AN INTEGRAL TRANSFORM FOR P–ADIC SYMMETRIC SPACES

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Introduction

One of the important tools in the study of the arithmetic of Shimura curves and other modular curves over a local field K is the theory of p–adic uniformization. Applying earlier work of Ihara and Mumford's general theory of uniformization of certain types of curves over K ([Mu]), Cerednik ([C]) showed that one could obtain certain Shimura curves as the quotient of a certain rigid analytic space, the "p–adic upper half plane," by appropriate cocompact arithmetic subgroups of $\mathrm{GL}_2(K)$. The p–adic upper half plane of this theory is the rigid analytic space over K obtained by deleting the K–rational points from the projective line over K.

Drinfeld ([D]) reconsidered Cerednik's results, working with the more general p–adic symmetric spaces $\Omega^{(d+1)}$ obtained by deleting all K–rational hyperplanes from projective d–space over K. He showed that $\Omega^{(d+1)}$ is a moduli space for a certain type of formal group, and constructed a tower of etale coverings of $\Omega^{(d+1)}$; these coverings are the topic of considerable interest in part because of Drinfeld's hope that their cohomology realizes all of the discrete series representations of $\mathrm{GL}_{d+1}(K)$.

In 1991, ([SS]) studied the cohomology of the spaces $\Omega^{(d+1)}$ in any cohomology theory which satisfied certain natural axioms. In particular, if the field K has characteristic zero they determined the De Rham cohomology of the spaces $\Omega^{(d+1)}$ and interpreted the results in representation–theoretic terms.

Applied to the one–dimensional p–adic upper half plane, the results of [SS] give an abstract isomorphism

$$\mathrm{H}^1_{DR}(\Omega^{(2)},K)\to \mathrm{Hom}(C^\infty(\mathbb{P}^1(K),\mathbb{Z})/\mathrm{constants},K),$$

where $C^\infty(\mathbb{P}^1(K),\mathbb{Z})$ is the space of locally constant functions on $\mathbb{P}^1(K)$. A number of versions of this result were known prior to the work in [SS]; this prior work

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX relied on analytic techniques available in the one dimensional case. Two of these techniques were, first, the explicit "residue" map of [S1]:

$$\mathrm{Res} : \Omega^1_K \to \mathrm{Hom}(C^\infty(\mathbb{P}^1(K), \mathbb{Z})/K, K),$$

and, second, an explicit inverse to this residue map (a "Poisson Kernel," see [T1])

$$I : \mathrm{Hom}(C^\infty(\mathbb{P}^1(K), \mathbb{Z})/K, \mathbb{Z}_p) \to \Omega^1_K.$$

In this paper we present generalizations of these two analytic constructions to the case of d–forms on $\Omega^{(d+1)}$.

In order to explain our results, let G denote $\mathrm{GL}_{d+1}(K)$, let P be a Borel subgroup, let S denote the set of fundamental reflections in the Weyl group, and let P_s denote the parabolic subgroup generated by P and $s \in S$. The Steinberg representation $\mathbf{St}$ is the G–space

$$\mathbf{St} := C^\infty(G/P, \mathbb{Z})/\sum C^\infty(G/P_s, \mathbb{Z}).$$

The computation of $H^d_{DR}(\Omega^{(d+1)}, K)$ in [SS] tells us that there is an isomorphism

$$H^d_{DR}(\Omega^{(d+1)}, K) \to \mathrm{Hom}(\mathbf{St}, K).$$

Our first main result is the construction of an explicit residue map

$$\mathrm{Res} : \Omega^d_K \to \mathrm{Hom}(\mathbf{St}, K)$$

which vanishes on exact forms and therefore induces a map on De Rham cohomology. To define this map, we exploit the reduction map r from $\Omega^{(d+1)}$ to the affine building for $\mathrm{PGL}_{d+1}(K)$. If Δ is a chamber (a top dimensional simplex) of the building, we restrict a form η to the fiber of the reduction map above Δ and define the residue $\mathrm{Res}_\Delta \eta$ of η at Δ to be an appropriate coefficient in a power series expansion of η. This associates to η a function $\mathrm{Res}\,\eta$ on the chambers of the building. We know (from [SS], for example) that such a function determines an element of $\mathrm{Hom}(C^\infty(G/P, \mathbb{Z}), K)$. Our first main theorem shows that $\mathrm{Res}\,\eta$ vanishes on $\sum C^\infty(G/P_s, \mathbb{Z})$. This amounts to showing that $\mathrm{Res}\,\eta$ is harmonic – in other words, that if F is a codimension one face of the affine building, then

$$\sum_{\Delta \mapsto F} \mathrm{Res}_\Delta \eta = 0$$

where the sum is over the chambers in the building which meet F. We establish this fact by studying the geometry of the reduction map, and eventually reducing the question to the one dimensional residue theorem.

Our second main result is the construction of an inverse to the residue map. We introduce a kernel function $k(g, q)$ on $G/P \times \Omega^{(d+1)}$, which, for fixed q, is continuous on G/P and vanishes off of the big cell. These results involve calculations with the Bruhat decomposition of G.

Having constructed our kernel function, we interpret elements $\lambda \in \mathrm{Hom}(\mathbf{St}, \mathbb{Z}) \otimes K$ as bounded p–adic distributions on the compact space G/P and show that the kernel function may be integrated against such a λ. The resulting function

$$F_\lambda(q) = \int_{G/P} k(g,q)d\lambda(g)$$

is rigid analytic on $\Omega^{(d+1)}$ and the differential form $\eta_\lambda = F_\lambda d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$ for appropriate coordinates Ξ_{β_i} on $\Omega^{(d+1)}$ satisfies $\mathrm{Res}(\eta_\lambda) = \lambda$.

We also show that the map $\lambda \to \eta_\lambda$ is G–equivariant on the level of forms, not just cohomology, so that $\eta_{g\lambda} = g(\eta_\lambda)$. Thus this map produces a canonical representative for each De Rham class (at least if the associated measure is bounded.) This result depends on a careful study of the transformation properties of the kernel function under the G–action. The behavior of $k(g,q)$ under P is easy to establish; $k(g,q)$ has the property that, for appropriate coordinates $\Xi_{\beta_0}, \ldots, \Xi_{\beta_{d-1}}$ on $\Omega^{(d+1)}$, the differential form $\kappa(g) = k(g,\cdot)d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$ satisfies

$$p_*\kappa(g) = \kappa(pg)$$

for $p \in P$. The situation is more complicated for general $h \in G$. Applying algebraic methods derived from Orlik and Terao's ([OT]) work on finite arrangements of hyperplanes, we show that

$$E(g) = s_*\kappa(g) - \kappa(sg)$$

satisfies $E(gs') = E(g)$, where $s' = wsw$ and w is the longest element in the Weyl group. This result is obtained by using Orlik and Terao to translate it into a statement about the geometry of the spherical building of flags of subspaces in projective space where it follows from considerations of convexity. Since the distributions we are integrating vanish on functions which are right invariant by some $s \in S$, this error term integrates to zero, and so we obtain the G–invariance of the kernel.

The integral kernel constructed in this paper is only a partial generalization of the one dimensional kernel. In this paper we treat only the case of d–forms and bounded measures, whereas in the one dimensional case we considered as well certain more general line bundles on the upper half plane and corresponding "admissible measures" on the boundary. A complete generalization to higher dimensions involves the study of boundary distributions derived from the p–adic discrete series representations ([S2],[T2]); we hope to return to these questions in later work.

Notation

We fix a non–archimedean local field K of characteristic zero. Let o be the ring of integers in K, π a fixed prime element of o, and $\omega : K^* \to \mathbb{Z}$ the valuation on K normalized so that $\omega(\pi) = 1$. Also let $\hat{\overline{K}}$ denote the completion of a fixed algebraic closure of K; the valuation ω extends uniquely to a valuation of $\hat{\overline{K}}$ again denoted by ω. Let $G = \mathrm{GL}_{d+1}(K)$ for a fixed integer d, which we assume greater than or equal to one.

The Symmetric Space. Drinfeld's p–adic symmetric space is the complement of the K–rational hyperplanes in d–dimensional projective space over K. More precisely, let V be a fixed $(d+1)$–dimensional vector space over K, viewed as a space of row vectors, with G–action on the left by:

$$g((a_0, \ldots, a_d)) := (a_0, \ldots, a_d)g^{-1}.$$

We let $\mathbb{A}(V)$ and $\mathbb{P}(V)$ denote the affine and projective spaces associated to V, with their group actions by G. In order to avoid confusion with the notation for the sheaf of differential forms, we discard the notation $\Omega^{(d+1)}$ for the p–adic symmetric space and instead we let

$$\mathcal{X} := \mathbb{A}(V)\backslash\text{union of all } K\text{–rational hyperplanes through } 0$$

and

$$\overline{\mathcal{X}} := \mathbb{P}(V)\backslash\text{union of all } K\text{–rational hyperplanes.}$$

$\mathcal{X}$ (resp. $\overline{\mathcal{X}}$) is an admissible open set in $\mathbb{A}(V)$ (resp. $\mathbb{P}(V)$), carrying a natural rigid topology which can be constructed in several ways (see [SS] and [D], for example.) Furthermore, $\mathcal{X}$ is a trivial $\mathbb{G}_m$–torsor over $\overline{\mathcal{X}}$.

The Building. The symmetric spaces $\mathcal{X}$ and $\overline{\mathcal{X}}$ are closely related to the Bruhat–Tits building of G. To fix ideas and notation, we review the construction of the building following ([BT], I).

The fundamental apartment. Let T be the torus of diagonal matrices in G and let $X^*(T)$ and $X_*(T)$ denote the groups of algebraic characters and cocharacters respectively. The fundamental apartment A is the real vector space

$$A := X_*(T) \otimes \mathbb{R}.$$

The fundamental cocharacters

$$e_i(t) = \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & t & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} \leftarrow i^{th} \text{ row}$$

form a basis for the vector space A. We write the natural pairing

$$A \times X^*(T) \to \mathbb{R}$$

as $(x, \mu) = \mu(x)$. We let $\{\epsilon_0, \dots, \epsilon_d\}$ denote the basis of the character group $X^*(T)$ dual to the $\{e_i\}$, and introduce the map $a : T \to A$ defined by

$$a(\begin{pmatrix} t_0 & & 0 \\ & \ddots & \\ 0 & & t_d \end{pmatrix}) = -\sum_{i=0}^{d} \omega(t_i) e_i,$$

which is characterized by the property $(a(t), \mu) = -\omega(\mu(t))$.

The normalizer N of T in G is the semidirect product of the group of permutation matrices with T. This group acts on A via affine transformations, with permutation matrices acting as permutation of coordinates and T via translation by $a(t)$.

Let

$$\mathbf{n} = \frac{1}{d+1} \sum_{i=0}^{d} e_i.$$

This element of A is preserved by the permutation matrices, and therefore N acts on the quotient $\overline{A} = A/\mathbb{R}\mathbf{n}$. We will denote by $\overline{x}$ the image of a point $x \in A$ in $\overline{A}$. The space $\overline{A}$ is spanned by $\overline{e}_i$ for $i = 1, \dots, d$, and can be naturally identified with $X_*(\overline{T}) \otimes \mathbb{R}$ where $\overline{T}$ is the maximal torus in $\overline{G} = G/K^*$.

Root Groups. Let

$$\Phi := \{\epsilon_i - \epsilon_j : 0 \le i, j \le d \text{ and } i \ne j\} \subset X^*(T)$$

be the set of roots of G relative to T. Corresponding to a root $\alpha = \epsilon_i - \epsilon_j$ we have a homomorphism

$$\tilde{\alpha} : K^+ \to G$$

sending $u \in K^+$ to the matrix (u_{rs}) with:

$$u_{rs} = \begin{cases} 1 & \text{if } r = s \\ u & \text{if } r = i \text{ and } s = j \\ 0 & \text{otherwise.} \end{cases}$$

The image U_α of $\tilde{\alpha}$ in G is called the root subgroup associated to α. It is filtered by the subgroups

$$U_{\alpha,r} := \tilde{\alpha}(\{u \in K : \omega(u) \ge r\}) \text{ for } r \in \mathbb{R}.$$

For a point $x \in A$ we define

$$U_x := \text{the subgroup of } G \text{ generated by all } U_{\alpha,-\alpha(x)} \text{ for } \alpha \in \Phi.$$

Let P denote the lower triangular Borel group of G. Let Φ^+ denote the set

$$\Phi^+ := \{\epsilon_i - \epsilon_j : i > j\};$$

these are the P–positive roots of G. Naturally, $\Phi^- = -\Phi^+$. We let $U^\pm$ denote the group generated by the root groups U_α for $\alpha \in \Phi^\pm$.

The groups U_x introduced above have the following properties ([BT], I, Sections 6 and 7):

1. $nU_xn^{-1} = U_{nx}$ for $n \in N$ and $x \in A$.
2. $U_\alpha \cap U_x = U_{\alpha,-\alpha(x)}$.
3. The product map
$$\prod_{\alpha\in\Phi^\pm} U_{\alpha,-\alpha(x)} \to U_x \cap U^\pm$$
is bijective (for any ordering of the terms on the left.)
4. $U_x = (U_x \cap U^+)(U_x \cap U^-)(U_x \cap N)$.
5. $G = U_xNU_y$ for any two points $x, y \in A$.
6. $U_x \cap N \subset N_{\overline{x}} := \{n \in N : n\overline{x} = \overline{x}\}$.

On the set $G \times A$ we introduce the relation:

$$(g, x) \sim (h, y) \Leftrightarrow \begin{cases} \text{There exists } u \in U_x \text{ and } n \in N \\ \text{such that } ny = x \text{ and } h = gun. \end{cases}$$

The same relation descends to a relation on $G \times \overline{A}$. It follows from property (1) above that $\sim$ is an equivalence relation. We define the buildings

$$X := G \times A/\sim$$
$$\overline{X} := G \times \overline{A}/\sim .$$

Each of these buildings carries a G–action through the first factor. The fundamental apartments A and $\overline{A}$ imbed injectively and N–equivariantly into X and $\overline{X}$ respectively by the map sending x to the class of $(1, x)$. Any set $gA \subset X$ and $g\overline{A} \subset \overline{X}$ is called an *apartment* of the respective buildings.

As a direct consequence of the definitions, the stabilizers B_x and $B_{\overline{x}}$ of points $x \in A \subset X$ and $\overline{x} \in \overline{A} \subset \overline{X}$ have the following description. Let N_x be the stabilizer in N of the point $x \in A$. Then $B_x = N_xU_x$ and $B_{\overline{x}} = N_{\overline{x}}U_x$; note that, by property (1), the groups N_x and $N_{\overline{x}}$ normalize U_x.

Valuations. The work of Iwahori–Goldman ([GI]) and Bruhat–Tits ([BrT]) gives an alternative approach to X and $\overline{X}$ by means of valuations.

Definition 1. A valuation γ on a K–vector space W is a map $\gamma : W \to \mathbb{R} \cup \{\infty\}$ such that

a. $\gamma(x+y) \geq \inf\{\gamma(x), \gamma(y)\}$
b. $\gamma(rx) = \gamma(x) + \omega(r)$ for $r \in K$.
c. $\gamma(x) = \infty$ if and only if $x = 0$.

Two valuations γ and γ' on W are *equivalent* if there is a constant $C \in \mathbb{R}$ such that

$$\gamma(x) - \gamma'(x) = C$$

for all $x \in W$.

We set

$$[\gamma] := \text{the equivalence class of } \gamma.$$

The Reduction Map. The reduction map relates the symmetric space $\mathcal{X}$ to the building X. This map arises through the interpretation of X as a space of norms on the dual space V^* to the fixed $(d+1)$–dimensional vector space V. To make this explicit, let Ξ_i be the element of V^* defined by

$$\Xi_i((a_0, \dots, a_d)) = a_i.$$

The elements Ξ_i span V^*; at the same time, they may be viewed as functions on the space $\mathcal{X}$.

An element $x \in A$ gives rise to a valuation on V^* by the formula

$$\gamma_x(\sum a_i \Xi_i) = \inf\{\omega(a_i) + \epsilon_i(x)\}.$$

This map extends in a unique way to a G–invariant bijection between X and the space of all norms on V^* ([BrT]). The space of equivalence classes of norms on V^* is identified in a compatible way with $\overline{X}$.

Definition 2. The reduction map

$$\mathrm{r} : \mathcal{X} \to X$$

is the map sending a point $q \in \mathcal{X}$ to the valuation $\mathrm{r}(q)$ defined by $\mathrm{r}(q)(\Xi) = \omega(\Xi(q))$ for $\Xi \in V^*$. This map descends to a corresponding reduction map

$$\mathrm{r} : \overline{\mathcal{X}} \to \overline{X}.$$

Lemma 3. For any point $q = (a_0, \dots, a_d)$ in $\mathcal{X}(\hat{\overline{K}})$ such that $\mathrm{r}(q) \in A$, we have
$$\mathrm{r}(q) = \omega(a_0)e_0 + \ldots + \omega(a_d)e_d.$$

Proof. Suppose that q reduces to $x \in A$. Then we know that
$$\gamma_x(b\Xi_i) = \omega(b) + \epsilon_i(x).$$
On the other hand, from the definition of the reduction map, we know that
$$\mathrm{r}(q)(b\Xi_i) = \omega(b) + \omega(a_i).$$
In particular, it follows that $\omega(a_i) = \epsilon_i(x)$, as claimed.

Suppose that $\mu \in X^*(T)$ is a weight. We associate to μ the function
$$\Xi_\mu := \prod_{i=0}^{d} \Xi_i^{m_i} \text{ where } \mu = \sum_i m_i \epsilon_i$$
on $\mathcal{X}$. For μ in the root lattice, Ξ_μ is actually a function on $\overline{\mathcal{X}}$.

Corollary 4. For any point $q \in \mathcal{X}(\hat{\overline{K}})$ such that $\mathrm{r}(q) \in A$, and any weight $\mu \in X^*(T)$, we have
$$\omega(\Xi_\mu(q)) = \mu(\mathrm{r}(q))$$

Chambers. The building $\overline{X}$, which we view as a space of norms, is also the geometric realization of a simplicial complex. The analytic structure of a fiber $\mathrm{r}^{-1}(z)$ of the reduction map depends to a great extent on where the point x lies relative to the faces of the simplices in this complex. Therefore we will devote some effort to identifying the simplicial structure in the space of norms.

Let $\overline{C} \subset \overline{A}$ denote the set
$$\overline{C} := \{\sum_{i=1}^{d} a_i \overline{e}_i : 0 \le a_1 \le \ldots \le a_d \le 1\}.$$
This set is a fundamental domain in $\overline{A}$ for the action of the affine group N.

The cocharacters of the torus $\overline{T} = T/K^*$ sit inside $\overline{A}$ as a lattice. We refer to an element of the G-orbit of $X_*(\overline{T})$ in $\overline{X}$ as a *vertex*. Let
$$G^+ := \{g \in G : \omega(\det(g)) = 0\}.$$
Since the stabilizer of a vertex $\overline{x}$ belongs to K^*G^+, we may define a function
$$\tau : X_*(\overline{T}) \to \mathbb{Z}/(d+1)\mathbb{Z}, \quad \tau(\sum_{i=1}^{d} a_i \overline{e}_i) = \sum_{i=1}^{d} a_i \bmod d+1.$$
and extend it to a function on the vertices in $\overline{X}$ by setting
$$\tau(g\overline{x}) = \tau(\overline{x}) + \omega(\det(g)) \bmod d+1$$
for a pair $(g, \overline{x}) \in G \times X_*(\overline{T})$. The value $\tau(g\overline{x})$ is called the *type* of $g\overline{x}$.

If $x \in A$, recall that the stabilizer of $\overline{x}$ in $\overline{A}$ is the group $B_{\overline{x}}$, which is the product $B_{\overline{x}} = N_{\overline{x}} U_x$. We may apply these properties to determine the stabilizer of $\overline{C}$:

Lemma 5. 1. An element of G fixes $\overline{C}$ pointwise if and only if it fixes each vertex in $\overline{C}$. 2. The subgroup of G fixing $\overline{C}$ pointwise is K^*B, where B is the Iwahori group of matrices within $\mathrm{GL}_{d+1}(o)$ which are lower triangular mod π.

Proof. Both parts follow from [BT] 10.2.9.

Definition 6. A pointed chamber in $\overline{X}$ is a pair $(g\overline{C}, z)$ consisting of a chamber (that is, a translate of $\overline{C}$) and a vertex z in $g\overline{C}$. The group G acts in a natural way on the pointed chambers.

Notice that the vertices of a chamber are ordered by type, and consequently a chamber carries an orientation.

Lemma 7. The map $gK^*B \mapsto (g\overline{C}, g\cdot 0)$ provides an isomorphism between G/K^*B and the set of pointed chambers.

Proof. The group G^+ permutes the chambers transitively and preserves the types of vertices, with K^*B fixing $\overline{C}$ pointwise. (By the Bruhat decomposition, any two chambers lie in a common apartment; then consideration of the affine group action on the apartment gives this fact.) The assertion then follows from the fact that the index of K^*G^+/K^* in G/K^* is $d+1$, with the difference accounted for by elements of N which fix $\overline{C}$ but cycle through the $d+1$ vertices.

Harmonic Functions and Distributions. In the one-dimensional theory of the Poisson Kernel, we make essential use of the correspondence between oriented edges of the tree of SL_2 and compact open subsets of the space of ends of the tree, which we identify with $\mathbb{P}^1(K)$. By means of this correspondence we may view harmonic functions on the tree as finitely additive distributions on the space of ends. In this section, following [SS], we will explain the generalization of this important relationship to yield a correspondence between pointed chambers of the building $\overline{X}$ and compact open subsets of the p–adic flag manifold G/P, which plays the role of the boundary of $\overline{X}$ in this theory.

From our discussion of the building $\overline{X}$, we know that the homogeneous space G/K^*B can be identified with the space of pointed chambers in $\overline{X}$ (See Lemma 7). Consequently, the space $C_c(G/K^*B, \mathbb{Z})$ of finitely supported functions on G/K^*B can be interpreted as the free abelian group on the pointed chambers of $\overline{X}$. We let $C^\infty(G/P, \mathbb{Z})$ denote the space of locally constant functions on G/P, and let χ_U be the characteristic function of an open set U in either of the two function spaces under consideration.

Following [SS], Section 4, and accounting for the change from upper to lower triangular Borel group, let y_i be the diagonal matrix

$$y_i = \mathrm{diag}(\overbrace{\pi, \ldots, \pi}^{i}, 1, \ldots, 1).$$

Proposition 8. ([SS], Proposition 11, page 80) Let I be the G–submodule of $C_c(G/K^*B, \mathbb{Z})$ generated by $\chi_{K^*By_iB} - \chi_{K^*B}$ (for $1 \le i \le d$). The map

$$H : C_c(G/K^*B, \mathbb{Z}) \to C^\infty(G/P, \mathbb{Z})$$

defined by $H(\phi) = \sum_{g \in G/K^*B} \phi(g) g(\chi_{BP/P})$ induces an isomorphism

$$C_c(G/K^*B, \mathbb{Z})/I \to C^\infty(G/P, \mathbb{Z}).$$

Remark. The map H in the statement of this proposition, which associates an open set in G/P to a pointed chamber in $\overline{X}$, can be given an elementary geometric interpretation in terms of the Borel–Serre compactification of $\overline{X}$. Let (Δ, z) be a pointed chamber of $\overline{X}$. Choose an apartment A_0 containing Δ. There is a unique sector in A_0 which is based at z and contains Δ. This sector determines a chamber Δ_∞ in the spherical building at infinity in the Borel–Serre compactification. (See [BS] or [B]). (Recall that the chambers of this spherical building correspond to G/P.) Let B_Δ be the Iwahori group which fixes Δ pointwise. Then the compact open subset corresponding to (Δ, z) is the orbit $B_\Delta \Delta_\infty$ of Δ_∞ in G/P.

Dualizing the map H in the above proposition, wee see that we may interpret finitely additive distributions on G/P as certain types of functions on pointed chambers in $\overline{X}$. The distributions which arise in our theory are those which correspond to harmonic functions in the following sense (compare [G], 3.15).

Definition 9. Let M be an abelian group. An element ψ of $\mathrm{Hom}(C_c(G/K^*B, \mathbb{Z}), M)$ will be called an M–valued harmonic function provided that

1. $\psi(\chi_{gnK^*B}) = (-1)^{d\omega(\det(n))} \psi(\chi_{gK^*B})$ for $g \in G$ and $n \in N$ such that $nBn^{-1} = B$. (Remark: The characteristic function on the left side determines the value $\omega(\det(n)) \bmod d+1$. If $d+1$ is odd, then d is even, so the sign in this expression is always one; if $d+1$ is even, then the sign is well–defined.)
2. $\psi(\chi_{K^*BsB}) + \psi(\chi_{K^*B}) = 0$ for s in the set of fundamental reflections in the group N. (These are the $d+1$ reflections in the walls of the fundamental chamber $\overline{C}$).

We let $\mathrm{C}_{\mathrm{har}}(M)$ denote the space of M–valued harmonic functions.

The two conditions of this definition have simple geometric interpretations. The first means that ψ transforms under the sign character of the group which acts on a pointed chamber (Δ, z) by fixing Δ and changing the chosen vertex z; this group is cyclic of order $d+1$. The second asserts that, if (Δ, z) is a pointed chamber and F is the codimension one face of Δ opposite to z, then

$$\sum_{(\Delta', z') \mapsto F} \psi(\Delta', z') = 0$$

where the sum is over the pointed chambers sharing the face F, each with selected vertex opposite to F.

To relate harmonic functions to distributions, we must, by the above proposition, prove the following lemma.

Lemma 10. Let ψ be an M–valued harmonic function. Then $\psi(\chi_{K^*By_iB}) = \psi(\chi_{K^*B})$.

Proof. Let $s_1, \ldots, s_d$ denote the fundamental reflections generating the Weyl group of G, where s_i permutes e_i and e_{i-1}. Let $\sigma = s_1 s_2 \cdots s_d$ and $\rho = y_1\sigma$. A direct matrix calculation shows that ρ generates the normalizer of K^*B modulo K^*B. Another simple calculation shows that $y_i = \rho^i\sigma^{-i}$. From this it follows that

$$\psi(\chi_{K^*By_iB}) = (-1)^{di}\psi(\chi_{K^*B\sigma^{-i}B}).$$

Applying the Bruhat decomposition and the harmonicity of ψ, we see that

$$\psi(\chi_{K^*B\sigma^{-i}B}) = (-1)^{\mathrm{length}(\sigma^{-i})}\psi(\chi_{K^*B}).$$

But σ has length d, so

$$\mathrm{length}(\sigma^{-i}) \equiv di \pmod 2.$$

This gives the desired relation.

Proposition 11. Let ψ be an element of $\mathrm{C_{har}}(M)$. The map

$$\chi_U \mapsto \int_U d\psi := \psi(H^{-1}(\chi_U))$$

gives a well–defined element of $\mathrm{Hom}(C^\infty(G/P, \mathbb{Z}), M)$ which vanishes on the subspace

$$\sum_s C^\infty(G/P_s, \mathbb{Z}) \subset C^\infty(G/P, \mathbb{Z})$$

where the sum is over the d fundamental reflections generating the Weyl group of G.

Proof. The definition of harmonicity shows that ψ vanishes on $\chi_{K^*BsB} + \chi_{K^*B}$. Using this, the proposition follows from the preceeding lemma and [SS], Corollary 17, page 83.

The G–space $C^\infty(G/P, \mathbb{Z})/\sum_s C^\infty(G/P_s, \mathbb{Z})$ is the important *Steinberg representation*, which we will denote in what follows by **St**. The map in Proposition 11 sets up a G–equivariant isomorphism between $\mathrm{C_{har}}(M)$ and $\mathrm{Hom}(\mathbf{St}, M)$.

We will conclude this section with some observations which will help us in the proof of our main theorem.

Let w_{d+1} denote the element of the Weyl group of G of maximal length. Every element in the the "big cell" $Pw_{d+1}P/P$ of G/P has a unique representation $uw_{d+1}P$ with u a unipotent matrix in P; this identifies the big cell with an affine space of dimension $d(d+1)/2$. As shown in [SS], Corollary 6, p. 76, the restriction map yields an isomorphism

$$\mathrm{Hom}(\mathbf{St}, M) \to \mathrm{Dist}(Pw_{d+1}P/P, M)$$

where Dist denotes the space of all finitely additive distributions. Consequently, a harmonic function ψ is determined by its values on those pointed chambers (Δ, z) which correspond to compact open sets in the big cell.

Let $t = y_1 y_2 \cdots y_d$ where the y_i are the matrices defined above. If u is a unipotent matrix in P and r is a positive integer, we let $\mathbf{B}(u,r)$ and $\Delta(u,r)$ denote respectively the compact open set $uw_{d+1}t^r BP/P$ in G/P and the pointed chamber $uw_{d+1}t^r(\overline{C}, 0)$.

Lemma 12. The set $\mathbf{B}(u,r)$ is the compact open corresponding to $\Delta(u,r)$. Every compact open subset of the big cell in G/P is, for any sufficiently large r, a finite disjoint union of sets of the form $\mathbf{B}(u,r)$.

Proof. The first assertion is an immediate consequence of G–equivariance of the map H giving the correspondence between pointed chambers and compact opens. For the second assertion, we refer to Proposition 8 of [SS], which asserts, in our notation, that every compact open in G/P is, given a sufficiently large $n \geq 0$, a finite disjoint union of sets of the form $bt^n BP/P$ for some $b \in \mathrm{GL}_{d+1}(o)$. Notice that $t^{-m}(P \cap B)t^m \subset (P \cap B)$ if $m \geq 0$. Any compact open subset of the set BP/P is a disjoint union of sets of the form $bt^m BP/P$ by [SS], and since $b = u^- p$ for u^- upper triangular and unipotent and p in $P \cap B$, we see that these sets are of the form $u^- t^m BP/P$. Applying the inversion w_{d+1} we see that any compact open subset of $w_{d+1}BP/P$ is a disjoint union of sets of the form $w_{d+1}u^- t^m BP/P$. Since $w_{d+1}u^- w_{d+1} \in P$, this gives the desired result.

Residues

In the first part of this section, we construct a "Residue map" which associates a K–valued function $\mathrm{Res}\,\eta$ on the pointed chambers in $\overline{X}$ to a rigid analytic d–form η on the symmetric space $\overline{\mathcal{X}}$. It is an immediate consequence of the construction of this residue map that it vanishes on exact forms. In the second part of this section, we prove the fundamental *residue theorem*, which asserts that the function $\mathrm{Res}\,\eta$ is harmonic. We will therefore obtain a well–defined map

$$\mathrm{Res} : \mathrm{H}^d_{DR}(\overline{\mathcal{X}}, K) \to \mathrm{C}_{\mathrm{har}}(K).$$

which generalizes the one dimensional residue map.

Definition of the Residue. In a manner analogous to the one–dimensional situation, we define the residue of a d–form η on spaces $\mathrm{r}^{-1}(\Delta)$, where Δ is a chamber in $\overline{X}$. In the one dimensional situation, the chambers are the edges in the tree of SL_2, and the fibers of reduction $\mathrm{r}^{-1}(\Delta)$ are annuli. Our first task is to understand the rigid geometry of the fiber $\mathrm{r}^{-1}(\overline{C})$ in the higher dimensional situation.

Fibers of Reduction. As canonical coordinates for $\overline{\mathcal{X}}$, we use the functions

$$\Xi_{\alpha_i} = \frac{\Xi_{i+1}}{\Xi_i} \quad \text{for } i = 0, \ldots, d-1$$

where $\alpha_i = \epsilon_{i+1} - \epsilon_i$ are the simple roots defined by our choice of the Borel subgroup P. Note that the α_i are a basis for the root lattice $X^*(\overline{T})$.

Now choose a set S of representatives for the primitive elements in $L/\pi^2 L$, where L is the lattice in V^* spanned by the Ξ_i. We may write any such element h in the form $h = \sum a_i(h)\Xi_i$. Since h is primitive, at least one of the $a_i(h)$ is a unit.

Consider the set U of points q in $\mathbb{P}(V)$ which satisfy the inequalities

$$\text{(i)} \qquad \begin{aligned} &\omega(\Xi_k(q)) \leq \omega(\Xi_{k+1}(q)) \quad \text{for } k = 0, \ldots, d-1 \\ &\omega(\Xi_d(q)) \leq 1 + \omega(\Xi_0(q)), \end{aligned}$$

and

$$\text{(ii)} \qquad \omega(h(q)) \leq \omega(a_k(h)) + \omega(\Xi_k(q)) \quad \text{for all } k = 0, \ldots, d \text{ and } h \in S.$$

Note that condition (i) implies that none of the $\Xi_k(q)$ vanish for $q \in U$. Moreover, any point $q \in \overline{\mathcal{X}}$ reducing to a point $\mathrm{r}(q) \in \overline{A}$ satisfies condition (ii).

Proposition 13. The set U defined by these inequalities is $\mathrm{r}^{-1}(\overline{C})$. U is an affinoid domain.

Proof. That U is affinoid is clear. It follows from Lemma 3 and the conditions in (i) that if $\mathrm{r}(U)$ is contained in $\overline{A}$, then $U = \mathrm{r}^{-1}(\overline{C})$. The inclusion $\mathrm{r}(U) \subset \overline{A}$ is proved in the following two lemmas.

Lemma 14. U is contained in $\overline{\mathcal{X}}$.

Proof. If $q \in U$ lies on a hyperplane defined over K, then we may find elements g and y in L and h in S such that $g(q) = 0$ and $g = h + \pi^2 y$. Then $h(q)/\Xi_0(q) = -\pi^2 y(q)/\Xi_0(q)$ and therefore $\omega(h(q)/\Xi_0(q)) \geq 2$. (Notice that $\omega(\Xi_0(q))$ is minimal among the $\omega(\Xi_i(q))$, so $\omega(y(q)/\Xi_0(q)) \geq 0$.) On the other hand, there is an i such that $a_i(h)$ is a unit. Condition (ii) then forces $\omega(\frac{h}{\Xi_i}(q) \leq 0$ and since $\omega(\frac{\Xi_i}{\Xi_0}(q)) \leq 1$, we have $\omega(\frac{h}{\Xi_0}(q)) \leq 1$. This contradicts the existence of g.

Lemma 15. If $q \in U$, then $\mathrm{r}(q) \in \overline{A}$.

Proof. Choose any $g \in L$. For purposes of analyzing the norm of g, we may assume g primitive mod π^2. Then $g = h + \pi^2 y$ as in the previous lemma. Writing $g = \sum g_i \Xi_i$, we have $\omega(g_i - a_i(h)) \geq 2$. As in the proof of the preceeding lemma, $\omega(\frac{h}{\Xi_0}(q)) \leq 1$, so that $\omega(\frac{g}{\Xi_0}(q)) = \omega(\frac{h}{\Xi_0}(q))$. From the defining inequalities for U,

$$\omega(\frac{h}{\Xi_0}(q)) = \inf_i \{\omega(a_i(h)) + \omega(\frac{\Xi_i}{\Xi_0}(q))\}.$$

Since there is at least one $a_i(h)$ of valuation zero, and the quantities $\omega(\frac{\Xi_i}{\Xi_0}(q))$ are at most one, any term with $\omega(a_i(h)) \geq 2$ may be neglected when computing the minimum. However, if $\omega(a_i(h)) < 2$, then $\omega(a_i(h)) = \omega(g_i)$, and therefore we can write this expression as

$$\omega(\frac{g}{\Xi_0}(q)) = \inf_i \{\omega(g_i) + \omega(\frac{\Xi_i}{\Xi_0}(q))\},$$

which shows that the norm associated to q belongs to the apartment $\overline{A}$.

The definition of the residue map depends on somewhat more information about the reduction map. Let $\overline{C}^\circ$ denote the interior of the closed chamber $\overline{C}$.

Lemma 16. The fiber of reduction $\mathrm{r}^{-1}(\overline{C}^\circ)$ is the admissible open set defined by the inequalities

$$\omega(\Xi_{\alpha_k}(q)) > 0 \quad \text{for } k = 0, \ldots, d-1,$$
$$\omega(\Xi_{\alpha_0} \cdots \Xi_{\alpha_{d-1}}(q)) < 1.$$

Proof. The set defined by these inequalities is admissible by [BGR],Proposition 5, page 343. If q is a point satisfying these inequalities, then the values $\omega(\Xi_i(q))$ are all distinct. Consequently, the additional inequalities defining $\mathrm{r}^{-1}(\overline{C})$ are automatically satisfied. This is sufficient to show that $\mathrm{r}(q)$ belongs to $\overline{A}$, and then to $\overline{C}^\circ$.

Construction of the Residue Map. The final ingredient in the definition of the residue of a form η at the pointed chamber $(\overline{C}, 0)$ is an understanding of the rigid functions on $U^\circ = \mathrm{r}^{-1}(\overline{C}^\circ)$. The set U° can be constructed as the direct limit of affinoid subdomains U_n° where a point q belongs to U_n° provided that

$$\omega(\Xi_{\alpha_i}(q)) \geq 1/n \quad \text{for } i = 0, \ldots, d-1.$$

and

$$\omega(\Xi_{\alpha_0} \cdots \Xi_{\alpha_{d-1}}(q)) \leq 1 - 1/n.$$

The coordinate ring of U_n° is the affinoid algebra

$$\mathcal{O}_n := K << \Xi_{\alpha_0}, \ldots, \Xi_{\alpha_{d-1}}, T_0, \ldots, T_{d-1}, X >> / J_n$$

where the ideal J_n is generated by $\Xi_{\alpha_i}^n - \pi T_i$ and $(\Xi_{\alpha_0} \cdots \Xi_{\alpha_{d-1}})^n X - \pi^{n-1}$.

In order to describe this algebra more explicitly we temporarily put

$$\alpha_d := \epsilon_0 - \epsilon_d = -\sum_{i=0}^{d-1} \alpha_i.$$

Any $\mu \in X^*(\overline{T})$ can be written in a unique way as

$$\mu = \sum_{i=0}^{d} m_i \alpha_i$$

with integers $m_i \geq 0$ of which at least one is equal to zero; put

$$|\mu| := \sum_{i=0}^{d} m_i \text{ and } \ell(\mu) := m_d.$$

The latter number can also be characterized by

$$\ell(\mu) = -\inf_{z \in \overline{C}} \mu(z).$$

Lemma 17. For $n > d$ we have

$$\mathcal{O}_n = \{ \sum_{\mu \in X^*(\overline{T})} a(\mu) \Xi_\mu \ : \ a(\mu) \in K, \omega(a(\mu)) + \frac{|\mu|}{n} - \ell(\mu) \to \infty \text{ if } |\mu| \to \infty \}.$$

Proof. This is an exercise in rigid analysis. The point to note is the following: Whenever $m_0, \ldots, m_d > 0$ are positive integers, then we have in $\mathcal{O}_n$ an equality

$$a T_0^{m_0} \ldots T_{d-1}^{m_{d-1}} X^{m_d} = b T_0^{m_0 - 1} \ldots T_{d-1}^{m_{d-1}-1} X^{m_d - 1}$$

where $\omega(b) = n - 1 - d + \omega(a) \geq \omega(a)$.

Remark. It follows that the ring of bounded analytic functions on U° has a particularly simple description. It is the ring of all series $\sum_{\mu \in X^*(\overline{T})} a(\mu) \Xi_\mu$ with $a(\mu) \in K$ for which the set $\{\omega(a(\mu)) + \inf_{z \in \overline{C}} \mu(z) \ : \ \mu \in X^*(\overline{T})\}$ is bounded below.

Lemma 18. $H^d_{DR}(U^\circ, K)$ is 1–dimensional, with basis the class of the d–form

$$\frac{d\Xi_{\alpha_{d-1}}}{\Xi_{\alpha_{d-1}}} \wedge \cdots \wedge \frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}}.$$

Proof. As we have seen, any d–form η on U° can be written as

$$\eta = \sum_{\mu \in X^*(\overline{T})} a(\mu)\Xi_\mu \, d\Xi_{\alpha_{d-1}} \wedge \cdots \wedge d\Xi_{\alpha_0}$$

where $\omega(a(\mu)) + \frac{|\mu|}{n} - \ell(\mu) \to \infty$ for any $n \in \mathbb{N}$ provided that $|\mu| \to \infty$. We have to show that η is exact if $a(\alpha_d) = 0$. Write

$$\mu = \sum_{i=0}^{d-1} n_i(\mu)\alpha_i.$$

It is sufficient to show that η is exact if it has the property that, for some fixed $0 \le j \le d-1$, all coefficients $a(\mu)$ with $n_j(\mu) = -1$ vanish. But if $n_j(\mu) \ne -1$, we have

$$a(\mu)\Xi_\mu \, d\Xi_{\alpha_{d-1}} \wedge \cdots \wedge d\Xi_{\alpha_0} = \frac{(-1)^{d-j-1} a(\mu)}{n_j(\mu)+1} d(\Xi_{\mu+\alpha_j} d\Xi_{\alpha_{d-1}} \wedge \cdots \wedge \widehat{d\Xi_{\alpha_j}} \wedge \cdots \wedge d\Xi_{\alpha_0}).$$

Hence it remains to check that

$$\omega(\frac{a(\mu)}{n_j(\mu)+1}) + \frac{|\mu+\alpha_j|}{n} - \ell(\mu+\alpha_j) \to \infty \text{ for any } n \in \mathbb{N}$$

provided that $|\mu + \alpha_j| \to \infty$. Since

$$|\mu+\alpha_j| = |\mu| + 1 \text{ or } |\mu| - d,$$
$$\ell(\mu+\alpha_j) = \ell(\mu) \text{ or } \ell(\mu) - 1,$$

this is the same as checking that

$$\omega(\frac{a(\mu)}{n_j(\mu)+1}) + \frac{|\mu|}{n} - \ell(\mu) \to \infty \text{ for any } n \in \mathbb{N}$$

provided $|\mu| \to \infty$. But we have that

$$\omega(a(\mu)) + \frac{|\mu|}{n+1} - \ell(\mu) \to \infty.$$

Hence our claim follows if the difference

$$-\omega(n_j(\mu)+1) + \frac{|\mu|}{n(n+1)}$$

is bounded below (for varying μ), or, in other words, if

$$\frac{\omega(n_j(\mu)+1)}{|\mu|} \to 0 \text{ as soon as } |\mu| \to \infty.$$

This sequence is of the form $\frac{\omega(n(\mu))}{m(\mu)+n(\mu)}$ where

$$n(\mu) := |n_j(\mu)+1| \ge 1 \text{ and } m(\mu) := |\mu| - n(\mu) \ge -1$$

and hence converges to 0 if the denominator goes to ∞.

Definition 19. Let η be a rigid analytic d–form on $\overline{\mathcal{X}}$. The residue $\operatorname{Res}_{(\overline{C},0)}\eta$ of η at $(\overline{C},0)$ is the unique element in K such that

$$\eta \equiv (\operatorname{Res}_{(\overline{C},0)}\eta)\frac{d\Xi_{\alpha_{d-1}}}{\Xi_{\alpha_{d-1}}}\wedge\cdots\wedge\frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}} \quad (\text{mod exact forms})$$

holds true on U°.

Lemma 20. Suppose that $g \in G$ stabilizes $(\overline{C},0)$. Then

$$\operatorname{Res}_{(\overline{C},0)} g_*\eta = \operatorname{Res}_{(\overline{C},0)}\eta.$$

Proof. It suffices to treat the d–form

$$\eta := \frac{d\Xi_{\alpha_{d-1}}}{\Xi_{\alpha_{d-1}}}\wedge\cdots\wedge\frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}}.$$

We have $g_*\eta = \eta$ for any $g \in T$. It therefore remains to consider an element $g \in B$ of the form $g = \tilde{\alpha}(u)$ for some root $\alpha = \epsilon_i - \epsilon_j$ and some $u \in K$. Then $\omega(u\Xi_\alpha(q)) > 0$ for $q \in U^\circ$ so that the function $1 + u\Xi_\alpha$ is invertible on U°. We now compute

$$\begin{aligned} g_*\eta &= \frac{1}{1+u\Xi_\alpha}\eta \\ &= \sum_{n\geq 0}(-u)^n\Xi_\alpha^n\eta \\ &= \sum_{n\geq 0}(-u)^n\Xi_{n\alpha-\alpha_0-\ldots-\alpha_{d-1}}\, d\Xi_{\alpha_{d-1}}\wedge\cdots\wedge d\Xi_{\alpha_0} \end{aligned}$$

and see that $\operatorname{Res}_{(\overline{C},0)} g_*\eta = 1 = \operatorname{Res}_{(\overline{C},0)}\eta$.

This property allows us to define the residue for any chamber in $\overline{X}$ by translating the chamber into $\overline{C}$.

Definition 21. (The Residue) Let (Δ, z) be any pointed chamber in $\overline{X}$, and let η be a d–form on $\overline{\mathcal{X}}$. Choose an element $g \in G$ such that $(g\Delta, gz) = (\overline{C}, 0)$. Then the residue of η at (Δ, z) is defined to be

$$\operatorname{Res}_{(\Delta,z)}\eta := \operatorname{Res}_{(g\Delta,gz)} g_*\eta.$$

Proposition 22. The Residue map induces a well–defined G–equivariant map

$$\operatorname{Res}: H^d_{DR}(\overline{\mathcal{X}}, K) \to \operatorname{Hom}(C_c(G/K^*B, \mathbb{Z}), K).$$

Remark. It is not hard to verify that the residue is invariant under any analytic automorphism of U° which preserves the ordering of the coordinates Ξ_{α_i} according to their valuation.

Later on, we will need the following fact.

Lemma 23. Let (Δ, z) be any pointed chamber, and let

$$\eta := \frac{d\Xi_{\alpha_{d-1}}}{\Xi_{\alpha_{d-1}}} \wedge \cdots \wedge \frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}};$$

we then have

$$\mathrm{Res}_{(\Delta,z)}\, \eta = \begin{cases} \pm 1 & \text{if } \Delta \subset \overline{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We noted already that η is invariant under T. Moreover, if s is a fundamental reflection in the Weyl group, then $s_*\eta = -\eta$. This proves the first half of the assertion. For the other half, choose a $g \in G$ such that $g(\Delta, z) = (\overline{C}, 0)$. Write $g = hwut$ with $h \in B$, $t \in T$, w a permutation matrix, and u a lower triangular unipotent matrix. Then $wut(\Delta, z) = (\overline{C}, 0)$, and

$$\mathrm{Res}_{(\Delta,z)}\, \eta = \mathrm{Res}_{(\overline{C},0)} (wut)_*\eta = \mathrm{Res}_{(\overline{C},0)} (wu)_*\eta.$$

Hence we have to show that

$$\mathrm{Res}_{(\overline{C},0)} (wu)_*\eta = 0 \text{ whenever } u^{-1}w^{-1}\overline{C} \not\subset \overline{A}.$$

Let u_{ij}, for $i > j$, denote the entries of the matrix u. We have

$$(wu)_*\eta = \mathrm{sign}(w) \left(\prod_{j=1}^{d-1} (1 + \sum_{i=j+1}^{d} u_{ij} \Xi_{w(\epsilon_i - \epsilon_j)})^{-1} \right) \eta.$$

Let us consider, for a moment, an arbitrary non–zero linear form

$$a_0 \frac{\Xi_0}{\Xi_j} + \cdots + a_d \frac{\Xi_d}{\Xi_j} \text{ with } a_0, \ldots, a_d \in K.$$

On U° we have

$$0 < \omega(\frac{\Xi_1}{\Xi_0}) < \ldots < \omega(\frac{\Xi_d}{\Xi_0}) < 1$$

and hence

$$\omega(\frac{\Xi_0}{\Xi_j}) < \ldots < \omega(\frac{\Xi_d}{\Xi_j}) \quad \text{and} \quad 0 < \omega(\frac{\Xi_d}{\Xi_j}) - \omega(\frac{\Xi_0}{\Xi_j}) < 1.$$

Let $0 \leq k \leq d$ be the smallest index such that

$$\omega(a_k) = \inf_{0 \leq i \leq d} \{\omega(a_i)\}.$$

With this choice, we have, on U°,

$$\omega(a_k \frac{\Xi_k}{\Xi_j}) < \omega(a_i \frac{\Xi_i}{\Xi_j}) \text{ for all } i \neq k,$$

and consequently the expansion

$$(a_0\frac{\Xi_0}{\Xi_j}+\ldots+a_d\frac{\Xi_d}{\Xi_j})^{-1}=(a_k\frac{\Xi_k}{\Xi_j})^{-1}\sum_{m\geq 0}(-1)^m(\sum_{\substack{0\leq i\leq d\\ i\neq k}}\frac{a_i}{a_k}\frac{\Xi_i}{\Xi_k})^m.$$

Let us say that an analytic function $\sum_{\mu\in X^*(\overline{T})}a(\mu)\Xi_\mu\in\mathcal{O}(U^\circ)$ is supported on the subset S of $X^*(\overline{T})$ if $a(\mu)=0$ for $\mu\notin S$. We see that $(a_0\frac{\Xi_0}{\Xi_j}+\ldots+a_d\frac{\Xi_d}{\Xi_j})^{-1}$ is supported on

$$(\epsilon_j-\epsilon_k)+\sum_{\substack{0\leq i\leq d\\ i\neq k\\ a_i\neq 0}}\mathbb{N}_0(\epsilon_i-\epsilon_k)$$

where $\mathbb{N}_0=\{0,1,\ldots,\}$. Applying all of this to the linear forms appearing in the above expression for $(wu)_*\eta$ we choose $k(j)$, for $0\leq j\leq d-1$, in such a way that $w(k(j))$ is the smallest index such that

$$\omega(u_{k(j),j})=\inf_{j\leq i\leq d}\{\omega(u_{ij})\}.$$

Here $u_{jj}:=1$ and w, as a permutation of $\{0,\ldots,d\}$, is defined by $w\epsilon_i=\epsilon_{w(i)}$. Note that $j\leq k(j)\leq d$. Then the function

$$(1+\sum_{i=j+1}^{d}u_{ij}\Xi_{w(\epsilon_i-\epsilon_j)})^{-1}$$

has support in

$$S'_j:=w\left[(\epsilon_j-\epsilon_{k(j)})+\sum_{\substack{j\leq i\leq d\\ i\neq k(j)}}\mathbb{N}_0(\epsilon_i-\epsilon_{k(j)})\right].$$

Put $S_j=w^{-1}S'_j$. It remains to show that

$$0\notin S_0+\ldots S_{d-1}\text{ provided that }u^{-1}w^{-1}\overline{C}\not\subset\overline{A}.$$

First, we observe the following simple facts:

– $S_j\subset\sum_{i\geq j}\mathbb{Z}\alpha_i$,

– $S_j\subset-\mathbb{N}\alpha_j+\sum_{i>j}\mathbb{Z}\alpha_i$ if $j<k(j)$, and

– $S_j\subset\mathbb{N}_0\alpha_j+\mathbb{N}_0(\alpha_j+\alpha_{j+1})+\ldots+\mathbb{N}_0(\alpha_j+\ldots+\alpha_{d-1})$ if $j=k(j)$.

From this, a little contemplation easily reveals that $S_0+\ldots+S_{d-1}$ contains 0 if and only if $k(j)=j$ for all $0\leq j\leq d-1$. However, if we assume that $u^{-1}w^{-1}\overline{C}\not\subset\overline{A}$, then $u\notin w^{-1}Bw$. Going back to the definition of $k(j)$, we see that this means that there is an index $0\leq j\leq d-1$ such that $k(j)>j$. Therefore 0 is not in $\sum_{i=0}^{d-1}S_i$ and the lemma is proved.

The Residue Theorem. The most important property of the residue map introduced in the preceeding section is the "residue theorem," which generalizes the one–dimensional residue theorem to our situation.

Theorem 24. (Residue Theorem) Let η be a rigid analytic d–form on $\overline{\mathcal{X}}$. The function

$$\operatorname{Res}\eta : (\Delta, z) \mapsto \operatorname{Res}_{(\Delta,z)} \eta$$

is a harmonic function.

Proof. We recall that the assertion of the theorem has two parts, corresponding to the two defining properties of harmonic functions (see Definition 9). The first of these properties requires that, if ρ generates the group stabilizing Δ modulo the group fixing Δ pointwise, then

$$\operatorname{Res}_{(\Delta,\rho z)} \eta = (-1)^d \operatorname{Res}_{(\Delta,z)} \eta.$$

By the G–equivariance, this is equivalent to the assertion that $\operatorname{Res}_{(\Delta,z)} \rho_*^{-1}\eta = (-1)^d \operatorname{Res}\eta$. It is straightfoward to verify that the permutation of the coordinates induced by ρ produces this sign change in the residue.

More significant is the verification of the second property of harmonic functions. This property asserts that, if F is a codimension one face of a chamber in $\overline{\mathcal{X}}$, then

$$\sum_{\Delta \mapsto F} \operatorname{Res}_{(\Delta,z)} \eta = 0$$

where the sum is over the chambers meeting F, with the distinguished vertex in each of the chambers Δ chosen opposite to F.

We will prove this part of the Residue Theorem over the balance of this section by analyzing the fiber of reduction at a codimension one face of $\overline{X}$. It turns out this fiber is essentially the cartesian product of a polyannulus with an affinoid subdomain of the projective line. Using this fact, we are able to reduce the residue theorem to the one–dimensional case.

Let F be the codimension one face of $\overline{C}$ opposite to 0; by G–equivariance, it suffices to consider the chambers meeting F. The open chamber $\overline{C}^\circ$ is determined by the conditions

$$1 > \omega(\frac{\Xi_d}{\Xi_0}(q)) > \omega(\frac{\Xi_{d-1}}{\Xi_0}(q)) > \cdots > \omega(\frac{\Xi_1}{\Xi_0}(q)) > 0.$$

In the apartment $\overline{A}$, the face F is determined by the condition

$$\omega(\frac{\Xi_d}{\Xi_0}(q)) = 1.$$

By applying the results on root groups, it is not hard to verify that the open chambers in $\overline{X}$ whose closures meet F consist of $\Delta_\infty^\circ := \overline{C}^\circ$ and the open chambers Δ_λ° defined by the conditions

$$1 > \omega(\frac{\pi^2\Xi_0}{\Xi_d + \lambda\pi\Xi_0}(q)) > \omega(\frac{\pi\Xi_{d-1}}{\Xi_d + \lambda\pi\Xi_0}(q)) > \cdots > \omega(\frac{\pi\Xi_1}{\Xi_d + \lambda\pi\Xi_0}(q)) > 0,$$

where λ runs through representatives for the residue field of K.

Fix an integer $n > 3$ and now let U be the open set defined by the following system of inequalities:

$$\text{(A)} \qquad 1 + 1/n > \omega(\frac{\pi^2\Xi_0}{\Xi_d + \pi\lambda\Xi_0}(q)) > 1 - 1/n \quad \text{for } \lambda \bmod \pi$$

and

$$\text{(B)} \qquad 1 - 1/n > \omega(\frac{\Xi_{d-1}}{\Xi_0}(q)) > \omega(\frac{\Xi_{d-2}}{\Xi_0}(q)) > \cdots > \omega(\frac{\Xi_1}{\Xi_0}(q)) > 1/n.$$

Lemma 25. The set U is an admissible open set in $\overline{\mathcal{X}}$. It has the following properties:

1. The reduction of U is contained in $F \cup \Delta_\infty^\circ \cup \bigcup_\lambda \Delta_\lambda^\circ$ and $U \cap \mathrm{r}^{-1}(\Delta_\lambda^\circ)$ is a non–empty admissible open set for each λ including $\lambda = \infty$.
2. U is isomorphic to the product of the sets

$$U^1 := \{(q_1, \ldots, q_{d-1}) \in \mathbb{A}^{d-1} : 1 - 1/n > \omega(q_{d-1}) > \cdots \omega(q_1) > 1/n\}$$

and

$$U^2 := \{q \in \mathbb{A}^1 : 1 + 1/n > \omega(q + \pi\lambda) > 1 - 1/n, \quad \lambda \bmod \pi\}.$$

Proof. The description of the set U in (A) and (B) shows that it is admissible and that it has the claimed product structure. Thus we need only compute the reduction of U.

Suppose first of all that, for some λ, we have

$$\omega(\frac{\Xi_d}{\Xi_0}(q) + \pi\lambda) < 1.$$

Then this must hold for all λ, and

$$\omega(\frac{\Xi_d}{\Xi_0}(q)) < 1.$$

It follows immediately that q reduces to $\overline{C}^{\circ}$.

Next, suppose that there is a λ such that

$$\omega(\frac{\Xi_d}{\Xi_0}(q) + \pi\lambda) > 1.$$

Then there is only one such λ, and we must have

$$1 > \omega(\frac{\pi^2\Xi_0}{\Xi_d + \pi\lambda\Xi_0}(q)) > 1 - 1/n.$$

Then, using (B), we see that

$$\omega(\frac{\pi\Xi_{d-1}}{\Xi_d + \pi\lambda\Xi_0}(q)) < 1 - 1/n,$$

as well as

$$\omega(\frac{\pi\Xi_i}{\Xi_d + \pi\lambda\Xi_0}(q)) < \omega(\frac{\pi\Xi_{i+1}}{\Xi_d + \pi\lambda\Xi_0}(q))$$

for $i = 1, \ldots, d-1$. Finally, we check that

$$\begin{aligned}\omega(\frac{\pi\Xi_1}{\Xi_d + \pi\lambda\Xi_0}(q)) &= 1 + \omega(\frac{\Xi_1}{\Xi_0}(q)) + \omega(\frac{\Xi_0}{\Xi_d + \pi\lambda\Xi_0}(q)) \\ &> 1 + 1/n + (-1 - 1/n) \\ &> 0.\end{aligned}$$

Thus in this case q reduces to Δ_λ°.

Finally, suppose that

$$\omega(\frac{\Xi_d}{\Xi_0}(q) + \pi\lambda) = 1 \text{ for all } \lambda \bmod \pi.$$

Then it is not hard to check that

$$\omega(\sum a_i\Xi_i(q)) = \inf_i\{\omega(a_i) + \omega(\Xi_i(q))\},$$

so that q reduces to the face F.

Now suppose that η is a rigid analytic d–form on $\overline{\mathcal{X}}$, which we restrict to the set U described in the preceeding lemma. From the product structure of U and the one dimensional Mittag–Leffler decomposition, we know that η can be written as a sum

$$\eta = (\sum_{\lambda\neq\infty}\sum_{i\geq 0}\frac{h_{\lambda,i}}{(\Xi+\pi\lambda)^i}\frac{d\Xi}{(\Xi+\pi\lambda)} + \sum_{i\geq 1}\Xi^i h_{\infty,i}\frac{d\Xi}{\Xi}) \wedge \frac{d\Xi_{\alpha_{d-2}}}{\Xi_{\alpha_{d-2}}} \wedge \cdots \wedge \frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}}.$$

where we have set $\Xi = \frac{\Xi_d}{\Xi_0}$. In this expression for η, each $h_{\lambda,i}$ is an analytic function on U^1 which "goes to zero uniformly with i" in a manner which will not play a role in our discussion.

In order to prove the residue theorem, we must relate the coordinate system consisting of Ξ and the Ξ_{α_i} to each Δ°_λ.

To perform this calculation, let us fix a choice of λ. Put

$$g_\lambda := \begin{pmatrix} \lambda & 0 & \cdots & 0 & \pi \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & 0 \\ \vdots & \vdots & \vdots & 1 & 0 \\ \pi^{-1} & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then

$$\text{(A)} \qquad \begin{aligned} (g_\lambda^{-1})_* \Xi_{\alpha_i} &= \Xi_{\alpha_i} \quad i = 1, \ldots, d-2, \\ (g_\lambda^{-1})_* \Xi_{\alpha_0} &= \frac{\pi}{\Xi_{\alpha_1} \cdots \Xi_{\alpha_{d-1}}}, \\ (g_\lambda^{-1})_*(\Xi + \pi\lambda) &= \frac{\pi^2}{\Xi_{\alpha_0} \cdots \Xi_{\alpha_{d-1}}}. \end{aligned}$$

A direct calculation using these formulae also shows that

$$\text{(B)} \quad (g_\lambda^{-1})_*\left(\frac{d\Xi}{(\Xi+\pi\lambda)} \wedge \frac{d\Xi_{\alpha_{d-2}}}{\Xi_{\alpha_{d-2}}} \wedge \cdots \wedge \frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}}\right) = -\frac{d\Xi_{\alpha_{d-1}}}{\Xi_{\alpha_{d-1}}} \wedge \frac{d\Xi_{\alpha_{d-2}}}{\Xi_{\alpha_{d-2}}} \wedge \cdots \wedge \frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}}.$$

Lemma 26. The residue $\mathrm{Res}_{(\Delta_\lambda, z_\lambda)} \eta$ (where z_λ is chosen opposite to the common face F) is:

$$-\mathrm{Res}_{(\Delta_\lambda, z_\lambda)} \eta = \text{constant term in } h_{\lambda,0} \text{ w.r.t. } \Xi_{\alpha_0}, \ldots, \Xi_{\alpha_{d-2}}.$$

Proof. On the region where U meets the fiber of reduction above Δ°_λ, we may replace the expansion of η given above with a "Laurent" expansion:

$$\eta = \sum_{i\in\mathbb{Z}} (\Xi + \pi\lambda)^i h^*_{\lambda,i} \frac{d\Xi}{(\Xi+\pi\lambda)} \wedge \frac{d\Xi_{\alpha_{d-2}}}{\Xi_{\alpha_{d-2}}} \wedge \cdots \wedge \frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}}$$

where $h^*_{\lambda,i} = h_{\lambda,i}$ if $i \leq 0$. A typical term in this expansion has the form

$$(\Xi + \pi\lambda)^i \Xi_{\alpha_{d-2}}^{i_{d-2}} \cdots \Xi_{\alpha_0}^{i_0}.$$

Applying g_λ^{-1} and using the relations (A), we see that such a term becomes a monomial in the Ξ_{α_i}, i running from 0 to $d-1$, and this monomial is non–trivial unless the term we began with had $i = 0$ and all $i_j = 0$. Conseqently, the only contribution from η to the residue at Δ_λ is from the term specified in the statement of the lemma. The sign comes from the relation (B) above.

To complete the proof of the residue theorem, we must consider η on the fiber above Δ_∞°. Note that $\Xi = \Xi_{\alpha_{d-1}} \cdots \Xi_{\alpha_0}$, and therefore

$$\frac{d\Xi_{\alpha_{d-1}}}{\Xi_{\alpha_{d-1}}} \wedge \cdots \wedge \frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}} = \frac{d\Xi}{\Xi} \wedge \frac{d\Xi_{\alpha_{d-2}}}{\Xi_{\alpha_{d-2}}} \wedge \cdots \wedge \frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}}.$$

We see easily that the terms in $\sum_i h_{\infty,i}\Xi^i$ cannot contribute to the residue of η at Δ_∞°. Furthermore, since $\omega(\Xi) < 1$ on the fiber above Δ_∞°, we may expand the terms

$$\frac{\Xi h_{\lambda,i}}{(\Xi + \pi\lambda)^{i+1}} = h_{\lambda,i}\Xi^{-i} + \text{terms with smaller powers of } \Xi.$$

Consequently, only the same constant terms of the various $h_{\lambda,0}$ as appear in Lemma 26 contribute to the residue. Since they appear with opposite sign, the residue theorem is proved.

The Integral Transform

We now begin the second main part of this paper, in which we construct an inverse to the residue map by means of an integral kernel. This integral kernel is a partial generalization of the Poisson Kernel for the p–adic upper half plane described in [T1], as explained in the introduction to this paper.

We first define our kernel function and establish its main properties. We then state our second main theorem, which summarizes the essential properties of the integral transform. The balance of the paper is taken up with establishing these properties.

Definition of the Kernel. In this section we introduce the function which serves as our integral kernel, and determine its fundamental properties.

As usual, we let P denote the lower triangular Borel subgroup of G, and let w_{d+1} denote the involution of maximal length in the Weyl group of G. Any element $g \in G$ which belongs to the big cell $Pw_{d+1}P$ of G can be written uniquely in the form

$$g = u_g w_{d+1} p$$

where $p \in P$ and u_g is a lower triangular unipotent matrix.

In this part of the paper, it will be more convenient to work with the slightly different set of coordinates Ξ_{β_i}, $i = 0, \ldots, d-1$, where $\beta_i = \epsilon_i - \epsilon_d$.

Note that

$$\frac{d\Xi_{\beta_0}}{\Xi_{\beta_0}} \wedge \cdots \wedge \frac{d\Xi_{\beta_{d-1}}}{\Xi_{\beta_{d-1}}} = (-1)^{d(d+1)/2} \frac{d\Xi_{\alpha_{d-1}}}{\Xi_{\alpha_{d-1}}} \wedge \cdots \wedge \frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}}.$$

Definition 27. Let k denote the function on $G \times \overline{\mathcal{X}}$ defined by

$$k(g,q) = \begin{cases} 0 \text{ if } g \text{ is not in the big cell of } G \\ (u_g)_* \frac{1}{\Xi_{\beta_0}(q)\cdots\Xi_{\beta_{d-1}}(q)} \text{ if } g = u_g w_{d+1} p \text{ is in the big cell.} \end{cases}$$

Notice that, by construction, we have $k(ugp, q) = k(g, u^{-1}q)$ for $p \in P$ and u a unipotent matrix in P.

Definition 28. Let $\mu(g,q)$ be the function on $G \times \overline{\mathcal{X}}$ defined by

$$\mu(g,q) = \det(g)\left(\frac{\Xi_d(q)}{g_*\Xi_d(q)}\right)^{d+1}.$$

This function has the property that

$$g_*(d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}) = \mu(g,\cdot) d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$$

for all $g \in G$.

Proposition 29. The kernel function $k(g,q)$ has the following properties:

1. For fixed q, $k(g,q)$ is continuous on G.
2. If $p \in P$, then $k(pg,q) = \mu(p,q)k(g,p^{-1}q)$.
3. Let s be a fundamental reflection in the Weyl group of G, and let $s' = w_{d+1}sw_{d+1}$, where w_{d+1} is the element of maximal length. Then the function

$$E(g,q) = k(sg,q) - \mu(s,q)k(g,sq)$$

satisfies $E(gs',q) = E(g,q)$ for all $g \in G$.

Corollary 30. The map

$$g \mapsto k(g,\cdot) d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$$

commutes with the action of P.

We will present proofs for each of the main properties of $k(g,q)$ in the following paragraphs.

Proof of 1: Continuity of the Kernel. By construction, $k(g,q)$ is continuous on the big cell of G. The crucial point, therefore, is to show that $k(g,q)$ approaches zero "at infinity."

We will begin by writing $k(g,q)$ more explicitly. Let g be an element of G viewed as a $d+1$ by $d+1$ matrix. For $i \geq j$, let c_{ij} be the $j+1$ by $j+1$ submatrix of g consisting of columns $d-j, \ldots, d$ and rows $0, \ldots, j-1$ together with row i. Let $\delta_{ij} = \det c_{ij}$.

Lemma 31. If g belongs to the big cell of G, then u_g is the matrix with entries u_{ij} where

$$u_{ij} = \begin{cases} 0 \text{ if } i < j, \\ \frac{\delta_{ij}}{\delta_{jj}} \text{ if } i \geq j. \end{cases}$$

Proof. It is straightforward to check that

$$\frac{\delta_{ij}(g)}{\delta_{jj}(g)} = \frac{\delta_{ij}(u_g w_{d+1})}{\delta_{jj}(u_g w_{d+1})}.$$

The matrix $u_g w_{d+1}$ has the form

$$u_g w_{d+1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & u_{10} \\ 0 & \cdots & u_{21} & u_{20} \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & u_{d1} & u_{d0} \end{pmatrix}.$$

For this matrix we have $\delta_{jj} = \pm 1$ and $\delta_{ij} = \pm u_{ij}$, so our expression must be correct for g.

Viewing the δ_{ij} as functions on G, we see that u_g is most naturally viewed as a (matrix–valued) rational function on G. It is also clear from this point of view that the big cell of G is defined as the Zariski open set where the rational function $\delta_{00} \cdots \delta_{d-1,d-1}$ is non–zero. We also obtain an explicit form for $k(g,q)$, as given in the following corollary.

Corollary 32. The function $k(g,q)$ is given explicitly by the formula

$$k(g,q) = \frac{\delta_{00}\delta_{11}\cdots\delta_{d-1,d-1}\Xi_d^d}{(\delta_{00}\Xi_0 + \cdots + \delta_{d0}\Xi_d)(\delta_{11}\Xi_1 + \cdots + \delta_{d1}\Xi_d)\cdots(\delta_{d-1,d-1}\Xi_{d-1} + \delta_{d,d-1}\Xi_d)}$$

Continuing with the proof that $k(g,q)$ is continuous, let us simplify matters somewhat by setting

$$\ell_i = \frac{\delta_{ii}\Xi_d}{\delta_{ii}\Xi_i + \cdots + \delta_{di}\Xi_d}$$

so that $k(g,q) = \prod \ell_i$.

Lemma 33. For any fixed $q \in \overline{\mathcal{X}}$, there is a constant C such that, for any i from 0 to $d-1$, and any g which is not a common zero of $\delta_{ii}, \ldots, \delta_{di}$, we have

$$\omega(\ell_i(g,q)) > C.$$

Proof. Fixing q and varying the δ's, the function ℓ_i can be viewed as a $\hat{\overline{K}}$–valued continuous function on the compact set $\mathbb{P}^{d-i}(K)$ (since q does not lie on any rational hyperplane.) Consequently, the valuation of ℓ_i is bounded below.

The final step in the proof of the continuity of $k(g,q)$ is a consequence of the following elementary lemma.

Lemma 34. Suppose that Y is a topological space and Z is a closed subset of Y. Let f be a continuous $\hat{\overline{K}}$–valued function on Y which vanishes along Z, and let h be a bounded, continuous $\hat{\overline{K}}$–valued function on $Y - Z$. Then fh is a continuous function on Y which vanishes along Z.

Proof of Proposition 29, Part (i). The δ_{i0} are simply the elements of the last column of g, and so have no common zero. Therefore the function ℓ_0 is continuous on G and vanishes on the locus where $\delta_{00} = 0$. We continue now by induction. Suppose that $\ell_0 \cdots \ell_i$ is continuous on G and vanishes on the locus Z_i where $\delta_{00} \cdots \delta_{ii} = 0$. The function ℓ_{i+1} is continuous except at the simultaneous zeros W_{i+1} of $\delta_{i+1,i+1}, \ldots, \delta_{d,i+1}$. We will prove below that W_{i+1} is contained in Z_i. Then by our elementary lemma, $\ell_0 \cdots \ell_{i+1}$ is continuous on G and vanishes on Z_i. However, it is clear that ℓ_{i+1} vanishes on $Z_{i+1} - (W_{i+1} \cup Z_i)$ and therefore on $Z_{i+1} - Z_i$. In this manner we are able to complete the induction.

The only missing step from our proof is the following consequence of the Bruhat decomposition.

Lemma 35. Let W_{i+1} be the simultaneous zeros of $\delta_{i+1,i+1}, \ldots, \delta_{d,i+1}$ on G and let Z_i be the zero locus of $\delta_{00} \cdots \delta_{ii}$. Then $W_{i+1} \subset Z_i$.

Proof. Viewing the right and left P–actions as row and column operations leads easily to the conclusion W_{i+1} and Z_i are each left and right P–invariant, and so each of these sets are unions of Bruhat cells. On the complement to Z_i, none of $\delta_{00}, \ldots, \delta_{ii}$ vanish. Thus the complement to Z_i is contained in a union of cells PwP where each permutation w has the property that $w(d) = 0, w(d-1) = 1, \ldots, w(d-i) = i$. On the other hand, if w' is not in Z_i, and therefore has this property, then $w'(d-i-1)$ must belong to the set $i+1, \ldots, d$. This means that column $d-i-1$ of w' has a one in one of the rows $i+1, \ldots, d$. It follows that one of the determinants $\delta_{i+1,i+1}, \ldots, \delta_{d,i+1}$ is non–zero, so that w' is also not in W_{i+1}.

Proof of 2: P–invariance. Suppose first that u is a unipotent matrix in P. In this case, $\mu(u, q) = 1$, and therefore we need to check that $k(ug, q) = k(g, u^{-1}q)$. Since $u_{ug} = uu_g$, this is immediate from the definition of k.

Next let $t \in T$ be a diagonal matrix. In this situation, we see that

$$\mu(t, q) = \frac{t_0 \cdots t_d}{t_d^{d+1}}.$$

On the other hand,

$$\begin{aligned} k(tg,q) &= (tu_g t^{-1})_* k(1,q) \\ &= \frac{t_0 \cdots t_{d-1}}{t_d^d}(tu_g)_* k(1,q) \\ &= \frac{t_0 \cdots t_{d-1}}{t_d^d} k(1, u_g^{-1} t^{-1} q) \\ &= \frac{t_0 \cdots t_{d-1}}{t_d^d} k(g, t^{-1} q) \\ &= \mu(t,q) k(g, t^{-1} q) \end{aligned}$$

which shows that k is T–invariant. Since P is generated by T and the unipotent matrices, this is sufficient.

Proof of 3: Action of the Weyl Group. The proof of this part of Proposition 29 requires the introduction of significant new ideas. Using the notion of "residue" for logarithmic d–forms on $\mathbb{P}(V)$, we will translate property (3) of the kernel function into an assertion about the geometry of the spherical building $\mathcal{T}$ of flags of subspaces of V. We will rely heavily on the results of Orlik and Terao ([OT]) on finite arrangements of hyperplanes.

Following [OT], let R be the $\mathbb{Z}$–subalgebra of $\Omega^{\cdot}(\mathcal{X})$ generated by 1 and by the logarithmic differential forms df/f where $f = a_0\Xi_0 + \ldots + a_d\Xi_d$ for $a_i \in K$. The algebra R carries an obvious grading, and we will let M denote the summand of R consisting of $(d+1)$–forms. The space M carries a natural G–action.

Similarly, let $\overline{R}$ be the $\mathbb{Z}$–subalgebra of $\Omega^{\cdot}(\overline{\mathcal{X}})$ generated by 1 and df/f where f is a rational function of the form

$$f = \frac{a_0\Xi_0 + \ldots + a_d\Xi_d}{b_0\Xi_0 + \ldots + b_d\Xi_d}$$

with $a_i, b_i \in K$. Let $\overline{M}$ denote the summand of $\overline{R}$ consisting of d–forms. It also carries a natural G–action. It follows from [OT], 3.57 and 3.126, together with a direct limit argument, that the map

$$\begin{aligned} \overline{M} &\to M \\ \eta &\mapsto \eta \wedge \frac{d\Xi_d}{\Xi_d} \end{aligned}$$

is an isomorphism. Moreover, a direct computation shows that this map is G–equivariant.

If h is a hyperplane in V defined over K, then any form η in M has at most a first order pole along h. Consequently ([OT]), we may write

$$(*) \qquad \eta = \eta' \wedge \frac{df}{f} + \theta$$

where f is a linear form in the Ξ_i defining h and η' and θ have no poles along h.

Definition 36. Suppose that η is a differential $(d+1)$–form with at most first order poles along a hyperplane h. Then the residue of η along h is the restriction of the form η' in the expression (*) to h.

As explained in [OT], for $\eta \in M$, the form $\operatorname{res}_h \eta$ has at most first order poles along hyperplanes in h; these poles lie along the intersections of the other poles of η with h. Consequently we may iterate this residue map and define the residue of η along a flag:

Definition 37. Suppose that

$$F : h_0 \supset h_1 \supset \cdots h_{d-1}$$

is a maximal flag of subspaces of V. We define

$$\operatorname{res}_F \eta := \operatorname{res}_{\{0\}} \operatorname{res}_{h_{d-1}} \cdots \operatorname{res}_{h_0} \eta.$$

Notice that $\operatorname{res}_F \eta$ is an element of K.

Let $C_{d-1}(\mathcal{T})$ denote the group of simplicial $d-1$ chains (with integer coefficients) on the $(d-1)$–dimensional spherical building $\mathcal{T}$ of flags in V. We generalize the notion of "residual divisor" of a differential form $\eta \in M$ by defining

$$\operatorname{div}(\eta) := \sum_F (\operatorname{res}_F \eta)[F]$$

where the sum is over all flags F of subspaces of V.

Lemma 38. For all $\eta \in M$, the $(d-1)$–chain $\operatorname{div}(\eta)$ is a cycle, and therefore div is a well–defined map

$$\operatorname{div} : M \to \tilde{H}_{d-1}(\mathcal{T}, \mathbb{Z})$$

commuting with the action of G.

Proof. The G–equivariance of the map is clear. By linearity, it suffices to show that generators of M map to cycles. M is generated by forms

$$\eta = \frac{df_0}{f_0} \wedge \cdots \wedge \frac{df_d}{f_d}.$$

Let h_i denote the hyperplane defined by $f_i = 0$. If $\eta \neq 0$, the set $\{h_0, \ldots, h_d\}$ of $d+1$ codimension one subspaces of V defines an apartment in $\mathcal{T}$, and a direct calculation shows that $\operatorname{div}(\eta)$ is simply the cycle in $\tilde{H}_{d-1}(\mathcal{T}, \mathbb{Z})$ defined by this apartment (see [BS], 2.5).

It is not hard to see that the map div is in fact surjective. By the Solomon–Tits theorem ([G] 2.2), the reduced homology $\tilde{H}_{d-1}(\mathcal{T}, \mathbb{Z})$ is spanned by the cycles corresponding to the set of apartments containing some fixed chamber. Choosing for our fixed chamber the flag of subspaces $F_0 : h_0 \supset \cdots \supset h_{d-1}$ where

$$h_i = \{\text{zeroes of } \Xi_d, \Xi_{d-1}, \ldots, \Xi_{d-i}\}$$

(that is, the flag stabilized by P), we see easily that the form

$$\eta = \frac{d\Xi_0}{\Xi_0} \wedge \cdots \wedge \frac{d\Xi_d}{\Xi_d}$$

has residual divisor

$$\operatorname{div}(\eta) = \sum_w (-1)^{l(w)} w \cdot F_0$$

where w runs through the elements of the Weyl group. If we denote this standard cycle by A_0, then any apartment/cycle A containing F_0 is of the form uA_0 for some unipotent u in P. Therefore $\operatorname{div}(u_*\eta) = A$ and div is surjective.

Theorem 39. The map $\operatorname{div} : M \to \tilde{H}_{d-1}(\mathcal{T}, \mathbb{Z})$ is an isomorphism.

Proof. It is obvious that M is the direct limit of the groups $M_{\mathcal{A}}$ where $\mathcal{A}$ runs through the finite central hyperplane arrangements in V defined over K. Correspondingly, the spherical building $\mathcal{T}$ is the direct limit of the Folkman complexes $F(\mathcal{A})$ ([OT], 4.97). The composite of the various isomorphisms in [OT] 3.110, 3.126, 4.112, and 4.116 is a bijection

$$M_{\mathcal{A}} \xrightarrow{\sim} \tilde{H}_{d-1}(F(\mathcal{A}), \mathbb{Z})$$

which is easily checked to be, up to sign, our residual divisor map.

Corollary 40. The d–forms $k(uw_{d+1}, \cdot) d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$, for u running through the lower triangular unipotent matrices, form a $\mathbb{Z}$–basis of $\overline{M}$.

Because of

$$\tilde{H}_{d-1}(\mathcal{T}, \mathbb{Z}) = \ker(\mathbb{Z}[G/P] \to \oplus_s \mathbb{Z}[G/P_s]),$$

there is an obvious injective map

$$\tilde{H}_{d-1}(\mathcal{T}, \mathbb{Z}) \to \operatorname{Hom}(\mathbf{St}, \mathbb{Z}).$$

Let

$$\operatorname{dis} : \overline{M} \to \operatorname{Hom}(\mathbf{St}, \mathbb{Z})$$

denote its composite with the residual divisor map. Also, for $g \in G$, let

$$\begin{aligned} \delta_g : C^\infty(G/P, \mathbb{Z}) &\to \mathbb{Z} \\ f &\mapsto f(g) \end{aligned}$$

denote the Dirac distribution. Clearly, the image of dis consists of appropriate linear combinations of Dirac distributions. More precisely, we have the following:

Remark. If u is a lower triangular unipotent matrix, then

$$\mathrm{dis}(k(uw_{d+1},\cdot)d\Xi_{\beta_0}\wedge\cdots\wedge d\Xi_{\beta_{d-1}})=\sum_{w}(-1)^{l(w)}\delta_{uw}.$$

Proof of Property 3 of Proposition 29. The proof of Property (3) is obtained by interpreting the property as a statement about the geometry of the spherical building using the map div defined above.

If h is a cycle in $\tilde{H}_{d-1}(\mathcal{T},\mathbb{Z})$, we let $\mathrm{Supp}(h)$ denote the set of chambers of $\mathcal{T}$ which occur in h with non–zero coefficient.

Let $A(g)$ be the cycle in $\mathcal{T}$ defined by

$$A(g)=\mathrm{div}(k(g,\cdot)d\Xi_{\beta_0}\wedge\cdots\wedge d\Xi_{\beta_{d-1}}).$$

By the injectivity of div, we see that Property 3 is equivalent to the assertion that $A(sg)-sA(g)$ (as a function of $g\in G$ with values in $\tilde{H}_{d-1}(\mathcal{T},\mathbb{Z})$) is invariant on the right by $s'=w_{d+1}sw_{d+1}$ for each fundamental reflection s in the Weyl group of G.

For purposes of this proof, let C denote the fundamental chamber in $\mathcal{T}$; that is, C is the unique chamber stabilized by P.

0. By continuity, it suffices to prove Property 3 under the hypothesis that g, sg, gs', and sgs' all belong to the big cell of G.

1. If gP is in the big cell of G/P, then $A(g)$ is the $(d-1)$–cycle in $\mathcal{T}$ determined by the unique apartment in which C and gC are opposite chambers. (See [BS], paragraph 2.5 for example.) We write $[C:C']$ for the cycle/apartment in which C and C' are opposite.

Proof: This is a simple calculation using the map div of the previous section.

2. Recall that we assume that gP and sgP are both in the big cell of G. Then $A(sg)=[C:sgC]$ and $sA(g)=[sC:sgC]$. These two cycles both contain the chamber sgC and the codimension one face $C\cap sC$.

3. Write $gP=u_gw_{d+1}P$ and $sgP=u_{sg}w_{d+1}P$. Then C, sC, and $u_{sg}sC$ are all distinct, and meet along the codimension one face $C\cap sC$.

Proof: The only way this can fail is for $u_{sg}sC = sC$. But $u_{sg}w_{d+1}C = sgC$ means that $su_{sg}w_{d+1}C = gC$. Choosing a representative for w_{d+1} which begins with s yields $gC \in PwC$ with $l(w) = l(w_{d+1}) - 1$. This contradicts the assumption that g is in the big cell.

4. A chamber D belongs to $\mathrm{Supp}([C : sgC]) \setminus \mathrm{Supp}([sC : sgC])$ if and only if $D = u_{sg}wC$ with $l(sw) = l(w) + 1$.

Proof: $D \in \mathrm{Supp}([C : sgC])$ means that $D = u_{sg}wC$ for some w. If $l(sw) = l(w) - 1$, then there is a minimal gallery from C to D crossing $C \cap sC$ (but, by (3), NOT passing through sC). This extends to a minimal gallery through C, D, and sgC. If we replace the initial chamber C by sC we obtain a gallery of the same length starting with sC, passing D, and ending in sgC. By convexity of apartments, the latter gallery lies in $[sC : sgC]$, and therefore so does D. Conversely, if $D \in \mathrm{Supp}([C : sgC]) \cap \mathrm{Supp}([sC : sgC])$, then we may find a minimal gallery in $[sC : sgC]$ through D passing through $C \cap sC$. Substituting C for sC in the initial stage yields a minimal gallery in $[C : sgC]$ through D crossing $C \cap sC$. This in turn implies that we may write $D = u_{sg}wC$ with w having initial term s, and therefore $l(sw) = l(w) - 1$.

5. The only chamber in $[C : sgC]$ adjacent to $u_{sg}w_{d+1}C$ but NOT in $[C : sgC] \cap [sC : sgC]$ is $u_{sg}sw_{d+1}C$.

Proof: The chambers adjacent to $u_{sg}w_{d+1}C$ are $u_{sg}w_{d+1}tC$ where $t \in S$. The adjacent chamber does not lie in the intersection precisely when $l(sw_{d+1}t) = l(w_{d+1}t) + 1 = l(w_{d+1})$ and by uniqueness of w_{d+1} we then have $sw_{d+1}t = w_{d+1}$ or $w_{d+1}t = sw_{d+1}$.

6. The chambers belonging to $[C : sgC]$ but not to $[sC : sgC]$ are precisely those belonging to a minimal gallery of length $l(w_{d+1}) - 1$ stretched from C to $u_{sg}w_{d+1}(C \cap s'C)$. The chambers belonging to $[sC : sgC]$ but not to $[C : sgC]$ are precisely those belonging to a minimal gallery of length $l(w_{d+1}) - 1$ stretched from sC to $u_{sg}w_{d+1}(C \cap s'C)$.

Proof: The first claim is a re-statement of the result in (5). The second follows from a parallel argument applied to the apartment $[sC : sgC]$.

7. The cycle $A(sg) - sA(g)$ is supported on precisely those chambers belonging to $A(sg)$ or $sA(g)$, but not both.

Proof: The orientation of a chamber is independent of the apartment to which it belongs.

Property 3: The difference $A(sg) - sA(g)$ is invariant on the right by s'.

Proof: The description of the difference in (6) and (7) shows that it depends only on s and on the codimension one face $u_{sg}w_{d+1}(C \cap s'C)$. But sgC and $sgs'C$ share this face, and consequently give rise to the same cycle.

The Main Theorem. In this section we present the main theorem of this paper. We begin with an elementary proposition from the theory of p–adic integration which will enable us to define our integral transform.

Proposition 41. Let λ be an element of $\mathrm{Hom}(C^\infty(G/P, \mathbb{Z}), o)$. Then the distribution λ extends to a continuous linear functional on the space $C(G/P, \hat{\overline{K}})$ of continuous $\hat{\overline{K}}$–valued functions on G/P. If λ vanishes on $\sum_s C^\infty(G/P_s, \mathbb{Z})$, then the extension of λ vanishes on $\sum_s C(G/P_s, \hat{\overline{K}})$.

In general, if λ is a distribution viewed as a linear map on continuous functions, we will write

$$\int_U f d\lambda := \lambda(\chi_U f)$$

for $f \in C(G/P, \hat{\overline{K}})$ and $U \subset G/P$ compact open.

We now state our main theorem.

Theorem 42. Suppose that λ is an element of $\mathrm{Hom}(\mathbf{St}, o)$. Then the function

$$F_\lambda(q) = \int_{G/P} k(g, q)\, d\lambda$$

is a rigid analytic function on $\overline{\mathcal{X}}$. In addition, the map

$$\lambda \mapsto \eta_\lambda := F_\lambda d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}$$

has the following properties:

1. $\eta_{h\lambda} = h_*(\eta_\lambda)$ for $h \in G$.

2. The harmonic function $\mathrm{Res}\,\eta_\lambda$ determines the distribution λ under the isomorphism of Proposition 11.

Proof. The integral exists by the continuity of the kernel function (property (1) of Proposition 29). The G–equivariance relation (1) follows directly from properties

(2) and (3) of the kernel function. Indeed, we know by properties (2) and (3) of the kernel that

$$k(hg,q) \equiv \mu(h,q)k(g,h^{-1}q) \pmod{\sum_s C(G/P_s, \hat{\overline{K}})}$$

for any h which lies in P or is a fundamental reflection. Therefore, for these h,

$$\begin{aligned} \eta_{h\lambda} &= \lambda(k(hg,q))d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} \\ &= \lambda(\mu(h,q)k(g,h^{-1}q))d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} \\ &= \lambda(k(g,h^{-1}q))(\mu(h,q)d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}) \\ &= h_*\eta_\lambda. \end{aligned}$$

as claimed. By the Bruhat decomposition, the same must then hold for any h in G.

Showing that F_λ is rigid analytic and that it has the correct residues will require more extended discussion in the next two paragraphs.

Analyticity. In order to show that F_λ is rigid analytic, it is most convenient to view the symmetric space $\overline{\mathcal{X}}$ as a direct limit of affinoid varieties $\overline{\mathcal{X}}_n$, as in [SS], Section 1. We recall very briefly the definition of the subdomains $\overline{\mathcal{X}}_n$. Let $\mathcal{H}$ denote the set of hyperplanes in $\mathbb{P}(V)$ which are defined over K, and for any $H \in \mathcal{H}$ let ℓ_H be a unimodular element of V^* such that H is the zero set of ℓ_H.

Definition 43. The set $\overline{\mathcal{X}}_n$ consists of the set of points $q \in \mathbb{P}(V)$ such that

$$\omega(\ell_H((q_0,\cdots,q_d))) \le n$$

for any $H \in \mathcal{H}$ whenever $[q_0 : q_1 : \cdots : q_d]$ is a unimodular representative for the homogeneous coordinates of q.

Proposition 44. The covering of $\overline{\mathcal{X}}$ by the sets $\{\overline{\mathcal{X}}_n\}$ has the following properties:

1. A function F on $\overline{\mathcal{X}}$ is rigid analytic if and only if its restriction to each $\overline{\mathcal{X}}_n$ is rigid analytic.

2. Let $f = \sum_{j=0}^{d-1} a_j \Xi_{\beta_j} + a_d$ with $a_0, \ldots, a_d$ elements of K. Then, for all $q \in \overline{\mathcal{X}}_n$, we have
$$|\omega(f(q)) - \inf_{j=0}^{d}\{\omega(a_j)\}| \le n.$$

Proof. Part (1) is a restatement of the results in [SS], Section 1. Part (2) is an immediate consequence of the definition of $\overline{\mathcal{X}}_n$.

The result in part (2) of this proposition enables us to improve our understanding of the kernel function $k(g,q)$.

For each non–negative integer m, let $U(m)$ be the compact open subset of the unipotent radical of P consisting of lower triangular matrices (u_{ij}) with $\omega(u_{ij}) \geq -m$ for all $i > j$. The sets $U(m)w_{d+1}P/P$ form an increasing sequence of compact open subsets of the big cell of G/P. Let $V(m)$ denote the complement to $U(m)w_{d+1}P/P$ in G/P.

Lemma 45. Fix $n > 0$. For any $N > 0$ there exists an m such that $\omega(k(g,q)) > N$ for all $q \in \overline{\mathcal{X}}_n$ and all $g \in V(m)$.

Proof. Since $k(g,q)$ vanishes off of the big cell of G, we need only consider cosets gP in $V(m)$ which belong to the big cell $Pw_{d+1}P$. From the definition of k, we may assume that g is lower triangular and unipotent, so that $u_g = g$. From our explicit representation for k, we see that

$$k(g,q) = \prod_i \frac{1}{f_i(g,q)}$$

where $f_i(g,q) = g_{di} + \sum_{j=i}^{d-1} g_{ji}\Xi_{\beta_j}$. Applying our earlier lemma for $q \in \overline{\mathcal{X}}_n$, we see that

$$\omega(f_i(g,q)) - \inf_j\{\omega(g_{ji})\} \leq n$$

uniformly for $q \in \overline{\mathcal{X}}_n$. The corresponding estimate for k yields

$$\omega(k(g,q)) + \inf_{i,j}\{\omega(g_{ij})\} \geq \omega(k(g,q)) + \sum_j \inf_i\{\omega(g_{ij})\} \geq -dn.$$

The first inequality comes from the fact that $g_{ii} = 1$, and therefore all the ocurring infima are non–positive. If $g \in V(m)$, then $\inf_{i,j}\{\omega(g_{ij})\} < -m$. Thus choosing m large enough so that $m - dn > N$, we obtain our desired estimate.

Thanks to this result, we can replace the integral over G/P in our transform with an integral over the big cell $Pw_{d+1}P$, which is simply an affine space.

Lemma 46. The function F_λ can be constructed as the sum

$$F_\lambda = \int_{U(0)} k\,d\lambda + \sum_{m=1}^{\infty} \int_{U(m)-U(m-1)} k\,d\lambda.$$

Furthermore, if each term in the sum is rigid analytic, then so is F_λ.

Proof. By the estimate in the preceeding lemma the above series converges uniformly for $q \in \overline{\mathcal{X}}_n$.

Since each of the sets U_0, $U_1 - U_0$, and so forth are compact open, Lemma 12 shows that each may be covered by finitely many open sets of the form $\mathbf{B}(u,r)$ for lower triangular unipotent u and r any fixed sufficiently large integer. Thus if we can show that the integral of k over sufficiently small sets $\mathbf{B}(u,r)$ is rigid analytic, then we are done.

Proposition 47. Choose r large enough that if $v = (v_{ij})$ belongs to $\mathbf{B}(u,r)$ where $u = (u_{ij})$, then $\omega(u_{ij} - v_{ij}) > 2n$. Then

$$F_\lambda^{u,r} = \int_{\mathbf{B}(u,r)} k d\lambda$$

is rigid analytic on $\overline{\mathcal{X}}_n$.

Proof. We know that k is a product of $1/f_i$ where each $f_i = \sum_{j=i}^{d-1} a_{ji}\Xi_{\beta_j} + a_{di}$, and the a_{ij} are coordinate functions on the affine space with $a_{ii} = 1$. On $\mathbf{B}(u,r)$, we may write

$$f_i = f_i(u,q) + \sum_{j=i+1}^{d-1} b_{ji}\Xi_{\beta_j} + b_{di}$$

where the $b_{ij} := a_{ij} - u_{ij}$ are functions on $\mathbf{B}(u,r)$ with $\omega(b_{ij}) > 2n$. Since we are assuming $q \in \overline{\mathcal{X}}_n$, we know that

$$\omega(f_i(u,q)) \le n + \inf_j\{\omega(u_{ji})\} \le n$$

and

$$\omega(\sum_{j=i+1}^{d-1} b_{ji}\Xi_{\beta_j} + b_{di}) \ge -n + \inf\{\omega(b_{ij})\} > n$$

uniformly on $\mathbf{B}(u,r)$. Consequently we may expand

$$(**) \qquad \frac{1}{f_i} = \frac{1}{f_i(u,q)}\left(1 + \frac{\sum_{j=i+1}^{d-1} b_{ji}\Xi_{\beta_j} + b_{di}}{f_i(u,q)}\right)^{-1}$$

in a uniformly convergent power series on $\mathbf{B}(u,r)$. The kernel function k, a product of such series, can be expressed as a uniformly convergent power series on $\mathbf{B}(u,r)$. This series may be integrated term by term exhibiting $F_\lambda^{u,r}$ as a rigid analytic function on $\overline{\mathcal{X}}_n$.

Corollary 48. F_λ is rigid analytic.

Proof. As observed above, each term in the convergent expansion for F_λ in Lemma 46 is a finite sum of terms of the form $F_\lambda^{u,r}$; each of these is rigid analytic by the Proposition.

Residue Property. To complete the proof of the main theorem, we must show that the function F_λ has the correct residues. Since the sets of the form $\mathbf{B}(u,r)$ cover the big cell, it suffices to check the residues of F_λ on chambers of the form $\Delta(u,r)$. Fix such a chamber $\Delta(u,r)$ and choose n large enough that $\mathrm{r}(\overline{\mathcal{X}}_n) \supset \Delta(u,r)$. As we saw in the proof of the analyticity of F_λ, on $\overline{\mathcal{X}}_n$ we may write

$$F_\lambda = \sum_{(u',r')} F_\lambda^{u',r'}$$

where the sum is over a collection of pairs (u',r') such that the balls $\mathbf{B}(u',r')$ cover the big cell and are sufficiently small. We will show that the residues of F_λ are correct by establishing the following two facts:

Fact A: If $\mathbf{B}(u,r)$ and $\mathbf{B}(u',r')$ are disjoint, then

$$\mathrm{Res}_{\Delta(u,r)} F_\lambda^{u',r'} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} = 0.$$

Fact B. If $\mathbf{B}(u',r') \subset \mathbf{B}(u,r)$ then

$$\mathrm{Res}_{\Delta(u,r)} F_\lambda^{u',r'} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} = \int_{\mathbf{B}(u',r')} d\lambda.$$

It is clear from the expression of F_λ as a sum of $F_\lambda^{u',r'}$ over disjoint sets $\mathbf{B}(u',r')$ that Facts A and B imply that

$$\mathrm{Res}_{\Delta(u,r)} \eta_\lambda = \int_{\mathbf{B}(u,r)} d\lambda$$

as required.

Before giving the proof of A and B, we reduce the matters at issue to questions about the geometry of the building $\overline{\mathcal{X}}$.

Lemma 49. Put

$$\xi := \frac{d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}}{\Xi_{\beta_0} \cdots \Xi_{\beta_{d-1}}}.$$

We have:

$$\mathrm{Res}_{\Delta(u,r)} F_\lambda^{u',r'} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} = (\mathrm{Res}_{\Delta(u,r)} u'_*\xi) \int_{\mathbf{B}(u',r')} d\lambda.$$

Proof. Using the series expansion of $1/f_i$ given in equation (**) above, we see that

$$\begin{aligned} F_\lambda^{u',r'}(q) &= \int_{\mathbf{B}(u',r')} k \, d\lambda \\ &= \sum_{n_0,\ldots,n_{d-1} \geq 0} \frac{\int_{\mathbf{B}(u',r')} h_0(g,q)^{n_0} \cdots h_{d-1}(g,q)^{n_{d-1}} \, d\lambda(g)}{f_0(u',q)^{n_0+1} \cdots f_{d-1}(u',q)^{n_{d-1}+1}} \end{aligned}$$

where

$$\begin{aligned} h_i(g,q) &= f_i(u',q) - f_i(g,q) \\ &= u'_* \Xi_{\beta_i}(q) - (u_g)_* \Xi_{\beta_i}(q). \end{aligned}$$

Notice that, since u_g and u' are unipotent matrices, $(u')_*^{-1} h_i(g,q)$ is a linear form in the Ξ_{β_j} for $i < j \le d-1$. To facilitate our later computations, set

$$c(n_0, \dots, n_{d-1}) := \int_{\mathbf{B}(u',r')} (u')_*^{-1} (h_0(g,q)^{n_0} \cdots h_{d-1}(g,q)^{n_{d-1}})\, d\lambda(g).$$

and observe that

$$df_0(u',q) \wedge \cdots \wedge df_{d-1}(u',q) = d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}.$$

We may now write

$$\begin{aligned} &\frac{\int_{\mathbf{B}(u',r')} h_0(g,q)^{n_0} \cdots h_{d-1}(g,q)^{n_{d-1}}\, d\lambda(g)}{f_0(u',q)^{n_0+1} \cdots f_{d-1}(u',q)^{n_{d-1}+1}} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} \\ &\qquad = u'_* \frac{c(n_0, \dots, n_{d-1})}{\Xi_{\beta_0}^{n_0+1} \cdots \Xi_{\beta_{d-1}}^{n_{d-1}+1}} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}. \end{aligned}$$

Suppose that n_k is not zero, but $n_j = 0$ for $0 \le j < k$. Then $c(0, \dots, 0, n_k, \dots, n_{d-1})$ involves only the variables Ξ_{β_i} for $k < i \le d-1$. Consequently

$$\begin{aligned} &\frac{(-1)^{k+1}}{n_k} du'_* \left(\frac{c(0,\dots,0,n_k,\dots,n_{d-1})}{\Xi_{\beta_0}\Xi_{\beta_1}\cdots\Xi_{\beta_{k-1}}\Xi_{\beta_k}^{n_k}\Xi_{\beta_{k+1}}^{n_{k+1}+1}\cdots\Xi_{\beta_{d-1}}^{n_{d-1}+1}} d\Xi_{\beta_0} \wedge \cdots \wedge \widehat{d\Xi_{\beta_k}} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} \right) \\ &\qquad = u'_* \frac{c(0,\dots,0,n_k,\dots,n_{d-1})}{\Xi_{\beta_0}\cdots\Xi_{\beta_{k-1}}\Xi_{\beta_k}^{n_k+1}\cdots\Xi_{\beta_{d-1}}^{n_{d-1}+1}} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}}. \end{aligned}$$

Since exact forms have zero residues, we see that

$$\begin{aligned} &\mathrm{Res}_{\Delta(u,r)} F_\lambda^{u',r'}\, d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} \\ &\quad = \sum_{n_0,\dots,n_{d-1} \ge 0} \mathrm{Res}_{\Delta(u,r)}\, u'_* \frac{c(n_0,\dots,n_{d-1})}{\Xi_{\beta_0}^{n_0+1}\cdots\Xi_{\beta_{d-1}}^{n_{d-1}+1}} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} \\ &\quad = \mathrm{Res}_{\Delta(u,r)}\, u'_* \frac{c(0,\dots,0)}{\Xi_{\beta_0}\cdots\Xi_{\beta_{d-1}}} d\Xi_{\beta_0} \wedge \cdots \wedge d\Xi_{\beta_{d-1}} \\ &\quad = c(0,\dots,0)\, \mathrm{Res}_{\Delta(u,r)}\, u'_*\xi \\ &\quad = \Big(\int_{\mathbf{B}(u',r')} d\lambda\Big)\, \mathrm{Res}_{\Delta(u,r)}\, u'_*\xi. \end{aligned}$$

as claimed.

In the light of this lemma, we reconsider A and B above.

Proof of A. We need to show that if $\mathbf{B}(u,r)$ and $\mathbf{B}(u',r')$ are disjoint, then

$$\operatorname{Res}_{\Delta(u,r)} u'_*\xi = 0.$$

Using equivariance, it suffices to consider the case $u' = 1$ and $r' = 1$. By Lemma 23, the residues of ξ are supported on the standard apartment $\overline{A}$, and therefore we need only show that if $\mathbf{B}(u,r)$ is disjoint from $\mathbf{B}(1,1)$ then $\Delta(u,r)$ does not belong to $\overline{A}$. Write $u = t^m u_0 t^{-m}$ with $u_0 \in B$. If $\Delta(u,r)$ DOES belong to $\overline{A}$, then so does $t^{-m}\Delta(u,r) = \Delta(u_0, m+r)$. But (by the Bruhat decomposition) this can only happen if $\Delta(u_0, m+r) = \Delta(1, m+r)$ which implies that $\Delta(u,r) = \Delta(1,r)$. Obviously $\mathbf{B}(1,r)$ and $\mathbf{B}(1,1)$ are not disjoint, so we have proved A.

Proof of B. As in the proof of A, we may reduce to the case where $u = 1$ and $r = 1$, so we must prove that if $\mathbf{B}(u,r) \subset \mathbf{B}(1,1)$ then

$$\operatorname{Res}_{\Delta(1,1)} u_*\xi = 1.$$

The assumption implies that u is contained in $\mathbf{B}(1,1)$. But $\mathbf{B}(1,1) = B'w_{d+1}P/P$ where the group $B' := w_{d+1}tBt^{-1}w_{d+1}$ stabilizes the pointed chamber $\Delta(1,1)$. Thus $u \in B'$ and the residue in question is:

$$\begin{aligned}
\operatorname{Res}_{\Delta(1,1)} \xi &= \operatorname{Res}_{(\overline{C},0)}(t^{-1}w_{d+1})_*\xi \\
&= (-1)^{d(d+1)/2} \operatorname{Res}_{(\overline{C},0)}(t^{-1}w_{d+1})_* \frac{d\Xi_{\alpha_{d-1}}}{\Xi_{\alpha_{d-1}}} \wedge \cdots \wedge \frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}} \\
&= (-1)^{d(d-1)/2} \operatorname{Res}_{(\overline{C},0)}(t^{-1})_* \frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}} \wedge \cdots \wedge \frac{d\Xi_{\alpha_{d-1}}}{\Xi_{\alpha_{d-1}}} \\
&= \operatorname{Res}_{(\overline{C},0)} \frac{d\Xi_{\alpha_{d-1}}}{\Xi_{\alpha_{d-1}}} \wedge \cdots \wedge \frac{d\Xi_{\alpha_0}}{\Xi_{\alpha_0}} \\
&= 1.
\end{aligned}$$

For forms $\eta \in \overline{M} \subset \Omega^d(\overline{\mathcal{X}})$, both $\operatorname{Res}\eta$ and $\operatorname{dis}(\eta)$ are linear forms on $\mathbf{St}$. There is a very simple relation between the two.

Corollary 50. For $\eta \in \overline{M}$, we have $\operatorname{Res}\eta = (-1)^{d(d+1)/2}\operatorname{dis}(\eta)$.

Proof. Because of Corollary 40, it suffices to consider $\eta = \xi$. Then

$$\operatorname{dis}(\xi) = \sum_w (-1)^{l(w)}\delta_w =: \lambda.$$

On the other hand, from the definitions we have

$$\eta_\lambda = (-1)^{d(d+1)/2} \cdot \xi$$

and hence, by Theorem 42,

$$(-1)^{d(d+1)/2} \operatorname{Res}\xi = \operatorname{Res}\eta_\lambda = \lambda.$$

Conclusion

This paper represents the first step in generalizing the extensive analytic theory of the p–adic upper half plane to the higher dimensional p–adic symmetric spaces. Two major questions which are not addressed in this paper concern the study of r–forms for $r < d$ and the theory of integral transforms for more general p–adic discrete series representations. In particular, the method of the Poisson Kernel in the one dimensional situation was used in [T2] to introduce an integral structure on the p–adic discrete series representations for SL_2 constructed by Morita [Mo], and to investigate the reductions mod p of these representations. Generalizing this work to the p–adic discrete series constructed in [S2] is a natural problem which will require a theory relating elements of these representation spaces to some type of admissible measure on G/P. To illustrate some of the difficulties in such a theory, the kernel function $k(g, x)$ which we introduce is not locally analytic on G/P (it fails to be locally analytic on a subset of the complement to the big cell) and so it is not clear how to integrate this function against an unbounded measure. These problems are the subject of our ongoing research, and we hope to return to them in a later paper.

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