Tempered representations of p-adic groups: Special idempotents and topology

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For the 70th birthday of Joseph Bernstein

Introduction

Throughout this paper k is a locally compact nonarchimedean field. Let G be the group of k-rational points of a connected reductive k-group. As usual we denote by $\mathcal{H} = \mathcal{H}(G)$ and $\mathcal{S} = \mathcal{S}(G)$ the Hecke and Schwartz algebra of G, respectively. The category of nondegenerate $\mathcal{H}$-modules, resp. $\mathcal{S}$-modules, is denoted by $\mathcal{M}(G)$, resp. $\mathcal{M}^t(G)$. We have the forgetful functor $\mathcal{M}^t(G) \longrightarrow \mathcal{M}(G)$. The second category $\mathcal{M}(G)$ coincides with the category of all smooth G-representations. A well known theory by Bernstein decomposes the category

$$\mathcal{M}(G) = \prod_{\Omega} \mathcal{M}(\Omega)$$

into the direct product of component categories $\mathcal{M}(\Omega)$ where Ω runs over the connected components of the center of $\mathcal{H}$ (i.e., this center is the direct product, indexed by Ω, of rings without nontrivial idempotents). In the papers [SZ1] and [SZ2] we had established a similar decomposition

$$\mathcal{M}^t(G) = \prod_{\Theta} \mathcal{M}^t(\Theta)$$

where Θ now runs over the connected components of the center of $\mathcal{S}$. In [BK] the theory of types for smooth G-representations was embedded into the following formalism. Whenever one has an idempotent $f \in \mathcal{H}$ one can form the full subcategory $\mathcal{M}_f(G)$ of all smooth G-representations V in $\mathcal{M}(G)$ such that $\mathcal{H} f V = V$. The idempotent f is called special if $\mathcal{M}_f(G)$ is an abelian subcategory. Types then can be seen as particular special idempotents in $\mathcal{H}$ which are traces of certain irreducible representations of compact open subgroups of G. Bernstein had shown that on the one hand the subcategory $\mathcal{M}_f(G)$, for a special idempotent f, necessarily is the direct product of certain full component categories $\mathcal{M}(\Omega)$ and that, on the other hand, every single component category $\mathcal{M}(\Omega)$ can be defined by a special idempotent. The first goal of this paper in section 1 is to establish a full analog of this theory for the algebra $\mathcal{S}$ and the category $\mathcal{M}^t(G)$. This is achieved in Prop. 1.2 and Prop. 1.4. Although we follow the strategy layed out in [BK] we have to use, of course, our theory from [SZ1] and [SZ2]. In addition the key part of the argument is different since the algebra $\mathcal{S}$ is related to C^∞-functions (on tori), whereas $\mathcal{H}$ is related to algebraic functions.

In our paper [SSZ] we had constructed, in the case of the groups $G = GL_n(k)$, actual types for certain finite products of component categories $\mathcal{M}^t(\Theta)$. In the present language these types correspond to idempotents $e \in \mathcal{H}$ which become special when considered in $\mathcal{S}$. This raises the following question for a general group G. Let $\mathcal{M}_e^t(G)$ be the abelian subcategory for a special idempotent $e \in \mathcal{S}$. From Prop. 1.3 we will know that there is a finite set θ_e of components Θ such that

$$\mathcal{M}_e^t(G) = \prod_{\Theta \in \theta_e} \mathcal{M}^t(\Theta) \ .$$

Which restriction on the sets θ_e does the condition that e lies in $\mathcal{H}$ impose? It turns out that this restriction is of a topological nature. Let $\mathrm{Irr}^t(G) \subseteq \mathrm{Irr}(G)$ denote the sets of isomorphism classes of irreducible tempered and irreducible smooth G-representations. We view $\mathrm{Irr}(G)$ as a topological space for the Jacobson topology defined by the algebra $\mathcal{H}$. Then the subset $\mathrm{Irr}^t(G)$ is dense in $\mathrm{Irr}(G)$. Any component Θ gives rise to a connected component of the subspace $\mathrm{Irr}^t(G)$, which we will denote by $(\nu^t)^{-1}(\Theta)$. Using the closure operation in $\mathrm{Irr}(G)$ we now introduce a preorder $\precsim$ on the set of all components Θ by the requirement that $\Theta' \precsim \Theta$ if there is a chain of components $\Theta' = \Theta_0, \ldots, \Theta_n = \Theta$ such that $(\nu^t)^{-1}(\Theta_i) \cap \overline{(\nu^t)^{-1}(\Theta_{i+1})} \neq \emptyset$ for all $i = 0, \ldots, n-1$. It turns out (Remark 1.8) that the set θ_e, if $e \in \mathcal{H}$, must be saturated in the sense that with Θ any $\Theta' \succsim \Theta$ also must lie in θ_e. This motivates our aim to better understand this preorder $\precsim$.

In section 2 we restrict to the case that our group G is k-split and has connected center. We also restrict attention to the Iwahori component $\mathcal{M}_{\epsilon_J}(G)$. This is the single component in the Bernstein decomposition of $\mathcal{M}(G)$ which is defined by the special idempotent $\epsilon_J \in \mathcal{H}$ which is the characteristic function of the Iwahori subgroup J of G. This will allow us to explore the Kazhdan-Lusztig classification of simple modules for the Iwahori-Hecke algebra in [KL] for our purposes. Let $\mathrm{Irr}_{\epsilon_J}(G) \subseteq \mathrm{Irr}(G)$ denote the subset of isomorphisms classes of irreducible representations in $\mathcal{M}_{\epsilon_J}(G)$, which is a connected component of the space $\mathrm{Irr}(G)$. We have

$$\mathrm{Irr}_{\epsilon_J}(G) \cap \mathrm{Irr}^t(G) = \dot{\bigcup}_{\Theta \in \theta_{\epsilon_J}} (\nu^t)^{-1}(\Theta) \ .$$

On the other hand let $\mathcal{O}_{\widehat{G}}$ denote the set of unipotent orbits in the connected Langlands dual group $\widehat{G}$ over $\mathbb{C}$. This finite set carries a natural partial order defined by the Zariski closure relation between unipotent orbits. The Kazhdan-Lusztig classification gives rise to a natural surjective map

$$\pi : \mathrm{Irr}_{\epsilon_J}(G) \longrightarrow \mathcal{O}_{\widehat{G}} \ . \tag{1}$$

Our main results are:

I. (Thm. 2.3) $\overline{\pi^{-1}(\mathcal{O})} \subseteq \bigcup_{\mathcal{O}' \geq \mathcal{O}} \pi^{-1}(\mathcal{O}')$ for any $\mathcal{O} \in \mathcal{O}_{\widehat{G}}$.

II. (Thm. 2.18) For any $\Theta \in \theta_{\epsilon_J}$, the map π is constant on the subset $(\nu^t)^{-1}(\Theta)$ (and we let $\mathcal{O}_\Theta$ denote the corresponding value).

III. (Thm. 2.18) If $\Theta' \precsim \Theta$, for any $\Theta, \Theta' \in \theta_{\epsilon_J}$, then $\mathcal{O}_{\Theta'} \geq \mathcal{O}_\Theta$.

We see that our topological preorder $\precsim$ is closely related to the partial order $\leq$ on $\mathcal{O}_{\widehat{G}}$.

In the final subsection we let $G = GL_n(k)$ be a general linear group. In this case the map $\theta_{\epsilon_J} \xrightarrow{\simeq} \mathcal{O}_{\widehat{G}}$ sending Θ to $\mathcal{O}_\Theta$ is a bijection. Note that the pair $(\mathcal{O}_{\widehat{G}}, \leq)$ can naturally be identified with the set of partitions of n equipped with the dominance partial order. It follows from III. that the preorder $\precsim$ (restricted to θ_{ϵ_J}), in fact, is a partial order. Somewhat unexpectedly we will show that, for general n, the image of $\precsim$ on $\mathcal{O}_{\widehat{G}}$ is strictly coarser than the reverse of the partial order $\leq$. On the other hand we also will show that this image is strictly finer than the reverse of the refinement partial order on $\mathcal{O}_{\widehat{G}}$. Indeed the latter reflects the stronger inclusion relation $(\nu^t)^{-1}(\Theta') \subseteq \overline{(\nu^t)^{-1}(\Theta)}$ (Cor. 2.25). The methods in this final subsection rely on the Bernstein-Zelevinsky classification of irreducible smooth representations of general linear groups in [Zel]. The precise form of the relation between the Kazhdan-Lusztig and Bernstein-Zelevinsky classifications must be known to the experts. But it is nowhere written in the literature. We therefore present the details of this comparison in an appendix.

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Notations: Let $|\ |_k$ denote the normalized absolute value of k. We fix a prime element ϖ of k, and we let q denote the cardinality of the residue class field of k (hence $|\varpi|_k = q^{-1}$). The multiplication in the algebras $\mathcal{H} \subseteq \mathcal{S}$ as well as their action on a module sometimes is denoted by a $*$ (for convolution). As a general convention we write the left and right translation action of a $g \in G$ on any locally constant function ϕ on G as $({}^g\phi)(h) := \phi(g^{-1}h)$ and $(\phi^g)(h) := \phi(hg^{-1})$.

For any compact open subgroup $U \subseteq G$ we let $\mathcal{H}(G,U)$, resp. $\mathcal{S}(G,U)$, denote the subalgebra of all U-bi-invariant functions in $\mathcal{H}(G)$, resp. $\mathcal{S}(G)$. Both these algebras are unital with the unit being the idempotent $\epsilon_U(g) = \mathrm{vol}_G(U)^{-1}$, resp. $= 0$, for $g \in U$, resp. $g \notin U$, corresponding to U. The map $g \mapsto g^{-1}$ on G induces on any of the rings $\mathcal{H}(G,U)$ and $\mathcal{S}(G,U)$ a canonical anti-involution so that, for these rings, we do not have to distinguish between left and right modules.

For any unital ring R we let $\mathcal{M}(R)$ be the category of left unital R-modules, and we let $Z(R)$ denote the center of R.

1 Special idempotents

We begin by recalling that a discrete pair (L,τ) of G consists of a Levi subgroup L of G and an irreducible discrete series representation τ of L. The group G acts by conjugation on the set of isomorphism classes of discrete pairs, and the Harish Chandra spectrum $\Omega^t(G)$ of G is defined to be the set of G-orbits of this action. For any discrete pair (L,τ) we have the map

$$\begin{aligned} X^1_{nr}(L) &\longrightarrow \Omega^t(G) \\ \chi &\longmapsto G\text{-orbit of } [(L,\chi\tau)] \end{aligned}$$

where $X^1_{nr}(L)$ denotes the compact torus of unitary unramified characters of L. The images of these maps partition $\Omega^t(G)$; they are called the connected components of $\Omega^t(G)$.

Let $\mathrm{Irr}^t(G)$ denote the set of isomorphism classes of all simple $\mathcal{S}$-modules, equivalently ([SSZ] App.), the set of isomorphism classes of all irreducible tempered G-representations. For any subset $B \subseteq \mathrm{Irr}^t(G)$ we introduce the full abelian subcategory

$$\mathcal{M}^t(G)_B := \begin{array}{l}\text{all nondegenerate } \mathcal{S}\text{-modules all of} \\ \text{whose simple subquotients lie in } B\end{array}$$

of $\mathcal{M}^t(G)$.

There is the discrete support map

$$\nu^t : \mathrm{Irr}^t(G) \longrightarrow \Omega^t(G)$$

characterized by the fact that if $\nu^t([V])$ is the G-orbit of $[(L,\tau)]$ then V is isomorphic to a subquotient of a G-representation (normalized) parabolically induced from (L,τ).

Given any connected component $\Theta \subseteq \Omega^t(G)$ we put

$$\mathcal{M}^t(\Theta) := \mathcal{M}^t(G)_{(\nu^t)^{-1}(\Theta)} \ .$$

It was shown in [SZ2] Thm. 4.1 that the primitive idempotents in the center of the category $\mathcal{M}^t(G)$ give rise to the decomposition of categories

$$\mathcal{M}^t(G) = \prod_{\Theta} \mathcal{M}^t(\Theta) \ .$$

We therefore refer to the $\mathcal{M}^t(\Theta)$ as the component categories of $\mathcal{M}^t(G)$. The goal of this section is to alternatively describe these component categories through idempotents in the algebra $\mathcal{S}(G)$. Since the central idempotents are not contained in $\mathcal{S}(G)$ there is no general formal reason that this should be possible. For any idempotent $e \in \mathcal{S}$ we consider the full subcategory $\mathcal{M}^t_e(G)$ of all $\mathcal{S}$-modules V in $\mathcal{M}^t(G)$ such that $\mathcal{S}eV = V$.

Definition 1.1. *The idempotent $e \in \mathcal{S}(G)$ is called special if the subcategory $\mathcal{M}^t_e(G)$ is closed under the formation of subquotients in $\mathcal{M}^t(G)$ (and hence is abelian).*

Proposition 1.2. *Let $e \in \mathcal{S}(G)$ be a special idempotent and $\Theta \subseteq \Omega^t(G)$ be a connected component; then the following assertions are equivalent:*

i. There is a $[V] \in \mathrm{Irr}^t(G)$ such that $\nu^t([V]) \in \Theta$ and $eV \neq 0$,

ii. $\mathcal{M}^t(\Theta) \subseteq \mathcal{M}^t_e(G)$.

Proof. Obviously ii. implies i. In the following we assume the existence of an irreducible V as in i. Its discrete support (M,τ) lies in Θ, and V is a subquotient of the parabolically induced representation $\mathrm{Ind}_P^G(E)$ where E is the vector space of the representation τ and P is some parabolic subgroup with Levi factor M. Let $C^\infty(X^1_{nr}(M))$ denote the ring of complex valued C^∞-functions on $X^1_{nr}(M)$. In [SZ2] Lemma 3.1 it is shown that $\Pi_0 := C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} E$ carries a natural structure of a nondegenerate $\mathcal{S}(M)$-module. By [SZ2] Cor. 5.3 its parabolic induction $\Pi := \mathrm{Ind}_P^G(\Pi_0)$ is a projective generator for the component category $\mathcal{M}^t(\Theta)$. Since the subcategory $\mathcal{M}^t_e(G)$ is closed under the formation of arbitrary direct sums and subquotients in $\mathcal{M}^t(G)$ it suffices, in order to obtain the assertion ii., to show that Π lies in $\mathcal{M}^t_e(G)$. The maximal ideals in the ring $C^\infty(X^1_{nr}(M))$ are given by $\mathfrak{m}_\chi := \{f \in C^\infty(X^1_{nr}(M)) : f(\chi) = 0\}$ with χ running over $X^1_{nr}(M)$ (cf. [SZ2] Remark 8.3). Let E_χ, for $\chi \in X^1_{nr}(M)$, denote the vector space E but viewed as the underlying vector space of the twisted representation $\chi\tau$. Starting from the obvious exact sequence

$$0 \longrightarrow \mathfrak{m}_\chi \otimes_{\mathbb{C}} E \xrightarrow{\subseteq} C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} E \xrightarrow{ev_\chi} E_\chi \longrightarrow 0$$

of $C^\infty(X^1_{nr}(M)) \times \mathcal{S}(M)$-modules where $ev_\chi(f \otimes v) := f(\chi)v$ we obtain by parabolic induction ([SZ1] §2) the exact sequence

$$0 \longrightarrow \mathfrak{m}_\chi \Pi \xrightarrow{\subseteq} \Pi \longrightarrow \mathrm{Ind}_P^G(E_\chi) \longrightarrow 0$$

of $C^\infty(X^1_{nr}(M)) \times \mathcal{S}(G)$-modules.

Claim 1: $e\,\mathrm{Ind}^G_P(E_\chi) \neq 0$ for any $\chi \in X^1_{nr}(M)$.
Fixing a good maximal compact subgroup $K \subseteq G$ we may identify

$$\Pi = \mathrm{Ind}^K_{P\cap K}(C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} E) = C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} \mathrm{Ind}^K_{P\cap K}(E) \ .$$

Hence the endomorphism of Π given by the action of e can be viewed as an element $e_\Pi \in C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} \mathrm{End}^\infty_{\mathbb{C}}(\mathrm{Ind}^K_{P\cap K}(E))$ such that

$$e_\Pi(\chi) = \left[\mathrm{Ind}^G_P(E_\chi) \xrightarrow{e*} \mathrm{Ind}^G_P(E_\chi)\right] \ .$$

Here the second factor in the tensor product denotes the smooth linear endomorphisms of $\mathrm{Ind}^K_{P\cap K}(E)$. They all are of finite rank so that one has the usual trace functional $tr : \mathrm{End}^\infty_{\mathbb{C}}(\mathrm{Ind}^K_{P\cap K}(E)) \longrightarrow \mathbb{C}$. In particular we obtain the C^∞-function

$$tr(e_\Pi) := (id \otimes tr)(e_\Pi) \in C^\infty(X^1_{nr}(M)) \ .$$

It satisfies

$$\begin{aligned} tr(e_\Pi)(\chi) &= \text{trace of } \mathrm{Ind}^G_P(E_\chi) \xrightarrow{e*} \mathrm{Ind}^G_P(E_\chi) \\ &= \dim_{\mathbb{C}}\, e\,\mathrm{Ind}^G_P(E_\chi) \end{aligned}$$

for any $\chi \in X^1_{nr}(M)$. Since an integral valued C^∞-function on $X^1_{nr}(M)$ must be constant we conclude that

$$\dim_{\mathbb{C}}\, e\,\mathrm{Ind}^G_P(E_\chi) = \dim_{\mathbb{C}}\, e\,\mathrm{Ind}^G_P(E) \geq \dim_{\mathbb{C}}\, eV \neq 0$$

for any $\chi \in X^1_{nr}(M)$.

For irreducible $\mathrm{Ind}^G_P(E_\chi)$ this means that $\mathcal{S}e\,\mathrm{Ind}^G_P(E_\chi) = \mathrm{Ind}^G_P(E_\chi)$ and hence by the above exact sequence that

$$\Pi = \mathcal{S}e\Pi + \mathfrak{m}_\chi\Pi \ .$$

Claim 2: There is a nonzero regular function $F \in \mathcal{O}(X_{nr}(M))$ on the algebraic torus $X_{nr}(M)$ of all unramified characters of M such that $\{\chi \in X^1_{nr}(M) : \mathrm{Ind}^G_P(E_\chi) \text{ is reducible}\}$ is contained in the zero set of F.
The Weyl group $N_G(M)/M$ acts on the isomorphism classes of irreducible smooth representations of M. The subset $\Xi \subseteq X_{nr}(M)$ of all characters χ such that $w_* E_\chi \not\cong E_\chi$ for any nontrivial $w \in N_G(M)/M$ is Zariski open in $X_{nr}(M)$. By [Cas] Thm. 6.6.1 or [Wal] Prop. IV.2.2(i) any $\mathrm{Ind}^G_P(E_\chi)$ with $\chi \in \Xi \cap X^1_{nr}(M)$ is irreducible. It therefore suffices to choose a nonzero $F \in \mathcal{O}(X_{nr}(M))$ such that the complement $D(F)$ of its zero set is contained in Ξ.

Let $U \subseteq G$ be an arbitrary compact open subgroup. From our earlier identity we deduce the equality of $C^\infty(X^1_{nr}(M))$-modules

$$e_U\Pi = e_U\mathcal{S}e\Pi + \mathfrak{m}_\chi(e_U\Pi)$$

for any $\chi \in X^1_{nr}(M)$ such that $F(\chi) \neq 0$. Using a good maximal compact subgroup $K \subseteq G$ the formula

$$e_U\Pi = e_U e_{U\cap K}\Pi = e_U[C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} e_{U\cap K}\,\mathrm{Ind}^K_{P\cap K}(E)]$$

exhibits $e_U\Pi$ as a finitely generated module over $C^\infty(X^1_{nr}(M))$. Hence, by the Nakayama lemma, we obtain the equality of localized modules

$$(e_U\Pi)_F = (e_U\mathcal{S}e\Pi)_F$$

over the localization $C^\infty(X^1_{nr}(M))_F$ of $C^\infty(X^1_{nr}(M))$ in F. This means, again by finite generation of $e_U\Pi$, that we find a natural number $n = n(U) \in \mathbb{N}$ such that

$$F^n \cdot \mathcal{S}e_U\Pi = \mathcal{S}(F^n \cdot e_U\Pi) \subseteq \mathcal{S}e_U\mathcal{S}e\Pi \subseteq \mathcal{S}e\Pi \ .$$

With $\mathcal{S}e\Pi$ therefore also $F^n \cdot \mathcal{S}e_U\Pi$ lies in the subcategory $\mathcal{M}^t_e(G)$. Furthermore, we know from the proof of [SZ2] Cor. 8.7 that Π has no F-torsion. Hence the map

$$\mathcal{S}e_U\Pi \xrightarrow{F^n\cdot} F^n \cdot \mathcal{S}e_U\Pi$$

is an isomorphism of $\mathcal{S}$-modules (remember that the $C^\infty(X^1_{nr}(M))$- and the $\mathcal{S}$-action commute). We see that $\mathcal{S}e_U\Pi$ lies in $\mathcal{M}^t_e(G)$. Hence

$$e_U\Pi \subseteq \mathcal{S}e_U\Pi = \mathcal{S}e\mathcal{S}e_U\Pi \subseteq \mathcal{S}e\Pi \ .$$

Since U was arbitrary we finally conclude that $\Pi = \mathcal{S}e\Pi$. □

The analog of the above proposition for the algebra $\mathcal{H}(G)$ is due to Bernstein. For any idempotent $f \in \mathcal{H}(G)$ we have the full subcategory $\mathcal{M}_f(G)$ of all $\mathcal{H}$-modules V in $\mathcal{M}(G)$ such that $\mathcal{H}fV = V$. Then f is called special if this subcategory $\mathcal{M}_f(G)$ is closed under the formation of subquotients in $\mathcal{M}(G)$. We followed the strategy of proof in [BK] (3:6) but had to deal with the extra difficulty that $C^\infty(X^1_{nr}(M))$ is not an integral domain.

For any V in $\mathcal{M}^t(G)$, let

$$V = \bigoplus_\Theta V_\Theta$$

with V_Θ in $\mathcal{M}^t(\Theta)$ be the decomposition corresponding to $\mathcal{M}^t(G) = \prod_\Theta \mathcal{M}^t(\Theta)$. In particular,

$$\mathcal{S}(G) = \bigoplus_\Theta \mathcal{S}(G)_\Theta$$

is a decomposition into two sided ideals. Let Z_Θ denote the central idempotent which defines the component category $\mathcal{M}^t(\Theta)$, i.e., multiplication by Z_Θ induces a projection $Z_{\Theta,V} : V \twoheadrightarrow V_\Theta$.

Proposition 1.3. *Let $e \in \mathcal{S}(G)$ be a special idempotent; then we have:*

i. The set of all connected components Θ such that $Z_{\Theta,\mathcal{S}}(e) \neq 0$ is finite;

ii. $\mathcal{M}^t_e(G) = \prod_{Z_{\Theta,\mathcal{S}}(e)\neq 0} \mathcal{M}^t(\Theta)$;

iii. $\mathcal{S}(G)e\mathcal{S}(G) = \sum_{Z_{\Theta,\mathcal{S}}(e)\neq 0} \mathcal{S}(G)_\Theta$;

iv. the functors

$$\begin{aligned} \mathcal{M}^t_e(G) &\longrightarrow \mathcal{M}(e\mathcal{S}e) \\ V &\longmapsto eV \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{M}(e\mathcal{S}e) &\longrightarrow \mathcal{M}^t_e(G) \\ M &\longmapsto \mathcal{S}e \otimes_{e\mathcal{S}e} M \end{aligned}$$

are quasi-inverse equivalences of categories.

Proof. i. Since $e = \sum_\Theta Z_{\Theta,\mathcal{S}}(e)$ the set of all connected components Θ such that $Z_{\Theta,\mathcal{S}}(e) \neq 0$ has to be finite.
ii. Obviously, with any V also all of its components V_Θ belong to $\mathcal{M}_e^t(G)$. Hence, by the proposition, $\mathcal{M}_e^t(G)$ is a product of full component categories. The formula

$$Z_{\Theta,V}(v) = Z_{\Theta,V}(ev) = Z_{\Theta,\mathcal{S}}(e)v \qquad \text{for any } v \in eV$$

shows that $Z_{\Theta,\mathcal{S}}(e) \neq 0$ if $\mathcal{M}^t(\Theta) \subseteq \mathcal{M}_e^t(G)$. On the other hand, if $Z_{\Theta,\mathcal{S}}(e) \neq 0$ then $(\mathcal{S}e)_\Theta = Z_{\Theta,\mathcal{S}}(\mathcal{S}e)$ is nonzero and lies in $\mathcal{M}_e^t(G)$. Hence any irreducible subquotient of $(\mathcal{S}e)_\Theta$ lies in $\mathcal{M}^t(\Theta)$ as well as in $\mathcal{M}_e^t(G)$. It then follows from the proposition that $\mathcal{M}^t(\Theta) \subseteq \mathcal{M}_e^t(G)$.
iii. Finally, if $\mathcal{M}^t(\Theta) \subseteq \mathcal{M}_e^t(G)$ then $\mathcal{S}_\Theta$ lies in $\mathcal{M}_e^t(G)$ which implies that

$$\mathcal{S}_\Theta = \mathcal{S}eZ_{\Theta,\mathcal{S}}(\mathcal{S}) = Z_{\Theta,\mathcal{S}}(\mathcal{S}e\mathcal{S}) \subseteq \mathcal{S}e\mathcal{S} \ .$$

It follows that

$$\mathcal{S}e\mathcal{S} = \sum_{Z_{\Theta,\mathcal{S}}(e)\neq 0} (\mathcal{S}e\mathcal{S})_\Theta = \sum_{Z_{\Theta,\mathcal{S}}(e)\neq 0} \mathcal{S}_\Theta \ .$$

iv. This is a completely formal consequence. Compare, for example, [BK] (3.12). □

Proposition 1.4. *For any finite set θ of connected components there is a special idempotent $e \in \mathcal{S}(G)$ such that $\mathcal{M}_e^t(G) = \prod_{\Theta\in\theta} \mathcal{M}^t(\Theta)$.*

Proof. This is literally the same argument as in [BK] (3.13) which we therefore only sketch. Choose a compact open subgroup $U \subseteq G$ such that $e_U V \neq 0$ for any irreducible representation in $\mathcal{M}^t(\theta) := \prod_{\Theta\in\theta} \mathcal{M}^t(\Theta)$. Write

$$e_U = e + e' \qquad \text{with } e \in \oplus_{\Theta\in\theta}\mathcal{S}_\Theta \text{ and } e' \in \oplus_{\Theta\notin\theta}\mathcal{S}_\Theta.$$

Then $\mathcal{S}eV \subseteq \oplus_{\Theta\in\theta} V_\Theta$ and hence $\mathcal{M}_e^t(G) \subseteq \mathcal{M}^t(\theta)$. On the other hand let V be in $\mathcal{M}^t(\theta)$ and suppose that $\mathcal{S}eV \neq V$. Any irreducible quotient V_0 of $V/\mathcal{S}eV$ lies in $\mathcal{M}^t(\theta)$ but satisfies $eV_0 = 0$. Since $e'V = e'V_0 = 0$ it follows that $e_U V_0 = 0$ which is a contradiction. In particular $\mathcal{M}_e^t(G) = \mathcal{M}^t(\theta)$ is abelian and the idempotent e therefore special. □

There is the following useful observation.

Lemma 1.5. *Any idempotent $f \in \mathcal{H}(G)$ which is special in $\mathcal{H}(G)$ also is special in $\mathcal{S}(G)$.*

Proof. Let V be in $\mathcal{M}_f^t(G)$, hence $\mathcal{S}fV = V$. As a G-representation we have, by the analog for $\mathcal{M}(G)$ of Prop. 1.3, the natural decomposition $V = V' \oplus \mathcal{H}fV$ where $\mathcal{H}fV$ lies in $\mathcal{M}_f(G)$ and $fV' = 0$. This decomposition is given by central idempotents for $\mathcal{H}$, i.e., by idempotents in $\varprojlim_U Z(\mathcal{H}(G,U))$ ([SZ2] Lemma 1.2.i). Since $\mathcal{H}(G,U)$ is dense in $\mathcal{S}(G,U)$ we have $Z(\mathcal{H}(G,U)) \subseteq Z(\mathcal{S}(G,U))$. It follows that the center of $\mathcal{M}(G)$ is contained in the center of $\mathcal{M}^t(G)$, and we obtain that the above decomposition is, in fact, $\mathcal{S}$-invariant. We deduce that $V = \mathcal{S}fV = \mathcal{H}fV$, i.e., that V lies in $\mathcal{M}_f(G)$. One easily concludes that with $\mathcal{M}_f(G)$ also $\mathcal{M}_f^t(G)$ is closed under the formation of subquotients. □

Let $\mathrm{Irr}(G)$ denote the set of isomorphism classes of irreducible smooth G-representations. It naturally is a topological space for the Jacobson topology. We recall that the Jacobson closed subsets are the subsets of the form

$$V(\mathcal{J}) := \{[V] \in \mathrm{Irr}(G) : \mathcal{J}V = 0\}$$

with $\mathcal{J}$ running over the two sided ideals in $\mathcal{H}(G)$.

Remark 1.6. *The subset* $\mathrm{Irr}^t(G)$ *is dense in* $\mathrm{Irr}(G)$*.*

Proof. We have to show that any $f \in \mathcal{H}(G)$ which annihilates all irreducible tempered V necessarily is zero. Suppose that $f \neq 0$. Because of the inclusions $\mathcal{H}(G) \subseteq \mathcal{S}(G) \subseteq C^*_{\mathrm{r}}(G)$ (see [Vig] Prop. 28 for the second inclusion into the reduced C^*-algebra of G) we may view f as a nonzero element in $C^*_{\mathrm{r}}(G)$. In the context of C^*-algebras it is a general fact that for any nonzero f there is a simple $C^*_{\mathrm{r}}(G)$-module X, which in particular is an irreducible unitary G-representation, such that $fX \neq 0$. As explained in the proof of [SSZ] App. Prop. 3.i the subspace V of smooth vectors in X is an irreducible tempered G-representation, and is dense in X. Hence $fV \neq 0$. □

We now introduce a preorder on the set of connected components of $\Omega^t(G)$. In the following $\overline{(.)}$ refers to the closure operation in the topological space $\mathrm{Irr}(G)$.

Definition 1.7. *For two connected components* Θ *and* Θ' *of* $\Omega^t(G)$ *we write* $\Theta' \lesssim \Theta$ *if there is a chain* $\Theta' = \Theta_0, \ldots, \Theta_n = \Theta$ *such that* $(\nu^t)^{-1}(\Theta_i) \cap \overline{(\nu^t)^{-1}(\Theta_{i+1})} \neq \emptyset$ *for all* $i = 0, \ldots, n-1$*.*

Remark 1.8. *i. If* θ *is a finite set of connected components of* $\Omega^t(G)$ *and* $e \in \mathcal{H}(G)$ *is an idempotent which is special in* $\mathcal{S}(G)$ *and such that* $\mathcal{M}^t_e(G) = \prod_{\Theta\in\theta} \mathcal{M}^t(\Theta)$ *then* θ *is saturated in the sense that with* Θ *any* $\Theta' \gtrsim \Theta$ *also lies in* θ*.*

ii. Any connected component Θ *lies in a finite saturated set.*

Proof. i. Suppose that $\Theta' \notin \theta$. Then $eV = 0$ for any V in $\mathcal{M}^t(\Theta')$. Let $\mathcal{J} \subseteq \mathcal{H}(G)$ denote the joint annihilator ideal of all irreducible representations in $\mathcal{M}^t(\Theta')$. In particular, $e \in \mathcal{J}$. But $V(\mathcal{J}) \supseteq \overline{(\nu^t)^{-1}(\Theta')}$. If $(\nu^t)^{-1}(\Theta) \cap \overline{(\nu^t)^{-1}(\Theta')} \neq \emptyset$ for some $\Theta \in \theta$ then $V(\mathcal{J}) \cap (\nu^t)^{-1}(\Theta) \neq \emptyset$. Hence there would be a representation $V \neq 0$ in $\mathcal{M}^t(\Theta) \subseteq \mathcal{M}^t_e(G)$ such that $\mathcal{J}V = 0$ and a fortiori $eV = 0$ which is a contradiction.

ii. We pick an irreducible tempered G-representation V in $\mathcal{M}^t(\Theta)$. By the analog for $\mathcal{H}(G)$ of Prop. 1.4 we find a special idempotent $f \in \mathcal{H}(G)$ such that V lies in $\mathcal{M}^t_f(G)$. According to Lemma 1.5 the idempotent f also is special in $\mathcal{S}(G)$. Hence we may apply Prop.s 1.2 and 1.3 and obtain a finite set θ of connected components such that $\mathcal{M}^t(\Theta) \subseteq \mathcal{M}^t_f(G) = \prod_{\Theta'\in\theta} \mathcal{M}^t(\Theta')$. By i. the finite set θ, which contains Θ, is saturated. □

2 Iwahori components

For the rest of the paper we will work under the assumption that our group G is k-split and has connected center. We fix a Borel subgroup $P \subseteq G$ with Levi decomposition $P = TN$ where T is a maximal k-split torus in P and N is the unipotent radical. As usual, $\overline{P} = T\overline{N}$ denotes the opposite Borel subgroup. We recall that the modulus character δ_P of P can be constructed as follows. We put

$$\Delta(t) := \det(\mathrm{ad}(t); \mathrm{Lie}(N))$$

where $\mathrm{ad}(t)$ denotes the adjoint action of $t \in T$ on $\mathrm{Lie}(N)$. Then δ_P is the composite

$$P \xrightarrow{pr} T \xrightarrow{|\Delta(.)|_k^{-1}} \mathbb{C}^\times .$$

Let $W := N_G(T)/T$ denote, as usual, the Weyl group of G.

We fix a hyperspecial vertex x_0 in the apartment corresponding to T and we let C denote the chamber in the direction of P with x_0 in its closure. The stabilizer K of x_0 is a good maximal compact subgroup of G, and the pointwise stabilizer $J \subseteq K$ of C is an Iwahori subgroup.

It is known from the work of Borel and Bernstein (cf. [BK] (9.2)) that the idempotent $\epsilon_J \in \mathcal{H}(G)$ is special in $\mathcal{H}(G)$. This means that the category $\mathcal{M}_{\epsilon_J}(G)$ of all smooth G-representations which are generated by their J-fixed vectors is abelian and is a direct factor of the full category $\mathcal{M}(G)$. According to Lemma 1.5 the idempotent ϵ_J is special in $\mathcal{S}(G)$ as well. Therefore, by Prop. 1.3, there are finitely many connected components $\Theta_1, \ldots, \Theta_r \subseteq \Omega^t(G)$ such that

$$\mathcal{M}^t_{\epsilon_J}(G) = \prod_{i=1}^{r} \mathcal{M}^t(\Theta_i) \ .$$

Let $\mathrm{Irr}^t_{\epsilon_J}(G) \subseteq \mathrm{Irr}_{\epsilon_J}(G)$ denote the sets of isomorphism classes of irreducible tempered and of irreducible smooth G-representations in $\mathcal{M}_{\epsilon_J}(G)$, respectively. Of course we have

$$\mathrm{Irr}^t_{\epsilon_J}(G) = (\nu^t)^{-1}(\Theta_1 \cup \ldots \cup \Theta_r) \ .$$

In the following we write $\theta_{\epsilon_J} := \{\Theta_1, \ldots, \Theta_r\}$. With respect to the Jacobson topology $\mathrm{Irr}_{\epsilon_J}(G)$ is an open and closed subset of $\mathrm{Irr}(G)$. We also recall that, by the result of Borel and Bernstein, the functor

$$\begin{aligned} \mathcal{M}_{\epsilon_J}(G) &\xrightarrow{\sim} \mathcal{M}(\mathcal{H}(G,J)) \\ V &\longrightarrow \epsilon_J V \end{aligned}$$

is an equivalence of categories. On the one hand, this Morita equivalence induces a bijection between $\mathrm{Irr}_{\epsilon_J}(G)$ and the set $\mathrm{Irr}(\mathcal{H}(G,J))$ of isomorphism classes of simple $\mathcal{H}(G,J)$-modules. On the other hand, it induces an inclusion preserving bijection between the sets of two sided ideals in $\mathcal{H}\epsilon_J\mathcal{H}$ and $\mathcal{H}(G,J) = \epsilon_J\mathcal{H}\epsilon_J$, respectively, which maps $\mathcal{J} \subseteq \mathcal{H}\epsilon_J\mathcal{H}$ to $\mathcal{I} := \epsilon_J\mathcal{J}\epsilon_J \subseteq \mathcal{H}(G,J)$; in this case we, moreover, have for any V in $\mathrm{Irr}_{\epsilon_J}(G)$ that $[V] \in V(\mathcal{J})$ if and only if $\mathcal{I}\epsilon_J V = 0$. We see that the Jacobson topology on $\mathrm{Irr}_{\epsilon_J}(G)$ coincides with the Jacobson topology on $\mathrm{Irr}(\mathcal{H}(G,J))$.

2.1 The Kazhdan-Lusztig classification

The simple $\mathcal{H}(G,J)$-modules have been classified by Kazhdan-Lusztig, and we have to review this classification in this section. We first note that the Bernstein presentation of the algebra $\mathcal{H}(G,J)$ as described in [HKP] Lemma 1.7.1 and (1.15.2) coincides with the generators and relations, specialized to $q \in \mathbb{C}^\times$, given in [KL] 2.12. Hence their generic Hecke algebra specialized to q is indeed the algebra $\mathcal{H}(G,J)$ (compare also [BM] (3.9)).

Let $\widehat{G} = {}^L G^\circ$ denote the (complex) connected Langlands dual group of G. We remark that our assumption that the center of G is connected is equivalent to the derived group of $\widehat{G}$ being simply connected ([Bo2] 2.2.(5)).

Let (u, s) be a pair of elements in $\widehat{G}$ where s is semisimple, u is unipotent, and which satisfy the relation $sus^{-1} = u^q$. The complex algebraic variety

$$\mathcal{B}_u := \{B : B \text{ Borel subgroup of } \widehat{G}, u \in B\}$$

is acted upon by the algebraic group

$$M(u) := \{(g, \lambda) \in \widehat{G} \times \mathbb{C}^\times : gug^{-1} = u^\lambda\}$$

through conjugation by the first component g. Inside $M(u)$ we consider the centralizer $M(u,s) := Z_{M(u)}((s,q))$ of the element (s,q) as well as the algebraic subgroup $M := M(s)$ generated by the element (s,q). By functoriality the group $M(u,s)$ acts on the M-equivariant K-group $K_0^M(\mathcal{B}_u)$. This action is $R(M)$-linear for the representation ring $R(M)$ of M. By general principles it factorizes through an action of the component group

$$\bar{M}(u,s) := M(u,s)/M(u,s)^\circ(Z(\widehat{G}) \times \{1\})$$

where $Z(\widehat{G})$ denotes the center of $\widehat{G}$. Using the character

$$\begin{aligned} R(M) &\longrightarrow \mathbb{C} \\ [E] &\longmapsto Tr((s,q);E) \end{aligned}$$

Kazhdan-Lusztig form

$$\mathcal{M}_{u,s} := \mathbb{C} \otimes_{R(M)} K_0^M(\mathcal{B}_u)$$

still carrying an $\bar{M}(u,s)$-action, and they show that $\mathcal{M}_{u,s}$ also carries a specific $\mathcal{H}(G,J)$-action which commutes with the $\bar{M}(u,s)$-action. For each irreducible representation ρ of the finite group $\bar{M}(u,s)$ one may therefore introduce the $\mathcal{H}(G,J)$-module

$$\mathcal{M}_{u,s,\rho} := (\rho^* \otimes_{\mathbb{C}} \mathcal{M}_{u,s})^{\bar{M}(u,s)} .$$

We obviously have

$$\mathcal{M}_{u,s} = \bigoplus_\rho (\rho \otimes_{\mathbb{C}} \mathcal{M}_{u,s,\rho}) .$$

The nonzero ones among the $\mathcal{M}_{u,s,\rho}$ are called the *standard $\mathcal{H}(G,J)$-modules*.

Theorem 2.1 ([KL] Thm. 7.12)**.** *i. The standard module $\mathcal{M}_{u,s,\rho}$ has a unique simple quotient $\mathcal{L}_{u,s,\rho}$.*

ii. Every simple $\mathcal{H}(G,J)$-module is isomorphic to some $\mathcal{L}_{u,s,\rho}$.

iii. $\mathcal{L}_{u,s,\rho}$ and $\mathcal{L}_{u',s',\rho'}$ are isomorphic if and only if the triples (u,s,ρ) and (u',s',ρ') are $\widehat{G}$-conjugate.

Let $\pi(u,s,\rho)$ be an irreducible smooth G-representation such that

$$\epsilon_J * \pi(u,s,\rho) \cong \mathcal{L}_{u,s,\rho}$$

as $\mathcal{H}(G,J)$-modules. The above theorem says that the association $\pi(u,s,\rho) \longleftrightarrow (u,s,\rho)$ sets up an injective map

$$\mathrm{Irr}_{\epsilon_J}(G) \hookrightarrow \text{set of } \widehat{G}\text{-orbits of triples } (u,s,\rho).$$

By [KL] 2.4.(g) there is a homomorphism of algebraic groups

$$\gamma_{u,s} : SL_2(\mathbb{C}) \longrightarrow \widehat{G}$$

such that $\gamma_{u,s}(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = u$ and

$$(2) \qquad s\gamma_{u,s}(A)s^{-1} = \gamma_{u,s}(\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} A \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}) \qquad \text{for any} \quad A \in SL_2(\mathbb{C}) \ .$$

In particular the semisimple elements s and $\gamma_{u,s}(\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix})$ commute. By [KL] 2.4.(h) the homomorphism $\gamma_{u,s}$ is uniquely determined up to conjugation by an element in the simultaneous centralizer $Z_{\widehat{G}}(u,s)$ of u and s in $\widehat{G}$.

Theorem 2.2. *Equivalent are:*

i. $\pi(u,s,\rho)$ *is tempered;*

ii. the semisimple element $\gamma_{u,s}(\begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix})s$ *in* $\widehat{G}$ *is compact.*

In this case we have $\mathcal{M}_{u,s,\rho} = \mathcal{L}_{u,s,\rho}$. *Also equivalent are:*

iii. $\pi(u,s,\rho)$ *is essentially discrete series;*

iv. there is no proper Levi subgroup of $\widehat{G}$ *which contains both* u *and* s.

Proof. This is [KL] Thm.s 8.2 and 8.3 provided G is semisimple. An argument which reduces the general case to the semisimple one can be found in [ABPS] Prop. 9.3. □

By this theorem the property of being tempered or discrete series for $\pi(u,s,\rho)$ does not depend on the parameter ρ. We therefore call in this case the pair (u,s) *tempered* and *discrete*, respectively.

2.2 The Jacobson topology on the smooth Iwahori component

Let $\widehat{G}_{unip}$ be the subset of unipotent elements in $\widehat{G}$. It is a closed irreducible subvariety of $\widehat{G}$ with respect to the Zariski topology ([SS] Thm. III.1.8). For any $u \in \widehat{G}_{unip}$ we denote by $O_u \subseteq \widehat{G}_{unip}$ the conjugacy class of u. Moreover, $\mathcal{O}_{\widehat{G}}$ denotes the set of all unipotent conjugacy classes in $\widehat{G}$. The closure $\overline{O}$ of a unipotent class O is understood with respect to the Zariski topology. We recall the following facts:

– The set $\mathcal{O}_{\widehat{G}}$ is finite ([SS] Thm. I.5.4).

– Each unipotent class O is open in its closure $\overline{O}$ ([Car] §1.5).

– The closure $\overline{O}$ is a union of unipotent classes.

On $\mathcal{O}_{\widehat{G}}$ one therefore can define the partial order

$$O_1 \leq O_2 \quad \text{if} \quad O_1 \subseteq \overline{O_2} \ .$$

We then have

$$\overline{O} = \bigcup_{O' \leq O} O' \qquad \text{and} \qquad \overline{O} \setminus O = \bigcup_{O' < O} O' \ .$$

Our goal in this section is to relate the Jacobson topology on $\mathrm{Irr}_{\epsilon_J}(G)$ to the partial order on $\mathcal{O}_{\widehat{G}}$ via the map

$$\begin{aligned} \mathrm{Irr}_{\epsilon_J}(G) &\longrightarrow \mathcal{O}_{\widehat{G}} \\ [\pi(u,s,\rho)] &\longmapsto O_u \ . \end{aligned}$$

For this purpose we define the subsets

$$\pi^{-1}(O) := \{[\pi(u,s,\rho)] : O_u = O\} \qquad \text{for any } O \in \mathcal{O}_{\widehat{G}}$$

in $\mathrm{Irr}_{\epsilon_J}(G)$.

Theorem 2.3. *For any $O \in \mathcal{O}_{\widehat{G}}$ we have*

$$\overline{\pi^{-1}(O)} \subseteq \bigcup_{O' \geq O} \pi^{-1}(O') \ .$$

To prepare for the proof of this theorem we recall a key tool from [KL] which is the Steinberg variety

$$\mathbf{Z} := \{(u,B,B') : u \in B \cap B'\}$$

consisting of triples where B, B' are Borel subgroups of $\widehat{G}$ and u is an unipotent element in their intersection. The group $\widehat{G} \times \mathbb{C}^\times$ acts on $\mathbf{Z}$ by

$$\begin{aligned} (\widehat{G} \times \mathbb{C}^\times) \times \mathbf{Z} &\longrightarrow \mathbf{Z} \\ ((g,\lambda),(u,B,B')) &\longmapsto (gu^{\lambda^{-1}}g^{-1}, gBg^{-1}, gB'g^{-1}) \ . \end{aligned}$$

(We recall that $u^{\lambda^{-1}} = \exp(\frac{1}{\lambda}X)$ if $u = \exp(X)$ for a nilpotent $X \in \mathrm{Lie}(\widehat{G})$.) For any locally closed $\widehat{G} \times \mathbb{C}^\times$-invariant subvariety X of $\mathbf{Z}$ we have the $(\widehat{G} \times \mathbb{C}^\times)$-equivariant K-group

$$\mathbf{K}(X) := K_0^{\widehat{G} \times \mathbb{C}^\times}(X) \otimes \mathbb{C} \ .$$

Such X of interest here arise as follows: For any locally closed subvariety $Y \subseteq \widehat{G}_{unip}$ which is invariant under conjugation by $\widehat{G}$ we define

$$\mathbf{Z}_Y := \{(u,B,B') \in \mathbf{Z} : u \in Y\}.$$

The canonical morphism $j = j_Y : \mathbf{Z}_Y \longrightarrow \mathbf{Z}_{\overline{Y}}$, resp. $i = i_{\overline{Y}} : \mathbf{Z}_{\overline{Y}} \longrightarrow \mathbf{Z}$, is an open, resp. closed, immersion. They give rise to homomorphisms

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbf{K}(\mathbf{Z}_{\overline{Y}}) & \xrightarrow{i_*} & \mathbf{K}(\mathbf{Z}) \\ & & \downarrow{\scriptstyle j^*} & & \\ & & \mathbf{K}(\mathbf{Z}_Y) & & \\ & & \downarrow & & \\ & & 0. & & \end{array}$$

For any $O \in \mathcal{O}_{\widehat{G}}$ the closed immersion $i_O : \overline{O} \setminus O \longrightarrow \overline{O}$ induces a homomorphism $i_O^* : \mathbf{K}(\mathbf{Z}_{\overline{O} \setminus O}) \longrightarrow \mathbf{K}(\mathbf{Z}_{\overline{O}})$. According to [KL] 5.3 one has:

a) i_* is injective.

b) j^* is surjective.

c) $\mathrm{im}\big(\mathbf{K}(\mathbf{Z}_{\overline{Y}}) \longrightarrow \mathbf{K}(\mathbf{Z})\big) = \sum_{O\in\mathcal{O}_{\widehat{G}}, O\subseteq Y} \mathrm{im}\big(\mathbf{K}(\mathbf{Z}_{\overline{O}}) \longrightarrow \mathbf{K}(\mathbf{Z})\big)$.

d) The sequence $0 \longrightarrow \mathbf{K}(\mathbf{Z}_{\overline{O}\setminus O}) \xrightarrow{i_O^*} \mathbf{K}(\mathbf{Z}_{\overline{O}}) \xrightarrow{j_O^*} \mathbf{K}(\mathbf{Z}_O) \longrightarrow 0$ is exact ([KL] 1.3(g) and Thm. 5.2).

Let $\mathbf{H}$ denote the (extended) affine Hecke algebra over the Laurent polynomial ring $\mathbb{C}[\mathbf{q}^{\pm1}]$ in the variable $\mathbf{q}$ of the root system of $\widehat{G}$ as described in [KL] 2.12-13. The algebra $\mathcal{H}(G,J)$ is the specialization of $\mathbf{H}$ with respect to $\mathbf{q}\mapsto q$.

In [KL] 3.4(b) a specific element $\mathbb{1}\in\mathbf{K}(\mathbf{Z})$ is exhibited and in Thm. 3.5 a distinguished $(\mathbf{H},\mathbf{H})$-bimodule structure on $\mathbf{K}(\mathbf{Z})$ is constructed such that $h\mathbb{1}=\mathbb{1}h$ for any $h\in\mathbf{H}$ and such that the map

$$\begin{aligned} \mathbf{H} &\xrightarrow{\cong} \mathbf{K}(\mathbf{Z}) \\ h &\longmapsto h\mathbb{1}=\mathbb{1}h \end{aligned} \tag{3}$$

is an isomorphism of bimodules. Again by [KL] 5.3 one has:

e) Each $\mathbf{K}(\mathbf{Z}_Y)$ carries an $(\mathbf{H},\mathbf{H})$-bimodule structure such that the maps i^* and j_* are homomorphisms of bimodules.

f) Each simple left $\mathbf{H}$-module is a (left) $\mathbf{H}$-module quotient of $\mathbf{K}(\mathbf{Z}_O)$ for some $O\in\mathcal{O}_{\widehat{G}}$.

Proposition 2.4. *The simple $\mathcal{H}(G,J)$-module $\mathcal{L}_{u,s,\rho}$ is a (left) $\mathbf{H}$-module quotient of $\mathbf{K}(\mathbf{Z}_O)$ if and only if $O=O_u$.*

Proof. We abbreviate $\mathcal{L}:=\mathcal{L}_{u,s,\rho}$. Recall that $\mathcal{L}$ is an $\mathbf{H}$-module quotient of the standard $\mathbf{H}$-module $\mathcal{M}_{u,s,\rho}$. On the other hand, the proof of [KL] Prop. 5.13 can be interpreted as saying that given any $\mathbf{H}$-module surjection $\mathbf{K}(\mathbf{Z}_O)\twoheadrightarrow\mathcal{L}$, for some O, there is another $\mathbf{H}$-module surjection $\mathcal{M}_{u',s',\rho'}\twoheadrightarrow\mathcal{L}$ with $u'\in O$ (since our q is fixed we suppress it in the notation of loc. cit.). The main Thm. 7.12 in [KL] then implies that the triples (u,s,ρ) and (u',s',ρ') must be $\widehat{G}$-conjugate and so, in particular, $O=O_u$. In view of property f) above this proves the assertion. □

We let $\mathrm{Irr}(\mathbf{H})$ be the set of isomorphism classes of simple $\mathbf{H}$-modules equipped, as always, with its Jacobson topology. Since $\mathcal{H}(G,J)$ is a quotient of $\mathbf{H}$ the space $\mathrm{Irr}(\mathcal{H}(G,J))$ is a closed subspace of $\mathrm{Irr}(\mathbf{H})$. For any $O\in\mathcal{O}_{\widehat{G}}$ we introduce the subset $\mathcal{L}^{-1}(O):=\{[\mathcal{L}_{u,s,\rho}]:O_u=O\}$ of $\mathrm{Irr}(\mathbf{H})$.

Because of properties a), d), and e) above we may identify, under the isomorphism (3), $\mathbf{K}(\mathbf{Z}_{\overline{O}\setminus O})$ and $\mathbf{K}(\mathbf{Z}_{\overline{O}})$ with two sided ideals $\mathcal{J}_O$ and $\mathcal{I}_{\overline{O}}$ in $\mathbf{H}$ such that $\mathcal{J}_O\subseteq\mathcal{I}_{\overline{O}}$ and $\mathcal{I}_{\overline{O}}/\mathcal{J}_O\cong\mathbf{K}(\mathbf{Z}_O)$ as bimodules.

Lemma 2.5. *For any simple $\mathcal{H}(G,J)$-module $\mathcal{L}=\mathcal{L}_{u,s,\rho}$ and any $O\in\mathcal{O}_{\widehat{G}}$ we have:*

i. $\mathcal{I}_{\overline{O}}\cdot\mathcal{L}\neq0$ *if and only if* $O_u\leq O$*;*

ii. $V(\mathcal{I}_{\overline{O}})\cap\mathrm{Irr}(\mathcal{H}(G,J))=\bigcup_{O'\not\leq O}\mathcal{L}^{-1}(O')$*;*

iii. $\bigcap_{O'' \not\geq O} V(\mathcal{I}_{\overline{O''}}) \cap \mathrm{Irr}(\mathcal{H}(G,J)) = \bigcup_{O' \geq O} \mathcal{L}^{-1}(O')$ *is closed in* $\mathrm{Irr}(\mathcal{H}(G,J))$.

iv. $\overline{\mathcal{L}^{-1}(O)} \subseteq \bigcup_{O' \geq O} \mathcal{L}^{-1}(O')$.

Proof. i. First we assume that O is minimal such that $\mathcal{I}_{\overline{O}} \cdot \mathcal{L} \neq 0$. Then $\mathcal{I}_{\overline{O}} \cdot v \neq 0$ for a certain $v \in \mathcal{L}$ and therefore $\mathcal{I}_{\overline{O}} \cdot v = \mathcal{L}$ because $\mathcal{L}$ is simple. So we obtain a surjective map $\mathcal{I}_{\overline{O}} \to \mathcal{L}$, which sends h to hv, of (left) $\mathbf{H}$-modules. The minimality of O implies that $\mathcal{I}_{\overline{O'}} \cdot \mathcal{L} = 0$ for any $O' < O$. On the other hand, applying property c) above with $Y := \overline{O} \setminus O$ we obtain

$$\mathcal{J}_O = \sum_{O' < O} \mathcal{I}_{\overline{O'}} \ .$$

It follows that $\mathcal{J}_O \cdot \mathcal{L} = 0$. Hence $\mathcal{L}$ is an $\mathbf{H}$-module quotient of $\mathcal{I}_{\overline{O}} / \mathcal{J}_O \cong \mathbf{K}(\mathbf{Z}_O)$. Prop. 2.4 then implies that $O = O_u$. Finally, if $O_u \leq O$ then $\mathcal{I}_{\overline{O_u}} \subseteq \mathcal{I}_{\overline{O}}$ and $0 \neq \mathcal{I}_{\overline{O_u}} \cdot \mathcal{L} \subseteq \mathcal{I}_{\overline{O}} \cdot \mathcal{L}$.

ii. is immediate from i.

iii. Using ii. we see that the left hand side is equal to

$$\bigcap_{O'' \not\geq O} \ \bigcup_{O' \not\leq O''} \mathcal{L}^{-1}(O') \ .$$

This is the union of all $\mathcal{L}^{-1}(O')$ such that $O' \not\leq O''$ whenever $O'' \not\geq O$, i.e., such that any $O'' \geq O'$ also satisfies $O'' \geq O$. The latter condition simply amounts to $O' \geq O$ which characterizes the right hand side of the asserted identity.

iv. is immediate from iii. $\square$

Under the natural homeomorphism of Jacobson topological spaces $\mathrm{Irr}_{\epsilon_J}(G) \simeq \mathrm{Irr}(\mathcal{H}(G,J))$ the subset $\pi^{-1}(O)$ of the left hand side corresponds to the subset $\mathcal{L}^{-1}(O)$ of the right hand side. Therefore Lemma 2.5.iv proves Thm. 2.3.

Remark 2.6. *For any* $O \in \mathcal{O}_{\widehat{G}}$ *we have* $\mathcal{L}^{-1}(O) = \mathrm{Irr}(\mathcal{H}(G,J)) \cap \big(V(\mathcal{J}_O) \setminus V(\mathcal{I}_{\overline{O}})\big)$*; in particular,* $\pi^{-1}(O)$ *is open in its closure.*

2.3 The cuspidal support

Let $X_{nr}(T)$ denote the complex torus of unramified characters of T. The cuspidal support map

$$\nu : \mathrm{Irr}_{\epsilon_J}(G) \longrightarrow W \backslash X_{nr}(T)$$

is characterized by the fact that if $\chi \in \nu([V])$ then V is isomorphic to a subquotient of the normalized parabolic induction $\mathrm{Ind}_P^G(\chi)$. Equivalently $\nu([V])$ can be viewed as the infinitesimal (or central) character of V as follows. For later purposes we need to recall this in a specific form. We put $T^0 := J \cap T$ which is the maximal compact subgroup of T. An element $t \in T$ is called dominant if $t(J \cap N)t^{-1} \subseteq J \cap N$ and $t^{-1}(J \cap \overline{N})t \subseteq J \cap \overline{N}$. For any compact open subset $X \subseteq G$ we let $char_X \in \mathcal{H}(G)$ denote its characteristic function. Bernstein has shown the following.

Lemma 2.7. *The map*

$$\begin{aligned} \mathbb{C}[T/T^0] &\longrightarrow \mathcal{H}(G,J) \\ tT^0 &\longmapsto \delta_P^{-1/2}(t) char_{Jt_1J} * char_{Jt_2J}^{-1} \end{aligned}$$

where $t = t_1 t_2^{-1}$ *with dominant* t_1 *and* t_2 *is a well defined embedding of unital algebras. It restricts to an isomorphism*

$$\mathbb{C}[T/T^0]^W \xrightarrow{\cong} Z(\mathcal{H}(G,J)) \ .$$

Proof. See [HKP] Remark 1.7.2, §1.4, and Lemma 2.3.1. □

On the other hand, by [BK] (9.2) the trivial representation on J is a G-cover of the trivial representation on T^0. Hence [BK] Cor. 7.12 applies and says that, for any Borel subgroup $T \subseteq Q \subseteq G$, there is a unique homomorpism of unital algebras

$$t_Q : \mathbb{C}[T/T^0] \longrightarrow \mathcal{H}(G,J)$$

such that

$$t_Q(tT^0) = \delta_Q^{-1}(t) char_{JtJ} \qquad \text{for any } Q\text{-dominant } t \in T.$$

(Note that by the formula before (7.8) in [BK] our δ_Q is the inverse of theirs.) Since [BK] work with unnormalized parabolic induction (whereas we work with the normalized one) we renormalize this algebra homomorphism by defining

$$t_Q^{norm}(tT^0) := \delta_Q^{1/2}(t) t_Q(tT^0) \ .$$

Then [BK] Cor. 8.4 says that, for any $\chi \in X_{nr}(T)$, we have an isomorphism of $\mathcal{H}(G,J)$-modules

$$\epsilon_J \operatorname{Ind}_Q^G(\chi) \cong \operatorname{Hom}_{\mathbb{C}[T/T^0]}(\mathcal{H}(G,J), \mathbb{C}_\chi)$$

where $\mathbb{C}[T/T^0]$ acts on $\mathcal{H}(G,J)$ through the homomorphism t_Q^{norm} and on $\mathbb{C}_\chi$ through the character χ. By comparing definitions we see that the homomorphism in Lemma 2.7 is equal to t_P^{norm}. It is shown in [Dat] Prop. 2.1 that the restriction $t_Q^{norm}|\mathbb{C}[T/T^0]^W$ is independent of the choice of the parabolic subgroup Q. Hence from now on we will treat the isomorphism $\mathbb{C}[T/T^0]^W \cong Z(\mathcal{H}(G,J))$ in Lemma 2.7 as an identification. The following description of the cuspidal support in terms of the action of the center is now an immediate consequence of the above isomorphism.

Corollary 2.8. *The cuspidal support map* ν *is continuous for the Jacobson and Zariski topology on the source and target, respectively.*

Proof. By the above discussion ν can be viewed as the map

$$\begin{aligned} \operatorname{Irr}(\mathcal{H}(G,J)) &\longrightarrow \operatorname{Spec}(Z(\mathcal{H}(G,J))) \\ [\mathcal{L}] &\longmapsto \text{annihilator ideal of } \mathcal{L} \text{ in } Z(\mathcal{H}(G,J)). \end{aligned}$$

In this form the map is visibly continuous. □

Proposition 2.9. *If* $\chi \in X_{nr}(T)$ *lies in the cuspidal support of some* $[V] \in \operatorname{Irr}_{\epsilon_J}(G)$ *then the center* $\mathbb{C}[T/T^0]^W$ *of* $\mathcal{H}(G,J)$ *acts on* $\epsilon_J V$ *through the character* χ*.*

By the very construction of the Langlands dual group we have

$$X_{nr}(T) = \widehat{T} \tag{4}$$

(cf. [Bo2] 9.5) which moreover is a maximal torus in $\widehat{G}$. Let $\chi_s \in X_{nr}(T)$ denote the unramified character which corresponds to an element $s \in \widehat{T}$ under this identification. We note that unitary unramified characters correspond to compact (= elliptic) elements in $\widehat{T}$.

Proposition 2.10. *Suppose that $s \in \widehat{T}$; then the center $\mathbb{C}[T/T^0]^W$ of $\mathcal{H}(G,J)$ acts on $\mathcal{M}_{u,s}$ through the character χ_s.*

Proof. Let $R(\widehat{G} \times \mathbb{C}^\times)$ denote the representation ring of the complex algebraic group $\widehat{G} \times \mathbb{C}^\times$. According to [KL] (2.14) the formal character provides an isomorphism

$$R(\widehat{G} \times \mathbb{C}^\times) \xrightarrow{\cong} \mathbb{Z}[\mathbf{q}^{\pm 1}][X^*(\widehat{T})]^W$$

between the representation ring and the center of the generic Hecke algebra; here $X^*(\widehat{T})$, of course, denotes the character group of the algebraic torus $\widehat{T}$. By [KL] (5.12) the center of the generic Hecke algebra acts, after making the above identification, on the module $\mathcal{M}_{u,s}$ through the character $Tr((s,q);.)$. This latter character corresponds on $\mathbb{Z}[\mathbf{q}^{\pm 1}][X^*(\widehat{T})]$ to the character sending $\mathbf{q}$ to q and $\xi \in X^*(\widehat{T})$ to $\xi(s)$. Using the formulas in [KL] (2.12) and [HKP] (1.15) one checks that under the identification

$$\begin{aligned} T/T^0 &\xrightarrow{\cong} X^*(\widehat{T}) \\ tT^0 &\longmapsto \xi_t(s') := \chi_{s'}(t) \end{aligned}$$

the center of the generic Hecke algebra specializes to the description of the center of $\mathcal{H}(G,J)$ given in Lemma 2.7. Moreover, the above character becomes on $\mathbb{C}[T/T^0]$ the character χ_s. □

Corollary 2.11. *Up to conjugation we may assume that the triple (u,s,ρ) satisfies $s \in \widehat{T}$; then the representation $\pi(u,s,\rho)$ has cuspidal support $W\chi_s$.*

Proof. Recalling that $\mathcal{L}_{u,s,\rho}$ is a quotient of $\mathcal{M}_{u,s}$ it remains to combine Prop. 2.9 and Prop. 2.10. □

Let $X_{nr}(G)$ denote the group of unramified characters of G. To be precise we recall that a character $\chi : G \longrightarrow \mathbb{C}^\times$ is called unramified if it is trivial on the subgroup $G^1 \subseteq G$ which is the simultaneous kernel of the $|\xi|_k$ with ξ running over all k-rational characters of G. Note that $X_{nr}(G)$ is a complex algebraic torus (cf. [Ca] III.3.2).

Lemma 2.12. $X_{nr}(G) = Z(\widehat{G})^\circ$.

Proof. Clearly $Z(\widehat{G})^\circ \subseteq \widehat{T}$. On the other hand it follows from the Cartan decomposition of G that the natural map $X_{nr}(G) \longrightarrow X_{nr}(T)$ is injective. Since all coroots $\check{\alpha}$ of G have their image in the derived group of G we have $\chi \circ \check{\alpha} = 1$ for any $\chi \in X_{nr}(G)$. As the coroots of G are the roots of $\widehat{G}$ this means that χ viewed, via (4), as an element of $\widehat{T}$ lies in the simultaneous kernel of all roots which is nothing else than the center $Z(\widehat{G})$. Hence under the identification (4) we have $X_{nr}(G) \subseteq Z(\widehat{G})$ and a fortiori $X_{nr}(G) \subseteq Z(\widehat{G})^\circ$. But both sides are tori of the same dimension so that equality must hold. □

2.4 The unipotent class of a tempered Iwahori component

In this section we will first of all show that in a tempered pair (u,s) the semisimple element s determines the unipotent orbit O_u. We begin with the following fact. Given a unipotent element $u \in \widehat{G}$ we choose, using the Jacobson-Morozov theorem, a homomorphism of algebraic groups $\gamma_u : SL_2(\mathbb{C}) \longrightarrow \widehat{G}$ such that $\gamma_u((\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})) = u$.

Proposition 2.13. *The map*

$$\mathcal{O}_{\widehat{G}} \longrightarrow \text{set of all semisimple conjugacy classes of } \widehat{G}$$
$$O_u \longmapsto \text{class of } \gamma_u(\left(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\right))$$

is injective.

Proof. The map is well defined since γ_u is determined by u up to conjugation by an element in the centralizer of u. Let now u and u' be two unipotent elements such that the homomorphisms $g\gamma_u g^{-1}$ and $\gamma_{u'}$ coincide in the element in $\left(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\right)$. Since this element generates a Zariski dense subgroup of the diagonal torus $\mathbb{C}^\times$ in $SL_2(\mathbb{C})$ it follows that we have $g\gamma_u g^{-1}|\mathbb{C}^\times = \gamma_{u'}|\mathbb{C}^\times$. By derivation we deduce that $Ad(g) \circ d\gamma_u|\mathbb{C} = d(g\gamma_u g^{-1})|\mathbb{C} = d\gamma_{u'}|\mathbb{C}$. Therefore the assumptions of [B-GAL] Chap. VIII §11.3 Prop. 6 are satisfied and we obtain that $d\gamma_u$ and $d\gamma_{u'}$ are $\widehat{G}$-conjugate. Since $SL_2(\mathbb{C})$ is simply connected this implies, by [B-GAL] Chap. III §6.1 Thm. 1, that γ_u and $\gamma_{u'}$ are $\widehat{G}$-conjugate. We conclude that $O_u = O_{u'}$. □

We now fix a semisimple element $s \in \widehat{G}$ and let $s = s_e s_h$ be its polar decomposition into the elliptic part s_e and the hyperbolic part s_h. Consider any unipotent $u \in \widehat{G}$ such that $sus^{-1} = s^q$ and define $\tau_u := \gamma_{u,s}(\left(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\right))$ with $\gamma_{u,s}$ as in section 2.1. We have:

1. τ_u is well defined up to conjugation by elements in $Z_{\widehat{G}}(u,s)$.
2. By (2) the image of $\gamma_{u,s}$ lies in the centralizer $Z_{\widehat{G}}(\tau_u^{-1}s)$ of $\tau_u^{-1}s$; in particular, τ_u and s commute.
3. τ_u is hyperbolic and satisfies $\tau_u u \tau_u^{-1} = u^q$.
4. u and s_e commute. (By 2. the elements $\tau_u^{-1}s$ and u, hence also $(\tau_u^{-1}s)_e$ and u commute; but $(\tau_u^{-1}s)_e = s_e$ by 3.)

Corollary 2.14. *In a tempered pair (u,s) the $\widehat{G}$-conjugacy class of s_h determines the unipotent orbit O_u.*

Proof. By Prop. 2.13 the class of τ_u determines O_u. On the other hand Thm. 2.2 says that $\tau_u^{-1}s$ is elliptic. Finally properties 2. and 3. above imply that $\tau_u = s_h$. □

We need two further properties of tempered pairs.

Lemma 2.15. *Let (u,s) and (u',s') be two tempered pairs; if $s' = s\sigma$ with an elliptic element $\sigma \in Z(\widehat{G})$ then $O_u = O_{u'}$.*

Proof. With (u',s') also (u',s) is a tempered pair, since $s'_h = s_h$. It remains to apply Cor. 2.14. □

Let $L \subseteq G$ be a Levi subgroup.

Remark 2.16. *For a general k-split reductive group G (with not necessarily connected center) we have:*

i. The centralizer of any k-split torus is a Levi subgroup in G, which is k-split;

ii. any Levi subgroup $L \subseteq G$ is the centralizer of its connected center $Z(L)^\circ$;

iii. the center $Z(L)$ of any Levi subgroup $L \subseteq G$ satisfies $Z(L) = Z(G) \cdot Z(L)^\circ$.

Proof. For i. see [BT] Thm. 4.15.a). For ii. see [Bo1] Prop. 14.18 (or Prop. 20.6(i); also note that $Z(L)$ and $Z(L)^\circ$ are defined over k by [Bo1] Prop. 1.2(b) and Thm. 18.2(ii)). In fact we need a few details of this argument to see iii. Suppose that the Levi subgroup L is standard with respect to the maximal torus T in G and the set of simple roots $\Delta \subseteq X^*(T)$. Then (cf. [Spr] 8.1.8(i))

$$Z(G) = \bigcap_{\alpha \in \Delta} \ker(\alpha) \ .$$

Moreover there is a subset $I \subseteq \Delta$ such that

$$Z(L) = \bigcap_{\alpha \in I} \ker(\alpha) \ .$$

The set I is a basis of the root system of L with respect to T. Hence $\dim(T/Z(G)) = |\Delta|$ and $\dim(T/Z(L)) = |I|$ so that $\dim(Z(L)^\circ/Z(G)^\circ) = |\Delta \setminus I|$. The map

$$Z(L) \xrightarrow{\prod_{\alpha \in \Delta \setminus I} \alpha} \prod_{\alpha \in \Delta \setminus I} \mathbb{G}_m$$

therefore restricts to a homomorphism $Z(L)^\circ/Z(G)^\circ \longrightarrow \prod_{\alpha \in \Delta \setminus I} \mathbb{G}_m$ of tori of the same dimension with a finite kernel. Hence the map $Z(L)^\circ \longrightarrow \prod_{\alpha \in \Delta \setminus I} \mathbb{G}_m$ is surjective which implies the identity in iii. □

The identity $Z(L) = Z(G) \cdot Z(L)^\circ$ shows that L inherits from G the assumption that its center is connected. The dual group $\widehat{L}$ can be viewed as a Levi subgroup of $\widehat{G}$.

Lemma 2.17. *Let (u, s) be a tempered pair in $\widehat{G}$ and (u', s') be a tempered pair in $\widehat{L}$; if s and s' are $\widehat{G}$-conjugate then u and u' are $\widehat{G}$-conjugate.*

Proof. Since (u', s') also is a tempered pair when considered in $\widehat{G}$ the assertion is immediate from Prop. 2.13. □

Theorem 2.18. *The map*

$$\begin{aligned} \mathrm{Irr}^t_{\epsilon_J}(G) &\longrightarrow \mathcal{O}_{\widehat{G}} \\ [\pi(u, s, \rho)] &\longmapsto O_u \end{aligned}$$

is constant on the components $(\nu^t)^{-1}(\Theta)$ for $\Theta \in \theta_{\epsilon_J}$ and hence induces a map

$$\begin{aligned} \theta_{\epsilon_J} &\longrightarrow \mathcal{O}_{\widehat{G}} \\ \Theta &\longmapsto O_\Theta \ . \end{aligned}$$

Moreover, we have:

$$\Theta' \lesssim \Theta \quad \Longrightarrow \quad O_{\Theta'} \geq O_\Theta \ .$$

Proof. Let $\pi(u,s,\rho)$ be any irreducible tempered G-representation in $\mathrm{Irr}^t_{\epsilon_J}(G)$. Its discrete support is represented by some pair $(L, \pi^L(u',s',\rho'))$. The cuspidal support of $\pi^L(u',s',\rho')$ is contained in the cuspidal support of $\pi(u,s,\rho)$. It therefore follows from Cor. 2.11 that s and s' are conjugate in $\widehat{G}$. Lemma 2.17 then implies that $O_u = O_{u'}$. Now let $\pi(u_1,s_1,\rho_1)$ be any other irreducible tempered G-representation in the same component category as $\pi(u,s,\rho)$. By construction its discrete support is represented by an L-representation of the form $\chi\pi^L(u',s',\rho')$ for some unitary unramified character χ of L. According to Lemma 2.12 we have $\chi = \chi_\sigma$ for some elliptic element $\sigma \in Z(\widehat{L})^\circ$. Comparing cuspidal supports again we see that the discrete support of $\pi(u_1,s_1,\rho_1)$ also can be represented by an L-representation of the form $\pi^L(u'',s'\sigma,\rho'')$. From our initial argument we know that $O_{u_1} = O_{u''}$. On the other hand Lemma 2.15 tells us that u' and u'' are conjugate already in L. Altogether it follows that $O_u = O_{u_1}$. The order reversing property of the map $\Theta \longmapsto O_\Theta$ is an immediate consequence of Thm. 2.3. □

This theorem, in particular, says that, for any $O \in \mathcal{O}_{\widehat{G}}$, the set of tempered components

$$\theta_O := \{\Theta \in \theta_{\epsilon_J} : O_\Theta \le O\}$$

is saturated.

2.5 The case $GL_n(k)$

In this section our group G always is assumed to be $G = GL_n(k)$. Then the map $\Theta \longmapsto O_\Theta$ in Thm. 2.18 is a bijection (cf. [SSZ] §2 Fact 2). With the notation introduced before Thm. 2.3 we have

$$(\nu^t)^{-1}(\Theta) = \pi^{-1}(\mathcal{O}_\Theta) \cap \mathrm{Irr}^t_{\epsilon_J}(G) \ .$$

It then follows from Thm. 2.18 that the preorder $\lesssim$ on θ_{ϵ_J} is a partial order. We will show that the image of this partial order $\lesssim$ for general n is strictly weaker than the reverse of the partial order $\le$ on $\mathcal{O}_{\widehat{G}}$.

The set $\mathcal{O}_{\widehat{G}}$ can and will be identified with the set of ordered partitions of n. Such a partition $\mathcal{P}$ is a non-increasing sequence $m_1 \ge \ldots \ge m_n$ of nonnegative integers such that $m_1 + \ldots + m_n = n$. It corresponds to the orbit $\mathcal{O}_\mathcal{P}$ of the unipotent element with Jordan blocks of size m_i. When writing a partition we usually will omit the m_i which are equal to zero. The partial order $\le$ on $\mathcal{O}_{\widehat{G}}$ corresponds to the dominance order on the set of partitions.

Lemma 2.19. *The relation $(l_1,\ldots,l_n) \ge (m_1,\ldots,m_n)$ is a cover for the dominance order if and only if there are $1 \le r,s \le n$ such that $l_r = m_r + 1$, $l_s = m_s - 1$, $l_i = m_i$ for any $i \ne r,s$ and either $s = r+1$ or $m_r = m_s$.*

Proof. See [Bry] Prop. 2.3. □

We remind the reader that, for any $\Theta \in \theta_{\epsilon_J}$, the closure of $(\nu^t)^{-1}(\Theta)$ viewed in $\mathrm{Irr}(\mathcal{H}(G,J))$ is $V(\mathcal{J}_\Theta)$ for the ideal $\mathcal{J}_\Theta := \{f \in \mathcal{H}(G,J) : fV^J = 0 \text{ for any } [V] \in (\nu^t)^{-1}(\Theta)\}$.

Lemma 2.20. *Let $M \subseteq G$ be a Levi subgroup, τ be a irreducible smooth M-representation, and denote by Ind_P^G the normalized parabolic induction functor for a fixed (but arbitrary) choice of parabolic subgroup P with Levi component M; then, for any $f \in \mathcal{H}(G,J)$, the map*

$$\begin{aligned} X_{nr}(M) &\longrightarrow \mathrm{End}_{\mathbb{C}}(\mathrm{Ind}_P^G(\chi\tau)^J) \\ \chi &\longmapsto f_{|\,\mathrm{Ind}_P^G(\chi\tau)^J} \end{aligned}$$

is a regular algebraic map on the algebraic torus $X_{nr}(M)$.

Proof. (We fix a good maximal compact subgroup K of G. The target of the map in the assertion naturally identifies with the vector space $\mathrm{End}_{\mathbb{C}}(\mathrm{Ind}_{K\cap P}^{K}(\tau)^J)$, which is independent of χ.) For the convenience of the reader we sketch the well known argument. Let $B := \mathcal{O}(X_{nr}(M))$ denote the ring of regular functions on the torus $X_{nr}(M)$. The group M acts on B via the universal unramified character $\chi_{univ}(m) := [\chi \mapsto \chi(m)]$. Then $B \otimes_{\mathbb{C}} \tau$ is a smooth M-representation via the diagonal action, which is B-linear. It follows that

$$\mathrm{Ind}_P^G(B \otimes_{\mathbb{C}} \tau)^J = \mathrm{Ind}_{K\cap P}^{K}(B \otimes_{\mathbb{C}} \tau)^J = B \otimes_{\mathbb{C}} \mathrm{Ind}_{K\cap P}^{K}(\tau)^J$$

is an $(\mathcal{H}(G,J), B)$-bimodule, which as a B-module is finitely generated free. We see that our element f, as an endomorphism of this B-module, is given by a matrix $(b_{ij})_{i,j}$ with entries in B. One easily checks that $f_{|\mathrm{Ind}_P^G(\chi\tau)^J}$ then is given by the matrix of complex numbers $(b_{ij}(\chi))_{i,j}$. □

We fix a connected component $\Theta \in \theta_{\epsilon_J}$. There is a discrete pair (M, τ) such that $(\nu^t)^{-1}(\Theta)$ consists of the isomorphism classes of all G-representations of the form $\mathrm{Ind}_P^G(\chi\tau)$ for $\chi \in X_{nr}^1(M)$. Recall for this that for the group $GL_n(k)$ unitary normalized parabolic induction is irreducible.

Lemma 2.21. *The closure $\overline{(\nu^t)^{-1}(\Theta)}$ contains all $[V]$ where V is an irreducible constituent of $\mathrm{Ind}_P^G(\chi\tau)$ for some $\chi \in X_{nr}(M)$.*

Proof. This is immediate from Lemma 2.20 since $X_{nr}^1(M)$ is Zariski dense in $X_{nr}(M)$. □

To go further we will use the Bernstein-Zelevinsky classification of irreducible representations in terms of multisets of segments. Since we only consider representations with an Iwahori fixed vector, a segment Δ of length r is a sequence of unramified characters of $k^\times$ of the form $\Delta = (\chi|\ |_k^j, \chi|\ |_k^{j+1}, \ldots, \chi|\ |_k^{j+r-1})$ where $j \in \mathbb{Z}$, $r \geq 1$, and χ is an unramified character. Bernstein-Zelevinsky associate with Δ a specific irreducible smooth representation $L(\Delta)$ of $GL_r(k)$ (we follow the notations in [Rod]). The representation $L(\Delta)$ is discrete series if and only if the segment Δ is centered, i.e., χ is unitary and $j = -(r-1)/2$ ([Rod] Prop. 11). Now consider a multiset of segments $(\Delta_1, \ldots, \Delta_s)$ such that the lengths of the Δ_i form a partition $m_1 \geq \ldots \geq m_s$ of n. We then view $\sigma := L(\Delta_1) \otimes \ldots \otimes L(\Delta_s)$ as a representation of the Levi subgroup $M = GL_{m_1}(k) \times \ldots \times GL_{m_s}(k)$ of G. Following [Rod] §2.1 we write $\mathrm{Ind}_P^G(\sigma) = L(\Delta_1) \times \ldots \times L(\Delta_s)$, where P is the upper block-diagonal parabolic subgroup with Levi M. Bernstein-Zelevinsky construct a specific irreducible subquotient $L(\Delta_1, \ldots, \Delta_s)$ of this parabolically induced representation (cf. [Rod] Thm. 3 and [Zel] Thm. 1.9). The classification theorem ([Rod] Thm. 3) says that this sets up a bijection between the set of all multisets of segments whose lengths add up to n and the set $\mathrm{Irr}_{\epsilon_J}(G)$. Moreover, the representation $L(\Delta_1, \ldots, \Delta_s)$ is tempered if and only if all segments Δ_i are centered, and $L(\Delta_1, \ldots, \Delta_s) = L(\Delta_1) \times \ldots \times L(\Delta_s)$ in this case. We emphasize that associated with a multiset of segments $a = (\Delta_1, \ldots, \Delta_s)$ we have

- the partition $\mathcal{P}(a)$ formed by the lengths of the Δ_i and
- the cuspidal support of $L(a)$, which is the multiset $\nu(a) := \Delta_1 + \ldots + \Delta_s$ of unramified characters of $k^\times$ (i.e., the sequence of unramified characters contained in the Δ_i) viewed as an element of $W\backslash X_{nr}(T)$.

For any partition $\mathcal{P}$ of n let $L^{-1}(\mathcal{P}) := \{[L(a)] \in \mathrm{Irr}_{\epsilon_J}(G) : \mathcal{P}(a) = \mathcal{P}\}$. Obviously $\mathrm{Irr}_{\epsilon_J}(G)$ is the disjoint union of these subsets $L^{-1}(\mathcal{P})$. If $\mathcal{P} = (m_1 \geq \ldots \geq m_s)$ then $L^{-1}(\mathcal{P})$ contains (the isomorphism class of) the representation $L(a(\mathcal{P}))$ where $a(\mathcal{P}) = (\Sigma_1(\mathcal{P}), \ldots, \Sigma_s(\mathcal{P}))$ is the multiset of segments $\Sigma_i(\mathcal{P}) := (|\ |_k^{-(m_i-1)/2}, \ldots, |\ |_k^{(m_i-1)/2})$. These segments are centered. Hence $L(a(\mathcal{P}))$ is tempered; in fact, it is the unique "real" tempered representation in $L^{-1}(\mathcal{P})$. Moreover, if M is the standard Levi subgroup with block sizes given by $\mathcal{P}$, then the map

$$\begin{aligned} Q_{\mathcal{P}} : X_{nr}(M) &\longrightarrow L^{-1}(\mathcal{P}) \\ (\chi_1, \ldots, \chi_s) &\longmapsto [L(\chi_1\Sigma_1(\mathcal{P}), \ldots, \chi_s\Sigma_s(\mathcal{P}))] \end{aligned}$$

as well as its restriction $Q^t_{\mathcal{P}} : X^1_{nr}(M) \longrightarrow L^{-1}(\mathcal{P}) \cap \mathrm{Irr}^t_{\epsilon_J}(G)$ are surjective. Note that the discrete support of a tempered $L(\chi_1\Sigma_1(\mathcal{P}), \ldots, \chi_s\Sigma_s(\mathcal{P}))$ is given by the discrete pair $(M, L(\chi_1\Sigma_1(\mathcal{P})) \otimes \ldots \otimes L(\chi_s\Sigma_s(\mathcal{P}))$.

Proposition 2.22. *For any partition $\mathcal{P}$ of n we have $L^{-1}(\mathcal{P}) = \pi^{-1}(\mathcal{O}_{\mathcal{P}})$.*

Proof. In the subsequent Appendix we will prove that $L(a) \cong \pi(u_a, s_a)$. Hence we must show that the unipotent class $\mathcal{O}_{u_a}$ corresponds to the partition $\mathcal{P}(a)$. But by definition, if $a = (\Delta_1, \ldots, \Delta_s)$, then u_a is the block-diagonal matrix with diagonal blocks $(u_{\Delta_1}, \ldots, u_{\Delta_s})$. Therefore the Jordan normal form $J(u_a)$ of u_a has diagonal blocks $(J(u_{\Delta_1}), \ldots, J(u_{\Delta_s}))$. Moreover the elements u_{Δ_i} are such that $\gamma_{u_{\Delta_i}, s_{\Delta_i}}$ is the irreducible representation of $SL_2(\mathbb{C})$ of dimension equal to the length of Δ_i. Passing to the Jordan normal form is a conjugation which preserves irreducibility. Hence the $J(u_{\Delta_i})$ indeed must be single Jordan blocks, which implies our claim. □

Zelevinsky introduces in [Zel] 7.1 a partial order $\leq$ on the set of all multisets of segments. It is the transitive hull of the following "elementary" relations. Let $a = (\Delta_1, \ldots, \Delta_s)$ be a multiset of segments. If two segments Δ_i and Δ_j are linked ([Zel] 4.1) then one may replace them by the segments $\Delta_i \cup \Delta_j$ and $\Delta_i \cap \Delta_j$ obtaining in this way a multiset of segments $b \leq a$. It is not difficult to deduce from this that whenever $b \leq a$ then $\mathcal{P}(b) \geq \mathcal{P}(a)$. Moreover, it is clear that, if $b \leq a$, then the two representations $L(b)$ and $L(a)$ have the same cuspidal support.

In the following let $\mathcal{P}(\Theta)$ be the unique partition of n such that $\mathcal{O}_\Theta = \mathcal{O}_{\mathcal{P}(\Theta)}$. It follows from Lemma 2.21 and Prop. 2.22 that

$$\overline{(\nu^t)^{-1}(\Theta)} = \overline{L^{-1}(\mathcal{P}(\Theta))} = \overline{\pi^{-1}(\mathcal{O}_\Theta)}\ ,$$

which can be viewed as a refinement of Remark 1.6.

Proposition 2.23. *$\overline{(\nu^t)^{-1}(\Theta)} \cap \mathrm{Irr}^t_{\epsilon_J}(G)$ is the set of all $[L(b)]$ such that $L(b)$ is tempered and $b \leq a$ for some a with $[L(a)] \in L^{-1}(\mathcal{P}(\Theta))$.*

Proof. First suppose that $b \leq a$ for some $[L(a)] \in L^{-1}(\mathcal{P}(\Theta))$. We have

$$a = (\chi_1\Sigma_1(\mathcal{P}(\Theta)), \ldots, \chi_s\Sigma_s(\mathcal{P}(\Theta))) \qquad \text{for some } \chi := (\chi_1, \ldots, \chi_s) \in X_{nr}(M).$$

By [Zel] Thm. 7.1 the relation $b \leq a$ implies that $L(b)$ is a constituent of $\mathrm{Ind}^G_P(\chi\sigma)$ where $\sigma := L(\Sigma_1(\mathcal{P}(\Theta))) \otimes \ldots \otimes L(\Sigma_s(\mathcal{P}(\Theta)))$. By Prop. 2.22 the G-orbit of the discrete pair (M, σ) lies in Θ. Hence it follows from Lemma 2.21 that $[L(b)] \in \overline{(\nu^t)^{-1}(\Theta)}$.

Now let, vice versa, $[L(b)]$ be an element in $\overline{(\nu^t)^{-1}(\Theta)} \cap \mathrm{Irr}^t_{\epsilon_J}(G) = \overline{L^{-1}(\mathcal{P}(\Theta))} \cap \mathrm{Irr}^t_{\epsilon_J}(G)$. It is straightforward to see that the composite map

$$X_{nr}(M) \xrightarrow{Q_{\mathcal{P}(\Theta)}} L^{-1}(\mathcal{P}(\Theta)) \xrightarrow{\nu} W\backslash X_{nr}(T)$$

has a Zariski closed image. Since the left arrow is surjective and ν is continuous it follows that

$$\nu(\overline{L^{-1}(\mathcal{P}(\Theta))}) = \nu(L^{-1}(\mathcal{P}(\Theta))) \ .$$

Hence we find an $[L(a)] \in L^{-1}(\mathcal{P}(\Theta))$ such that $L(b)$ and $L(a)$ have the same cuspidal support. For a fixed cuspidal support $(\chi_1, \ldots, \chi_n) \in W\backslash X_{nr}(T)$ let S denote the (finite) set of all multisets of segments with this cuspidal support. We then have:

- S is the set of all multisets of segments c such that $c \leq (\{\chi_1\}, \ldots, \{\chi_n\})$ ([Zel] Thm. 7.1).
- S contains a unique multiset d such that $d \leq c$ for any $c \in S$; it is characterized as being the multiset $d = (\Sigma'_1, \ldots, \Sigma'_r)$ such that no pair of segments Σ'_i, Σ'_j, for any $1 \leq i \neq j \leq r$, is linked ([Zel] Lemma 9.10).

As $L(b)$ is tempered the segments in the multiset b are centered. Hence b satisfies this latter characterizing condition, and it follows that $b \leq a$. □

Corollary 2.24. *Equivalent are for any two $\Theta', \Theta \in \theta_{\epsilon_J}$:*

i. $(\nu^t)^{-1}(\Theta') \cap \overline{(\nu^t)^{-1}(\Theta)} \neq \emptyset$;

ii. $[L(a(\mathcal{P}(\Theta')))] \in \overline{(\nu^t)^{-1}(\Theta)}$;

iii. there exists a multiset of segments $a = (\Delta'_1, \ldots, \Delta'_s)$ such that $\mathcal{P}(a) = \mathcal{P}(\Theta)$ and $\nu(a) = \nu(a(\mathcal{P}(\Theta')))$.

Proof. By definition $b_1 := a(\mathcal{P}(\Theta')) = (\Sigma_1, \ldots, \Sigma_r)$ with $\Sigma_i := \Sigma_i(\mathcal{P}(\Theta'))$. As a consequence of Prop. 2.22 we have

$$(\nu^t)^{-1}(\Theta') = \{[L(\chi_1\Sigma_1, \ldots, \chi_r\Sigma_r)] : \chi_i \in X^1_{nr}(k^\times)\}.$$

In particular, $[L(b_1)] \in (\nu^t)^{-1}(\Theta')$.

Now let $[L(b)] \in (\nu^t)^{-1}(\Theta') \cap \overline{(\nu^t)^{-1}(\Theta)}$. We have $b = (\chi_1\Sigma_1, \ldots, \chi_r\Sigma_r)$ for some $\chi_i \in X^1_{nr}(k^\times)$. By Prop. 2.23 there is an $[L(a)] \in L^{-1}(\mathcal{P}(\Theta))$ such that $b \leq a$. Let $a = (\Delta_1, \ldots, \Delta_s)$. The representations $L(b)$ and $L(a)$ have the same cuspidal support, i.e., b and a have the same underlying multiset of unramified characters of $k^\times$. Suppose that $\xi_i \in \chi_i\Sigma_i$ and $\xi_j \in \chi_j\Sigma_j$ both are contained in some Δ_l; then $\xi_i \in \xi_j|\ |_k^{\mathbb{Z}}$; since χ_i and χ_j are unitary it follows that $\chi_i = \chi_j$. We see that, for any $1 \leq l \leq s$, there is a unique $\chi(l) \in \{\chi_1, \ldots, \chi_r\}$ such that $\Delta_l \subseteq \chi(l)(\Sigma_1 \cup \ldots \cup \Sigma_r)$. Hence we may define new segments Δ'_l by

$$\chi(l)\Delta'_l = \Delta_l \ .$$

For any $\chi \in \mathfrak{x} := \{\chi_1, \ldots, \chi_r\}$ we define the multisets of segments $b_\chi := (\chi_i\Sigma_i : \chi_i = \chi)$ and $a_\chi := (\Delta_l : \chi(l) = \chi)$. Then

$$b = (\chi_1\Sigma_1, \ldots, \chi_r\Sigma_r) = \sum_{\chi\in\mathfrak{x}} b_\chi \qquad \text{and} \qquad a = (\Delta_1, \ldots, \Delta_s) = \sum_{\chi\in\mathfrak{x}} a_\chi \ .$$

By construction b_χ and a_χ, for any $\chi \in \mathfrak{x}$, have the same underlying multiset of unramified characters. It follows that $b_1 = (\Sigma_1, \dots, \Sigma_r)$ and $a_1 := (\Delta'_1, \dots, \Delta'_s)$ have the same multiset of unramified characters. As we have seen already in the proof of Prop. 2.23, the fact that the segments Σ_i are centered then implies that necessarily $b_1 \leq a_1$. Since $\mathcal{P}(a_1) = \mathcal{P}(a) = \mathcal{P}(\Theta)$ it finally follows from Prop. 2.23 that $[L(b_1)] \in \overline{(\nu^t)^{-1}(\Theta)}$. ☐

Corollary 2.25. *Equivalent are for any two $\Theta', \Theta \in \theta_{\epsilon_J}$:*

i. $(\nu^t)^{-1}(\Theta') \subseteq \overline{(\nu^t)^{-1}(\Theta)}$*;*

ii. the partition $\mathcal{P}(\Theta)$ is a refinement of the partition $\mathcal{P}(\Theta')$.

Proof. We again observe that $(\nu^t)^{-1}(\Theta') = \{[L(\chi_1\Sigma_1, \dots, \chi_r\Sigma_r)] : \chi_i \in X^1_{nr}(k^\times)\}$ with the centered segments $\Sigma_i := \Sigma_i(\mathcal{P}(\Theta'))$. If ii. holds true then we may replace in any multiset $b = (\chi_1\Sigma_1, \dots, \chi_r\Sigma_r)$ some pairs of segments by their sum obtaining a multiset $a = (\Delta_1, \dots, \Delta_s)$ such that $b \leq a$ and $[L(a)] \in L^{-1}(\mathcal{P}(\Theta))$. It therefore follows from Prop. 2.23 that $[L(b)] \in \overline{(\nu^t)^{-1}(\Theta)}$. On the other hand, if i. holds true then any $[L(b)]$, where the unitary characters χ_i in $b = (\chi_1\Sigma_1, \dots, \chi_r\Sigma_r)$ are pairwise different, lies in $\overline{(\nu^t)^{-1}(\Theta)}$. Again by Prop. 2.23 there must exist a multiset $a = (\Delta_1, \dots, \Delta_s) \geq b$ such that $\mathcal{P}(a) = \Theta$. Since $\nu(b) = \nu(a)$ the additional assumption on the χ_i implies that the segments Δ_j only can arise by subdividing some of the segments $\chi_i\Sigma_i$. Hence necessarily $\mathcal{P}(a)$ is a refinement of $\mathcal{P}(b)$. ☐

We see that the image of $\lesssim$ is finer than the reverse of the refinement order on partitions. In order to show that the image of $\lesssim$, in general, is strictly in between the reverse of the refinement order and the reverse of the dominance order we consider $\Theta', \Theta \in \theta_{\epsilon_J}$ such that

$$\mathcal{P}(\Theta') = (n - m \geq m) = (m_1 + m_2 - m \geq m) > \mathcal{P}(\Theta) = (m_1 \geq m_2) \tag{5}$$

for some $m_2 > m \geq 0$. By Lemma 2.19 this is a cover for the dominance order if and only if $m = m_2 - 1$.

Proposition 2.26. *In the above situation* (5) *we have*

$$(\nu^t)^{-1}(\Theta') \cap \overline{(\nu^t)^{-1}(\Theta)} \neq \emptyset \quad \textit{if and only if } m_1 = m_2 \textit{ or } m = 0.$$

Proof. First suppose that the left hand intersection is nonempty and that $m \neq 0$. By Cor. 2.24 we have $[L(\Sigma_1, \Sigma_2)] \in \overline{(\nu^t)^{-1}(\Theta)}$ where $\Sigma_1 = (|\ |_k^{-(n-m-1)/2}, |\ |_k^{(n-m-1)/2})$ and $\Sigma_2 = (|\ |_k^{(-(m-1)/2}, |\ |_k^{(m-1)/2})$. By Prop. 2.23 we must have a multiset of segments $(\Delta_1, \Delta_2) \geq (\Sigma_1, \Sigma_2)$ such that Δ_i has length m_i. Comparing cuspidal supports we have:

1) $\Delta_i = (|\ |_k^{l_i}, \dots, |\ |_k^{l_i+m_i-1})$ for some $l_i \in \frac{1}{2}\mathbb{Z}$.

2) $\Sigma_2 = \Sigma_1 \cap \Sigma_2 = \Delta_1 \cap \Delta_2$.

The other case being analogous we may assume that $l_1 \leq l_2$. Then $2l_1 = -n+m+1$ by 1) and $2l_2 = -m+1$ by 2). Since $m_2 > m$ we deduce that $2l_2 + 2m_2 - 2 = -m + 2m_2 - 1 > m - 1$. It follows that $2l_2 + 2m_2 - 2 = n - m - 1$ and $2l_1 + 2m_1 - 2 = m - 1$. We conclude that $2m_1 = -2l_1 + m + 1 = n = -2l_2 + n - m + 1 = 2m_2$.

Vice versa, let us first suppose that $m_1 = m_2$ and $m \neq 0$. Then $a(\mathcal{P}(\Theta')) = (\Sigma_1, \Sigma_2)$ with Σ_i as above. We define $a = (\Delta_1, \Delta_2)$ by $\Delta_1 = (|\ |_k^{-(n-m-1)/2}, \dots, |\ |_k^{(m-1)/2})$ and

$\Delta_2 = (|\ |_k^{-(m-1)/2}, \ldots, |\ |_k^{(n-m-1)/2})$. Both segments Δ_i have length $m_1 = m_2$ and are linked. Moreover, $\Sigma_1 = \Delta_1 \cup \Delta_2$ and $\Sigma_2 = \Delta_1 \cap \Delta_2$. Hence $a(\mathcal{P}(\Theta')) \leq a$ and Prop. 2.23 implies that $[L(\Sigma_1, \Sigma_2)] \in (\nu^t)^{-1}(\Theta') \cap \overline{(\nu^t)^{-1}(\Theta)}$. Finally, suppose that $m = 0$. Then $a(\mathcal{P}(\Theta'))$ is the single centered segment $\Sigma = (|\ |_k^{-(n-1)/2}, |\ |_k^{(n-1)/2})$ of length n. We define $a = (\Delta_1, \Delta_2)$ by $\Delta_1 = (|\ |_k^{-(n-1)/2}, \ldots, |\ |_k^{-\frac{n-1}{2}+m_1-1})$ and $\Delta_2 = (|\ |_k^{-\frac{n-1}{2}+m_1}, \ldots, |\ |_k^{(n-1)/2})$ and conclude as before. □

This result can partially be generalized as follows. Suppose that we have

$$(6) \quad \mathcal{P}(\Theta') = (\ldots \geq m_i + m_j - m \geq \ldots \geq \widehat{m_i} \geq \ldots \geq \widehat{m_j} \geq \ldots \geq m \geq \ldots) \\ > \mathcal{P}(\Theta) = (\ldots \geq m_i \geq \ldots \geq m_j \geq \ldots)$$

for some $i < j$ and some $m_j > m \geq 0$, and consider the following assertions:

a) $\Theta' \lesssim \Theta$.

b) $(\nu^t)^{-1}(\Theta') \cap \overline{(\nu^t)^{-1}(\Theta)} \neq \emptyset$.

c) $m_1 = m_2$ or $m = 0$.

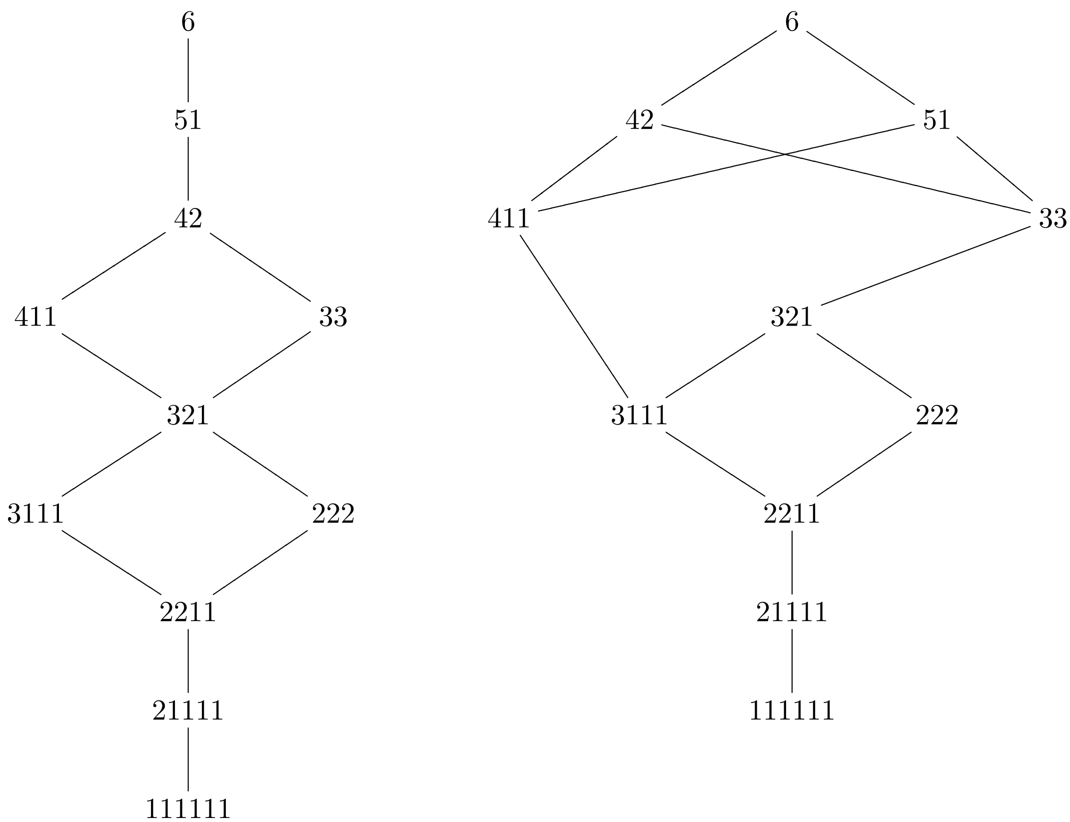
Lemma 2.27. *In the above situation* (6) *we have the implications c)* $\Longrightarrow$ *b) and b)* $\Longrightarrow$ *a).*

Proof. The implication from b) to a) is trivial. The implication from c) to b) is shown by an argument exactly analogous to the corresponding part of the proof of Prop. 2.26. □

Proposition 2.28. *In the above situation* (6) *suppose in addition that* $\mathcal{P}(\Theta') > \mathcal{P}(\Theta)$ *is a cover for the dominance order (so that, in particular,* $m = m_j - 1$*). Then all three assertions a), b), and c) are equivalent.*

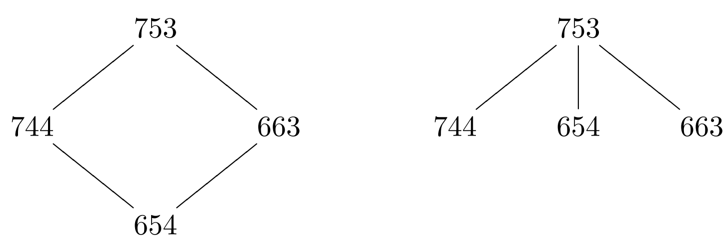
Proof. Since $\mathcal{P}(\Theta') > \mathcal{P}(\Theta)$ is a cover the relation $\Theta' \lesssim \Theta$ in a) must be a cover as well. This implies b) by the definition of $\lesssim$. Because of Lemma 2.27 it remains to show that b) implies c). By Cor. 2.24 we have $[L(a(\mathcal{P}(\Theta')))] \in \overline{(\nu^t)^{-1}(\Theta)}$. Moreover, by Prop. 2.23 we have a multiset of segments a such that $\mathcal{P}(a) = \mathcal{P}(\Theta)$ and $b := a(\mathcal{P}(\Theta')) \leq a$. By our cover assumption this latter relation must be elementary. This means that b and a only differ in pairs of segments $(\Sigma_1, \Sigma_2) < (\Delta_1, \Delta_2)$ where Δ_1 and Δ_2 have length m_i and m_j, respectively. From this point on the further argument again is exactly analogous to the first part of the proof of Prop. 2.26. □

Example 2.29. For $n = 6$ we below display the Hasse diagram for the dominance order on the left and for (the reverse of) the partial order $\lesssim$ on the right:



Prop. 2.28 tells us, for general n, exactly which covers for the dominance order disappear in the Hasse diagram for $\lesssim$. But the partial order also has new covers.

Example 2.30. For $n = 15$ we have the dominance covers on the left and the corresponding covers for the reverse of $\lesssim$ on the right:



We finish by an example that a) above does not, in general, imply b).

Example 2.31. For $n = 7$ we have the covers $\Theta_{421} \lesssim \Theta_{331} \lesssim \Theta_{322}$, where $\Theta_{\mathcal{P}}$ denotes the component corresponding to the partition $\mathcal{P}$. But the intersection $(\nu^t)^{-1}(\Theta_{421}) \cap \overline{(\nu^t)^{-1}(\Theta_{322})} = \emptyset$ is empty. The reason is that, because of the different parities occurring in (421), an elementary relation $b \leq a$ for multisets of segments with $\mathcal{P}(b) = (421)$ can only affect the two segments of length 4 and 2.

Appendix: Comparing Bernstein-Zelevinsky and Kazhdan-Lusztig parameters

Although it must be well known to the experts how the Bernstein-Zelevinsky and Kazhdan-Lusztig parameters for representations with Iwahori fixed vector compare to each other we could not find any proof in the literature. We therefore, for the convenience of the reader, sketch the argument in the following.

First we consider a single segment $\Delta = (\chi, \chi|\ |_k, \ldots, \chi|\ |_k^{n-1})$ with $n \geq 1$ and an unramified character χ of $k^\times$. We put $\lambda := \chi(\varpi) \in \mathbb{C}^\times$ and define $s_\Delta \in GL_n(\mathbb{C})$ to be the diagonal matrix with diagonal entries $(\lambda, \lambda q^{-1}, \ldots, \lambda q^{1-n})$. Let $\rho_n : SL_2(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$ be the irreducible representation given by the $(n-1)$th symmetric power of the natural two dimensional representation. The elements

$$u_2 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \tau_2 := \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}$$

satisfy $\tau_2 u_2 \tau_2^{-1} = u_2^q$. One easily checks that $s_\Delta = (\lambda q^{-(1-n)/2}) \cdot \rho_n(\tau_2)$. We define the unipotent element $u_\Delta := \rho_n(u_2)$. It immediately follows that $s_\Delta u_\Delta s_\Delta^{-1} = u_\Delta^q$, and one checks that ρ_n satisfies the properties of $\gamma_{u_\Delta, s_\Delta}$. Hence $\tau_{u_\Delta} = \rho_n(\tau_2)$ and $\tau_{u_\Delta}^{-1} s_\Delta = (\lambda q^{-(n-1)/2}) \cdot \mathrm{Id}_n$. We see that the pair (u_Δ, s_Δ) is a Kazhdan-Lusztig parameter for the group $GL_n(k)$, and this pair is tempered (cf. Thm. 2.2.i) if and only if $|\lambda| = q^{(1-n)/2}$ if and only if the segment Δ is centered.

Now let $a = (\Delta_1, \ldots, \Delta_t)$ be a multiset of segments of unramified characters of $k^\times$. Let n be the sum of the lengths of these segments. We define the element u_a, resp. s_a, in $GL_n(\mathbb{C})$ to be the block-diagonal matrix with diagonal blocks $(u_{\Delta_1}, \ldots, u_{\Delta_t})$, resp. $(s_{\Delta_1}, \ldots, s_{\Delta_t})$. We obviously have $s_a u_a s_a^{-1} = u_a^q$ so that (u_a, s_a) is a Kazhdan-Lusztig parameter for $GL_n(k)$. Moreover, if $M \subseteq GL_n(k)$ denotes the Levi subgroup corresponding to $\mathcal{P}(a)$ then we have

$$\gamma_{u_a, s_a} = \gamma_{u_{\Delta_1}, s_{\Delta_1}} \times \ldots \times \gamma_{u_{\Delta_t}, s_{\Delta_t}} : SL_2(\mathbb{C}) \longrightarrow \widehat{M} \subseteq GL_n(\mathbb{C}) \ ,$$

and therefore, using Prop. 2.2, we see that (u_a, s_a) is tempered if and only if all the segments Δ_i are centered. We obtain that the representation $L(a)$ is tempered if and only if the representation $\pi(u_a, s_a)$ is tempered. Since unitary induction is irreducible for the groups $GL_n(k)$ a tempered representation is (up to isomorphism) uniquely determined by its cuspidal support. But it follows from Cor. 2.11 that $L(a)$ and $\pi(u_a, s_a)$ have the same cuspidal support. This proves the following intermediate result.

Lemma 2.32. *If $L(a)$ is tempered then $L(a) \cong \pi(u_a, s_a)$.*

The general case will be reduced to the tempered case by using the Langlands classification. We proceed in several steps.

Step 1: Let $T \subseteq G = GL_n(k)$ be the torus of diagonal matrices. Its character group $X^*(T)$ has the standard basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ where ε_i maps a diagonal matrix to its ith diagonal entry. Let $P_\emptyset$ denote the upper triangular Borel subgroup of G. The corresponding simple roots are the $\varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n-1$. For any subset $I \subseteq \Delta := \{1, \ldots, n-1\}$ we define the subtorus $T_I := \{t \in T : \varepsilon_i(t) = \varepsilon_{i+1}(t) \text{ for any } i \in I\}$. Its centralizer M_I in G is a Levi subgroup, which is a Levi component of a unique parabolic subgroup P_I containing $P_\emptyset$. We have the obvious injective restriction maps for characters $X^*(M_I) \hookrightarrow X^*(T)$ and $X^*(M_I) \hookrightarrow X^*(T_I)$. The cokernel of the composite map $X^*(M_I) \hookrightarrow X^*(T) \to X^*(T_I)$ is finite. Hence we obtain

$$\begin{array}{ccc} X^*(M_I) \otimes \mathbb{R} & & \\ & \searrow & \\ \downarrow \cong & & X^*(T) \otimes \mathbb{R} \\ & \swarrow & \\ X^*(T_I) \otimes \mathbb{R}. & & \end{array}$$

The simple coroots induce, for any $j \in \Delta$, the linear form

$$\begin{aligned} \lambda_j : X^*(T) \otimes \mathbb{R} &\longrightarrow \mathbb{R} \\ \sum_{i=1}^{n} r_i \varepsilon_i &\longmapsto r_j - r_{j+1} \ . \end{aligned}$$

The image of the upper oblique arrow is $\{\mu \in X^*(T) \otimes \mathbb{R} : \lambda_i(\mu) = 0 \text{ for any } i \in I\}$. We define the subset

$$(X^*(M_I) \otimes \mathbb{R})_+ := \{\mu \in X^*(M_I) \otimes \mathbb{R} : \lambda_i(\mu|T) > 0 \text{ for any } i \in \Delta \setminus I\}.$$

Finally we need the injective map

$$\begin{aligned} X^*(M_I) \otimes \mathbb{R} &\longrightarrow X_{nr}(M_I) \\ \nu = \mu \otimes r &\longmapsto \nu^{nr}(m) := |\mu(m)|_k^r \ . \end{aligned}$$

The Langlands classification now says the following (cf. [BW] XI§2 or [Sil]).

Proposition 2.33. *For any smooth irreducible representation π of G there is a unique subset $I \subseteq \Delta$, a (up to isomorphism) unique tempered irreducible representation σ of M_I, and a unique element $\nu \in (X^*(M_I) \otimes \mathbb{R})_+$ such that π is the unique irreducible quotient of the normalized parabolic induction $\mathrm{Ind}_{P_I}^G(\nu^{nr}\sigma)$.*

We call (M_I, σ, ν) the Langlands triple associated with π.

To make this even more explicit let $M_I = GL_{n_1}(k) \times \ldots \times GL_{n_s}(k)$ (with $n = n_1 + \ldots + n_s$ and $\Delta \setminus I = \{n_1, n_1 + n_2, \ldots, n_1 + \ldots + n_{s-1}\}$). The character group $X^*(M_I)$ has the basis $\det_1, \ldots, \det_s$ where $\det_l$ is the projection onto the factor $GL_{n_l}(k)$ followed by the determinant map. We then have

$$(X^*(M_I) \otimes \mathbb{R})_+ = \{\sum_{l=1}^{s} r_l \det_l : r_1 > \ldots > r_s\}.$$

Step 2: We now suppose that $\pi = L(a) = L(\Delta_1, \ldots, \Delta_t)$ corresponds to the multiset of segments $a = (\Delta_1, \ldots, \Delta_t)$, and we will explain how to compute the Langlands triple of π in terms of the multiset. In a unique way we may write $\Delta_i = |\ |_k^{r_i'} \Delta_i^0$ where $r_i' \in \mathbb{R}$ and the segment Δ_i^0 is centered. Let $\{r_1', \ldots, r_t'\} = (r_1 > \ldots > r_s)$. For each $1 \leq l \leq s$ we form the multiset of centered segments $a_l := (\Delta_i^0 : r_i' = r_l)$. We then have $a = |\ |_k^{r_1} a_1 + \ldots + |\ |_k^{r_s} a_s$. By [Rod] Prop. 12 each $L(a_l)$ is a tempered representation of $GL_{n_l}(k)$ for some $n_l \geq 1$, and $n_1 + \ldots + n_s = n$.

Lemma 2.34. *The Langlands triple of $\pi = L(a)$ is*

$$(GL_{n_1}(k) \times \ldots \times GL_{n_s}(k), L(a_1) \otimes \ldots \otimes L(a_s), r_1 \det_1 + \ldots + r_s \det_s) \; .$$

Proof. It is clear that the triple in the assertion satisfies the conditions of a Langlands triple. It belongs to the unique irreducible quotient of the parabolic induction

$$|\det|_k^{r_1} L(a_1) \times \ldots \times |\det|_k^{r_s} L(a_s) \; .$$

On the other hand, $L(a)$ is the unique irreducible quotient of the parabolic induction

$$L(\Delta_1) \times \ldots \times L(\Delta_t) = |\det|_k^{r'_1} L(\Delta_1^0) \times \ldots \times |\det|_k^{r'_t} L(\Delta_t^0) \; ,$$

where the segments $\Delta_1, \ldots, \Delta_t$ have to be numbered in such a way that, for any $i < j$, the segment Δ_i does not precede the segment Δ_j ([Rod] Thm. 3.a). We do this enumeration in the following way. For any $1 \le l \le s$ let $a_l = (\Delta_{l,1}^0, \ldots, \Delta_{l,t_l}^0)$. We then take the enumeration

$$a = (|\;|_k^{r_1} \Delta_{1,1}^0, \ldots, |\;|_k^{r_1} \Delta_{1,t_1}^0, |\;|_k^{r_2} \Delta_{2,1}^0, \ldots, |\;|_k^{r_2} \Delta_{2,t_2}^0, \ldots, |\;|_k^{r_s} \Delta_{s,1}^0, \ldots, |\;|_k^{r_s} \Delta_{s,t_s}^0) \; .$$

According to [Rod] Prop. 12 we have $L(a_l) = L(\Delta_{l,1}^0) \times \ldots \times L(\Delta_{l,t_l}^0)$ and hence

$$\begin{aligned} &|\det|_k^{r_1} L(a_1) \times \ldots \times |\det|_k^{r_s} L(a_s) \\ &= |\det|_k^{r_1} L(\Delta_{1,1}^0) \times \ldots \times |\det|_k^{r_1} L(\Delta_{1,t_1}^0) \times \ldots \times |\det|_k^{r_s} L(\Delta_{s,1}^0) \times \ldots \times |\det|_k^{r_s} L(\Delta_{s,t_s}^0) \; . \end{aligned}$$

It therefore remains to show that, for any $1 \le i < j \le s$, $1 \le b \le t_i$, and $1 \le c \le t_j$, the segment $|\;|^{r_i} \Delta_{i,b}^0$ does not precede the segment $|\;|^{r_j} \Delta_{j,c}^0$. We have

$$\begin{aligned} |\;|_k^{r_i} \Delta_{i,b}^0 &= (\chi_{i,b} |\;|_k^{r_i - \frac{\ell_{i,b}-1}{2}}, \ldots, \chi_{i,b} |\;|_k^{r_i + \frac{\ell_{i,b}-1}{2}}) \quad \text{and} \\ |\;|_k^{r_j} \Delta_{j,c}^0 &= (\chi_{j,c} |\;|_k^{r_j - \frac{\ell_{j,c}-1}{2}}, \ldots, \chi_{j,c} |\;|_k^{r_j + \frac{\ell_{j,c}-1}{2}}) \end{aligned}$$

for appropriate integers $\ell_{i,b}, \ell_{j,c} \ge 0$ and unitary unramified characters $\chi_{i,b}, \chi_{j,c}$ of $k^\times$. If $\chi_{i,b} \neq \chi_{j,c}$ then obviously none of the two segments can precede the other. Hence we may assume that $\chi_{i,b} = \chi_{j,c} = 1$. Suppose that the left segment precedes the right segment. Then

$$r_i - \frac{\ell_{i,b}-1}{2} < r_j - \frac{\ell_{j,c}-1}{2} = (r_i - \frac{\ell_{i,b}-1}{2}) + m \le r_i + \frac{\ell_{i,b}-1}{2} < r_j + \frac{\ell_{j,c}-1}{2}$$

for some integer $m \ge 1$. Recall that we have $r_i > r_j$. The last inequality therefore implies $\frac{\ell_{i,b} - \ell_{j,c}}{2} < r_j - r_i < 0$, whereas the equality implies $\frac{\ell_{i,b} - \ell_{j,c}}{2} = r_i - r_j + m > m > 0$. This is a contradiction. $\square$

Step 3: Next we suppose that $\pi = \pi(u, s)$ corresponds to the $\widehat{G}$-orbit of the pair (u, s) in the Kazhdan-Lusztig classification. This time we will compute the Langlands triple of π in terms of (u, s). By conjugation we may assume that the semisimple element $s' := \tau_u^{-1} s$ (cf. properties 1.-4. after Prop. 2.13) lies in $\widehat{T}$. We let $s' = s'_e s'_h$ be the polar decomposition and we consider the centralizer $Z_{\widehat{G}}(s'_h)$, which is a Levi subgroup of $\widehat{G}$ containing $\widehat{T}$. We have $Z_{\widehat{G}}(s'_h) = \widehat{M}$ for a Levi subgroup M of G which contains T. Using those properties 1.-4. one sees that u and the semisimple element $s_{\mathcal{L}} := \tau_u s'_e$ both lie in $\widehat{M}$ and satisfy $s_{\mathcal{L}} u s_{\mathcal{L}}^{-1} = u^q$. Moreover

$(u, s_{\mathcal{L}})$, by Thm. 2.2, is a tempered pair in $\widehat{M}$. Hence we have the tempered M-representation $\pi(u, s_{\mathcal{L}})$. On the other hand, the element s'_h lies in the (connected) center of $\widehat{M}$ and therefore, by Lemma 2.12, corresponds to an unramified character $\chi_{s'_h}$ of M. Since s'_h is hyperbolic the character $\chi_{s'_h}$ has positive real values and therefore is of the form $\chi_{s'_h} = \nu^{nr}$ for a unique $\nu \in X^*(M) \otimes \mathbb{R}$. Since $\widehat{M}$ is the maximal Levi subgroup with s'_h in its center, the point ν viewed in the Coxeter complex $X^*(T) \otimes \mathbb{R}$ lies in a unique facet which is open in $X^*(M) \otimes \mathbb{R}$ and hence corresponds to a unique parabolic subgroup P of G with Levi component M. By conjugation by an appropriate Weyl group element we may assume that $P = P_I$ and hence $M = M_I$ and $\nu \in (X^*(M_I) \otimes \mathbb{R})_+$.

Proposition 2.35. *The Langlands triple of* $\pi = \pi(u, s)$ *is* $(M_I, \pi(u, s_{\mathcal{L}}), \nu)$.

Proof. See [BM] after Thm. 6.2. □

We now will combine these steps to establish the general case.

Proposition 2.36. *For any multiset of segments* a *we have* $L(a) \cong \pi(u_a, s_a)$.

Proof. We have to compute the Langlands triple of $\pi(u_a, s_a)$ and match it to the Langlands triple of $L(a)$. For a single segment $\Delta = (\chi, \ldots) = |\ |_k^{r'} \Delta^0$, where the segment Δ^0 is centered, we have

$$s'_\Delta = \tau_{u_\Delta}^{-1} s_\Delta = (q^{-r'} \frac{\chi(\varpi)}{|\chi(\varpi)|}) \cdot \mathrm{Id}_{\mathrm{length}(\Delta)} \quad \text{and hence} \quad (s_\Delta)'_h = q^{-r'} \cdot \mathrm{Id}_{\mathrm{length}(\Delta)} \ .$$

Our decomposition $a = |\ |_k^{r_1} a_1 + \ldots + |\ |_k^{r_s} a_s$ then implies

$$(s_a)'_h = \mathrm{diag}(q^{-r_1} \cdot \mathrm{Id}_{n_1}, \ldots, q^{-r_s} \cdot \mathrm{Id}_{n_s}) \ ,$$

where here and in the following diag(...) denotes the (block-)diagonal matrix with (block-) entries .... Therefore $\widehat{M} = Z_{\widehat{G}}((s_a)'_h)$ corresponds to $M = GL_{n_1}(k) \times \ldots \times GL_{n_s}(k)$. Moreover

$$s_a = \mathrm{diag}(s_{|\ |_k^{r_1} a_1}, \ldots, s_{|\ |_k^{r_s} a_s}) = \mathrm{diag}(q^{-r_1} s_{a_1}, \ldots, q^{-r_s} s_{a_s})$$

implies

$$(s_a)_{\mathcal{L}} = ((s_a)'_h)^{-1} s_a = \mathrm{diag}(s_{a_1}, \ldots, s_{a_s}) \ ,$$

and obviously $u_a = \mathrm{diag}(u_{a_1}, \ldots, u_{a_s})$ since u_a only depends on the lengths of the segments. Therefore in the Langlands triple of $\pi(u_a, s_a)$ we will have $M = M_I$ and

$$\pi(u_a, (s_a)_{\mathcal{L}}) = \pi(\mathrm{diag}(u_{a_1}, \ldots, u_{a_s}), \mathrm{diag}(s_{a_1}, \ldots, s_{a_s})) \ .$$

The latter is isomorphic to $L(a_1) \otimes \ldots \otimes L(a_s)$ by Lemma 2.32. Finally we have to determine $\nu_a \in (X^*(M_I) \otimes \mathbb{R})_+$ such that $\nu_a^{nr} = \chi_{(s_a)'_h} \in X_{nr}(M_I)$. For a general $z \in Z(\widehat{M_I})^\circ \subseteq \widehat{T}$ the unramified character χ_z is determined via

$$\chi_z(\varepsilon^\vee(\varpi)) = \varepsilon(z) \quad \text{whenever } \varepsilon^\vee \in X_*(T) \text{ corresponds to } \varepsilon \in X^*(\widehat{T}).$$

In our case, with the input $z = (s_a)'_h = \mathrm{diag}(q^{-r_1} \cdot \mathrm{Id}_{n_1}, \ldots, q^{-r_s} \cdot \mathrm{Id}_{n_s})$, we see that $\chi_z = \nu_a^{nr}$ for $\nu_a = r_1 \det_1 + \ldots + r_s \det_s \in (X^*(M_I) \otimes \mathbb{R})_+$. Now use Lemma 2.34. □

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