Stable Groups 2: attack of the stabilizers

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Following section 6.2 in Palacin, we introduce connected components, stabilizers and prove the *fundamental theorem of stable groups*. For any mistake you find in these notes, feel free to throw a rock at my window[1], Room 110.021 in the Cluster building (or send me an email at sramello [at] wwu [dot] de).

1 The main ingredients

Recall that we work with a type-definable group G (whose operation is relatively definable) acting type-definably and transitively on a type-definable set S. If you grow tired of repeating *type-definable*, you can just assume that everything is definable and already reap most of the fruits of this theory. Moreover, most of the time it is a smart move to just take $G = S$ together with the action on itself by multiplication. If you happen to come from the right neighbourhood of Port Model Theory, you might want to keep in mind the example of the additive or multiplicative group of a field acting on itself (this is indeed the starting point of the theory of stable fields).

Every time that a topology is available on a group G, one can define the connected component of the identity G^0; the first aim for today is to recover this notion somewhat abstractly.

Lemma 1.1. — There is a minimal type-definable subgroup of G with bounded index.

Proof. We work with $G = S$. Note that any type-definable $H \leq G$ of bounded index is generic as a partial type – in fact, any $H \subset A$ relatively definable must

[1] Make sure you aim at the side with the blackboard, since poor Marco hasn't done anything wrong.

be generic (by compactness). Thus, there is a global generic type $\mathfrak{p}$ that extends H, so that $g \cdot \mathfrak{p}$ extends the type of gH, and hence the index of H in G is bounded by the number of translates of $\mathfrak{p}$. On the other hand, $\mathfrak{p}$ does not fork over $\emptyset$, and thus there are at most $2^{\#T}$ of them. In particular, the *ideal* candidate for this minimal type-definable subgroup, namely $\bigcap_{H \leq G \text{ type-def.}} H$ is a small intersection, and thus it is type-definable. ■

Definition 1.2. — Let G^0 be the intersection of all relatively definable *finite-index* subgroups of G.

Note that then G^0 is normal, has bounded index in G (in fact, bounded by $2^{\#T}$[2]) and it is generic (as a partial type). As an example, if G is an algebraic group over some algebraically closed field $\mathbb{C}$, then G^0 is exactly the connected component in the Zariski topology.

The *second* aim of today is introducing some machinery to prove the fundamental theorem of stable groups, which characterizes global generic types through the action of the group G (or, more precisely, through the action of G^0).

Definition 1.3. — For any global type $\mathfrak{p}$ extending S, define

$$\mathrm{Stab}(\mathfrak{p}) = \{g \in G \mid g \cdot \mathfrak{p} = \mathfrak{p}\}.$$

More precisely, for any ϕ,

$$\mathrm{Stab}(\mathfrak{p}, \phi) = \{g \in G \mid \forall y (\mathrm{d}_{\mathfrak{p}}\phi(x,y) \iff \mathrm{d}_{\mathfrak{p}}\phi(g^{1} \cdot x, y))\},$$

then $\mathrm{Stab}(\mathfrak{p}) = \cap_{\phi} \mathrm{Stab}(\mathfrak{p}, \phi)$. Note that if $\mathfrak{p}$ is definable over A, then $g \in \mathrm{Stab}(\mathfrak{p})$ if and only if for any $a \vDash \mathfrak{p}|_{A,g}$ then $g \cdot a \vDash \mathfrak{p}|_{A,g}$.

2 The fundamental theorem

Recall that a group action is *regular* if for every $a, b \in S$ there exists a unique $h \in G$ such that $h \cdot a = b$.

2.1. Intermezzo: a standard argument

I recall here an argument that will be used repeatedly in the next theorem. If M is a small model and $\mathfrak{p}$ is a global type, and $a \vDash \mathfrak{p}|_M$, then every time that some other $b \perp_M a$, $a \vDash \mathfrak{p}|_{M,b}$. This is a consequence of stationarity: in fact, both $\mathfrak{p}|_{M,b}$ and $\mathrm{tp}(a/M, b)$ are non-forking extensions of $\mathfrak{p}|_M$, hence they coincide.

2.2. The proof

Theorem 2.1. — The following holds,

1. for every global generic type $\mathfrak{p}$, $G^0 \subseteq \mathrm{Stab}(\mathfrak{p})$,

2. G acts transitively on global generic types,

[2]The equality can be realized: take $\mathbb{Z} \leq G$ saturated, so $G^0 = \bigcap_{n \in \mathbb{N}} nG$ and so $G/G^0 \cong \hat{\mathbb{Z}}$.

3. if G acts regularly on S and $\mathfrak{p}$ is a global type extending S, $\mathfrak{p}$ is generic if and only if $G^0 = \mathrm{Stab}(\mathfrak{p})$.

Proof. Let $\mathcal{G}$ be the set of generic global types. To prove 1, we take a bit of a longer road: suppose

$$F = \bigcap_{\mathfrak{p}\in\mathcal{G}} \mathrm{Stab}(\mathfrak{p}).$$

Then G/F acts faithfully on $\mathcal{G}$, so in particular $G/F \hookrightarrow \mathrm{Symm}(\mathcal{G})$, and hence G/F has bounded cardinality. In other words, $[G:F]$ is bounded. Keep this in mind. Now,

$$F = \bigcap_{\mathfrak{p}\in\mathcal{G}} \bigcap_{\phi\in L} \mathrm{Stab}(\mathfrak{p},\phi) \leq G,$$

and hence by compactness, because the index of F is bounded, each of $\mathrm{Stab}(\mathfrak{p},\phi)$ – which are relatively definable – must have finite-index. And so, $G^0 \subseteq F$.

As for 2, let $\mathfrak{p},\mathfrak{q} \in \mathcal{G}$. Consider a small model M and let $a \models \mathfrak{p}|_M$, $b \models \mathfrak{q}|_M$. Since the action of G is transitive, there is $g \in G$ such that $g \cdot a = b$. We will prove that $g \cdot \mathfrak{p} = \mathfrak{q}$. To do so, choose $h \in G^0$ generic (for the action by right multiplication) and independent from M,a,b,g. In particular, this means that $h \mathop{\smile\!\!\!\!\!\!|}_{M,a} g$ and hence, by genericity, $hg \mathop{\smile\!\!\!\!\!\!|}_M a$. Thanks to the standard argument as above, this means $a \models \mathfrak{p}|_{M,hg}$. Then, $(hg)\cdot a \models (hg)\cdot(\mathfrak{p}|_{M,hg}) = ((hg)\cdot\mathfrak{p})|_{M,hg}$. Note that, by 1, $h \in \mathrm{Stab}(\mathfrak{q})$, and moreover since $h \mathop{\smile\!\!\!\!\!\!|}_M b$ we have $b \models \mathfrak{q}|_{M,h}$, and so $h\cdot b = (hg)\cdot a \models \mathfrak{q}|_M$. Hence $((hg)\cdot\mathfrak{p})|_M = \mathfrak{q}|_M$, so by stationarity $(hg)\cdot\mathfrak{p} = \mathfrak{q}$. We have already won, but actually, since $h \in \mathrm{Stab}(g\cdot\mathfrak{p})$, we even have that $g\cdot\mathfrak{p} = \mathfrak{q}$.

Finally, for 3, assume first that $\mathfrak{p}$ is generic. We already know that $G^0 \subseteq \mathrm{Stab}(\mathfrak{p})$; on the other hand, note that for any $H \leq G$ relatively definable of finite index there is $a \in S$ such that $\mathfrak{p} \longrightarrow x \in H\cdot a$[3]. By compactness, we can find $b \in S$ such that $\mathfrak{p} \longrightarrow x \in G^0 \cdot b$, thus

$$g\cdot\mathfrak{p} \longrightarrow x \in g\cdot(G^0\cdot b)$$

and the latter is equal to $G^0\cdot(g\cdot b)$. Hence if $g \in \mathrm{Stab}(\mathfrak{p})$, then b and $g\cdot b$ are in the same orbit. Take $h_1,h_2 \in G$ such that $(h_1 g)\cdot b = h_2\cdot b$, so that by regularity $g \in G^0$ and hence we know $\mathrm{Stab}(\mathfrak{p}) \subseteq G^0$.

Viceversa, take M model on which $\mathfrak{p}$ does not fork. It is enough to show that $\mathfrak{p}|_M$ is generic – in fact, we know that for any element a and sets $B \subseteq A$, a is generic over A if and only if a is generic over B and $a \mathop{\smile\!\!\!\!\!\!|}_B A\setminus B$. Now, since $\mathfrak{p}$ is generic if and only if $\mathfrak{p}|_{M,b}$ is generic for any tuple b, we get that $\mathfrak{p}$ is generic if and only if $\mathfrak{p}|_M$ is generic and $\mathfrak{p}|_{M,b}$ does not fork over M. The latter is true by assumption. We show the former.

Choose any $c \models \mathfrak{p}|_M$ and $g \in G^0$ generic over M,a, so that we get $(g\cdot\mathfrak{p})|_{M,g} = \mathfrak{p}|_{M,g}$ and $g \mathop{\smile\!\!\!\!\!\!|}_M a$, hence $a \models (g\cdot\mathfrak{p})|_{M,g}$ so $a \models g\cdot\mathfrak{p}|_M$ and so $g^{-1}\cdot a \models \mathfrak{p}|_M$. But then, $\mathfrak{p}|_M = \mathrm{tp}(g^{-1}\cdot a/M)$ is a generic type (recall that Thomas proved that g generic over A,a implies $g\cdot a$ generic over A for any A,g,a). ■

[3] Because H only has finitely many orbits.

2.3. Some consequences

Remark *(see Chernikov, 4.30)*. — For those who might be interested in topological dynamics, the set of generics *should* form the unique minimal flow for the action of G on types extending S, but I couldn't find a reference for this.

Corollary 2.2. — Every coset of G^0 contains a unique generic type.

Proof. We begin by finding a complete generic type in G^0. Notice that $G^0(x)$ is a generic type, so we can extend it to a global complete generic type $\mathfrak{p}$. Now, if $\mathfrak{q}$ is another generic global type extending G^0, take some $a \models \mathfrak{p}$ and some $b \models \mathfrak{q}$. Since $G^0 = \mathrm{Stab}(\mathfrak{p}) = \mathrm{Stab}(\mathfrak{q})$, then

$$a \models \mathfrak{p} \iff b = (ba)^{-1} \cdot a \models (ba)^{-1} \cdot \mathfrak{p} = \mathfrak{p},$$

and viceversa. So $\mathfrak{p} = \mathfrak{q}$. By translating, we obtain a unique generic type in every coset of G^0. ■

As a way of example, we look at what happens with fields. Let K be a field definable in T. An *additive* generic is a generic for $(K,+)$, while a *multiplicative* generic is a generic for $(K^\times, \cdot)$. A type *concentrates* on a definable set if it implies its defining formula.

Corollary 2.3. — There is a unique additive generic p_+, a unique multiplicative generic $p_\times$, and they coincide.

Proof. We first show that there is a unique additive generic, i.e. we show that $(K,+)$ is connected. Suppose $H = G^0$ is the connected component of the identity: for any $b \in K^\times$, the coset $b^{-1}H$ is definable and has finite index, so $H \subseteq b^{-1}H$ and so $bH \subseteq H$. It follows that H is a non-empty ideal of K, hence $H = K$. Let p_+ be the unique additive generic. If $p_\times$ is a multiplicative generic and $p_\times \neq p_+$, there is a definable $X \in \mathrm{Def}(K^\times)$ such that $p_\times$ concentrates on X but p_+ does not. Write

$$K^\times \subseteq \bigcup_{i=1}^{n} a_i X$$

and notice that each $a_i X$ is obtained from X through a definable automorphism of $(K,+)$, so they are not additively generic and so p_+ does not concentrate on $K^\times$, which is a contradiction. It follows that $p_+ = p_\times$ and it is unique. ■