finite over M^{n-1}.

Let $Y = \partial_n A_1 \cup \dots \cup \partial_n A_\ell$: it is still finite over M^{n-1}, so by **(UFn)** there is a $N \geq 0$ such that $|Y_x| \leq N$ for each $x \in M^{n-1}$.

For every $i = 1, \dots N$ let
$$B_i = \{x \in M^{n-1} : |Y_x| = i\}.$$

For each $x \in B_i$, consider $Y_x = \{y_1, \dots y_i\}$ and define $f_{ij}(x) := y_j$. This gives $f_{ij} : B_i \to M$ with $f_{i1} < \dots < f_{ii}$; define $f_{i0} = -\infty$ and $f_{i,i+1} = +\infty$.

Finally, let

$$C_{i,\ell,j} = \{ x \in B_i : f_{ij}(x) \in (A_\ell)_x \},$$

$$D_{\ell,i,j} = \{ x \in B_i : (f_{ij}(x), f_{i,j+1}(x)) \subseteq (A_\ell)_x \},$$

for each i, ℓ, j.

We use (CD_{n-1}) on all $C_{i,\ell,j}$ and $D_{\ell,i,j}$ and refine the cell decomposition with (PC_{n-1}) to make the f_{ij} continuous on the cells.

Let $\{E_1, \dots E_m\}$ be the resulting cell decomposition of M^{n-1}. The cell decomposition of M^n we are looking for is given by

$(f_{i0}, f_{i1})_C, \dots (f_{ii}, f_{i,i+1})_C,$ open cells

$\Gamma(f_{i1}|_C), \dots \Gamma(f_{ii}|_C)$ non-open cells

whenever $C \subseteq B_i$.

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(PC$_n$) suppose $f: A \subseteq M^n \to M$ is definable, there is a cell decomposition of M^n adapted to A such that for each cell $C \subseteq A$ we have that

$f|_C$ is continuous.

We can assume A is already a cell.

- if A is not open, there is a $d \geq 0$ such that $\pi: A \xrightarrow{\sim} \pi(A) \subseteq M^d$. We use (PC$d$) and "pull back" the needed decomposition.

- if A is open,

Def. Say $(x,r) \in W \subseteq A$ if there is a box $(x,r) \in B \times (a,b) \subseteq A$ such that $f(x,\cdot)$ is continuous and monotone on (a,b) and $f(\cdot,r)$ is continuous at x.

$W \subseteq A$ is definable and dense (*). Use (CDn) on A and W: if a cell is open, say $D \subseteq A$, then $D \cap W \neq \emptyset$ and hence $D \subseteq W$. Now we can apply the following lemma:

lemma: if $(R_1, <)$ and $(R_2, <)$ are dense linear orders without endpoints, X is a topological space and

$$f : X \times R_1 \to R_2$$

is such that

- $f(x, \cdot)$ is continuous and monotone,
- $f(\cdot, r)$ is continuous,

for each $(x, r) \in X \times R_1$, then

f is continuous.

Apply the lemma locally to D, so f is continuous on D.

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(when it makes sense, i.e. M expands a field)

NOTE: (PC_n) can be improved. For each $k \in \mathbb{N}$ we can find a cell decomposition such that f restricted to each cell is k-times continuously differentiable.

§4. SOME CONSEQUENCES

theorem: if M is o-minimal, every model of $Th(M)$ is o-minimal.

proof (sketch): recall o-minimality is equivalent to every definable $\emptyset \neq A \subsetneq M$ having finite boundary. Suppose $\varphi(x,\bar{y})$ is a formula and

$$\psi(M,\bar{b}) = \partial\varphi(M,\bar{b}).$$

For $M' \succ M$ we can find a finite bound on $|\psi(M,\bar{b})|$ and write

$$\forall \bar{b}\,(\exists x\, \varphi(x,\bar{b}) \wedge \exists x\, \neg\varphi(x,\bar{b})) \to 1 \leq |\{x \mid \varphi(x,\bar{b})\}| \leq k.$$

Use $M' \succ M$, we're done. ▨

facts: (w/ proofs)

1. every definable set has finitely many definably connected components.
2. if $X \subseteq M^{n+1}$ is definable, then there is $k \in \mathbb{N}$ such that for every $z \in M^n$, X_z has at most k definably connected components.

THANK YOU