Resolutions for smooth representations of the general linear group over a local field

P.Schneider, U.Stuhler

Let K be a non-archimedean local field with ring of integers o and uniformizing element π. We fix a $d \geq 1$ and put $G := GL_{d+1}(K)$. A smooth (or algebraic) representation V of G is a complex vector space V together with a linear action of G such that the stabilizer of each vector is open in G. Let $\mathrm{Alg}(G)$ denote the category of those smooth representations. Furthermore let U be the principal congruence subgroup of level n in G. In this paper we will construct for any representation in $\mathrm{Alg}(G)$ which is generated by its U-fixed vectors a resolution which has nice finiteness and projectivity properties. This applies in particular to any irreducible representation in $\mathrm{Alg}(G)$ if the level n of U is chosen only large enough. As a consequence we will obtain finiteness results for certain Ext-groups in $\mathrm{Alg}(G)$ and also for the homology of cocompact (mod center) discrete subgroups with certain infinite dimensional coefficients.

In the first Paragraph we observe that any smooth representation of G gives rise to a coefficient system on the Bruhat-Tits building of G. The homological complex of (oriented) chains naturally belonging to such a coefficient system will be our candidate for the searched for resolution. Among the representations in $\mathrm{Alg}(G)$ which are generated by their U-fixed vectors there is an "universal" one. For this one we show that the exactness of the corresponding chain complex can be reduced to the contractibility of certain subcomplexes in the Bruhat-Tits building. This contractibility will be established in the second Paragraph. In the third Paragraph we use the Bernstein-Borel-Matsumoto theory which describes an arbitrary smooth representation which is generated by its U-fixed vectors in terms of the universal one in order to deduce the exactness of our chain complex in general. We then discuss the mentioned finiteness and projectivity properties and their consequences.

§1 Representations as coefficient systems

The Bruhat-Tits building $\mathcal{BT}$ of G is the simplicial complex whose vertices are the similarity classes $[L]$ of o-lattices in the vector space K^{d+1} and whose q-simplices are given by families $\{[L_0,\ldots,[L_q]\}$ of similarity classes such that

$$L_0 \underset{\neq}{\subseteq} L_1 \underset{\neq}{\subseteq} \ldots \underset{\neq}{\subseteq} L_q \underset{\neq}{\subseteq} \pi^{-1} L_0 \quad .$$

Let $\mathcal{BT}_q$ denote the set of all q-simplices of $\mathcal{BT}$. In order to make notations simpler we fix an orientation of $\mathcal{BT}$ with the corresponding incidence numbers $[\tau : \sigma]$. A coefficient system (of complex vector spaces) $\underline{\underline{V}}$ on $\mathcal{BT}$ consists of

— complex vector spaces V_σ for each simplex σ of $\mathcal{BT}$, and

— linear maps $r_\sigma^{\sigma'} : V_{\sigma'} \to V_\sigma$ for each pair $\sigma \subseteq \sigma'$ of simplices of $\mathcal{BT}$ such that $r_\sigma^\sigma = \mathrm{id}$ and $r_\sigma^{\sigma''} = r_\sigma^{\sigma'} \circ r_{\sigma'}^{\sigma''}$ whenever $\sigma \subseteq \sigma' \subseteq \sigma''$.

In an obvious way the coefficient systems form a category which we denote by Coeff$(\mathcal{BT})$. A standard computation shows that the boundary map

$$\begin{aligned} \partial : \bigoplus_{\tau \in \mathcal{BT}_{q+1}} V_\tau &\longrightarrow \bigoplus_{\sigma \in \mathcal{BT}_q} V_\sigma \\ (v_\tau)_\tau &\longmapsto (\sum_{\tau \supseteq \sigma} [\tau : \sigma] r_\sigma^\tau (v_\tau))_\sigma \end{aligned}$$

fulfills $\partial^2 = 0$ so that we obtain a homological chain complex.

For any simplex σ of $\mathcal{BT}$ we have the compact open subgroup

$$B_\sigma := \{g \in G : g\sigma = \sigma \ \text{ and } \ \det g \in o^\times\}$$

in G; it has a unique maximal normal pro-p-subgroup $U_\sigma^{(1)}$ which itself is compact open. More generally we fix a natural number $n \geq 1$ and let

$$U^{(n)} := \{g \in GL_{d+1}(o) : g \equiv 1 \bmod \pi^n\}$$

denote the principal congruence subgroup of level n in G. For any vertex σ of $\mathcal{BT}$ define

$$U_\sigma^{(n)} := gU^{(n)}g^{-1} \ \text{ if } \sigma = g([o^{d+1}]) \ \text{ for some } g \in G \quad ;$$

this is well-defined since $U^{(n)}$ is a normal subgroup in the stabilizer of $[o^{d+1}]$ in G. For an arbitrary simplex $\sigma = \{\sigma_0, \ldots, \sigma_q\}$ of $\mathcal{BT}$ we then put

$$U_\sigma^{(n)} := \text{subgroup in } G \text{ generated by } U_{\sigma_0}^{(n)} \cup \ldots \cup U_{\sigma_q}^{(n)} \quad .$$

It was shown in [4] §6 Lemma 2 that in case $n = 1$ this definition agrees with the previous one. Since n is fixed throughout the paper we will abbreviate

$$U := U^{(n)} \ \text{ and } \ U_\sigma := U_\sigma^{(n)} \quad .$$

Because of $U_\sigma \subseteq U_\tau$ if $\sigma \subseteq \tau$ we can form, for any representation V in Alg(G), the coefficient system $\underline{\underline{V}} := (V^{U_\sigma})$ of subspaces of fixed vectors

$$V_\sigma := V^{U_\sigma} := \{v \in V : gv = v \ \text{ for all } \ g \in U_\sigma\}$$

with the obvious inclusions as transition maps. Since the U_σ are profinite groups the functor

$$\begin{aligned} \mathrm{Alg}(G) &\longrightarrow \mathrm{Coeff}(\mathcal{BT}) \\ V &\longmapsto (V^{U_\sigma}) \end{aligned}$$

is exact. The homological chain complex of this coefficient system is augmented in a natural way:

$$(*) \qquad 0 \longrightarrow \bigoplus_{\tau\in\mathcal{BT}_d} V^{U_\tau} \stackrel{\partial}{\longrightarrow} \dots \stackrel{\partial}{\longrightarrow} \bigoplus_{\sigma\in\mathcal{BT}_0} V^{U_\sigma} \longrightarrow V \longrightarrow 0$$
$$(v_\sigma) \longmapsto \Sigma v_\sigma \quad .$$

Our aim in this paper is to study the exactness properties of this complex. For example, if $V = \mathbb{C}$ is the trivial representation then $(*)$ simply is the complex of chains of the Bruhat-Tits building $\mathcal{BT}$; it is exact since $\mathcal{BT}$ is contractible.

First we consider what could be called the “universal” case. The method in this case will be analogous to that which was developed in [4] §6.

Notation: If T is any set then $C_c(T)$ denotes the space of complex valued functions with finite support on T. Any map $f : S \to T$ which is finite, i.e., has finite fibers induces a map

$$C_c(T) \longrightarrow C_c(S)$$
$$\psi \longmapsto \psi \circ f \quad .$$

Clearly $T := G/U$ is a discrete set and $C_c(T)$ is a smooth representation of G which acts by left translations. For any simplex σ we put

$$T_\sigma := U_\sigma\backslash T \quad .$$

It follows from the definition of the U_σ that

$$T_\sigma = T_{\sigma_0} \coprod_T \dots \coprod_T T_{\sigma_q} \quad \text{if } \sigma = \{\sigma_0, \dots, \sigma_q\} \quad .$$

The natural projection $T \to T_\sigma$ is finite and induces an isomorphism

$$C_c(T_\sigma) \xrightarrow{\cong} C_c(T)^{U_\sigma}$$

which we will view as an identification. More generally we have the simplicial set

$$T.^{(\sigma)} : \dots \overrightarrow{\overrightarrow{\overrightarrow{\longrightarrow}}} T \underset{T_\sigma}{\times} T \underset{T_\sigma}{\times} T \overrightarrow{\overrightarrow{\longrightarrow}} T \underset{T_\sigma}{\times} T \overrightarrow{\longrightarrow} T$$

all face maps of which are finite. There are obvious commutative diagrams

$$\begin{array}{ccccc} & & T.^{(\sigma)} & & \\ & \nearrow & & \searrow & \\ T & & \longrightarrow & & T_\sigma \end{array}$$

and

$$\begin{array}{ccc} T_\cdot^{(\sigma)} & \longrightarrow & T_\sigma \\ \downarrow & & \downarrow \\ T_\cdot^{(\tau)} & \longrightarrow & T_\tau \end{array} \qquad \text{for } \sigma \subseteq \tau \quad .$$

By passing to functions we obtain the double complex

(A)

$$\begin{array}{ccccccc}
 & 0 & & 0 & 0 & \\
 & \downarrow & & \downarrow & \downarrow & \\
0 \to & \bigoplus\limits_{\tau \in \mathcal{BT}_d} C_c(T_\tau) & \overset{\partial}{\to} \dots \overset{\partial}{\to} & \bigoplus\limits_{\sigma \in \mathcal{BT}_0} C_c(T_\sigma) & \to C_c(T) \to 0 \\
 & \downarrow & & \downarrow & \| & \\
0 \to & \bigoplus\limits_{\tau \in \mathcal{BT}_d} C_c(T) & \to \dots \to & \bigoplus\limits_{\sigma \in \mathcal{BT}_0} C_c(T) & \to C_c(T) \to 0 \\
 & \downarrow & & \downarrow & \downarrow 0 & \\
0 \to & \bigoplus\limits_{\tau \in \mathcal{BT}_d} C_c(T \underset{T_\tau}{\times} T) & \to \dots \to & \bigoplus\limits_{\sigma \in \mathcal{BT}_0} C_c(T \underset{T_\sigma}{\times} T) & \to C_c(T) \to 0 \\
 & \downarrow & & \downarrow & \| & \\
0 \to & \bigoplus\limits_{\tau \in \mathcal{BT}_d} C_c(T \underset{T_\tau}{\times} T \underset{T_\tau}{\times} T) & \to \dots \to & \bigoplus\limits_{\sigma \in \mathcal{BT}_0} C_c(T \underset{T_\sigma}{\times} T \underset{T_\sigma}{\times} T) & \to C_c(T) \to 0 \\
 & \downarrow & & \downarrow & \downarrow 0 & \\
 & \vdots & & \vdots & \vdots &
\end{array}$$

All its columns are exact. This follows from the fact that each $T_\cdot^{(\sigma)}$ is the disjoint union

$$T_\cdot^{(\sigma)} = \dot{\bigcup_{t \in T_\sigma}} T_\cdot^{(t)}$$

of the simplicial finite sets

$$T_\cdot^{(t)} : \dots \overrightarrow{\overrightarrow{\overrightarrow{\longrightarrow}}} T_t \times T_t \times T_t \overrightarrow{\overrightarrow{\longrightarrow}} T_t \times T_t \overrightarrow{\longrightarrow} T_t$$

where T_t denotes the fiber of the map $T \to T_\sigma$ in t. Simplicial sets of the form $T_\cdot^{(t)}$ are well-known to be contractible. The top row in (A) is the complex $(*)$ for the representation $V = C_c(T)$. Next we have to study, for a fixed $m \geq 0$, the row

$$(A_m) \qquad 0 \to \bigoplus_{\tau \in \mathcal{BT}_d} C_c(T_m^{(\tau)}) \to \dots \to \bigoplus_{\sigma \in \mathcal{BT}_0} C_c(T_m^{(\sigma)}) \to C_c(T) \to 0$$

from (A). If we view each $T_m^{(\sigma)}$, resp. T, as a subset of $T^{m+1} := T \times \ldots \times T$ ($m+1$ factors) in the obvious way, resp. diagonally, then the differentials in the above complex are induced by the inclusions

$$T \subseteq T_m^{(\sigma)} \subseteq T_m^{(\tau)} \quad \text{for } \sigma \subseteq \tau \quad .$$

In order to rewrite this row in a more suitable form we introduce certain subcomplexes of the Bruhat-Tits building. For a fixed $(t_0, \ldots, t_m) \in T^{m+1}$ we put

$$\begin{aligned} \mathcal{BT}^{(t_0,\ldots,t_m)} :=& \text{subcomplex of all simplices } \sigma \text{ in } \mathcal{BT} \\ & \text{such that } (t_0, \ldots, t_m) \notin T_m^{(\sigma)} \text{ (i.e.,} \\ & t_0, \ldots, t_m \text{ are not mapped to the same} \\ & \text{element in } T_\sigma). \end{aligned}$$

We then have

$$\bigoplus_{\sigma \in \mathcal{BT}.} C_c(T_m^{(\sigma)}) = \bigoplus_{(t_0,\ldots,t_m) \in T^{m+1}} C_c(\mathcal{BT}.\backslash \mathcal{BT}.^{(t_0,\ldots,t_m)})$$

and the decomposition on the right hand side is compatible with the differentials. Furthermore the next Lemma implies that

$$\bigcap_{\sigma \in \mathcal{BT}_0} T_m^{(\sigma)} = T \quad ,$$

i.e., that $\mathcal{BT}^{(t_0,\ldots,t_m)}$ is empty if and only if $t_0 = \ldots = t_m$.

Lemma:

The natural map $G/U \to \prod_{\sigma \in \mathcal{BT}_0} U_\sigma \backslash G/U$ *is injective.*

Proof: The map in the assertion is a map of G-sets. Since G acts transitively on G/U it therefore suffices to show that any coset gU such that $U_\sigma g U = U_\sigma 1 U$ for all $\sigma \in \mathcal{BT}_0$ must be the trivial coset. But this is clear since U occurs among the U_σ.

As a result of this discussion we obtain that

$$(A_m) = \bigoplus_{(t_0,\ldots,t_m) \in T^{m+1}} \begin{array}{l} \text{(augmented complex of relative} \\ \text{chains of the pair } (\mathcal{BT}, \mathcal{BT}^{(t_0,\ldots,t_m)}) \end{array} \quad .$$

Since $\mathcal{BT}$ is contractible this means that we have to compute the homology of the simplicial complexes $\mathcal{BT}^{(t_0,\ldots,t_m)}$.

§2 The contractibility of $\mathcal{BT}^{(t_0,\dots,t_m)}$

Any element $t \in T$ defines a vertex σ_t of $\mathcal{BT}$ by

$$\sigma_t := g([o^{d+1}]) \quad \text{if } t = gU \quad .$$

We now fix once and for all in this Paragraph elements $t_0, \dots, t_m \in T$ such that the set $\{t_0, \dots, t_m\}$ has cardinality at least 2.

Remark 1:

The vertices $\sigma_{t_0}, \dots, \sigma_{t_m}$ are contained in $\mathcal{BT}^{(t_0,\dots,t_m)}$.

Proof: By symmetry it suffices to consider σ_{t_0}. If σ_{t_0} is not contained in $\mathcal{BT}^{(t_0,\dots,t_m)}$ then we have

$$U_{\sigma_{t_0}} g_i U = U_{\sigma_{t_0}} g_0 U \quad \text{for all } 0 \leq i \leq m$$

where $g_i U = t_i$. Since $U_{\sigma_{t_0}} = g_0 U g_0^{-1}$ we obtain $U g_0^{-1} g_i U = U$ which means $t_i = t_0$ for all $0 \leq i \leq m$. This is a contradiction.

A basic property of the topological realization $|\mathcal{BT}|$ of the Bruhat-Tits building is that its topology is defined by a metric d which restricted to any apartment is the Euclidean metric on the affine space. Moreover any two points $x, y \in |\mathcal{BT}|$ are joined by a unique geodesic $[xy]$ with respect to d which lies in every apartment containing x and y. (Introductory texts to the theory of Bruhat-Tits buildings are [2] and [3].) The topological realization $|\mathcal{BT}^{(t_0,\dots,t_m)}|$ of the subcomplex $\mathcal{BT}^{(t_0,\dots,t_m)}$ in $\mathcal{BT}$ is a closed subspace of $|\mathcal{BT}|$. Our aim in this Paragraph is to establish the following result.

Proposition 2:

With any point $x \in |\mathcal{BT}^{(t_0,\dots,t_m)}|$ also the geodesics $[\sigma_{t_i} x]$ for $0 \leq i \leq m$ are contained in $|\mathcal{BT}^{(t_0,\dots,t_m)}|$.

Corollary 3:

$\mathcal{BT}^{(t_0,\dots,t_m)}$ is contractible.

Proof: $|\mathcal{BT}^{(t_0,\dots,t_m)}|$ even is "geodesically contractible"; compare [3] p. 184.

In order to prove Proposition 2 we have to translate geometric relations between simplices σ in $\mathcal{BT}$ into group theoretic relations between the corresponding groups U_σ, resp. B_σ. First of all we need more notations. Let τ be a simplex in $\mathcal{BT}$. Its topological realization $|\tau|$ is a closed subset of $|\mathcal{BT}|$. The interior of $|\tau|$ is defined to be

$$|\tau|^0 := |\tau| \setminus \bigcup_{\substack{\sigma \subset \tau \\ \neq}} |\sigma| \quad .$$

If τ is a vertex then we usually do not distinguish between τ and $|\tau|$ so that we have $\tau = |\tau| = |\tau|^0$ in this case. Let $\mathcal{A} \subseteq \mathcal{BT}$ be an apartment. Then there is a basis $e_1, \ldots, e_{d+1}$ of the vector space K^{d+1} such that $\mathcal{A}$ is the full subcomplex in $\mathcal{BT}$ generated by the vertices

$$[\pi^{m_1} o e_1 + \ldots + \pi^{m_{d+1}} o e_{d+1}] \quad \text{with } m_1, \ldots, m_{d+1} \in \mathbb{Z} \quad .$$

The topological realization $|\mathcal{A}|$ is an apartment in $|\mathcal{BT}|$. Its affine structure can be described in the following way (compare [2] p. 148): The map

$$\begin{aligned} \text{vertices in } \mathcal{A} &\longrightarrow \mathbb{R}^{d+1}/\mathbb{R} \cdot (1, \ldots, 1) \\ \left[\sum_{\mu=1}^{d+1} \pi^{m_\mu} o e_\mu \right] &\longmapsto (m_1, \ldots, m_{d+1}) \,\mathrm{mod} \ldots \end{aligned}$$

extends linearly to a topological isomorphism

$$|\mathcal{A}| \xrightarrow{\cong} \mathbb{R}^{d+1}/\mathbb{R} \cdot (1, \ldots, 1)$$

under which the metric d restricted to $|\mathcal{A}|$ corresponds to the Euclidean metric on the right hand side. Whenever convenient we view this isomorphism as an identification. The simplicial structure on $|\mathcal{A}|$ then is given by the set $H(\mathcal{A})$ of affine hyperplanes

$$\{(\xi_1, \ldots, \xi_{d+1}) \in \mathbb{R}^{d+1}/\mathbb{R} \cdot (1, \ldots, 1) : \xi_\mu - \xi_\nu = k\}$$

in $|\mathcal{A}|$ where $1 \leq \mu, \nu \leq d+1$ and $k \in \mathbb{Z}$. Let τ be a simplex in $\mathcal{A}$ and let $H \in H(\mathcal{A})$ be a hyperplane which does not contain $|\tau|$. If H is given by the equation $\xi_\mu - \xi_\nu = k$ then there is a unique sign $\varepsilon \in \{\pm 1\}$ such that

$$|\tau|^0 \subseteq \{x \in |\mathcal{A}| : \xi_\mu(x) - \xi_\nu(x) - k \in \varepsilon \cdot \mathbb{R}_+^\times\} =: H(\tau) \quad .$$

The half space $H(\tau)$ in $|\mathcal{A}|$ is called the τ-side of H.

Now we consider a pair $\sigma \underset{\neq}{\subseteq} \tau$ of simplices in $\mathcal{BT}$. We fix an apartment $\mathcal{A} \subseteq \mathcal{BT}$ which contains τ and we define

$$|\mathcal{A}|(\sigma;\tau) := \bigcap_{\substack{H\in H(\mathcal{A})\\ |\sigma|\subseteq H, |\tau|\not\subseteq H}} H(\tau)$$

which is an open convex subset of $|\mathcal{A}|$. For trivial reasons we have the inclusion $U_\sigma \subseteq U_\tau$. The following technical key result measures the difference between these two groups. In order to formulate the assertion it is convenient to call an element in G $\mathcal{A}$-elementary, resp. $\mathcal{A}$-diagonal, if written as a matrix with respect to the basis $e_1, \ldots, e_{d+1}$ it is an elementary, resp. a diagonal, matrix. This notion only depends on $\mathcal{A}$ since $\mathcal{A}$ determines the basis in question up to scalar multiples. The $\mathcal{A}$-elementary elements in G are given by

$$E_{\mu\nu}(a) : e_\lambda \longmapsto \begin{cases} e_\nu + ae_\mu & \text{if } \lambda = \nu \ , \\ e_\lambda & \text{if } \lambda \neq \nu \end{cases}$$

where $1 \leq \mu \neq \nu \leq d+1$ and $a \in K$.

Lemma 4:

Let $G' \subseteq G$ be a subgroup such that $U_\tau \cap G'$ is generated by the $\mathcal{A}$-elementary and the $\mathcal{A}$-diagonal elements contained in it; then we have

$$\begin{aligned} U_\tau \cap G' &\subseteq (U_\sigma \cap G') \cdot \bigcap \{U_\rho : \rho \textit{ vertex in } |\mathcal{A}|(\sigma;\tau)\} \\ &= (U_\sigma \cap G') \cdot \bigcap \{U_\rho : \rho \textit{ simplex with } |\rho|^0 \subseteq |\mathcal{A}|(\sigma;\tau)\} \quad . \end{aligned}$$

Proof: The equality in the assertion is rather obvious. The subgroups U_τ and U_σ contain the same $\mathcal{A}$-diagonal elements. For the asserted inclusion it therefore suffices to consider an $\mathcal{A}$-elementary element $E_{\mu\nu}(a) \in U_\tau \cap G'$ which is not contained in U_σ. We want to show that then

$$E_{\mu\nu}(a) \in U_\rho \ \text{ for any vertex } \rho \in |\mathcal{A}|(\sigma;\tau) \quad .$$

Without loss of generality we may assume that our simplices are of the form

$$\tau = \{[L_0], \ldots, [L_q]\} \underset{\neq}{\supseteq} \sigma = \{[L_0] = [L_{i_0}], [L_{i_1}], \ldots, [L_{i_r}]\}$$

with

$$L_i = \sum_{\lambda=1}^{\lambda_i} \pi^{-1} oe_\lambda + \sum_{\lambda=\lambda_i+1}^{d+1} oe_\lambda$$

and

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_q < d+1 \quad .$$

We formally put $\lambda_{q+1} := d+1$ and $i_{r+1} := q+1$. Since $U_\tau \subseteq U^{(n-1)}$ we have $a \in \pi^{n-1}o$. By explicit computation one checks that the assumption $E_{\mu\nu}(a) \in U_\tau \backslash U_\sigma$ means that

$$\lambda_{i_j} + 1 \leq \mu < \lambda_k + 1 \leq \nu \leq \lambda_{i_{j+1}}$$

for some $0 \leq j \leq r$ and $0 \leq k \leq q$ (and that $a \in \pi^{n-1}o^\times$). The equation $\xi_\mu - \xi_\nu = 0$ then defines a hyperplane $H \in H(\mathcal{A})$ such that

$$|\sigma| \subseteq H, \quad \text{but } |\tau| \not\subseteq H \quad .$$

The vertex $[L_k]$ obviously lies in the τ-side of H. Because of $\xi_\mu([L_k]) - \xi_\nu([L_k]) = -1$ we see that the τ-side $H(\tau)$ of H is given by the sign $\varepsilon = -1$. If now $\rho = [\pi^{m_1}oe_1 + \ldots + \pi^{m_{d+1}}oe_{d+1}]$ is any vertex in $H(\tau)$ we have $m_\mu - m_\nu = \xi_\mu(\rho) - \xi_\nu(\rho) < 0$. On the other hand another simple computation shows that an arbitrary $\mathcal{A}$-elementary element $E_{\mu\nu}(b)$ is contained in U_ρ if and only if $b \in \pi^{m_\mu - m_\nu + n}o$. Since $a \in \pi^{n-1}o$ we obtain $E_{\mu\nu}(a) \in U_\rho$.

Next we consider three arbitrary simplices σ, τ, σ' in $\mathcal{BT}$ (of possibly different dimension).

Definition:

The simplex τ lies between σ and σ' if there are points $x \in |\sigma|^0, y \in |\tau|^0$, and $x' \in |\sigma'|^0$ such that $y \in [xx']$.

Proposition 5:

If τ lies between σ and σ' then we have

$$\mathrm{im}(U_\tau \cap B_{\sigma'} \longrightarrow B_{\sigma'}/U_{\sigma'}) \subseteq \mathrm{im}(U_\sigma \cap B_{\sigma'} \longrightarrow B_{\sigma'}/U_{\sigma'}) \quad .$$

Proof: We first treat the special case that $\sigma \subsetneqq \tau$. Choose an apartment $\mathcal{A} \subseteq \mathcal{BT}$ which contains σ, τ and σ'. The assumption that τ lies between σ and σ' in particular means that

$$|\sigma'|^0 \subseteq |\mathcal{A}|(\sigma; \tau) \quad .$$

Therefore Lemma 4 implies

$$U_\tau \cap B_{\sigma'} \subseteq (U_\sigma \cap B_{\sigma'}) \cdot U_{\sigma'}$$

provided the group $U_\tau \cap B_{\sigma'}$ is generated by the $\mathcal{A}$-elementary and the $\mathcal{A}$-diagonal elements contained in it. But to check this is a simple exercise using elementary matrix transformations.

In the general case we fix points $x \in |\sigma|^0$, $y \in |\tau|^0$, and $x' \in |\sigma'|^0$ such that $y \in [xx']$ and we consider the set

$$\mathcal{S} := \{\rho \text{ simplex in } \mathcal{BT} : |\rho|^0 \cap [xx'] \neq \emptyset\} \quad .$$

For any $\rho \in \mathcal{S}$ we put

$$s_\rho := |\rho|^0 \cap [xx'] \quad .$$

We have the following elementary facts:

— The sets s_ρ are convex subsets of $[xx']$.

— Each set s_ρ consists of one point or is open in $[xx']$.

(Choose an apartment $|\mathcal{A}| \subseteq |\mathcal{BT}|$ which contains $[xx']$ and therefore also $|\rho|$. Let $|\mathcal{A}|(\rho)$ denote the affine subspace in $|\mathcal{A}|$ spanned by $|\rho|$. If the intersection $|\mathcal{A}|(\rho) \cap [xx']$ consists of more than one point then necessarily $[xx']$ is contained in $|\mathcal{A}|(\rho)$. Since $|\rho|^0$ is open in $|\mathcal{A}|(\rho)$ we obtain in the latter case that s_ρ is open in $[xx']$.)

— The geodesic $[xx']$ is the union of all s_ρ with $\rho \in \mathcal{S}$.
(Any point of $|\mathcal{BT}|$ lies in the interior of some simplex.)

— For two different simplices ρ_1 and ρ_2 in $\mathcal{S}$ the sets s_{ρ_1} and s_{ρ_2} are disjoint.
(A simplex is uniquely determined by any point in its interior.)

— The set $\mathcal{S}$ is finite.

(Consider an apartment $|\mathcal{A}| \subseteq |\mathcal{BT}|$ which contains $[xx']$ and therefore $|\rho|$ for all $\rho \in \mathcal{S}$.)

It is clear from these facts that the elements $\rho_1, \ldots, \rho_r$ in $\mathcal{S}$ can be enumerated in such a way that

$$d(x, y_i) < d(x, y_{i+1}) \ \text{ whenever } y_i \in s_{\rho_i} \ \textit{ and } y_{i+1} \in s_{\rho_{i+1}} \quad .$$

In particular we have

$$\sigma = \rho_1, \tau = \rho_{i_0} \ \text{ for some } 1 \leq i_0 \leq r \ , \ \text{ and } \sigma' = \rho_r \quad .$$

It also follows that if s_{ρ_i}, for some $1 \leq i < r$, consists of one point, resp. is open in $[xx']$, then $s_{\rho_{i+1}}$ is open in $[xx']$, resp. consists of one point. This means that for $1 \leq i < r$ we have

$$\text{either } \ \rho_i \underset{\neq}{\supseteq} \rho_{i+1} \ \text{ or } \rho_i \underset{\neq}{\subseteq} \rho_{i+1} \quad .$$

We claim that in both cases the inclusion

$$\mathrm{im}(U_{\rho_{i+1}} \cap B_{\sigma'} \longrightarrow B_{\sigma'}/U_{\sigma'}) \subseteq \mathrm{im}(U_{\rho_i} \cap B_{\sigma'} \longrightarrow B_{\sigma'}/U_{\sigma'})$$

holds true: This is trivial if $\rho_i \underset{\neq}{\supset} \rho_{i+1}$. But in the other case we are with the tripel $(\rho_i, \rho_{i+1}, \sigma')$ in the special situation which we have treated first. The assertion now follows by induction.

Proof of Proposition 2: For reasons of symmetry it suffices to consider the geodesic $[x\sigma_{t_0}]$. Let σ denote the unique simplex in $\mathcal{BT}$ such that $x \in |\sigma|^0$. The simplices σ_{t_0} and σ are contained in $\mathcal{BT}^{(t_0,\ldots,t_m)}$. In particular there is a $1 \leq i \leq m$ such that

$$gt_i \neq t_0 \quad \text{for all } g \in U_\sigma \quad .$$

We will show that any simplex τ which lies between σ and σ_{t_0} is contained in $\mathcal{BT}^{(t_0,\ldots,t_m)}$, too. By a similar argument as in the proof of Proposition 5 it suffices to treat the case where $\sigma \underset{\neq}{\subset} \tau$. Let us assume that τ is not contained in $\mathcal{BT}^{(t_0,\ldots,t_m)}$ so that there is a $g_0 \in U_\tau$ with $g_0 t_i = t_0$. From Proposition 5 applied to the tripel $(\sigma_{t_0}, \tau, \sigma)$ we obtain

$$U_\tau/U_\sigma \subseteq \mathrm{im}(U_{\sigma_{t_0}} \cap B_\sigma \longrightarrow B_\sigma/U_\sigma)$$

so that

$$g_0 U_\sigma = h_0 U_\sigma \quad \text{for some } h_0 \in U_{\sigma_{t_0}} \cap B_\sigma \quad .$$

Of course, we also have $h_0 \in U_\tau$. Applying Proposition 5 this time to the tripel $(\sigma, \tau, \sigma_{t_0})$ and the element $h_0^{-1} \in U_\tau \cap B_{\sigma_{t_0}}$ gives

$$h_0^{-1} = hu \quad \text{for some } h \in U_\sigma \cap B_{\sigma_{t_0}} \quad \text{and } u \in U_{\sigma_{t_0}} \quad .$$

Put $g := ug_0$. Then we have $gt_i = t_0$. Because of $g = h^{-1}h_0^{-1}g_0 \in h^{-1}U_\sigma = U_\sigma$ this leads to a contradiction.

§3 Resolutions

An ordered q-simplex of $\mathcal{BT}$ is a sequence $(\sigma_0, \ldots, \sigma_q)$ of vertices such that $\{\sigma_0, \ldots, \sigma_q\}$ is a q-simplex in $\mathcal{BT}$. Two such ordered q-simplices are called equivalent if they differ by an even permutation of the vertices; the corresponding equivalence classes are called oriented q-simplices and are denoted by $< \sigma_0, \ldots, \sigma_q >$. Let $\mathcal{BT}_{(q)}$ be the set of all oriented q-simplices of $\mathcal{BT}$. In

Paragraph 1 we had associated with any representation V in $\mathrm{Alg}(G)$ the coefficient system $\underline{\underline{V}} = (V^{U_\sigma})$ on $\mathcal{BT}$. The space of oriented q-chains of $\underline{\underline{V}}$ by definition is

$$\begin{aligned} C_c^{or}(\mathcal{BT}_{(q)}, V) :=& \mathbb{C}\text{-vector space of all maps } \omega : \mathcal{BT}_{(q)} \longrightarrow V \\ & \text{such that} \\ & \text{— } \omega \text{ has finite support,} \\ & \text{— } \omega(< \sigma_0, \dots, \sigma_q >) \in V^{U_{\{\sigma_0,\dots,\sigma_q\}}} \text{ , and} \\ & \text{— } \omega(< \sigma_{\iota(0)}, \dots, \sigma_{\iota(q)} >) = sgn(\iota) \cdot \omega(< \sigma_0, \dots, \sigma_q >) \\ & \text{for any permutation } \iota \ . \end{aligned}$$

The group G acts smoothly on $C_c^{or}(\mathcal{BT}_{(q)}, V)$ via

$$(g\omega)(< \sigma_0, \dots, \sigma_q >) := g(\omega(< g^{-1}\sigma_0, \dots, g^{-1}\sigma_q >)) \ .$$

With respect to the G-equivariant boundary map

$$\begin{aligned} \partial : C_c^{or}(\mathcal{BT}_{(q+1)}, V) &\longrightarrow C_c^{or}(\mathcal{BT}_{(q)}, V) \\ \omega &\longmapsto (< \sigma_0, \dots, \sigma_q > \mapsto \sum_{\substack{\{\sigma,\sigma_0,\dots,\sigma_q\} \\ \in \mathcal{BT}_{q+1}}} \omega(< \sigma, \sigma_0, \dots, \sigma_q >)) \end{aligned}$$

we then have the augmented homological complex

$$\begin{aligned} C_c^{or}(\mathcal{BT}_{(d)}, V) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_c^{or}(\mathcal{BT}_{(0)}, V) &\longrightarrow V \\ \omega &\longmapsto \sum_{\sigma \in \mathcal{BT}_{(0)}} \omega(\sigma) \end{aligned}$$

in $\mathrm{Alg}(G)$. It is easy to see that this complex is isomorphic to the complex $(*)$ which we considered in Paragraph 1 (but which is not G-equivariant). We have remarked already that

$$\begin{aligned} \mathrm{Alg}(G) &\longrightarrow \text{homological complexes in } \mathrm{Alg}(G) \\ V &\longmapsto C_c^{or}(\mathcal{BT}_{(.)}, V) \end{aligned}$$

is an exact functor. Since U_σ, for any simplex σ in $\mathcal{BT}$, contains a subgroup which is conjugate to U it is clear that
— the complex $C_c^{or}(\mathcal{BT}_{(.)}, V)$ is zero if and only if $V^U = 0$, and that
— the augmentation map $C_c^{or}(\mathcal{BT}_{(0)}, V) \to V$ is surjective if and only if V as a G-representation is generated by the subspace V^U.

Therefore we are mainly interested in representations which lie in the full subcategory

$$\begin{aligned}\mathrm{Alg}^U(G) :=& \text{category of those smooth } G\text{-representations}\\ & \text{which (as } G\text{-representation) are generated}\\ & \text{by their } U\text{-fixed vectors}\end{aligned}$$

of $\mathrm{Alg}(G)$.

Theorem:

For any representation V in $\mathrm{Alg}^U(G)$ the augmented complex

$$C_c^{or}(\mathcal{BT}_{(.)}, V) \longrightarrow V$$

is an exact resolution of V in $\mathrm{Alg}^U(G)$.

Proof: In the case $V = C_c(G/U)$ the complex in question was exhibited in Paragraph 1 as the top row of a double complex (A) whose columns are exact. Moreover the exactness of the other rows (A_m) was shown to be a consequence of the contractibility of the subcomplexes $\mathcal{BT}^{(t_0,\ldots,t_m)}$ in $\mathcal{BT}$. Since this contractibility is established in §2 Corollary 3 the rows (A_m) and therefore by a standard argument also the top row are exact.
In the general case we use the Bernstein-Borel-Matsumoto theory which says that the kernel of the surjective G-homomorphism

$$\begin{aligned}C_c(G/U) \otimes V^U &\longrightarrow V\\ \psi \otimes v &\longmapsto \sum_{g \in G/U} \psi(g) \cdot g(v)\end{aligned}$$

again lies in $\mathrm{Alg}^U(G)$ ([1] Thm. 4.2). We therefore have an exact resolution in $\mathrm{Alg}^U(G)$ of the form

$$\ldots \longrightarrow \bigoplus_{I_1} C_c(G/U) \longrightarrow \bigoplus_{I_0} C_c(G/U) \longrightarrow V \longrightarrow 0$$

with appropriate index sets $I_0, I_1, \ldots$ Again a standard double complex argument reduces now our assertion in case V to the "universal" case $C_c(G/U)$ which we have settled already.

This resolution has various nice properties. We recall that a representation V in $\mathrm{Alg}(G)$ is admissible if the subspace V^N, for any open subgroup $N \subseteq G$, is finite-dimensional.

Proposition 1:

If V in $\mathrm{Alg}^U(G)$ is an admissible representation then the resolution $C_c^{or}(\mathcal{BT}_{(.)}, V)$ consists of finitely generated G-representations.

Proof: The number of orbits of G in $\mathcal{BT}_{(q)}$, for a fixed $0 \leq q \leq d$, is finite. Therefore we find a finite subset $\Sigma_q \subseteq \mathcal{BT}_{(q)}$ which is closed with respect to changing the orientation of a simplex and which contains a representative of each G-orbit in $\mathcal{BT}_{(q)}$. Obviously the subspace

$$C^{or}(\Sigma_q, V) := \{\omega \in C_c^{or}(\mathcal{BT}_{(q)}, V) : \omega \text{ has support in } \Sigma_q\}$$

generates $C_c^{or}(\mathcal{BT}_{(q)}, V)$ as a G-representation. But the finiteness of Σ_q together with the admissibility of V guarantee that $C^{or}(\Sigma_q, V)$ is finite-dimensional.

Let now $Z \cong K^\times$ denote the center of G. For any continuous character $\chi : Z \to \mathbb{C}^\times$ we define the full subcategory

$$\begin{aligned} \mathrm{Alg}_\chi(G) :=& \text{category of those smooth } G\text{-representations } V \\ & \text{such that } gv = \chi(g) \cdot v \text{ for all } g \in Z \text{ and } v \in V \end{aligned}$$

of $\mathrm{Alg}(G)$; also put

$$\mathrm{Alg}_\chi^U(G) := \mathrm{Alg}_\chi(G) \cap \mathrm{Alg}^U(G) \quad .$$

Since Z acts trivially on $\mathcal{BT}$ we see that the complex $C_c^{or}(\mathcal{BT}_{(.)}, V)$ lies in $\mathrm{Alg}_\chi^U(G)$ if V does.

Proposition 2:

For any representation V in $\mathrm{Alg}_\chi^U(G)$ the augmented complex

$$C_c^{or}(\mathcal{BT}_{(.)}, V) \longrightarrow V$$

is a projective (exact) resolution of V in $\mathrm{Alg}_\chi(G)$.

Proof: We have to show that the functors $\mathrm{Hom}_G(C_c^{or}(\mathcal{BT}_{(q)}, V), .)$ are exact on $\mathrm{Alg}_\chi(G)$. Let $<\sigma> \in \mathcal{BT}_{(q)}$ be an oriented q-simplex with underlying simplex $\sigma \in \mathcal{BT}_q$; also let $\overline{<\sigma>}$ denote that oriented q-simplex with the same underlying simplex σ but with the reversed orientation (if $q = 0$ we have $\sigma = <\sigma> = \overline{<\sigma>}$). Then

$$C_c^{or}(\sigma, V) := \{\omega \in C_c^{or}(\mathcal{BT}_{(q)}, V) : \omega \text{ has support in } G<\sigma> \cup\ G\overline{<\sigma>}\}$$

is a G-subrepresentation of $C_c^{or}(\mathcal{BT}_{(q)}, V)$. Since G has only finitely many orbits in $\mathcal{BT}_q$ we see that $C_c^{or}(\mathcal{BT}_{(q)}, V)$ is a finite direct sum of such subrepresentations. It therefore suffices to show that the functor $\mathrm{Hom}_G(C_c^{or}(\sigma, V), .)$ is exact on $\mathrm{Alg}_\chi(G)$. Let G_σ, resp. $G_{<\sigma>}$, be the stabilizer in G of σ, resp. $<\sigma>$. Then $G_{<\sigma>}$ is a subgroup of index ≤ 2 in G_σ; let $\varepsilon : G_\sigma \to \{\pm 1\}$ be the unique character such that $\ker(\varepsilon) = G_{<\sigma>}$. For any $v \in V^{U_\sigma}$ we define an oriented q-chain $\omega_v \in C_c^{or}(\sigma, V)$ by

$$\omega_v(.) := \begin{cases} +v & \text{if } . = <\sigma> \ , \\ -v & \text{if } q > 0 \text{ and } . = \overline{<\sigma>} \ , \\ 0 & \text{otherwise } . \end{cases}$$

Note that U_σ is a normal subgroup in G_σ and that V^{U_σ} is a G_σ-submodule of V. It is straightforward to check that, for any W in $\mathrm{Alg}(G)$, the linear map

$$\begin{aligned} \mathrm{Hom}_G(C_c^{or}(\sigma, V), W) &\xrightarrow{\cong} \{A \in \mathrm{Hom}_{\mathbf{C}}(V^{U_\sigma}, W) : g(A(g^{-1}v)) = \varepsilon(g) \cdot A(v) \\ &\qquad\qquad \text{for all } g \in G_\sigma \text{ and } v \in V^{U_\sigma}\} \\ E &\longmapsto (v \mapsto E(\omega_v)) \end{aligned}$$

is an isomorphism. So we are reduced to show that

$$\mathrm{Hom}_{G_{<\sigma>}}(V^{U_\sigma}, W) = \mathrm{Hom}_{G_{<\sigma>}/U_\sigma}(V^{U_\sigma}, W^{U_\sigma})$$

is exact in $W \in \mathrm{Alg}_\chi(G)$. But for W in $\mathrm{Alg}_\chi(G)$ any linear map $V^{U_\sigma} \to W^{U_\sigma}$ necessarily is a Z-module homomorphism. We therefore obtain

$$\mathrm{Hom}_{G_{<\sigma>}}(V^{U_\sigma}, W) = \mathrm{Hom}_{\mathbf{C}}(V^{U_\sigma}, W^{U_\sigma})^{G_{<\sigma>}/ZU_\sigma} \ .$$

The right hand side is an exact functor in $W \in \mathrm{Alg}_\chi(G)$ since U_σ is profinite and $G_{<\sigma>}/ZU_\sigma$ is finite.

Corollary 3:

Let V, resp. W, be an admissible representation in $\mathrm{Alg}_\chi^U(G)$, resp. $\mathrm{Alg}_\chi(G)$; then the vector spaces $\mathrm{Ext}^*_{\mathrm{Alg}_\chi(G)}(V, W)$ are finite-dimensional and vanish for $* > d$.

Proof: This is a standard consequence of Propositions 1 and 2 once one observes that whenever V' is a finitely generated and W is an admissible representation in $\mathrm{Alg}(G)$ then the vector space $\mathrm{Hom}_G(V', W)$ is finite-dimensional. Namely let $v_1, \ldots, v_m$ be generators of V' as a G-representation and let $N \subseteq G$ be a compact open subgroup which fixes these generators; then the map

$$\begin{aligned} \mathrm{Hom}_G(V', W) &\longrightarrow W^N \times \ldots \times W^N \quad (m \text{ times}) \\ E &\longmapsto (E(v_1), \ldots, E(v_m)) \end{aligned}$$

is injective into a finite-dimensional right hand side.

If χ is the trivial character then $\mathrm{Alg}_\chi(G)$ is the category of smooth $PGL_{d+1}(K)$-repesentations. Let $\Gamma \subseteq PGL_{d+1}(K)$ be a cocompact discrete subgroup.

Proposition 4:

Let V be an admissible smooth $PGL_{d+1}(K)$-representation which is generated by V^U; then, as a Γ-module, V has a projective (exact) resolution by finitely generated free $\mathbb{C}[\Gamma]$-modules; in particular the vector spaces $H_*(\Gamma, V)$ and $\mathrm{Ext}^*_{\mathbb{C}[\Gamma]}(V, M)$, for any finite-dimensional Γ-module M, are finite-dimensional.

Proof: Compare the proof of §6 Prop. 16 in [4].

References

[1] Bernstein, I.N., Zelevinskii, A.V.: Representations of the group $GL(n, F)$ where F is a non-archimedean local field. Russ. Math. Surv. 31 (3), 1-68 (1976)

[2] Brown, K.S.: Buildings. Berlin-Heidelberg-New York: Springer 1989

[3] Ronan, M.: Lectures on Buildings. Boston: Academic Press 1989

[4] Schneider, P., Stuhler, U.: The cohomology of p-adic symmetric spaces. Invent. math. 105, 47-122 (1991)