Points of rigid analytic varieties

by Peter Schneider

On a rigid analytic variety the usual stalk functors are not very useful for other than coherent sheaves; e.g., they do not detect the exactness of a sequence of sheaves. The reason for this of course is the semiglobal character of the analytic Grothendieck topology. In his paper [Put] van der Put made the basic observation that on an affinoid space one can define using valuation theory a larger family of stalk functors which has the usual properties for a reasonably big category of sheaves. As a main result he then could prove a cohomological base change theorem for those sheaves. Berkovich in [Ber] associated with any affinoid algebra a compact Hausdorff space the points of which correspond bijectively to van der Put's stalk functors. Taking those as building blocks he actually created a whole new theory of analytic spaces but which is more restrictive than the theory of rigid analytic varieties.

In the first Paragraph we will combine the ideas of van der Put and Berkovich in the affinoid case and will develop systematically the notion of an analytic point of an affinoid variety. Instead of working with valuations or multiplicative seminorms as Berkovich we borrow from algebraic geometry the idea of an F-valued point where F here is any complete field lying above the base field. One has to be aware of the fact that contrary to the field of complex numbers a non-archimedean complete field has arbitrarily large such extensions F. Although these analytic points are not visible in the topological space which underlies an affinoid variety $X = Sp(A)$ they live in the structure sheaf $\mathcal{O}_X$ and therefore give rise to stalk functors. We construct the subcategory of what we call conservative sheaves on X as the image of a natural functor from the sheaves on Berkovich' compact space $\mathcal{M}(X)$ into the sheaves on X. The analytic points of X then form a conservative family of points in the topos theoretic sense for the category of conservative sheaves. This approach leads to a very conceptual proof of van der Put's base change theorem.

In the second Paragraph we generalize this theory including the base change theorem to general rigid analytic varieties. This is possible since it turns out that the notion of a conservative sheaf is of a local nature. It demonstrates in particular that by working with sheaf categories instead of spaces the ideas of Berkovich can be fully saved in the rigid analytic context.

In the third Paragraph we introduce the notion of a wide open subset of an affinoid variety X. The wide open subsets together with the admissible coverings form a Grothendieck topology on X which is coarser than the analytic Grothendieck topology. We show that the category of conservative sheaves on X coincides with the full category of sheaves with respect to this "wide open topology". It is not unlikely that this topology is the correct conceptual framework for Coleman's theory of p-adic integration. Actually "wide" is a relative notion. We hope to explore this in a future paper on rigid etale cohomology.

In all our notations we follow very closely the book [BGR] the reader is assumed to be familiar with. One word of apology that we change the language introduced by van der Put might be necessary: His closed geometric points are our quasi-geometric points whereas our geometric points simply are the points of the underlying topological spaces. Also our conservative sheaves are his constructible sheaves. The latter notion already has a fixed meaning in topology and in algebraic geometry which has nothing to do with the situation here.

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§1 Analytic points of affinoid varieties

Throughout this paper k denotes a field which is complete with respect to a non-trivial non-archimedean absolute value $|\ |$.

Definition:

A complete extension field F of k is an extension field F of k together with an absolute value $|\ |_F$ which restricts to $|\ |$ on k and such that F is complete with respect to $|\ |_F$.

Let A be an affinoid k-algebra; it is a topological k-algebra in a natural way. Put $X := Sp(A)$.

Definition:

An analytic point x of X is a continuous k-algebra homomorphism $x : A \to F$ into a complete extension field F of k; then x is said to be F-valued and F is called the field of values of x.

Points $x \in X$ henceforth will be called *geometric* points. If $\mathfrak{m}_x \subseteq A$ is the maximal ideal corresponding to some $x \in X$ then $A/\mathfrak{m}_x$ is a finite extension of k; therefore x determines a unique analytic point $x : A \xrightarrow{pr} A/\mathfrak{m}_x$. Let us fix an analytic point $x : A \to F$ of X.

Remark 1:

Let $g, f_1, \ldots, f_r \in A$ be elements which generate the unit ideal. Then

$$Sp(A\langle \frac{f_1}{g}, \ldots, \frac{f_r}{g} \rangle) = X(\frac{f_1}{g}, \ldots, \frac{f_r}{g}) = \{y \in X : |g(y)| \geq \max_{1 \leq i \leq r} |f_i(y)|\}$$

is a rational subdomain of X. By [BGR] 6.1.4 Prop. 3 the map x extends to a continuous k-algebra homomorphism $\tilde{x} : A\langle \frac{f_1}{g}, \ldots, \frac{f_r}{g} \rangle \to F$ (i.e., the analytic point x "lies in the subdomain $X(\frac{f_1}{g}, \ldots, \frac{f_r}{g})$ of X") if and only if $|x(g)|_F \geq \max_{1 \leq i \leq r} |x(f_i)|_F$; moreover, if $\tilde{x}$ exists then it is unique.

Lemma 2:

i. $|x(f)|_F \leq \|f\|_X$ *for all* $f \in A$;
ii. if $X = Sp(A_1) \cup \ldots \cup Sp(A_r)$ *is a finite covering by affinoid subdomains then* x *extends to a continuous* k*-algebra homomorphism* $A_{j_0} \to F$ *for some* $1 \leq j_0 \leq r$;
iii. let $Sp(B) \subseteq X$ *be an affinoid subdomain; if* x *extends to a continuous* k*-algebra homomorphism* $B \to F$ *then there is a unique such extension.*

Proof: i. Recall that $\|f\|_X = \sup_{y \in X} |f(y)|$ denotes the supremum or spectral semi-norm on A. Fixing a residue norm $|\ |_\alpha$ on A the continuity of x implies $|x(f)|_F \leq |f|_\alpha$. But according to [BGR] 6.2.3 Prop. 3 we have

$$\|f\|_X = \inf_{i \in \mathbb{N}} |f^i|_\alpha^{1/i} \quad .$$

We therefore obtain

$$\|f\|_X \geq \inf_{i \in \mathbb{N}} |x(f^i)|_F^{1/i} = |x(f)|_F \quad .$$

ii. By refining the given covering we may assume ([BGR] 7.3.5 Cor. 3) that the $Sp(A_j)$ are rational subdomains of X. The A_j even can be assumed to be of the form

$$A_j = A\langle \frac{f_1}{f_j}, \ldots, \frac{f_r}{f_j} \rangle \qquad \text{for all } 1 \leq j \leq r$$

with appropriately chosen $f_1, \ldots, f_r \in A$ generating the unit ideal ([BGR] 8.2.2 Lemma 2). Now choose j_0 in such a way that $|x(f_{j_0})|_F = \max_{1 \leq j \leq r} |x(f_j)|_F$ and apply the Remark 1.
iii. Assume that $\tilde{x}, \tilde{\tilde{x}} : B \to F$ are two continuous extensions of x. Using again [BGR] 7.3.5 Cor. 3 together with the previous assertion we obtain that $\tilde{x}$ extends continuously to $A\langle \frac{f_1}{g}, \ldots, \frac{f_r}{g} \rangle$ for some rational subdomain $X(\frac{f_1}{g}, \ldots, \frac{f_r}{g}) \subseteq Sp(B)$. By Remark 1 this means that $|x(g)|_F \geq \max_{1 \leq i \leq r} |x(f_i)|_F$. Viewing now $g, f_1, \ldots, f_r$ as elements in B we have $A\langle \frac{f_1}{g}, \ldots, \frac{f_r}{g} \rangle = B\langle \frac{f_1}{g}, \ldots, \frac{f_r}{g} \rangle$ with $|\tilde{\tilde{x}}(g)|_F \geq \max_{1 \leq i \leq r} |\tilde{\tilde{x}}(f_i)|_F$. Applying Remark 1 again we see that $\tilde{\tilde{x}}$ extends continuously to $B\langle \frac{f_1}{g}, \ldots, \frac{f_r}{g} \rangle$ and that this extension has to be equal to the extension of $\tilde{x}$. Hence $\tilde{x} = \tilde{\tilde{x}}$.

Example:

Let $A = k\langle T\rangle$ be the Tate algebra over an algebraically closed field k. Any subset in X of the form

$$D = \{y \in X : |y - a| \leq \rho\}$$

for some $a \in X$ and some $0 < \rho \leq 1$ (not necessarily contained in $|k^\times|$) defines a multiplicative norm

$$|f|_D := \sup_{y \in D} |f(y)| \leq \|f\|_X$$

on A. Then

$$x_D : A \overset{\subseteq}{\hookrightarrow} F_D := \text{ completion of Quot(A) w.r.t. } |\ |_D$$

is an analytic point of X.

Definition:

i. An affinoid subdomain $U = Sp(B) \subseteq X$ is called an affinoid neighbourhood of x if we have a factorization

$$\begin{array}{ccc} A & \xrightarrow{can} & B \\ & {}_x\searrow \quad \swarrow & \\ & F & \end{array}$$

with a continuous k-algebra homomorphism $B \to F$ which being unique by Lemma 2 iii will usually be denoted by x, too.
ii. An admissible open subset $U \subseteq X$ is called a neighbourhood of x if U contains an affinoid neighbourhood of x.

If x is geometric then these neighbourhoods of course are the usual "topological" ones.

Lemma 3:

i. With $U_1, U_2 \subseteq X$ also $U_1 \cap U_2$ is a neighbourhood of x;
ii. if $U \subseteq X$ is a neighbourhood of x and $U = \bigcup_{i \in I} U_i$ is an admissible covering then U_{i_0} is a neighbourhood of x for some $i_0 \in I$.

Proof: i. We may assume that the $U_i = Sp(B_i)$ are affinoid. Let $x_i : B_i \to F$ be the continuous extensions of x. Then $U_1 \cap U_2 = Sp(B_1 \hat{\otimes}_A B_2)$ and $x_1 \otimes x_2 : B_1 \otimes_A B_2 \to F$ extends to $B_1 \hat{\otimes}_A B_2 \to F$ by the completeness of F.
ii. Let $Sp(B) \subseteq U$ be an affinoid neighbourhood of x. The covering $Sp(B) = \bigcup_{i \in I} Sp(B) \cap U_i$ has a finite affinoid refinement

$$Sp(B) = Sp(B_1) \cup \ldots \cup Sp(B_r) \quad .$$

By Lemma 2 ii we find $1 \leq j_0 \leq r$ such that $Sp(B_{j_0})$ is an affinoid neighbourhood of x. But $Sp(B_{j_0}) \subseteq U_{i_0}$ for some $i_0 \in I$.

Lemma 4:

Let $x' : A \to F'$ be another analytic point of X such that $|x(.)|_F \neq |x'(.)|_{F'}$; then we find neighbourhoods U of x and U' of x' such that $U \cap U' = \emptyset$.

Proof: Fix an element $f \in A$ with $|x(f)|_F \neq |x'(f)|_{F'}$, say $|x(f)|_F < |x'(f)|_{F'}$. Choose numbers $\varepsilon, \varepsilon' \in \sqrt{|k^\times|}$ such that

$$|x(f)|_F \leq \varepsilon < \varepsilon' \leq |x'(f)|_{F'}$$

and consider the rational subdomains $U := X(\varepsilon^{-1} f)$ and $U' := X(\varepsilon' f^{-1})$.

Corollary 5:

The intersection of all neighbourhoods of x is empty unless x is a geometric point.

For any sheaf $\mathcal{F}$ on X we now define its stalk at x by

$$\mathcal{F}_x := \varinjlim_{\substack{U \text{ neighbour-} \\ \text{hood of } x}} \mathcal{F}(U) = \varinjlim_{\substack{U \text{ affinoid} \\ \text{neighbourhood of } x}} \mathcal{F}(U) \quad .$$

Remark 6:

$\mathcal{O}_{X,x}$ is a local ring.

Proof: By definition x induces a homomorphism $x : \mathcal{O}_{X,x} \longrightarrow F$ the kernel $\mathfrak{m}_x$ of which is a maximal ideal in $\mathcal{O}_{X,x}$. Let U be an affinoid neighbourhood of x and consider any $f \in \mathcal{O}_X(U)$ such that the germ f_x is not contained in $\mathfrak{m}_x$. The latter implies $|x(f)|_F \neq 0$. Multiplying f by an appropriate element in $k^\times$ we may assume that $|x(f)|_F \geq 1$. Then the rational subdomain $U(f^{-1})$ also is a neighbourhood of x. But f restricted to $U(f^{-1})$ is invertible. This shows that $\mathcal{O}_{X,x} \backslash \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}^\times$.

Even with x running through all analytic points of X the stalks $\mathcal{F}_x$ as defined above still have the defect of not detecting the vanishing of $\mathcal{F}$, i.e., all stalks $\mathcal{F}_x$ can be trivial without $\mathcal{F}$ being trivial. We have to analyze this phenomenon in more detail.

Definition:

Let U be a neighbourhood of x. We call x inner in U, resp. U a wide neighbourhood of x, if U contains an affinoid neighbourhood $V = Sp(B)$ of x for which there is an affinoid generating system $f_1, \ldots, f_r$ of B over A such that $|x(f_i)|_F < 1$ for all $1 \leq i \leq r$.

Note that $|x(f_i)|_F \leq 1$ always holds true in this situation. There is the following more invariant criterion.

Proposition 7:

For an affinoid neighbourhood $V = Sp(B)$ of x the following assertions are equivalent:
i. There is an affinoid generating system $f_1, \ldots, f_r$ of B over A such that $|x(f_i)|_F < 1$ for all $1 \leq i \leq r$;
ii. for every $f \in \mathring{B}$ there exists a monic polynomial $P_f(T) \in \mathring{A}[T]$ such that $|x(P_f(f))|_F < 1$.

Proof: (Recall that $\mathring{A}$ denotes the subring of all power-bounded elements in A.) This is contained in [Ber]2.5.2. Since the argument can be somewhat simplified in our context we give the proof. Let us first assume that i. holds true. If φ denotes the epimorphism

$$\begin{array}{rccc} \varphi: & A\langle T_1, \ldots, T_r\rangle & \longrightarrow & B \\ & T_i & \longmapsto & f_i \end{array}$$

then, according to [BGR]6.3.5 Thm. 1, $\mathring{\varphi}$ is integral. This means that there is a natural number $m \geq 1$ and power series $h_1, \ldots, h_m \in \mathring{A}\langle T_1, \ldots, T_r\rangle = (A\langle T_1, \ldots, T_r\rangle)^\circ$ such that

$$f^m + h_1(f_1, \ldots, f_r)f^{m-1} + \ldots + h_m(f_1, \ldots, f_r) = 0$$

for an arbitrary but fixed $f \in \mathring{B}$. If we define

$$P_f(T) := T^m + h_1(0)T^{m-1} + \ldots + h_m(0) \in \mathring{A}[T]$$

then it is easily seen that $|x(P_f(f))|_F < 1$ (observe that by the continuity of x we have $|x(g)|_F \leq 1$ for any power-bounded element g in A or B). Now assume that ii. holds true and let $g_1, \ldots, g_s$ be some affinoid generating system of B over A. For every $1 \leq i \leq s$ fix a monic polynomial $P_i(T) \in \overset{\circ}{A}[T]$ of degree $m_i \geq 1$ such that $|x(P_i(g_i))|_F < 1$. Also fix some $a \in k^\times$ with $|a| < 1$. Consider the system

$$C := \{ag_i : 1 \leq i \leq s\} \cup \{g_i^{\mu_i} P_i(g_i) : 1 \leq i \leq s, 0 \leq \mu_i < m_i\}$$

of power-bounded elements in B. Clearly any element f in C fulfills $|x(f)|_F < 1$. We claim that C is an affinoid generating system of B over A. For that we have to show that the map

$$\begin{aligned} A\langle\{T_f : f \in C\}\rangle &\longrightarrow B \\ T_f &\longmapsto f \end{aligned}$$

is surjective. Let $\tilde{B}$ denote its image; this is an affinoid algebra which is equal to B if and only if

— $g_1, \ldots, g_s \in \tilde{B}$, and

— $g_1, \ldots, g_s$ are power-bounded in $\tilde{B}$.

We use the following general fact the proof of which is straightforward:

> Let $P \in R[T]$ be a monic polynomial of degree $m \geq 1$ over a commutative ring with unit R; we then have
> $R[T] = R[P, T \cdot P, \ldots, T^{m-1} \cdot P] + R \cdot T + \ldots + R \cdot T^{m-1}$.

We obtain

$$g_i^\nu \in \overset{\circ}{A}[P_i(g_i), g_i P_i(g_i), \ldots, g_i^{m_i-1} P_i(g_i)] + a^{-m_i+1} \cdot \overset{\circ}{A}[ag_i]$$

for each $1 \leq i \leq s$ and each $\nu \geq 0$. This shows that $g_i \in \tilde{B}$ and that $\|g_i^\nu\|_{Sp(\tilde{B})} \leq |a|^{-m_i+1}$ for each $\nu \geq 0$, i.e., that $\|g_i\|_{Sp(\tilde{B})} \leq 1$.

Corollary 8:

Let $U = Sp(B)$ be an affinoid wide neighbourhood of x. Then there is an affinoid generating system $f_1, \ldots, f_r$ of B over A such that $|x(f_i)|_F < 1$ for all $1 \leq i \leq r$.

It is clear (compare the proof of Lemma 3 i) that the intersection of two wide neighbourhoods of x is still a wide neighbourhood of x. We now define a modification of the stalk $\mathcal{F}_x$ by

$$\mathcal{F}_x^{\text{mod}} := \varinjlim_{\substack{U \text{ wide neigh-} \\ \text{bourhood of } x}} \mathcal{F}(U) \quad .$$

There is an obvious map

$$\mathcal{F}_x^{\mathrm{mod}} \longrightarrow \mathcal{F}_x$$

but which need not to be bijective.

Remark 9:

If $x \in X$ is a geometric point then any neighbourhood of x is wide; in particular we then have $\mathcal{F}_x = \mathcal{F}_x^{\mathrm{mod}}$.

Proof: If $\mathfrak{m}_x \subseteq A$ is the maximal ideal in A corresponding to x then the Weierstraß domains $X(f_1, \ldots, f_r)$ for $f_1, \ldots, f_r \in \mathfrak{m}_x$ form a fundamental system of neighbourhoods of x with respect to the canonical topology on X ([BGR]7.2.1 Prop. 4); they, of course, are wide neighbourhoods of x.

Example:

Let $A = k\langle T \rangle$ be the Tate algebra in one variable over k so that $X = Sp(A)$ is the closed unit disk. If F denotes the completion of the quotient field of A with respect to the Gauß norm then the inclusion $x : A \xrightarrow{\subseteq} F$ is an analytic point of X. The annulus $X(T^{-1}) = \{y \in X : |T(y)| = 1\}$ is a neighbourhood of x which is not wide.

We will see that the $\mathcal{F}_x^{\mathrm{mod}}$ have a different description in quite "classical" terms.

Lemma 4':

Let $x' : A \to F'$ be another analytic point of X such that $|x(.)|_F \neq |x'(.)|_{F'}$; then we find wide neighbourhoods U of x and U' of x' with $U \cap U' = \emptyset$.

Proof: This is the same argument as for Lemma 4; only choose the numbers $\varepsilon, \varepsilon'$ in such a way that $|x(f)|_F < \varepsilon < \varepsilon' < |x'(f)|_{F'}$.

Corollary 10:

For two analytic points $x : A \to F$ and $x' : A \to F'$ the following assertions are equivalent:
i. $|x(.)|_F = |x'(.)|_{F'}$;
ii. x and x' have the same system of neighbourhoods;
iii. x and x' have the same system of wide neighbourhoods.
If these assertions are fulfilled then $\mathcal{F}_x = \mathcal{F}_{x'}$ and $\mathcal{F}_x^{\mathrm{mod}} = \mathcal{F}_{x'}^{\mathrm{mod}}$ for any sheaf $\mathcal{F}$ on X.

Proof: Remark 1 and Lemmata 4 and 4'.

We call x and x' congruent if they fulfill the equivalent assertions in the above Corollary. Put

$$\mathcal{M}(X) := \text{ set of all congruence classes of analytic points of } X \quad .$$

If it is clear from the context what is meant we will not distinguish notationally between an analytic point and its congruence class in $\mathcal{M}(X)$. There is an obvious injective map

$$\begin{aligned} X &\longrightarrow \mathcal{M}(X) \\ x &\longmapsto \text{congruence class of } x \quad ; \end{aligned}$$

its image consists of precisely those analytic points x for which the residue class field of $\mathcal{O}_{X,x}$ is a finite extension of k. We equip $\mathcal{M}(X)$ with the coarsest topology such that all maps

$$\begin{aligned} \mathcal{M}(X) &\longrightarrow \mathbb{R} \qquad\qquad \text{for } f \in A \\ (x : A \to F) &\longmapsto |x(f)|_F \end{aligned}$$

are continuous.

Lemma 11:

i. $\mathcal{M}(X)$ *is compact;*
ii. with respect to the canonical topology on X *the natural map* $X \to \mathcal{M}(X)$ *induces a homeomorphism onto an everywhere dense subset in* $\mathcal{M}(X)$.

Proof: [Ber]1.2.1 and 2.1.15.

Let $U, V \subseteq X$ be affinoid subdomains. It follows from Lemma 11 that

$$\mathcal{M}(U) = \text{closure of } U \text{ in } \mathcal{M}(X) \quad .$$

We also have

$$U = X \cap \mathcal{M}(U) \quad \text{and} \quad \mathcal{M}(U) \cap \mathcal{M}(V) = \mathcal{M}(U \cap V) \quad .$$

Lemma 12:

i. An affinoid neighbourhood $U \subseteq X$ *of* x *is wide if and only if* $\mathcal{M}(U)$ *is a neighbourhood of* x *in* $\mathcal{M}(X)$;
ii. the $\mathcal{M}(U)$ *with* $U \subseteq X$ *running through the affinoid wide neighbourhoods of* x *form a fundamental system of compact neighbourhoods of* x *in* $\mathcal{M}(X)$.

Proof: [Ber]2.5.13 (ii) and 2.2.3 (iii) (those proofs can be literally translated into our slightly different language).

We call special subsets in X those which are finite unions of affinoid subdomains. If $U \subseteq X$ is special let $\overline{U}$ denote its closure in $\mathcal{M}(X)$. In this way we obtain a map

$$\begin{aligned} \text{special subsets in } X &\longrightarrow \text{compact subsets in } \mathcal{M}(X) \\ U &\longmapsto \overline{U} \end{aligned}$$

which is injective and compatible with forming finite unions and intersections; its image contains a fundamental system of compact neighbourhoods of any point in $\mathcal{M}(X)$. In the following we think of X as being equipped with the "special" G-topology given by the special subsets and the finite coverings. It gives the same category of sheaves on X as the strong G-topology since the latter is slightly finer. For any sheaf $\mathcal{F}$ on X we define a presheaf $\overline{\mathcal{F}}$ on $\mathcal{M}(X)$ by

$$\overline{\mathcal{F}}(M) := \varprojlim_{\substack{\overline{U} \subseteq M \\ U \subseteq X \text{ special}}} \mathcal{F}(U) \qquad \text{for } M \subseteq \mathcal{M}(X) \text{ open} \quad .$$

Lemma 13:

$\overline{\mathcal{F}}$ is a sheaf.

Proof: This is straightforward from the following observation. Let $M = \bigcup_{i \in I} M_i$ be an open covering of some open subset $M \subseteq \mathcal{M}(X)$ and let $U \subseteq X$ be a special subset such that $\overline{U} \subseteq M$. Then there exists a finite covering

$$U = V_1 \cup \ldots \cup V_r$$

by special subsets $V_\rho \subseteq X$ such that $\overline{V}_\rho \subseteq M_{i(\rho)}$ for some $i(\rho) \in I$. Namely by Lemma 12 ii, for any $x \in M_i$, there is an open neighbourhood $N_x \subseteq \mathcal{M}(X)$ of x and a special subset $V_x \subseteq X$ such that

$$x \in N_x \subseteq \overline{V}_x \subseteq M_i \quad .$$

We obtain

$$\overline{U} \subseteq \bigcup_{x \in \overline{U}} N_x \subseteq \bigcup_{x \in \overline{U}} \overline{V}_x \quad .$$

Since $\overline{U}$ is compact we already have

$$\overline{U} \subseteq N_{x_1} \cup \ldots \cup N_{x_r} \subseteq \overline{V}_{x_1} \cup \ldots \cup \overline{V}_{x_r}$$

for finitely many appropriately chosen points x_ρ. Now define $V_\rho := U \cap V_{x_\rho}$.

Lemma 14:

For any analytic point x of X we have $\mathcal{F}_x^{\text{mod}} = (\overline{\mathcal{F}})_x$.

Proof: There is an obvious map

$$(\overline{\mathcal{F}})_x = \varinjlim_{\substack{M \subseteq \mathcal{M}(X) \text{ open} \\ x \in M}} \varprojlim_{\substack{\overline{U} \subseteq M \\ U \subseteq X \text{ special}}} \mathcal{F}(U) = \varinjlim_{\substack{M \subseteq \mathcal{M}(X) \text{ open} \\ x \in M}} \varprojlim_{\substack{\overline{U} \subseteq M \\ U \subseteq X \text{ special} \\ \text{neighbourhood of } x}} \mathcal{F}(U)$$

$$\downarrow$$

$$\mathcal{F}_x^{\text{mod}} = \varinjlim_{\substack{V \subseteq X \text{ affinoid} \\ \text{wide neighbourhood of } x}} \mathcal{F}(V).$$

As for the surjectivity fix a section $s \in \mathcal{F}(V)$ for some affinoid wide neighbourhood V of x. Since V is open in X we find by Lemmata 11 ii and 12 i an open neighbourhood $M \subseteq \mathcal{M}(X)$ of x such that $M \cap X = V$. For any special subset $U \subseteq X$ with $\overline{U} \subseteq M$ we then have $U \subseteq V$. Therefore the germ in x of the element $(s|U)_U \in \overline{\mathcal{F}}(M)$ is a preimage of the germ s_x.

For the injectivity let $M \subseteq \mathcal{M}(X)$ be an open neighbourhood of x and let

$$t = (t_U)_U \in \varprojlim_{\substack{\overline{U} \subseteq M \\ U \subseteq X \text{ special} \\ \text{neighbourhood of } x}} \mathcal{F}(U) = \overline{\mathcal{F}}(M)$$

be a section such that $t_V = 0$ for some affinoid wide neighbourhood $V \subseteq X$ of x with $\overline{V} \subseteq M$. Similarly as before let $N \subseteq M$ be an open neighbourhood such that $N \cap X = V$. Then clearly $t|N = 0$ holds true.

There is also a functor in the opposite direction. For any sheaf $\mathcal{G}$ on $\mathcal{M}(X)$ we define a presheaf $\hat{\mathcal{G}}$ on X by

$$\hat{\mathcal{G}}(U) := \varinjlim_{\substack{\overline{U} \subseteq M \\ M \subseteq \mathcal{M}(X) \text{ open}}} \mathcal{G}(M) \qquad \text{for } U \subseteq X \text{ special.}$$

Lemma 15:

$\hat{\mathcal{G}}$ is a sheaf.

Proof: This is straightforward from the following fact which holds in any normal space: Let $\overline{U}, \overline{V} \subseteq \mathcal{M}(X)$ be closed subsets and let L be an open neighbourhood of $\overline{U} \cap \overline{V}$. Then there are open neighbourhoods M of $\overline{U}$ and N of $\overline{V}$ such that $M \cap N = L$.

Lemma 16:

The functor

$$\text{sheaves on } \mathcal{M}(X) \longrightarrow \text{sheaves on } X$$
$$\mathcal{G} \longmapsto \hat{\mathcal{G}}$$

is exact and left adjoint to the functor $\mathcal{F} \mapsto \overline{\mathcal{F}}$*.*

Proof: The adjointness property is easy to check. Any functor with a right adjoint is right exact. The left exactness is immediate from the definition.

Corollary 17:

The functor $\mathcal{F} \mapsto \overline{\mathcal{F}}$ *preserves injective sheaves.*

Lemma 18:

For any analytic point x *of* X *we have* $(\hat{\mathcal{G}})_x^{\mathrm{mod}} = (\hat{\mathcal{G}})_x = \mathcal{G}_x$*.*

Proof: This follows from Lemma 12 ii.

We see that the adjunction map

$$\hat{\overline{\mathcal{F}}} \to \mathcal{F} \quad ,$$

for any sheaf $\mathcal{F}$ on X, realizes in the stalks the natural map $\mathcal{F}_x^{\mathrm{mod}} \to \mathcal{F}_x$. On the other hand, the adjunction map $\mathcal{G} \xrightarrow{\sim} \overline{\hat{\mathcal{G}}}$ is an isomorphism for any sheaf $\mathcal{G}$ on $\mathcal{M}(X)$.

Definition:

A sheaf $\mathcal{F}$ *on* X *is called conservative if the adjunction map* $\hat{\overline{\mathcal{F}}} \xrightarrow{\sim} \mathcal{F}$ *is an isomorphism.*

We obtain equivalences of categories

$$\begin{array}{ccc} \text{category of all} & \overset{\mathcal{G} \mapsto \hat{\mathcal{G}}}{\longrightarrow} & \text{full subcategory of all} \\ \text{sheaves on } \mathcal{M}(X) & \underset{\overline{\mathcal{F}} \leftarrow\!\shortmid\, \mathcal{F}}{\longleftarrow} & \text{conservative sheaves on } X \end{array}$$

which are quasi-inverse to each other.

Recall that a sheaf $\mathcal{G}$ on $\mathcal{M}(X)$, resp. $\mathcal{F}$ on X, is called flabby if the restriction map $\mathcal{G}(\mathcal{M}(X)) \to \mathcal{G}(M)$, resp. $\mathcal{F}(X) \to \mathcal{F}(U)$, is surjective for every open subset $M \subseteq \mathcal{M}(X)$, resp. every special open subset $U \subseteq X$. In both cases flabby sheaves are acyclic for cohomology and higher direct images. Since in our special G-topology on X only finite coverings are allowed all the standard arguments about flabby sheaves can easily be saved.

Lemma 19:

The functor $\mathcal{G} \mapsto \hat{\mathcal{G}}$ preserves flabby sheaves.

Proof: Obvious from the definition.

Corollary 20:

i. For any sheaf $\mathcal{G}$ on $\mathcal{M}(X)$ we have $H^*(\mathcal{M}(X), \mathcal{G}) = H^*(X, \hat{\mathcal{G}})$;
ii. for any conservative sheaf $\mathcal{F}$ on X we have $H^*(X, \mathcal{F}) = H^*(\mathcal{M}(X), \overline{\mathcal{F}})$.

There is a corresponding result for Čech cohomology.

Remark 21:

For any presheaf Q on X or $\mathcal{M}(X)$ let $\check{H}^i(Q)$ denote the corresponding presheaves of Čech cohomology groups. The functor $\mathcal{P} \mapsto \hat{\mathcal{P}}$ obviously makes sense for presheaves $\mathcal{P}$ on $\mathcal{M}(X)$. Using the topological facts observed in the proofs of Lemma 13 and Lemma 15 one checks that

$$[\check{H}^i(\mathcal{P})]^{\wedge} = \check{H}^i(\hat{\mathcal{P}})$$

holds true. This implies:
i. The functor $\mathcal{P} \mapsto \hat{\mathcal{P}}$ commutes with sheafification;
ii. $\check{H}^*(X, \hat{\mathcal{P}}) = \check{H}^*(\mathcal{M}(X), \mathcal{P})$;
iii. $\check{H}^*(X, \mathcal{F}) = \check{H}^*(\mathcal{M}(X), \overline{\mathcal{F}})$ for any conservative sheaf $\mathcal{F}$ on X.

One can give a more direct characterization of conservative sheaves which does not involve the space $\mathcal{M}(X)$. As before let $\| \|_X$ denote the spectral semi-norm on A.

Lemma 22:

There are finitely many analytic points $x_i : A \to F_i$ of X, for $1 \leq i \leq m$, such that

$$\|.\|_X = \max_{1 \leq i \leq m} |x_i(.)|_{F_i} \quad .$$

Proof: The following argument is extracted from [Ber]2.4.4. Let $\tilde{X} = \mathrm{Spec}(\tilde{A})$ denote the reduction of X. If $\tilde{A}$ is an integral domain then it follows from [BGR]6.2.3 Prop. 5 that there is an analytic point $x : A \to F$ such that $|x(.)|_F$ is the spectral semi-norm on A. For general A let $\tilde{\wp}_1, \ldots, \tilde{\wp}_m$ be the minimal prime ideals in $\tilde{A}$ (remember that $\tilde{A}$ is of finite type over $\tilde{k}$). Choose elements $f_1, \ldots, f_m \in \overset{\circ}{A}$ such that

$$\tilde{f}_i \notin \tilde{\wp}_i \quad \text{and} \quad \tilde{f}_i \in \tilde{\wp}_j \quad \text{for } j \neq i \quad .$$

In [BGR]7.2.6 Prop. 3 it is shown that the Laurent domain $X(f_i^{-1}) \subseteq X$ has reduction $\mathrm{Spec}(\tilde{A}[\tilde{f}_i^{-1}]) \subseteq \tilde{X}$. But $\mathrm{Spec}(\tilde{A}[\tilde{f}_i^{-1}])$ by construction is irreducible. We have already noted that in this situation (remember that reductions always are reduced) we find an analytic point $x_i : A \to F_i$ such that

$$X(f_i^{-1}) \text{ is a neighbourhood of } x_i \text{ and } |x_i(.)|_{F_i} = \|.\|_{X(f_i^{-1})} \quad .$$

It remains to show that

$$\|.\|_X = \max_{1 \leq i \leq m} \|.\|_{X(f_i^{-1})} \quad .$$

The right hand side for trivial reasons is less than or equal to the left hand side. Since spectral semi-norms are power-multiplicative it therefore suffices to check that for any $f \in A$ with $\|f\|_X = 1$ we find some i such that $\|f\|_{X(f_i^{-1})} = 1$. The element f is contained in $\overset{\circ}{A}$ and does not reduce to $0 \in \tilde{A}$. The ring $\tilde{A}$ being reduced there must be some i such that $\tilde{f} \notin \tilde{\wp}_i$; the latter implies $\|f\|_{X(f_i^{-1})} = 1$.

Remark:

In [Ber]2.4.4 it is shown that there is a unique minimal system of points in $\mathcal{M}(X)$ which has the property in the above Lemma; this system corresponds under the reduction map bijectively to the system of generic points of the irreducible components of the reduction $\tilde{X}$.

Proposition 23:

Let $V = Sp(C) \subseteq U = Sp(B) \subseteq X$ be affinoid subdomains; then the following assertions are equivalent:
i. Any analytic point x of V is inner in U;
ii. $\mathcal{M}(U)$ is a neighbourhood of $\mathcal{M}(V)$ in $\mathcal{M}(X)$;
iii. there is an affinoid generating system $f_1, \ldots, f_r$ of B over A such that

$$V \subseteq \{y \in U : |f_i(y)| < 1 \text{ for } 1 \leq i \leq r\} \quad ;$$

iv. there is an affinoid generating system $f_1, \ldots, f_r$ of B over A and an $\varepsilon \in \sqrt{|k^\times|}, \varepsilon < 1$, such that

$$\|f_i\|_V \leq \varepsilon \text{ for } 1 \leq i \leq r \text{ , i.e., } V \subseteq U(\varepsilon^{-1} f_1, \ldots, \varepsilon^{-1} f_r) \quad .$$

Proof: The equivalence of i. and ii. follows from Lemma 12 i. The equivalence of iii. and iv. follows from the Maximum Modulus Principle ([BGR]6.2.1 Prop. 4). If $x : C \to F$ is an analytic point then the continuity of x implies that

$$|x(f)|_F \leq \|f\|_V \quad \text{for all } f \in C \quad .$$

This shows that i. follows from iv. Finally let us prove that iii. is a consequence of i. From Proposition 7 we know that for any $f \in \overset{\circ}{B}$ and any analytic point $x : A \to F$ of V there exists a monic polynomial $P_{f,x}(T) \in \overset{\circ}{A}[T]$ such that

$$|x(P_{f,x}(f))|_F < 1 \quad .$$

According to Lemma 22 there are finitely many analytic points $x_1, \ldots, x_m$ of V such that

$$\|.\|_V = \max_{1 \leq i \leq m} |x_i(.)|_{F_i} \quad .$$

Defining

$$P_f(T) := \prod_{i=1}^{m} P_{f,x_i}(T) \in \overset{\circ}{A}[T]$$

we obtain

$$\|P_f(f)\|_V < 1 \quad .$$

Knowing this the wanted affinoid generating system of B over A can be constructed in exactly the same way as in the second half of the proof of Proposition 7.

Definition:

Let $V \subseteq U \subseteq X$ be two admissible open subsets. We call V inner in U, resp. U a wide neighbourhood of V, and write $V \subset\subset U$ if any analytic point of X of which V is a neighbourhood is inner in U.

It is clear that the intersection of two wide neighbourhoods of V again is a wide neighbourhood of V.

Remark 24:

Let $V \subseteq X$ be a special subset; the $\overline{U}$ with $U \subseteq X$ running through the special wide neighbourhoods of V form a fundamental system of compact neighbourhoods of $\overline{V}$ in $\mathcal{M}(X)$.

Proof: This follows from Lemma 12 ii and the compactness of $\overline{V}$.

Lemma 25:

For any sheaf $\mathcal{F}$ on X and any special subset $V \subseteq X$ we have

$$\hat{\overline{\mathcal{F}}}(V) = \varinjlim_{\substack{V \subset\subset U \\ U \subseteq X \text{ special}}} \mathcal{F}(U) \quad .$$

Proof: Similar to the proof of Lemma 14.

We see that our conservative sheaves coincide with the constructible sheaves in [Put]. But we want to reserve the latter notion for a use which is analogous to the classical meaning of the word.

Proposition 26:

i. If $\alpha : \mathcal{F} \to \mathcal{F}'$ is a homomorphism between conservative sheaves on X then $\ker(\alpha)$, $\operatorname{im}(\alpha)$, and $\operatorname{coker}(\alpha)$ are conservative, too;
ii. if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of conservative sheaves on X then the sequence $0 \to \overline{\mathcal{F}'} \to \overline{\mathcal{F}} \to \overline{\mathcal{F}''} \to 0$ is exact, too;
iii. a conservative sheaf $\mathcal{F}$ on X is trivial if and only if $\mathcal{F}_x = 0$ for all analytic points of X;
iv. any conservative sheaf $\mathcal{F}$ on X possesses a resolution by conservative and flabby sheaves.

Proof: (Compare [Put]1.4.8, 11, 12; but our proof will avoid any tedious computations.) i. Since the functor $\mathcal{G} \mapsto \hat{\mathcal{G}}$ is exact by Lemma 16 we have $\ker(\alpha) = \ker(\overline{\alpha})\hat{}$ and $\operatorname{coker}(\alpha) = \operatorname{coker}(\overline{\alpha})\hat{}$ which establishes the assertion for kernels and cokernels. Because of $\operatorname{im}(\alpha) = \ker(\mathcal{F}' \to \operatorname{coker}(\alpha))$ it then also follows for the image. ii. The functor $\mathcal{F} \mapsto \mathcal{F}_x$ is exact by the usual argument (built upon Lemma 3 ii). On the other hand, on $\mathcal{M}(X)$ the exactness of sequences of sheaves can be tested stalkwise. Therefore the assertion follows from Lemma 14. iii. If $\mathcal{F}_x = 0$ for all $x \in \mathcal{M}(X)$ we obtain from Lemma 14 that $\overline{\mathcal{F}} = 0$; but then also $\mathcal{F} = \hat{\overline{\mathcal{F}}} = 0$. iv. By assertion i. it suffices to show that $\mathcal{F}$ can be embedded into a conservative and flabby sheaf. Let

$$\mathcal{F} \longrightarrow \mathcal{J}$$

be a monomorphism into an injective sheaf $\mathcal{J}$ on X. Then

$$\mathcal{F} = \hat{\overline{\mathcal{F}}} \longrightarrow \hat{\overline{\mathcal{J}}}$$

is a monomorphism into a conservative sheaf. But according to Corollary 17 and Lemma 19 $\hat{\overline{\mathcal{J}}}$ also is flabby.

Now let

$$\pi : Y = Sp(B) \longrightarrow X = Sp(A)$$

be a morphism of k-affinoid varieties. It comes from a unique continuous homomorphism of affinoid k-algebras $a : A \to B$. The map

$$\begin{array}{rccc} \overline{\pi}: & \mathcal{M}(Y) & \longrightarrow & \mathcal{M}(X) \\ & y & \longmapsto & y \circ a \end{array}$$

is continuous and makes the diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{M}(Y) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\overline{\pi}} \\ X & \longrightarrow & \mathcal{M}(X) \end{array}$$

commutative.

Remark 27:

If $U \subseteq X$ is a special subset then $\pi^{-1}(U)$ is a special subset in Y, and we have

$$\overline{\pi^{-1}(U)} = \overline{\pi}^{-1}(\overline{U}) \quad .$$

Proof: We may assume $U = Sp(C)$ to be an affinoid subdomain. Then $\pi^{-1}(U) = Sp(C \hat{\otimes}_A B)$ is an affinoid subdomain of Y. The inclusion $\overline{\pi^{-1}(U)} \subseteq \overline{\pi}^{-1}(\overline{U})$ is clear for topological reasons. Therefore let $y : B \to F$ be an analytic point of Y such that U is a neighbourhood of $\overline{\pi}(y)$. This means that we have a commutative diagram of continuous homomorphisms

$$\begin{array}{ccc} B & \stackrel{y}{\longrightarrow} & F \\ {\scriptstyle a}\uparrow & & \uparrow \\ A & \longrightarrow & C \end{array} \quad .$$

By the universal property of the completed tensor product (and the completeness of F) this diagram can be extended to

$$\begin{array}{ccc} B & \stackrel{y}{\longrightarrow} & F \\ \| & & \uparrow \\ B & \longrightarrow & B \hat{\otimes}_A C \\ {\scriptstyle a}\uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

saying that $\pi^{-1}(U)$ is a neighbourhood of y.

Lemma 28:

For any sheaf $\mathcal{F}$ on X and any analytic point y of Y we have

$$(\pi^*\mathcal{F})_y = \mathcal{F}_{\overline{\pi}(y)} \quad .$$

Proof: This is the usual computation based upon the fact that forming stalks in arbitrary analytic points commutes with sheafification ([Put]1.2.1).

Proposition 29:

The functor $\mathcal{G} \mapsto \hat{\mathcal{G}}$ commutes with the formation of inverse images and higher direct images of sheaves. The functor $\mathcal{F} \mapsto \overline{\mathcal{F}}$ commutes with the formation of direct images.

Proof: First let $\mathcal{G}$ be a sheaf on $\mathcal{M}(Y)$ and $U \subseteq X$ be a special subset. We have

$$(\overline{\pi}_*\mathcal{G})\hat{}(U) = \varinjlim_{\substack{\overline{U}\subseteq M \\ M\subseteq\mathcal{M}(X)\ open}} (\overline{\pi}_*\mathcal{G})(M) = \varinjlim_{\substack{\overline{U}\subseteq M \\ M\subseteq\mathcal{M}(X)\ open}} \mathcal{G}(\overline{\pi}^{-1}M)$$

and

$$(\pi_*\hat{\mathcal{G}})(U) = \hat{\mathcal{G}}(\pi^{-1}U) = \varinjlim_{\substack{\overline{\pi^{-1}U}\subseteq N \\ N\subseteq\mathcal{M}(Y)\ open}} \mathcal{G}(N) = \varinjlim_{\substack{\overline{\pi}^{-1}\overline{U}\subseteq N \\ N\subseteq\mathcal{M}(Y)\ open}} \mathcal{G}(N) \quad .$$

Since the map $\overline{\pi}$ is closed (as a continuous map between compact spaces) the system of neighbourhoods $\overline{\pi}^{-1}M$ with M an open neighbourhood of $\overline{U}$ is cofinal in the system of all open neighbourhoods of $\overline{\pi}^{-1}\overline{U}$. We therefore obtain

$$(\overline{\pi}_*\mathcal{G})\hat{} = \pi_*\hat{\mathcal{G}} \quad .$$

Since the functor $\mathcal{G} \mapsto \hat{\mathcal{G}}$ is exact and preserves flabby sheaves this implies that

$$(R^i\overline{\pi}_*\mathcal{G})\hat{} = R^i\pi_*\hat{\mathcal{G}}$$

holds true for all $i \geq 0$.
Next let $\mathcal{F}$ be a sheaf on Y. We have

$$(\overline{\pi}_*\overline{\mathcal{F}})(M) = \overline{\mathcal{F}}(\overline{\pi}^{-1}M) = \varprojlim_{\substack{\overline{V}\subseteq\overline{\pi}^{-1}M \\ V\subseteq Y\ special}} \mathcal{F}(V)$$

and

$$(\overline{\pi_*\mathcal{F}})(M) = \varprojlim_{\substack{\overline{U}\subseteq M \\ U\subseteq X\ special}} \pi_*\mathcal{F}(U) = \varprojlim_{\substack{\overline{U}\subseteq M \\ U\subseteq X\ special}} \mathcal{F}(\pi^{-1}U) \quad .$$

In order to see that the two right hand sides are equal we have to find, given $V \subseteq Y$ special with $\overline{V} \subseteq \overline{\pi}^{-1}M$, a special subset $U \subseteq X$ such that $\overline{U} \subseteq M$ and $V \subseteq \pi^{-1}U$. By Lemma 12 ii there is for any analytic point $x \in \overline{\pi}(\overline{V})$ an affinoid wide neighbourhood $U_x \subseteq X$ of x such that $\overline{U_x}$ is a neighbourhood of x in M. The map $\overline{\pi}$ being closed we have $\overline{\pi}(\overline{V}) = \overline{\pi(V)}$ and this set is compact. We therefore obtain

$$\overline{\pi(V)} \subseteq \overline{U}_{x_1} \cup \ldots \cup \overline{U}_{x_m} \subseteq M$$

for finitely many points $x_1, \ldots, x_m$. Then $U := U_{x_1} \cup \ldots \cup U_{x_m}$ has the required properties.
Finally using the adjointness of the two functors (Lemma 16) we compute

$$\begin{aligned}\mathrm{Hom}_Y((\overline{\pi}^*\mathcal{G})\hat{}, \mathcal{F}) &= \mathrm{Hom}_{\mathcal{M}(Y)}(\overline{\pi}^*\mathcal{G}, \overline{\mathcal{F}}) = \mathrm{Hom}_{\mathcal{M}(X)}(\mathcal{G}, \overline{\pi}_*\overline{\mathcal{F}}) \\ &= \mathrm{Hom}_{\mathcal{M}(X)}(\mathcal{G}, \overline{\pi_*\mathcal{F}}) = \mathrm{Hom}_X(\hat{\mathcal{G}}, \pi_*\mathcal{F}) = \mathrm{Hom}_Y(\pi^*\hat{\mathcal{G}}, \mathcal{F})\end{aligned}$$

for any sheaf $\mathcal{G}$ on $\mathcal{M}(X)$ and $\mathcal{F}$ on Y, respectively. This implies

$$(\overline{\pi}^*\mathcal{G})\hat{} = \pi^*\hat{\mathcal{G}} \quad .$$

Corollary 30:

For conservative sheaves $\mathcal{F}$ the functor $\mathcal{F} \mapsto \overline{\mathcal{F}}$ commutes with the formation of inverse images and higher direct images.

Corollary 31:

Inverse images and higher direct images of conservative sheaves are conservative.

For the last result see also [Put]1.4.9 and 1.4.10.

Fibers:

As before we fix an analytic point $x : A \to F$ of X of which we assume in addition that $x(A)$ generates a dense subfield in F (any congruence class of points contains an essentially unique point which has this additional property). Then $B\hat{\otimes}_A F$ is an affinoid F-algebra (see [Put]2.1). We call

$$Y_x := Sp(B\hat{\otimes}_A F)$$

the fiber of π at x. By contemplating the diagram

$$\begin{array}{ccccc} B & \longrightarrow & B\hat{\otimes}_A F & \underset{y}{\longrightarrow} & F' \\ {\scriptstyle a}\uparrow & & \uparrow & & \\ A & \underset{x}{\longrightarrow} & F & & \end{array}$$

where y is an analytic point of Y_x we see that because of our additional assumption on x

$$\mathcal{M}(Y_x) \xrightarrow{\sim} \overline{\pi}^{-1}(x) \subseteq \mathcal{M}(Y)$$
$$y \longmapsto y(.\hat{\otimes}1)$$

is a homeomorphism. Therefore we may define a restriction functor

$$\text{sheaves on } Y \longrightarrow \text{conservative sheaves on } Y_x$$
$$\mathcal{F} \longmapsto \mathcal{F}|Y_x := (\overline{\mathcal{F}}|\overline{\pi}^{-1}(x))^{\wedge} \quad .$$

Actually we will consider this functor only on the subcategory of conservative sheaves $\mathcal{F}$ on Y where it is exact by Lemma 16 and Proposition 26 ii.

Remark 32:

If $x \in X$ is a geometric point then, of course, we have a morphism of k-affinoid varieties $\iota : Y_x \to Y$. In this case it follows from Proposition 29 and Corollary 30 that

$$\mathcal{F}|Y_x = \iota^* \mathcal{F}$$

for conservative sheaves $\mathcal{F}$ on Y.

Theorem 1: (Base change)

For any conservative sheaf $\mathcal{F}$ on Y and any analytic point x of X as above we have

$$(R^i \pi_* \mathcal{F})_x = H^i(Y_x, \mathcal{F}|Y_x) \quad \textit{for } i \geq 0 \quad .$$

Proof: The subsequent identities

$$(R^i \pi_* \mathcal{F})_x = (\overline{R^i_* \mathcal{F}})_x = (R^i \pi_* \overline{\mathcal{F}})_x = H^i(\mathcal{M}(Y_x), \overline{\mathcal{F}}|\mathcal{M}(Y_x)) = H^i(Y_x, \mathcal{F}|Y_x)$$

consecutively numbered are justified as follows: 1) by Lemma 14 and Corollary 31; 2) by Corollary 30; 3) by the topological base change theorem ([God]II.4.17.1); 4) by Proposition 29.

This Theorem (with a different proof) is due to van der Put ([Put]2.3). Our reduction to the topological base change theorem makes the proof very simple and conceptual and avoids the technical steps in [Put]. The next Lemma will show that our restriction functor $\mathcal{F} \mapsto \mathcal{F}|Y_x$ is the same, indeed, as that used in [Put]. Since any analytic point of Y_x in a natural way also is an analytic point of Y it is convenient first to make the following definition.

Definition:

Let V, resp. U, be an admissible open subset of Y_x, resp. Y. We call U a (wide) neighbourhood of V and write $V \leq U (V << U)$ provided for any analytic point y of Y_x the following holds: If V is a neighbourhood of y then U is a (wide) neighbourhood of y viewed as an analytic point of Y.

In case V, resp. U, is a special subset of Y_x, resp. Y, we have $V \leq U$ ($V << U$) if and only if the subset $\overline{U}$ in $\mathcal{M}(Y)$ contains (is a neighbourhood of) the subset $\overline{V}$ in $\mathcal{M}(Y_x) \subseteq \mathcal{M}(Y)$. Similarly as before we then obtain that the $\overline{U}$ with $V << U$ form a fundamental system of compact neighourhoods of $\overline{V}$ in $\mathcal{M}(Y)$.

Lemma 33:

For any conservative sheaf $\mathcal{F}$ on Y and any special $V \subseteq Y_x$ we have

$$(\mathcal{F}|Y_x)(V) = \varinjlim_{\substack{V<<U \\ U\subseteq Y \text{ special}}} \mathcal{F}(U) = \varinjlim_{\substack{V\leq U \\ U\subseteq Y \text{ special}}} \mathcal{F}(U) \quad .$$

Proof: Let $i : \mathcal{M}(Y_x) \to \mathcal{M}(Y)$ be the inclusion map and let i^p denote the inverse image functor for presheaves. If $\mathcal{G}$ is any sheaf on $\mathcal{M}(Y)$ we know from Remark 21 i that $(\mathcal{G}|\mathcal{M}(Y_x))\hat{} = (i^*\mathcal{G})\hat{}$ is the sheafification of $(i^p\mathcal{G})\hat{}$. The latter presheaf is given by

$$\begin{aligned}(i^p\mathcal{G})\hat{}(V) &= \varinjlim_{\substack{\overline{V}\subseteq N \\ N\subseteq \mathcal{M}(Y_x) \text{ open}}} (i^p\mathcal{G})(N) = \varinjlim_{\substack{\overline{V}\subseteq N \\ N\subseteq \mathcal{M}(Y_x) \text{ open}}} \varinjlim_{\substack{N\subseteq M \\ M\subseteq \mathcal{M}(Y) \text{ open}}} \mathcal{G}(M) \\ &= \varinjlim_{\substack{\overline{V}\subseteq M \\ M\subseteq \mathcal{M}(Y) \text{ open}}} \mathcal{G}(M) \quad .\end{aligned}$$

But by the same argument as in the proof of Lemma 15 this already is a sheaf. We obtain

$$(\mathcal{G}|\mathcal{M}(Y_x))\hat{}(V) = \varinjlim_{\substack{\overline{V}\subseteq M \\ M\subseteq \mathcal{M}(Y) \text{ open}}} \mathcal{G}(M)$$

and in particular

$$(\mathcal{F}|Y_x)(V) = \varinjlim_{\substack{\overline{V}\subseteq M \\ M\subseteq \mathcal{M}(Y) \text{ open}}} \overline{\mathcal{F}}(M) = \varinjlim_{\substack{\overline{V}\subseteq M \\ M\subseteq \mathcal{M}(Y) \text{ open}}} \varprojlim_{\substack{\overline{U}\subseteq M \\ U\subseteq Y \text{ special}}} \mathcal{F}(U) \quad .$$

Exactly in the same way as in the proofs of Lemma 14 and Lemma 25 one shows that the right hand side is equal to

$$\varinjlim_{\substack{V<<U \\ U\subseteq Y \text{ special}}} \mathcal{F}(U) \quad .$$

The second identity in the assertion is an immediate consequence, by Lemma 25, of the assumption that $\mathcal{F}$ is conservative.

Corollary 34:

The functor $\mathcal{F} \mapsto \mathcal{F}|Y_x$ preserves flabby conservative sheaves.

Remark 35:

The map

$$\begin{aligned} \textit{special subsets in } Y &\longrightarrow \textit{special subsets in } Y_x \\ U &\longmapsto \overline{U} \cap Y_x \end{aligned}$$

is well-defined and surjective; furthermore, one has $(\overline{\overline{U} \cap Y_x}) = \overline{U} \cap \mathcal{M}(Y_x)$.

Proof: [Ber]2.2.3 (i) and [Put]2.4 and 2.6.

§2 Analytic points of analytic varieties

The notion of an analytic point generalizes in an obvious way to arbitrary analytic varieties. Let X be a k-analytic variety.
A pointed subdomain of X is a pair (U, x) consisting of an affinoid open subset $U \subseteq X$ and an analytic point x of U; if x is of the form $x : \mathcal{O}_X(U) \to F$ we say that (U, x) has values in the complete extension field F of k.
Let (U, x) and (U', x') be two pointed subdomains of X with values in the same complete extension field F of k. Then (U, x) and (U', x') are called equivalent if there exists a third pointed subdomain (V, y) of X with values in F such that

$$V \subseteq U \cap U' \quad \text{and}$$

$$\begin{array}{ccccc} \mathcal{O}_X(U) & \longrightarrow & \mathcal{O}_X(V) & \longleftarrow & \mathcal{O}_X(U') \\ & {\scriptstyle x} \searrow & \downarrow {\scriptstyle y} & \swarrow {\scriptstyle x'} & \\ & & F & & \end{array} \quad \text{is commutative.}$$

It is easy to see that this is actually an equivalence relation: If

$$\begin{array}{ccccc} \mathcal{O}_X(U') & \longrightarrow & \mathcal{O}_X(V') & \longleftarrow & \mathcal{O}_X(U'') \\ & {\scriptstyle x'} \searrow & \downarrow {\scriptstyle y'} & \swarrow {\scriptstyle x''} & \\ & & F & & \end{array} \quad \text{with } V' \subseteq U' \cap U''$$

is another such commutative diagram then $V \cap V'$ is affinoid open in X, too, and the diagram

$$\begin{array}{ccccc} \mathcal{O}_X(U) & \longrightarrow & \mathcal{O}_X(V \cap V') & \longleftarrow & \mathcal{O}_X(U'') \\ & {\scriptstyle x} \searrow & \downarrow {\scriptstyle y \hat{\otimes} y'} & \swarrow {\scriptstyle x''} & \\ & & F & & \end{array}$$

is commutative showing that (U, x) and (U'', x'') are equivalent.

Definition:

An analytic point of X is an equivalence class of pointed subdomains (U,x) of X with respect to the above defined equivalence relation.

By abuse of notation we denote the analytic point given by the equivalence class of (U,x) simply by x and call U an (open) affinoid neighbourhood of x; we say that x has values in F if (U,x) does. An admissible open subset in X is called a neighbourhood of x if it contains an affinoid neighbourhood of x.
It is clear again that any *geometric* point $x \in X$ determines a unique analytic point of X. In the case that X is affinoid we obtain our previous notion of an analytic point: The equivalence class representing the analytic point x then contains a unique pointed subdomain of the form (X,x).
Let x be a fixed analytic point of X.

Lemma 1:

i. The intersection of two neighbourhoods of x is a neighbourhood of x;
ii. if $U = \bigcup_{i\in I} U_i$ is an admissible covering of the neighbourhood U of x then U_i, for some $i \in I$, is a neighbourhood of x;
iii. $\bigcap\{U \subseteq X : U \text{ neighbourhood of } x\} = \begin{cases} \{x\} & \text{if } x \in X \text{ is geometric }, \\ \emptyset & \text{otherwise }. \end{cases}$

Proof: This is either obvious or follows from the corresponding results in the affinoid case.

For any sheaf $\mathcal{F}$ on X we again define its stalk at x by

$$\mathcal{F}_x = \varinjlim_{\substack{U \text{ neighbour-} \\ \text{hood of } x}} \mathcal{F}(U) = \varinjlim_{\substack{U \text{ affinoid} \\ \text{neighbourhood of } x}} \mathcal{F}(U) \quad .$$

Because of Lemma 1 the functor $\mathcal{F} \mapsto \mathcal{F}_x$ is exact. The stalk $\mathcal{O}_{X,x}$ of the structure sheaf $\mathcal{O}_X$ always is a local ring.

It is tempting to try now to define for our general X the space $\mathcal{M}(X)$. But one soon realizes that it seems hard to produce such an object having satisfying properties. Berkovich who pursues this point of view in [Ber] ends up by defining a different category of analytic spaces. Our further investigations will be built instead upon the observation that the crucial notion of a conservative sheaf generalizes nicely. The reason for this is, as we will see presently, that this notion is of a local nature.

Lemma 2:

Let Y be a k-affinoid variety and let $Y = Y_1 \cup \ldots \cup Y_m$ be a covering by affinoid subdomains Y_i. A sheaf $\mathcal{F}$ on Y is conservative if and only if all the restrictions $\mathcal{F}|Y_i$ for $1 \leq i \leq m$ are conservative.

Proof: It follows from §1 Corollary 31 that with $\mathcal{F}$ all the $\mathcal{F}|Y_i$ are conservative. Let us assume on the other hand that all the $\mathcal{F}|Y_i$ are conservative. We have the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \prod_{i=1}^{m} \varphi_{i*}(\mathcal{F}|Y_i) \rightrightarrows \prod_{i,j=1}^{m} \varphi_{ij*}(\mathcal{F}|Y_i \cap Y_j)$$

of sheaves on Y where $\varphi_i : Y_i \to Y$ and $\varphi_{ij} : Y_i \cap Y_j \to Y$ denote the inclusion maps. Again it follows from §1 Corollary 31 (and from the obvious fact that finite direct sums of conservative sheaves are conservative) that the middle and the right hand term in this sequence are conservative. But then $\mathcal{F}$ is conservative, too, by §1 Proposition 26 i.

Definition:

A sheaf $\mathcal{F}$ on X is called conservative if there is an admissible covering $X = \bigcup_{i \in I} X_i$ by open affinoid subvarieties X_i such that $\mathcal{F}|X_i$ is conservative for all $i \in I$.

The following properties are quite immediate from §1 Proposition 26 and Corollary 31.

Proposition 3:

i. Finite direct sums of conservative sheaves on X are conservative;
ii. if $\alpha : \mathcal{F} \to \mathcal{F}'$ is a homomorphism between conservative sheaves on X then $\ker(\alpha)$, $\mathrm{im}(\alpha)$, *and* $\mathrm{coker}(\alpha)$ *are conservative;*
iii. a sequence $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ of conservative sheaves on X is exact if and only if the stalk sequences $\mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x$ are exact for all analytic points x of X;
iv. if $\pi : Y \to X$ is a morphism of k-analytic varieties then the inverse image $\pi^*\mathcal{F}$ of every conservative sheaf on X is conservative.

We recall that
— X is quasi-compact if it has a finite admissible covering by open affinoid subvarieties,
— X is quasi-separated if the intersection of any two open affinoid subvarieties in X is quasi-compact, and

— a morphism $\pi : Y \to X$ of k-analytic varieties is quasi-compact, resp. quasi-separated, if $\pi^{-1}(U)$ is quasi-compact, resp. quasi-separated, for any open affinoid subvariety U in X.
Any separated morphism is quasi-separated.

Lemma 4:

If $\pi : Y \to X$ is a quasi-compact and quasi-separated morphism of k-analytic varieties then the direct image $\pi_*\mathcal{F}$ of every conservative sheaf $\mathcal{F}$ on Y is conservative.

Proof: Since the assertion is local in X we may assume X to be affinoid. Then we find finite admissible coverings

$$Y = \bigcup_i Y_i \quad \text{and} \quad Y_i \cap Y_j = \bigcup_\ell Y_{ij\ell}$$

by open affinoid subvarieties. Put $\pi_i := \pi|Y_i$ and $\pi_{ij\ell} := \pi|Y_{ij\ell}$ and consider the exact sequence

$$0 \longrightarrow \pi_*\mathcal{F} \longrightarrow \prod_i \pi_{i*}(\mathcal{F}|Y_i) \rightrightarrows \prod_{i,j,\ell} \pi_{ij\ell*}(\mathcal{F}|Y_{ij\ell})$$

of sheaves on X. The middle and the right hand term are conservative by Proposition 3 iv and §1 Corollary 31. Therefore $\pi_*\mathcal{F}$ is conservative by Proposition 3 ii.

We call special subsets in X those admissible open subsets which are quasi-compact.

Remark 5:

If X is quasi-separated we have:
i. The intersection of two special subsets in X is a special subset;
ii. the union $U = U_1 \cup \ldots \cup U_m$ of finitely many special subsets $U_i \subseteq X$ is a special subset and $\{U_i\}_{1 \leq i \leq m}$ is an admissible covering of U.

Proof: Exercise.

If X is quasi-compact and quasi-separated we sometimes consider the "special" G-topology on X which is given by the special subsets and the finite coverings. The analytic G-topology on X is slightly finer than the special one so that both give the same category of sheaves on X. In particular, flabbiness of a sheaf on X always is meant with respect to the special G-topology.

Lemma 6:

If X is quasi-compact and quasi-separated then any conservative sheaf $\mathcal{F}$ on X possesses a resolution by conservative and flabby sheaves.

Proof: By Proposition 3 ii it suffices to show that $\mathcal{F}$ can be embedded into a conservative and flabby sheaf. Let $X = \bigcup_i X_i$ be a finite covering by open affinoid subvarieties. By §1 Proposition 26 iv we find an embedding

$$\mathcal{F}|X_i \longrightarrow \mathcal{J}_i$$

into a conservative and flabby sheaf $\mathcal{J}_i$ on X_i. If $\varphi_i : X_i \to X$ denotes the inclusion map we have the monomorphism

$$\mathcal{F} \longrightarrow \prod_i \varphi_{i*}(\mathcal{F}|X_i) \longrightarrow \prod_i \varphi_{i*}\mathcal{J}_i \quad .$$

Obviously the right hand term is flabby. But by Lemma 4 it is also conservative since φ_i is quasi-compact and quasi-separated.

Proposition 7:

If $\pi : Y \to X$ is a quasi-compact and quasi-separated morphism of k-analytic varieties then the higher direct images $\mathcal{R}^i\pi_*\mathcal{F}$, for $i \geq 0$, of every conservative sheaf $\mathcal{F}$ on Y are conservative.

Proof: We may assume X to be affinoid. According to Lemma 6 we then find a conservative and flabby resolution $\mathcal{F} \xrightarrow{\sim} \mathcal{J}^\cdot$. Since $\mathcal{R}^i\pi_*\mathcal{F} = h^i(\pi_*\mathcal{J}^\cdot)$ the assertion follows from Lemma 4 and Proposition 3 ii (observe that a quasi-compact morphism is continuous with respect to the special G-topologies).

Fibers:

We fix a quasi-separated morphism $\pi : Y \to X$ of k-analytic varieties and we fix an analytic point x of X with values in F.

Remark 8:

The following assertions are equivalent:
i. For any affinoid neighbourhood $U \subseteq X$ of x the subfield generated by $x(\mathcal{O}_X(U))$ is dense in F;
ii. the residue class field $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ is dense in F.

Proof: Let $U = Sp(A)$ be an affinoid neighbourhood of x. If the prime ideal $\wp_x \subseteq A$ denotes the kernel of $x : A \to F$ we obviously have the commutative diagram

$$\begin{array}{ccc} A/\wp_x & \xrightarrow{x} & F \\ & \searrow \quad \nearrow & \\ & \mathcal{O}_{X,x}/\mathfrak{m}_x = k(x) & \end{array} .$$

This shows that i. implies ii. On the other hand since any affinoid subdomain of U is a finite union of rational subdomains it follows from §1 Lemma 3 ii that

$$\mathcal{O}_{X,x} = \varinjlim_{\substack{V \subseteq U \\ rational\ sub- \\ domain}} \mathcal{O}_X(V) \quad .$$

Now assume ii. to hold true and fix an element $b \in F$. For any $\varepsilon > 0$ we find a rational subdomain $U\left(\frac{f_1}{g}, \ldots, \frac{f_n}{g}\right)$ in U (which is a neighbourhood of x) and an element $\tilde{f} \in A\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \rangle$ such that $|x(\tilde{f}) - b|_F \leq \varepsilon$. But the localization A_g is dense in $A\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \rangle$ with respect to the Banach topology on the latter ([BGR]6.1.4). Therefore there is a $m \in \mathbb{N}$ and an element $f \in A$ such that

$$|x\left(\frac{f}{g^m}\right) - x(\tilde{f})|_F \leq \varepsilon \quad , \quad \text{resp.} \quad |\frac{x(f)}{x(g)^m} - b|_F \leq \varepsilon \quad .$$

This shows i.

Definition:

A point x is called quasi-geometric if it fulfills the equivalent conditions in the previous Remark.

We assume that our fixed point x is quasi-geometric. Using the field extension functor $.\hat{\otimes}_k F$ for quasi-separated k-analytic varieties constructed in [BGR]9.3.6 we can define the fiber Y_x of π at x as follows:
Let $U \subseteq X$ be an affinoid neighbourhood of x. Then $Y_U := \pi^{-1}U$ is quasi-separated and we have an obvious morphism of F affinoid varieties $Sp(F) \to U\hat{\otimes}_k F$. We put

$$Y_x := (Y_U \hat{\otimes}_k F) \underset{(U\hat{\otimes}_k F)}{\times} Sp(F) \quad .$$

In more explicit terms this definition amounts to the following. Let

$$Y_U = \bigcup_i V_i$$

be an admissible affinoid covering and let

$$V_i \cap V_j = \bigcup_{\ell} V_{ij\ell}$$

be finite affinoid coverings. For the maps $V_i \to U$ and $V_{ij\ell} \to U$ the fibers $(V_i)_x$ and $(V_{ij\ell})_x$ at x were defined in §1. We obtain Y_x by pasting the $(V_i)_x$ along the $\bigcup_{\ell}(V_{ij\ell})_x$. It is not difficult to derive the subsequent properties of Y_x from this description:
— Y_x does not depend on the choice of the neighbourhood U;
— Y_x is a quasi-separated F-analytic variety;
— if π is quasi-compact then Y_x is quasi-compact;
— the map

$$\text{special subsets in } Y_U \longrightarrow \text{special subsets in } Y_x$$
$$V \longmapsto V_x$$

is well-defined, compatible with finite unions and intersections, and surjective (for the latter use §1 Remark 35).
Also:
— If $x \in X$ is geometric then $Y_x = Y \underset{X}{\times} Sp(k(x))$.

Remark 9:

If $V \to W$ is a morphism of k-affinoid varieties, $U \subseteq V$ is an affinoid subdomain, w is a quasi-geometric point of W, and $\mathcal{F}$ is a conservative sheaf on V then we have

$$(\mathcal{F}|U)|U_w = (\mathcal{F}|V_w)|U_w \quad .$$

Proof: §1 Proposition 29 and Corollary 30.

Let $\mathcal{F}$ be a conservative sheaf on Y. It follows from Remark 7 that the sheaves $(\mathcal{F}|V_i)|(V_i)_x$ can be pasted together to a conservative sheaf $\mathcal{F}|Y_x$ on Y_x. In this way we obtain a natural functor

$$\text{conservative sheaves on } Y \longrightarrow \text{conservative sheaves on } Y_x$$
$$\mathcal{F} \longmapsto \mathcal{F}|Y_x$$

which is exact according to §1 and which fulfills:
— If $V \subseteq Y_U$ is a special subset then $(\mathcal{F}|V)|V_x = (\mathcal{F}|Y_x)|V_x$;
— if $x \in X$ is geometric with corresponding closed immersion $\iota : Y_x \to Y$ then

$$\mathcal{F}|Y_x = \iota^* \mathcal{F} \quad .$$

Lemma 10:

If Y is quasi-compact and quasi-separated then the functor $\mathcal{F} \mapsto \mathcal{F}|Y_x$ preserves flabby sheaves.

Proof: (The stronger assumption that Y itself instead of π is quasi-compact and quasi-separated only has to be made since we have chosen such a simple minded approach to flabby sheaves.) It is easily shown that flabbiness with respect to the special G-topologies is a local notion. Therefore the assertion follows from §1 Corollary 34.

Theorem 2: (Base change)

Let $\pi : Y \to X$ be a quasi-compact and quasi-separated morphism of k-analytic varieties, let x be a quasi-geometric point of X, and let $\mathcal{F}$ be a conservative sheaf on Y. Then we have

$$(\mathcal{R}^i \pi_* \mathcal{F})_x = H^i(Y_x, \mathcal{F}|Y_x) \quad \textit{for } i \geq 0 \quad .$$

Proof: Since the assertion is local in X we may assume X to be affinoid so that Y is quasi-compact and quasi-separated. Because of Lemma 6 and Lemma 10 the case of arbitrary $i \geq 0$ then follows from the case $i = 0$ applied to a conservative and flabby resolution of $\mathcal{F}$. In order to settle the case $i = 0$ we fix finite affinoid coverings

$$Y = \bigcup_i V_i \quad \text{and} \quad V_i \cap V_j = \bigcup_\ell V_{ij\ell} \quad .$$

Setting $\pi_i := \pi|V_i$ and $\pi_{ij\ell} := \pi|V_{ij\ell}$ we have the exact sequence of stalks

$$0 \longrightarrow (\pi_* \mathcal{F})_x \longrightarrow \prod_i (\pi_{i*}(\mathcal{F}|V_i))_x \rightrightarrows \prod_{i,j,\ell} (\pi_{ij\ell *}(\mathcal{F}|V_{ij\ell}))_x \quad .$$

On the other hand the sheaf property of $\mathcal{F}|Y_x$ gives the exact sequence

$$\begin{array}{cccc} 0 \to H^0(Y_x, \mathcal{F}|Y_x) \to & \prod_i H^0((V_i)_x, \mathcal{F}|Y_x) & \rightrightarrows & \prod_{i,j,\ell} H^0((V_{ij\ell})_x, \mathcal{F}|Y_x) \\ & \| & & \| \\ & \prod_i H^0((V_i)_x, (\mathcal{F}|V_i)|(V_i)_x) & \rightrightarrows & \prod_{i,j,\ell} H^0((V_{ij\ell})_x, (\mathcal{F}|V_{ij\ell})|(V_{ij\ell})_x). \end{array}$$

Theorem 1 in §1 says that the middle and right hand terms in these two sequences are naturally isomorphic. Therefore we obtain

$$(\pi_* \mathcal{F})_x = H^0(Y_x, \mathcal{F}|Y_x) \quad .$$

§3 Another look at conservative sheaves

We have defined conservative sheaves on a k-affinoid variety $X = Sp(A)$ by using the compact space $\mathcal{M}(X)$. The introduction of this space was very useful since through it certain results can be proved by reduction to purely topological facts. Nevertheless we will show in this Paragraph that those sheaves also have a conceptual characterization in intrinsically analytic terms. It is this second description which we hope to generalize in a later paper to the etale topological setting.

Let $X = Sp(A)$ be a k-affinoid variety. For any admissible open subset $\Omega \subseteq X$ we put

$$\mathcal{M}(\Omega) := \{x \in \mathcal{M}(X) : \Omega \text{ is a neighbourhood of } x\} \quad .$$

In this way we obtain an injective map

$$\begin{array}{ccc} \text{admissible open} & \longrightarrow & \text{subsets} \\ \text{subsets in } X & & \text{in } \mathcal{M}(X) \\ \Omega & \longmapsto & \mathcal{M}(\Omega) \end{array}$$

which preserves inclusions and finite intersections and which fulfills

$$\mathcal{M}(\Omega) \cap X = \Omega \quad .$$

We have already explored the fact that $\mathcal{M}(U)$, for a special subset $U \subseteq X$, is equal to the closure $\overline{U}$ of U in $\mathcal{M}(X)$.

Definition:

An admissible open subset $\Omega \subseteq X$ is called wide open (in X) if Ω is a wide neighbourhood of any of its analytic points $x \in \mathcal{M}(\Omega)$.

Lemma 1:

Let $\Omega \subseteq X$ be an admissible open subset; then Ω is wide open in X if and only if $\mathcal{M}(\Omega)$ is open in $\mathcal{M}(X)$.

Proof: This follows from §1 Lemma 12.

Lemma 2:

Let $\pi : Sp(B) \to Sp(A)$ be a morphism of k-affinoid varieties; if Ω is wide open in $Sp(A)$ then $\pi^{-1}(\Omega)$ is wide open in $Sp(B)$.

Proof: Since with Ω also $\pi^{-1}(\Omega)$ is admissible open it suffices to show that, for any admissible open subset $\Omega \subseteq \mathcal{M}(X)$, we have

$$\overline{\pi}^{-1}(\mathcal{M}(\Omega)) = \mathcal{M}(\pi^{-1}(\Omega))$$

where $\overline{\pi} : \mathcal{M}(Sp(B)) \to \mathcal{M}(X)$ denotes, as before, the induced map. Let x be an analytic point of $Sp(B)$ and let $\Omega = \bigcup_{i \in I} U_i$ be an admissible covering by affinoid open subsets. Using §2 Lemma 1 we conclude that Ω is a neighbourhood of $\overline{\pi}(x)$ if and only if some U_i is a neighbourhood of $\overline{\pi}(x)$ if and only if some $\pi^{-1}(U_i)$ is a neighbourhood of x if and only if $\pi^{-1}(\Omega)$ is a neighbourhood of x.

Proposition 3:

i. Let $Sp(B) \subseteq X$ be an affinoid subdomain and let $f_1, \ldots, f_r \in B$ be an affinoid generating system of B over A; then

$$\Omega := \{y \in Sp(B) : |f_i(y)| < 1 \quad \textit{for } 1 \leq i \leq r\}$$

is wide open in X;
ii. let $U \subseteq X$ be a special subset; then $\Omega := X \backslash U$ is wide open in X and $\mathcal{M}(\Omega) = \mathcal{M}(X) \backslash \overline{U}$;
iii. any Zariski open subset is wide open in X.

Proof: i. By [BGR]9.1.4 Prop. 5 the subset Ω is admissible open in $Sp(B)$ and therefore in X. That Ω even is wide open in X is a consequence of §1 Proposition 23.
ii. The following argument due to Fresnel shows that Ω is admissible open: Since any special subset is a finite union of rational subdomains we may assume that U is of the form $U = X\left(\frac{f_1}{g}, \ldots, \frac{f_r}{g}\right)$ with elements $g, f_1, \ldots, f_r \in A$ generating the unit ideal. Since g is invertible on U the Maximum Modulus Principle implies the existence of an $\varepsilon \in |k^\times|$ such that $|g(x)| > \varepsilon$ for all $x \in U$. We put

$$X_+ := X(\varepsilon g^{-1}) \quad \text{and} \quad X_- := X(\varepsilon^{-1} g)$$

so that $X = X_+ \cup X_-$ is an admissible covering. Since g is invertible even on X_+ we may furthermore consider the subsets

$$X_+^i := \{x \in X_+ : |\frac{f_i(x)}{g(x)}| > 1\} \quad \text{for } 1 \leq i \leq r \quad .$$

According to [BGR]9.1.4 Prop. 5 the subset $\bigcup\limits_{1\le i\le r} X_+^i$ is admissible open in X_+ and consequently in X. Therefore both

$$\Omega \cap X_- = X_- \quad \text{and} \quad \Omega \cap X_+ = \bigcup_{1\le i\le r} X_+^i$$

are admissible open which means that Ω itself is admissible open.
Because of Lemma 1 it remains to prove the equality $\mathcal{M}(\Omega) = \mathcal{M}(X)\backslash\overline{U}$. Clearly we have $\overline{U} \cap \mathcal{M}(\Omega) = \emptyset$. Therefore let x be an analytic point of X of which U is not a neighbourhood. Since x then is contained in the open set $\mathcal{M}(X)\backslash\overline{U}$ it follows from §1 Lemma 12 that x has an affinoid (wide) neighbourhood $V \subseteq X$ such that $\overline{V} \subseteq \mathcal{M}(X)\backslash\overline{U}$. This implies $\overline{U \cap V} = \overline{U} \cap \overline{V} = \emptyset$ so that $V \subseteq \Omega$ and $x \in \mathcal{M}(\Omega)$.
iii. Let $\Omega \subseteq X$ be Zariski open. This means that $\Omega = X\backslash Sp(A/\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq A$. It is proved in [BGR]9.1.4 Cor. 7 that Ω is admissible open in X. In the same way as above one shows that $\mathcal{M}(\Omega) = \mathcal{M}(X)\backslash\mathcal{M}(Sp(A/\mathfrak{a}))$ holds true. This together with Lemma 1 implies that Ω is wide open in X.

Proposition 4:

Let $C \subseteq \mathcal{M}(X)$ be a compact subset; the subsets $\mathcal{M}(\Omega)$, for Ω wide open in X, which contain C form a fundamental system of open neighbourhoods of C in $\mathcal{M}(X)$.

Proof: From §1 Lemma 12 and a compactness argument we know that the subsets $\overline{U}$ with $U \subseteq X$ running through the special subsets which are wide neighbourhoods of every analytic point in C form a fundamental system of (compact) neighbourhoods of C in $\mathcal{M}(X)$. Therefore fix such a special subset $U \subseteq X$. By the very definition of the topology on $\mathcal{M}(X)$ we find finitely many elements $f_{i\ell}, g_{j\ell} \in A$ and $\varepsilon_{i\ell}, \delta_{j\ell} \in \sqrt{|k^\times|}$ such that

$$\begin{aligned} C &\subseteq \bigcup_{\ell} \{A \xrightarrow{y} F \text{ in } \mathcal{M}(X) : |y(f_{i\ell})|_F < \varepsilon_{i\ell} \text{ and } |y(g_{j\ell})|_F > \delta_{j\ell} \text{ for all } i,j\} \\ &\subseteq \overline{U} \quad . \end{aligned}$$

Choose numbers $\varepsilon'_{i\ell}, \varepsilon''_{i\ell}, \delta'_{j\ell}, \delta''_{j\ell} \in \sqrt{|k^\times|}$ such that

$$\begin{aligned} &\max_{y\in C} |y(f_{i\ell})|_F \le \varepsilon''_{i\ell} < \varepsilon'_{i\ell} < \varepsilon_{i\ell} \quad \text{and} \\ &\min_{y\in C} |y(g_{j\ell})|_F \ge \delta''_{j\ell} > \delta'_{j\ell} > \delta_{j\ell} \end{aligned}$$

and put

$$\Omega := \bigcup_\ell \{x \in X : |f_{i\ell}(x)| < \varepsilon'_{i\ell} \text{ and } |g_{j\ell}(x)| > \delta'_{j\ell} \text{ for all } i, j\} \ ,$$

$$V := \bigcup_\ell X(\varepsilon''^{-1}_{i\ell} f_{i\ell}, \ldots, \delta''_{j\ell} g^{-1}_{j\ell}, \ldots) \ \text{, and}$$

$$W := \bigcup_\ell X(\varepsilon'^{-1}_{i\ell} f_{i\ell}, \ldots, \delta'_{j\ell} g^{-1}_{j\ell}, \ldots) \ .$$

Obviously V and W are special subsets in X and Ω being the complement of a special subset is wide open in X by Proposition 3 ii. We furthermore have

$$C \subseteq \overline{V} \subseteq \mathcal{M}(\Omega) \subseteq \overline{W} \subseteq \overline{U}$$

which in particular means $\Omega \subseteq U$.

Proposition 5:

i. The intersection of two wide open subsets in X is wide open in X;
ii. let $\Omega = \bigcup_{i \in I} \Omega_i$ be a covering of an admissible open subset $\Omega \subseteq X$ by wide open subsets $\Omega_i \subseteq X$; this covering is admissible if and only if $\mathcal{M}(\Omega) = \bigcup_{i \in I} \mathcal{M}(\Omega_i)$; furthermore, if the covering is admissible then Ω is wide open in X;
iii. the wide open subsets form a basis for the canonical topology on X;
iv. the subsets $\mathcal{M}(\Omega)$ with Ω running through the wide open subsets in X form a basis for the topology on $\mathcal{M}(X)$.

Proof: i. This is clear. ii. First assume that the covering is admissible. It then follows from §2 Lemma 1 that $\mathcal{M}(\Omega) = \bigcup_{i \in I} \mathcal{M}(\Omega_i)$ holds and this in turn implies by the above Lemma 1 that Ω is wide open in X. Now let us assume that we have $\mathcal{M}(\Omega) = \bigcup_{i \in I} \mathcal{M}(\Omega_i)$. For the admissibility of the covering it suffices, according to [BGR]9.1.4 Prop. 2, to prove that for any morphism $\pi : Sp(B) \rightarrow Sp(A)$ of k-affinoid varieties such that $\pi(Sp(B)) \subseteq \Omega$ the covering $Sp(B) = \bigcup_{i \in I} \pi^{-1}(\Omega_i)$ can be refined into a finite covering by affinoid open subsets. By Lemma 2 and its proof we may assume that $\pi = \text{id}$ and $\Omega = Sp(A)$. Given $y \in \mathcal{M}(X)$ we then find an $i(y) \in I$ such that $y \in \mathcal{M}(\Omega_{i(y)})$. Since $\Omega_{i(y)}$ is wide open we furthermore find an affinoid wide neighbourhood U_y of y which is contained in $\Omega_{i(y)}$. Then $\overline{U}_y$ is a neighbourhood of y in $\mathcal{M}(X)$. But $\mathcal{M}(X)$ is compact so that we obtain

$$\mathcal{M}(X) = \overline{U}_{y_1} \cup \ldots \cup \overline{U}_{y_m}$$

for appropriately chosen analytic points $y_1, \ldots, y_m$. Obviously $\{U_{y_1}, \ldots, U_{y_m}\}$ is the required refinement. iii. and iv. follow from Proposition 4.

We see that the wide open subsets in X together with the admissible coverings form a G-topology on X which we call the wide open G-topology and which is coarser than the analytic G-topology. The identity map induces a continuous morphism

$$w : \text{ analytic } G\text{-topology on } X \longrightarrow \text{wide open } G\text{-topology on } X \quad .$$

Proposition 5 also implies that the functor

$$\begin{array}{ccl} \text{sheaves on} & \xrightarrow{\sim} & \text{sheaves for the} \\ \mathcal{M}(X) & & \text{wide open } G\text{-topology on } X \\ \mathcal{G} & \longmapsto & \tilde{\mathcal{G}}(\Omega) := \mathcal{G}(\mathcal{M}(\Omega)) \end{array}$$

is an equivalence of categories. Using Proposition 4 one easily derives the following connection to the functors considered in §1:

$$\begin{array}{rcll} \tilde{\tilde{\mathcal{F}}} & = & w_*\mathcal{F} & \text{for any sheaf } \mathcal{F} \text{ on } X, \text{ and} \\ \hat{\mathcal{G}} & = & w^*\tilde{\mathcal{G}} & \text{for any sheaf } \mathcal{G} \text{ on } \mathcal{M}(X). \end{array}$$

This makes it possible to restate all the results from §1 solely in terms of the functors w_* and w^*. We only list the most basic ones:
A sheaf $\mathcal{F}$ on X is conservative if and only if the adjunction map

$$w^*w_*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$$

is an isomorphism. The functor w^* is exact and induces an equivalence of categories

$$\begin{array}{ccc} \text{sheaves for the} & \xrightarrow{\sim} & \text{conservative} \\ \text{wide open } G\text{-topology on } X & & \text{sheaves on } X \\ \mathcal{H} & \longmapsto & w^*\mathcal{H} \end{array} \quad .$$

We have

$$(w^*\mathcal{H})| \text{ special } G\text{-topology} = (w^p\mathcal{H})| \text{ special } G\text{-topology}$$

where w^p denotes the inverse image functor for presheaves; this means that for any special subset $U \subseteq X$ we have

$$(w^*\mathcal{H})(U) = \varinjlim_{\substack{U \subseteq \Omega \subseteq X \\ \Omega \text{ wide open}}} \mathcal{H}(\Omega) \quad .$$

This latter fact is the reason that w^* transforms injective sheaves into flabby sheaves (recall that flabbiness always is meant w.r.t. the special G-topology).

For any analytic point x of X we may define the stalk at x of a sheaf $\mathcal{H}$ for the wide open G-topology by

$$\mathcal{H}_x := \varinjlim_{\substack{\Omega \subseteq X \\ \text{wide open} \\ \text{neighbourhood of } x}} \mathcal{H}(\Omega) \quad .$$

These stalk functors have the usual properties: They are exact and form a conservative family, i.e., they detect the exactness of a sequence of sheaves for the wide open G-topology (the reason for the latter is that, by Proposition 5 ii, a covering $X = \bigcup_{i \in I} \Omega_i$ by wide open subsets Ω_i is admissible if every analytic point of X has a neighbourhood among the Ω_i). We have

$$(w^*\mathcal{H})_x = \mathcal{H}_x, (w_*\mathcal{F})_x = \mathcal{F}_x^{\text{mod}} \quad , \quad \text{and in particular}$$
$$(w_*\mathcal{F})_x = \mathcal{F}_x \quad \text{for conservative } \mathcal{F} \quad .$$

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Peter Schneider
Mathematisches Institut
der Universität zu Köln
Weyertal 86-90
5000 Köln 41
Germany