The algebraic theory of tempered representations of p-adic groups,

Part II: Projective generators

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Dedicated to Joseph Bernstein

As the title indicates this paper is a sequel to [SZ]. In that paper we began the systematic investigation of the category $\mathcal{M}^t(G)$ of nondegenerate modules over Harish Chandra's Schwartz algebra $\mathcal{S}(G)$ of a reductive p-adic group G. We fully developed the formalism of parabolic induction and restriction for the category $\mathcal{M}^t(G)$. In the present sequel we use this formalism to construct explicit projective generators and to explicitly compute the center of the category $\mathcal{M}^t(G)$. As a further application we obtain a (largely) algebraic proof of the Plancherel isomorphism for G.

After recalling some well known basic facts about the center of idempotented rings in section 1 we show in section 2 that irreducible discrete series representations of G give rise to component categories splitting $\mathcal{M}^t(G)$. In section 3 we construct projective generators and compute the center of these so called discrete component categories. In section 4 we decompose all of $\mathcal{M}^t(G)$ into component categories. The formalism of parabolic induction and restriction from [SZ] then allows us in section 5 to produce in a rather straightforward way, by parabolic induction, projective generators for any component category. The full computation of the center of $\mathcal{M}^t(G)$ is achieved in section 6. It takes the form of the ring of C^∞-functions on the Harish Chandra spectrum of G viewed as an orbifold. Before addressing the Plancherel isomorphism we have to complement, in section 7, the results of [SZ] by extending the so called geometric lemma (the analog for parabolic induction and restriction of Mackey's formula) to the framework of the category $\mathcal{M}^t(G)$.

Having explicit projective generators at our disposal we get a first form of the Plancherel isomorphism almost for free. It identifies $\mathcal{S}(G)$ with the direct sum of the endomorphism rings of these projective generators. To further compute each individual such endomorphism ring in section 8 we need, as an additional analytic input, the existence and basic properties of specific intertwining operators which we take from [Wal]. But apart from this the rest of the computation is purely algebraic as well.

To make clear to the reader to what extent our treatment in section 8 of the Plancherel isomorphism is independent of Waldspurger's paper [Wal] we point out that besides the basic theory in chapters I – III we only use the following results from that paper: the existence and rationality of the intertwining operators in IV.1.1 and IV.2.2 as well as the consequence V.1.1(1); the special case

$M = G$ in VII.1.3 which only uses basic facts from the initial chapters. On the other hand we make use of a finiteness result of Silberger ([Sil] 5.4.5.1).

In the last section 9 we establish a conjecture by U. Stuhler and the first author which said that, for admissible tempered representations of G, their Ext-groups formed in $\mathcal{M}^t(G)$ coincide with those formed in the category of all smooth G-representations. This relies very much on a recent result of R. Meyer in [Mey].

It will be entirely obvious to the alert reader that the extent to which our treatment in this paper is influenced by the beautiful work of Joseph Bernstein on smooth representations can hardly be overestimated. We cannot thank him enough for all the inspiration he has given us.

Notations

Throughout this paper k is a locally compact nonarchimedean field with absolute value $|\ |_k$. Let G be the group of k-rational points of a connected reductive k-group. As usual we denote by $\mathcal{H} = \mathcal{H}(G)$ and $\mathcal{S} = \mathcal{S}(G)$ the Hecke and Schwartz algebra of G, respectively. The category of nondegenerate $\mathcal{H}$-modules, resp. $\mathcal{S}$-modules, is denoted by $\mathcal{M}(G)$, resp. $\mathcal{M}^t(G)$. We have the forgetful functor $\mathcal{M}^t(G) \longrightarrow \mathcal{M}(G)$. The second category $\mathcal{M}(G)$ coincides with the category of all smooth G-representations. The multiplication in each of these two algebras as well as their action on a module always is denoted by a $*$ (for convolution). As a general convention we write the left and right translation action of a $g \in G$ on any locally constant function ϕ on G as $({}^g\phi)(h) := \phi(g^{-1}h)$ and $(\phi^g)(h) := \phi(hg^{-1})$.

For any compact open subgroup $U \subseteq G$ we let $\mathcal{H}(G,U)$, resp. $\mathcal{S}(G,U)$, denote the subalgebra of all U-bi-invariant functions in $\mathcal{H}(G)$, resp. $\mathcal{S}(G)$. Both these algebras are unital with the unit being the idempotent $\epsilon_U(g) = \mathrm{vol}_G(U)^{-1}$, resp. $= 0$, for $g \in U$, resp. $g \notin U$, corresponding to U. The map $g \mapsto g^{-1}$ on G induces on any of the rings $\mathcal{H}(G,U)$ and $\mathcal{S}(G,U)$ a canonical anti-involution so that, for these rings, we do not have to distinguish between left and right modules.

For any unital ring R we let $\mathcal{M}^r(R)$ be the category of right unital R-modules, and we let $Z(R)$ denote the center of R.

1. The center of idempotented rings

In this brief section we recall without proofs the formalism introduced in [BeD] 1.1-5 which is at the base of our later investigation. Let A be an idempotented ring which means that for any finite subset $\{a_1, \ldots, a_m\} \subseteq A$ there is an idempotent $e \in A$ such that $ea_i = a_i e = a_i$ for any $1 \leq i \leq m$. A (left) A-module V is called nondegenerate if for any $v \in V$ there is an idempotent $e \in A$ such that $ev = v$. Let $\mathcal{M}(A)$ denote the abelian category of all nondegenerate A-modules. The center $\mathcal{Z}_A$ of $\mathcal{M}(A)$ is defined to be the ring of natural endomorphisms of the identity functor on the category $\mathcal{M}(A)$. For any $z \in \mathcal{Z}_A$ we denote by z_V the A-module endomorphism induced by z on any given nondegenerate A-module V. The ring A is a nondegenerate (A, A)-bimodule by left and right multiplication, and any z_A, by definition, is a bimodule endomorphism.

Lemma 1.1: *i. The natural map*

$$\begin{array}{ccc} \mathcal{Z}_A & \xrightarrow{\cong} & \mathrm{End}_{(A,A)}(A) \\ z & \longmapsto & z_A \end{array}$$

is an isomorphism of unital rings;

ii. we have $z_V(av) = z_A(a)v$ *for any* V *in* $\mathcal{M}(A)$, $v \in V$, *and* $a \in A$.

For any idempotent $e \in A$ the subring eAe is unital with unit element e. By definition, A is the filtered union

$$A = \bigcup_e eAe$$

where e runs over the set of all idempotents in A and where this set is partially ordered by $e \leq e'$ if $e = e'ee'$. Correspondingly any nondegenerate A-module V is the filtered union

$$V = \bigcup_e eV \ .$$

For any $z \in \mathcal{Z}_A$ and any idempotent $e \in A$ we have

$$z_A(eAe) \subseteq eAe \qquad \text{and} \qquad z_A(e) \in Z(eAe) \ .$$

Whenever $e \leq e'$ the map

$$\begin{array}{ccc} Z(e'Ae') & \longrightarrow & Z(eAe) \\ a & \longmapsto & ea = ae \end{array}$$

is a unital ring homomorphism, and we have

$$ez_A(e') = z_A(e) \ .$$

Lemma 1.2: *i. The natural map*

$$\begin{array}{ccc} \mathcal{Z}_A & \xrightarrow{\cong} & \varprojlim Z(eAe) \\ z & \longmapsto & (z_A(e))_e \end{array}$$

is an isomorphism of unital rings;

ii. we have $z_V(v) = z_A(e)v$ *for any idempotent* e*, any* V *in* $\mathcal{M}(A)$*, and any* $v \in eV$*.*

We finally consider the case that $z \in \mathcal{Z}_A$ is an idempotent. Then we have, for any V in $\mathcal{M}(A)$, the functorial decomposition

$$V = \mathrm{im}(z_V) \oplus \ker(z_V) \ .$$

This means that the category $\mathcal{M}(A)$ decomposes into the direct product

$$\mathcal{M}(A) = \mathcal{M}(A)_z \times \mathcal{M}(A)_z^{\perp}$$

of the full subcategories

$$\mathcal{M}(A)_z := \text{ all } V \text{ in } \mathcal{M}(A) \text{ such that } z_V = \mathrm{id}_V$$

and

$$\mathcal{M}(A)_z^{\perp} := \text{ all } V \text{ in } \mathcal{M}(A) \text{ such that } z_V = 0 \ .$$

Obviously there are no nonzero homomorphisms between an object in $\mathcal{M}(A)_z$ and an object in $\mathcal{M}(A)_z^{\perp}$.

We point out that the category $\mathcal{M}(A)$ has arbitrary direct sums and products. The former are the obvious ones whereas the nondegenerate direct product is defined by

$$\prod_{i \in I}{}' V_i := A \cdot \prod_{i \in I} V_i \ .$$

The subcategories $\mathcal{M}(A)_z$ and $\mathcal{M}(A)_z^{\perp}$ are closed under the formation of arbitrary direct sums and products.

All of the above is applicable to the Schwartz algebra $A = \mathcal{S}(G)$ in which "sufficiently many" idempotents are given by the ϵ_U for U running over the compact open subgroups in G. We put $\mathcal{Z}^t(G) := \mathcal{Z}_{\mathcal{S}(G)}$. One of the aims in this paper is the computation of this center $\mathcal{Z}^t(G)$. It is immediate from Lemma 1.2.ii that, for any $z \in \mathcal{Z}^t(G)$, the endomorphism $z_{\mathcal{S}(G)}$ of $\mathcal{S}(G)$ is continuous.

2. Discrete components

We recall that a discrete series representation of G is an admissible smooth G-representation which is preunitary and whose matrix coefficients are square-integrable modulo the center of G. Each such discrete series representation is tempered in the traditional sense ([Sil] 4.5.10) and hence is in a canonical way an $\mathcal{S}(G)$-module ([SSZ] App. Prop. 1).

At first we fix an irreducible discrete series representation E of G. Let $X^1 := X^1_{nr}(G)$ denote the compact torus of unitary unramified characters of G. For each $\chi \in X^1$ we have the irreducible discrete series representation E_χ of G on the same underlying vector space but with the G-action twisted by χ. Another way to say this in terms of $\mathcal{S}(G)$-modules is that E_χ is the pull-back of the nondegenerate $\mathcal{S}(G)$-module E along the homomorphism of rings

$$\begin{array}{ccc} \mathcal{S}(G) & \longrightarrow & \mathcal{S}(G) \\ \psi & \longmapsto & \psi\chi \ . \end{array}$$

We introduce the full subcategories

$$\mathcal{M}^t(G)_E \quad := \quad \begin{array}{l} \text{all nondegenerate } \mathcal{S}(G)\text{-modules} \\ \text{all of whose simple subquotients} \\ \text{are isomorphic to some } E_\chi \end{array}$$

and

$$\mathcal{M}^t(G)_E^\perp \quad := \quad \begin{array}{l} \text{all nondegenerate } \mathcal{S}(G)\text{-modules} \\ \text{none of whose simple subquotients} \\ \text{is isomorphic to any } E_\chi \end{array}$$

of $\mathcal{M}^t(G)$. Obviously, there is no nonzero $\mathcal{S}(G)$-module homomorphism between an object in the first and an object in the second subcategory. Our first aim is to establish the direct product decomposition of abelian categories

$$\mathcal{M}^t(G) = \mathcal{M}^t(G)_E \times \mathcal{M}^t(G)_E^\perp \ .$$

Let G^0 denoted the subgroup of G generated by all compact open subgroups. We have (cf. [Ber] Prop.s 22 and 25):

– G^0 is open and normal in G;

– G^0 has compact center;

– G/G^0 is a free abelian group of finite rank;

– if Z denotes a maximal split torus in the center of G then G^0Z is of finite index in G;

– as a G^0-representation E is semisimple and of finite length; more precisely, there are finitely many pairwise nonisomorphic irreducible G^0-representations $E_1, \ldots, E_m$ such that

$$E \cong (E_1 \oplus \ldots \oplus E_m) \oplus \ldots \oplus (E_1 \oplus \ldots \oplus E_m) \ .$$

We introduce the subalgebra

$$\mathcal{S}(G^0) := \{\psi \in \mathcal{S}(G) : \mathrm{supp}(\psi) \subseteq G^0\}$$

of $\mathcal{S}(G)$. By a continuity argument (cf. [SSZ] App. Prop. 1) each E_i is an $\mathcal{S}(G^0)$-submodule of E. If $\widetilde{E}_i$ denotes the smooth dual of E_i then, according to [Sil] 4.4.5, the matrix coefficients of E_i provide an injective (G^0, G^0)-bimodule map

$$\begin{array}{rcl}\gamma_i : E_i \otimes_{\mathbb{C}} \widetilde{E}_i & \longrightarrow & \mathcal{S}(G^0) \\ v \otimes \widetilde{v} & \longmapsto & \gamma_{\tilde{v},v}(g) := \widetilde{v}(g^{-1}v) \ .\end{array}$$

The left $\mathcal{S}(G^0)$-action on E_i induces by functoriality a nondegenerate right $\mathcal{S}(G^0)$-action on $\widetilde{E}_i$. We claim that the above map in fact is an $(\mathcal{S}(G^0), \mathcal{S}(G^0))$-bimodule homomorphism. The other case being entirely analogous we consider only the left $\mathcal{S}(G^0)$-action. We have to show that

$$\gamma_{\tilde{v},\psi * v}(g) = (\psi * \gamma_{\tilde{v},v})(g)$$

holds true for any $(v, \widetilde{v}) \in E_i \times \widetilde{E}_i$, $g \in G^0$, and $\psi \in \mathcal{S}(G^0)$. The formula certainly holds in case $\psi \in \mathcal{H}(G^0)$. It therefore suffices to observe that both sides as linear forms in $\psi \in \mathcal{S}(G^0)$ are continuous. This is obvious for the right hand side and follows from [SSZ] App. Prop. 1 for the left hand side.

On the other hand the space $\mathrm{End}(E_i)$ of linear endomorphisms of E_i carries the obvious $(\mathcal{S}(G^0), \mathcal{S}(G^0))$-bimodule structure by composition of endomorphisms. Its smooth part $\mathrm{End}^0(E_i)$ (as a $G^0 \times G^0$-representation) is a nondegenerate $(\mathcal{S}(G^0), \mathcal{S}(G^0))$-bimodule. According to [Sil] §1.11 or [Ber] Lemma 10 the map of $(\mathcal{S}(G^0), \mathcal{S}(G^0))$-bimodules

$$\begin{array}{rcl}\alpha_i : E_i \otimes_{\mathbb{C}} \widetilde{E}_i & \longrightarrow & \mathrm{End}^0(E_i) \\ v_0 \otimes \widetilde{v}_0 & \longmapsto & [v \mapsto \widetilde{v}_0(v)v_0]\end{array}$$

is bijective. Moreover, the $\mathcal{S}(G^0)$-action on E_i can be viewed as a homomorphism of $(\mathcal{S}(G^0), \mathcal{S}(G^0))$-bimodules

$$\rho_i : \mathcal{S}(G^0) \longrightarrow \mathrm{End}^0(E_i) \ .$$

The composite $\alpha_i^{-1}\rho_i\gamma_i$ is an endomorphism of the irreducible $G^0 \times G^0$-representation $E_i \otimes_{\mathbb{C}} \widetilde{E}_i$ and therefore has to be the multiplication by a scalar which is known to be nonzero ([Dix] 14.3.3); its inverse $d(E_i)$ is called the *formal degree* of E_i. It follows that the normalized composite

$$Z_{E_i} := d(E_i) \cdot \gamma_i \alpha_i^{-1} \rho_i : \mathcal{S}(G^0) \longrightarrow \mathcal{S}(G^0)$$

is an $(\mathcal{S}(G^0), \mathcal{S}(G^0))$-bimodule projector. It also follows that

$$Z_{E_i} \circ Z_{E_j} = 0 \qquad \text{for any } i \neq j \ .$$

Hence

$$Z'_E := Z_{E_1} + \ldots + Z_{E_m} : \mathcal{S}(G^0) \longrightarrow \mathcal{S}(G^0)$$

is a projector as well. We leave it to the reader to verify the following straightforward facts:

– The projector Z_{E_i} only depends on the isomorphism class of the $\mathcal{S}(G^0)$-module E_i.

– The projector Z'_E commutes with the action of G by conjugation on $\mathcal{S}(G^0)$ (observe that the G-action permutes the E_i up to isomorphism).

Since $\mathcal{S}(G)$ is a nondegenerate left $\mathcal{S}(G^0)$-module the idempotent Z'_E in the center of $\mathcal{S}(G^0)$ induces an $\mathcal{S}(G^0)$-module projector

$$Z_E := (Z'_E)_{\mathcal{S}(G)} : \mathcal{S}(G) \longrightarrow \mathcal{S}(G) \ .$$

As a consequence of Lemma 1.2.ii the projector Z_E is continuous and commutes with the right multiplication by $\mathcal{S}(G)$ on $\mathcal{S}(G)$. The same formula implies that, for any $g \in G$ and $\psi \in \mathcal{S}(G,U)$, we have

$$\begin{aligned} Z_E({}^g\psi) &= Z'_E({}^g\epsilon_U^{g^{-1}}) * {}^g\psi \\ &= {}^gZ'_E(\epsilon_U)^{g^{-1}} * {}^g\psi \\ &= {}^g(Z'_E(\epsilon_U) * \psi) \\ &= {}^g(Z_E(\psi)) \ . \end{aligned}$$

This means that Z_E commutes with the left translation action of G on $\mathcal{S}(G)$ and hence with the left multiplication action of $\mathcal{H}(G)$ on $\mathcal{S}(G)$. By continuity Z_E then even has to commute with the left multiplication of $\mathcal{S}(G)$ on itself. We see that Z_E is an $(\mathcal{S}(G), \mathcal{S}(G))$-bimodule endomorphism of $\mathcal{S}(G)$ and therefore lies in the center $\mathcal{Z}^t(G)$. The corresponding decomposition of the category $\mathcal{M}^t(G)$ is

$$\mathcal{M}^t(G) = \mathcal{M}^t(G)_{Z_E} \times \mathcal{M}^t(G)^{\perp}_{Z_E} \ .$$

Lemma 2.1: *$\mathcal{M}^t(G)_{Z_E} = \mathcal{M}^t(G)_E$ and $\mathcal{M}^t(G)^{\perp}_{Z_E} = \mathcal{M}^t(G)^{\perp}_E$.*

Proof: It suffices to establish the inclusions

$$\mathcal{M}^t(G)_{Z_E} \subseteq \mathcal{M}^t(G)_E \quad \text{and} \quad \mathcal{M}^t(G)^{\perp}_{Z_E} \subseteq \mathcal{M}^t(G)^{\perp}_E \ .$$

We begin by computing

$$\rho_i \circ Z_{E_i} = d(E_i) \cdot \rho_i\gamma_i\alpha_i^{-1}\rho_i = d(E_i) \cdot \alpha_i(\alpha_i^{-1}\rho_i\gamma_i)\alpha_i^{-1}\rho_i = \alpha_i\alpha_i^{-1}\rho_i = \rho_i \ .$$

Together with Lemma 1.2.ii this implies that

$$(Z_{E_i})_{E_i} = \mathrm{id}_{E_i} \ .$$

In addition we obtain

$$(Z_{E_j})_{E_i} = (Z_{E_j})_{E_i} \circ (Z_{E_i})_{E_i} = (Z_{E_j} \circ Z_{E_i})_{E_i} = 0$$

for $j \neq i$. It follows that

$$(Z_E)_{E_\chi} = (Z'_E)_{E_\chi} = \mathrm{id}_{E_\chi}$$

for any $\chi \in X^1$ since $(Z'_E)_{E_\chi}$ only depends on E_χ as an $\mathcal{S}(G^0)$-module. Hence if $(Z_E)_V = 0$ for some V in $\mathcal{M}^t(G)$ then no simple subquotient of V can be isomorphic to some E_χ. This proves the second inclusion

$$\mathcal{M}^t(G)^{\perp}_{Z_E} \subseteq \mathcal{M}^t(G)^{\perp}_{E} \ .$$

For the first inclusion

$$\mathcal{M}^t(G)_{Z_E} \subseteq \mathcal{M}^t(G)_E$$

we first show that any V in $\mathcal{M}^t(G)_{Z_E}$ as an $\mathcal{S}(G^0)$-module is semisimple with all simple subquotients being isomorphic to one of the $E_1, \ldots, E_m$. For this it suffices to observe that we have a G^0-equivariant surjection of the form

$$\bigoplus_{i=1}^{m} \bigoplus_{v \in V} E_i \otimes_{\mathbb{C}} \widetilde{E}_i \twoheadrightarrow \bigoplus_{i=1}^{m} \mathrm{im}(\gamma_i) * V = \bigoplus_{i=1}^{m} \mathrm{im}(Z_{E_i}) * V = \bigoplus_{i=1}^{m} \mathrm{im}(Z_{E_i})_V = V$$

and that the left hand side has the wanted property. Suppose now that E' is a simple $\mathcal{S}(G)$-subquotient of V. The simple constituents of E' as an $\mathcal{S}(G^0)$-module then must occur among the E_i. It therefore follows from [Ber] Prop. 25 that $E' \cong E_\chi$ for some unramified character χ of G. Since E' and E both are tempered the character χ has to be unitary. This proves that V lies in $\mathcal{M}^t(G)_E$.

Proposition 2.2: $\mathcal{M}^t(G) = \mathcal{M}^t(G)_E \times \mathcal{M}^t(G)^{\perp}_E$.

Proof: According to the previous lemma the asserted decomposition coincides with the decomposition with respect to the central idempotent Z_E.

Proposition 2.3: *As an $\mathcal{S}(G^0)$-module E is projective; moreover, there are finitely many smooth linear forms $\widetilde{v}_1, \ldots, \widetilde{v}_r$ on E such that the map*

$$\begin{array}{ccc} E & \longrightarrow & \mathcal{S}(G^0)^r \\ v & \longmapsto & (\gamma_{\tilde{v}_1,v}, \ldots, \gamma_{\tilde{v}_r,v}) \end{array}$$

is a split monomorphism of $\mathcal{S}(G^0)$-modules.

Proof: For the first part of the assertion it suffices to show that any E_i is a projective $\mathcal{S}(G^0)$-module. Let $W \twoheadrightarrow E_i$ be any surjection of $\mathcal{S}(G^0)$-modules. We fix a vector $0 \neq v_0 \in E_i$ and a smooth linear form $\widetilde{v}_0$ on E_i such that

$\widetilde{v}_0(v_0) = d(E_i)$; we also fix a preimage $w_0 \in W$ of v_0. We claim that the $\mathcal{S}(G^0)$-module homomorphism

$$\begin{array}{ccc} E_i & \longrightarrow & W \\ v & \longmapsto & \gamma_i(v \otimes \widetilde{v}_0) * w_0 \end{array}$$

splits the given surjection. This follows from

$$\begin{aligned} \gamma_i(v \otimes \widetilde{v}_0) * v_0 &= \rho_i(\gamma_i(v \otimes \widetilde{v}_0))(v_0) \\ &= d(E_i)^{-1} \cdot \alpha_i(v \otimes \widetilde{v}_0)(v_0) = d(E_i)^{-1} \widetilde{v}_0(v_0) v \\ &= v \ . \end{aligned}$$

The second part of the assertion is a consequence of the split monomorphism

$$\overset{m}{\underset{i=1}{\oplus}} E_i \otimes_{\mathbb{C}} \widetilde{E}_i \xrightarrow[\oplus \gamma_i]{\cong} \mathrm{im}(Z'_E) \hookrightarrow \mathcal{S}(G^0) \ .$$

There will be a refinement of the decomposition in Prop. 2.2 which takes all irreducible discrete representations of G simultaneously into account. For this it is convenient to first introduce some general notation. Let $\mathrm{Irr}^t(G)$ denote the set of isomorphism classes of all simple $\mathcal{S}(G)$-modules, equivalently ([SSZ] App.), the set of isomorphism classes of all irreducible tempered G-representations. In $\mathrm{Irr}^t(G)$ we define Dis to be the subset of isomorphism classes of irreducible discrete series representations, and we put $Ind := \mathrm{Irr}^t(G) \setminus Dis$. The compact torus X^1 acts on Dis by the twist $(\chi, [E]) \longmapsto [E_\chi]$. For any subset $\Pi \subseteq \mathrm{Irr}^t(G)$ we introduce the full abelian subcategory

$$\mathcal{M}^t(G)_\Pi \ := \ \begin{array}{l} \text{all nondegenerate } \mathcal{S}(G)\text{-modules all of} \\ \text{whose simple subquotients lie in } \Pi \end{array}$$

of $\mathcal{M}^t(G)$. If D is the X^1-orbit of $[E] \in Dis$ where E is an irreducible discrete series representation then

$$\mathcal{M}^t(G)_D = \mathcal{M}^t(G)_E \ .$$

It follows immediately from Prop. 2.2 that for any finite number of pairwise distinct X^1-orbits $D_1, \ldots, D_r$ in Dis we have the decomposition

$$\mathcal{M}^t(G) = \mathcal{M}^t(G)_{D_1} \times \ldots \times \mathcal{M}^t(G)_{D_r} \times \mathcal{M}^t(G)_{D'}$$

where $D' := \mathrm{Irr}^t(G) \setminus (D_1 \cup \ldots \cup D_r)$. But in fact the following stronger result holds true.

Proposition 2.4: *We have*

$$\mathcal{M}^t(G) = (\prod_D \mathcal{M}^t(G)_D) \times \mathcal{M}^t(G)_{Ind}$$

where D runs over all X^1-orbits in Dis.

Proof: Consider any V in $\mathcal{M}^t(G)$. According to Prop. 2.2 we have, for any D, a functorial decomposition

$$V = V_D \oplus V_D^\perp$$

with V_D in $\mathcal{M}^t(G)_D$ and $V_D^\perp$ in $\mathcal{M}^t(G)_{\mathrm{Irr}^t(G)\setminus D}$. We put

$$V_{Ind} := \bigcap_D V_D^\perp \ .$$

Clearly V_{Ind} lies in $\mathcal{M}^t(G)_{Ind}$, and the subspaces V_D, for all D, and V_{Ind} are linearly independent in V. It remains to show that

$$V = (\sum_D V_D) + V_{Ind}$$

holds true. Let $v \in V$ be any vector and write $v = v_D + v_D^\perp$ according to the first decomposition above. We choose a compact open subgroup $U \subseteq G^0$ such that $v \in V^U$ is a U-fixed vector. Then $v_D \in V_D^U$ for any D. Hence the subsequent proposition implies that $v_D = 0$ for all but finitely many orbits $D_1, \ldots, D_r \subseteq Dis$. It therefore remains to check that $v' := v - (v_{D_1} + \ldots + v_{D_r})$ lies in V_{Ind}. By construction we have $v' \in V_{D_i}^\perp$ for any $1 \leq i \leq r$. On the other hand, for any $D \neq D_1, \ldots, D_r$ the projection of $v' \in V$ to V_D must be zero since V_D, by assumption, has no nonzero U-fixed vectors. Hence $v' \in V_D^\perp$.

Proposition 2.5: *Given any compact open subgroup $U \subseteq G^0$ there are only finitely many X^1-orbits $D \subseteq Dis$ such that the nonzero modules in $\mathcal{M}^t(G)_D$ have nonzero U-fixed vectors.*

Proof: We have to show that up to isomorphism and up to twist by X^1 there are only finitely many irreducible discrete series representations of G which have a nonzero U-fixed vector. For that we consider the decomposition

$$\mathcal{M}(G) = \prod_\Omega \mathcal{M}(G)_\Omega$$

of the category of all smooth G-representations according to the connected components of the center of the Hecke algebra $\mathcal{H}(G)$ ([BeD]). As a consequence of [Sil] 5.4.5.1 any given component category $\mathcal{M}(G)_\Omega$ contains, up to isomorphism and twist by X^1, only finitely many irreducible discrete series representations.

Moreover, by [BeD] Cor. 3.9(i), there are only finitely many components Ω such that the representations in $\mathcal{M}(G)_\Omega$ can have nonzero U-fixed vectors.

Corresponding to the decomposition in Prop. 2.4 we have a decomposition

$$\mathcal{Z}^t(G) = (\prod_D \mathcal{Z}^t(G)_D) \times \mathcal{Z}^t(G)_{Ind}$$

of the center $\mathcal{Z}^t(G)$ into a direct product of ideals such that

$$\mathcal{Z}^t(G)_D = \mathcal{Z}^t(G) \cdot Z_E$$

if D is the orbit of the irreducible discrete series representation E.

3. Projective generators for discrete components

We again fix an irreducible discrete series representation E of G. Our goal in this section is to exhibit an explicit projective generator for the abelian category $\mathcal{M}^t(G)_E$. Let $C^\infty(X^1)$ denote the ring of complex valued C^∞-functions on X^1. It is a smooth G-representation via ${}^g f(\chi) := \chi(g) \cdot f(\chi)$. We define the space of E-valued C^∞-functions on X^1 to be

$$C^\infty(X^1, E) := C^\infty(X^1) \otimes_{\mathbb{C}} E \ .$$

The diagonal G-action makes this a smooth representation. Viewing a vector $\widetilde{f} \in C^\infty(X^1, E)$ as a map $\widetilde{f} : X^1 \longrightarrow E$ the G-action can be written as

$${}^g \widetilde{f}(\chi) = \chi(g) \cdot g(\widetilde{f}(\chi)) \ .$$

This suggests that we may extend this G-action to an $\mathcal{S}(G)$-module structure by

$$(*) \qquad (\psi * \widetilde{f})(\chi) := \psi\chi * \widetilde{f}(\chi) \quad \text{for } \psi \in \mathcal{S}(G) \ .$$

The only thing to check is that the new map $\psi * \widetilde{f} : X^1 \longrightarrow E$ again lies in $C^\infty(X^1, E)$. As a special case of [Wal] VII.1.3 we have the following fact:

Given a $\psi \in \mathcal{S}(G)$ there are finitely many $f_1, \ldots, f_r \in C^\infty(X^1)$ and $A_1, \ldots, A_r \in \mathrm{End}_{\mathbb{C}}(E)$ such that

$$\psi\chi * v = \sum_{i=1}^r f_i(\chi) \cdot A_i(v) \quad \text{for any } \chi \in X^1 \text{ and } v \in E \ .$$

For $\widetilde{f} = f \otimes v \in C^\infty(X^1, E) = C^\infty(X^1) \otimes E$ we therefore obtain

$$\psi * \widetilde{f} = \sum_{i=1}^{r} f f_i \otimes A_i(v) \in C^\infty(X^1, E) \ .$$

In the following we always will view $C^\infty(X^1, E)$ as a nondegenerate $\mathcal{S}(G)$-module via $(*)$. Note that the evaluation maps

$$\begin{array}{rcl} C^\infty(X^1, E) & \longrightarrow & E_\chi \\ \widetilde{f} & \longmapsto & \widetilde{f}(\chi) \end{array}$$

are $\mathcal{S}(G)$-module homomorphisms. This property in fact uniquely characterizes the $\mathcal{S}(G)$-module structure on $C^\infty(X^1, E)$.

Lemma 3.1: *$C^\infty(X^1, E)$ lies in $\mathcal{M}^t(G)_E$.*

Proof: Suppose that E' is a simple $\mathcal{S}(G)$-subquotient module of $C^\infty(X^1, E)$. Since G^0 acts trivially on $C^\infty(X^1)$ the two irreducible G-representations E' and E as G^0-representations have a common irreducible constituent. We therefore ([Ber] Prop. 25) must have $E' \cong E_\chi$ for some unramified character χ of G. Since E' and E both are tempered the character χ has to be unitary.

All further properties of the $\mathcal{S}(G)$-module $C^\infty(X^1, E)$ will be consequences of an isomorphism

$$C^\infty(X^1, E) \cong \mathcal{S}(G) \otimes_{\mathcal{S}(G^0)} E$$

which we are going to establish. Here $\mathcal{S}(G^0)$ denotes the subalgebra

$$\mathcal{S}(G^0) := \{\psi \in \mathcal{S}(G) : \mathrm{supp}(\psi) \subseteq G^0\}$$

of $\mathcal{S}(G)$. For any $v \in E$ we let $const_v \in C^\infty(X^1, E)$ denote the constant function with value v. We then have the $\mathcal{S}(G)$-module homomorphism

$$\begin{array}{rcl} I : \mathcal{S}(G) \otimes_{\mathcal{S}(G^0)} E & \longrightarrow & C^\infty(X^1, E) \\ \psi \otimes v & \longmapsto & \psi * const_v \ . \end{array}$$

It is well defined since

$$\phi * const_v(\chi) = \phi\chi * v = \phi * v = const_{\phi * v}(\chi)$$

for any $\phi \in \mathcal{S}(G^0)$. In order to prove that I in fact is an isomorphism we first point out that the two $\mathcal{S}(G)$-modules involved carry an additional structure. The target $C^\infty(X^1, E)$ obviously is, by pointwise multiplication of functions, a

$C^\infty(X^1)$-module. The corresponding structure on the source $\mathcal{S}(G) \otimes_{\mathcal{S}(G^0)} E$ is more involved. We begin by looking at the smooth left G-action

$$\begin{array}{rcl} G \times (\mathcal{S}(G) \otimes_{\mathbb{C}} E) & \longrightarrow & \mathcal{S}(G) \otimes_{\mathbb{C}} E \\ (g, \psi \otimes v) & \longmapsto & \psi^{g^{-1}} \otimes gv \ . \end{array}$$

For $\phi \in \mathcal{S}(G^0)$ we compute

$$\psi^{g^{-1}} \otimes g(\phi * v) = \psi^{g^{-1}} \otimes ({}^g\phi^{g^{-1}} * gv) = (\psi^{g^{-1}} * {}^g\phi^{g^{-1}}) \otimes gv = (\psi * \phi)^{g^{-1}} \otimes gv$$

in $\mathcal{S}(G) \otimes_{\mathcal{S}(G^0)} E$. It follows that this G-action descends to a smooth G-action on $\mathcal{S}(G) \otimes_{\mathcal{S}(G^0)} E$ which factorizes through G/G^0.

Let $\mathcal{S}(G/G^0)$ denote the Schwartz algebra of the discrete group G/G^0. Using an isomorphism $G/G^0 \cong \mathbb{Z}^d$ this algebra $\mathcal{S}(G/G^0)$ identifies with the Schwartz algebra $\mathcal{S}(\mathbb{Z}^d)$ of all functions $\varphi : \mathbb{Z}^d \longrightarrow \mathbb{C}$ which satisfy

$$\nu_k(\phi) := \sup_{(i_1,\ldots,i_d)\in\mathbb{Z}^d} |\phi(i_1,\ldots,i_d)|(1 + |i_1| + \ldots |i_d|)^k < \infty$$

for any $k \in \mathbb{N}$. For technical reasons we later need the following alternative characterization of $\mathcal{S}(G/G^0)$. Let Z be the maximal split torus in the center of G and put $Z^0 := Z \cap G^0$. Then Z^0 is compact and Z/Z^0 is a subgroup of finite index in G/G^0. Let $\mathcal{S}(Z)$ be, as always, the Schwartz algebra of the reductive group Z in the sense of Harish Chandra. We then have (cf. the proof of Lemma 4.12 in [SZ])

$$\begin{array}{rl} \mathcal{S}(G/G^0) = & \text{all functions } \varphi : G/G^0 \longrightarrow \mathbb{C} \text{ such that} \\ & ({}^g f)|Z \in \mathcal{S}(Z) \text{ for any } g \in G \ . \end{array}$$

We now want to extend the above G/G^0-action on $\mathcal{S}(G)\otimes_{\mathcal{S}(G^0)}E$ to an $\mathcal{S}(G/G^0)$-module structure by

$$\varphi \circ (\psi \otimes v) := \sum_{g\in G/G^0} \varphi(g)(\psi^{g^{-1}} \otimes gv)$$

for $\varphi \in \mathcal{S}(G/G^0)$. Of course, as such the sum on the right hand side does not make sense in $\mathcal{S}(G) \otimes_{\mathcal{S}(G^0)} E$.

Lemma 3.2: *The pairing*

$$\begin{array}{rcl} \mathcal{S}(Z) \times \mathcal{S}(G) & \longrightarrow & \mathcal{S}(G) \\ (\varphi, \psi) & \longmapsto & \varphi *_Z \psi(h) := \int_Z \varphi(z)\psi(hz)dz \end{array}$$

is well defined and is a separately continuous action of $\mathcal{S}(Z)$ *on* $\mathcal{S}(G)$.

Proof: This is a variant of the proof of [Sil] 4.4.2. Note that, since Z is central in G, with ψ also $\varphi *_Z \psi$ obviously is bi-invariant under some compact open subgroup of G.

Let $\omega : G \longrightarrow \mathbb{C}^\times$ be the unitary central character of E. Furthermore, we fix a set $R \subseteq G$ of representatives for the finitely many cosets in G/G^0Z. We may then formally compute

$$\begin{aligned}\varphi \circ (\psi \otimes v) &= \sum_{g \in G/G^0} \varphi(g)(\psi^{g^{-1}} \otimes gv) \\ &= \sum_{g \in R} \sum_{z \in Z/Z^0} \varphi(zg)(\psi^{g^{-1}z^{-1}} \otimes \omega(z)gv) \\ &= \mathrm{vol}(Z^0)^{-1} \cdot \sum_{g \in R} (\omega(\varphi^{g^{-1}}|Z) *_Z \psi^{g^{-1}}) \otimes gv \ .\end{aligned}$$

By Lemma 3.2 the convolutions $\omega(\varphi^{g^{-1}}|Z) *_Z \psi^{g^{-1}}$ are well defined elements in $\mathcal{S}(G)$. Hence we may formally define

$$\varphi \circ (\psi \otimes v) := \mathrm{vol}(Z^0)^{-1} \cdot \sum_{g \in R} (\omega(\varphi^{g^{-1}}|Z) *_Z \psi^{g^{-1}}) \otimes gv \ .$$

It is straightforward to check that this "$\circ$" is independent of the choice of R and is an action of $\mathcal{S}(G/G^0)$ on $\mathcal{S}(G) \otimes_{\mathcal{S}(G^0)} E$ which commutes with the obvious $\mathcal{S}(G)$-action through the first factor.

We now consider the map

$$\begin{aligned} J : \mathcal{S}(G/G^0) \otimes_{\mathbb{C}} E &\longrightarrow \mathcal{S}(G) \otimes_{\mathcal{S}(G^0)} E \\ \varphi \otimes v &\longmapsto \varphi \circ (\epsilon_U \otimes v) \end{aligned}$$

where, on the right hand side, $U \subseteq G^0$ is some compact open subgroup such that $v \in E^U$.

Lemma 3.3: *The map J is surjective.*

Proof: Let $\psi \in \mathcal{S}(G)$ and $v \in E$. We fix a compact open subgroup $U \subseteq G^0$ such that ψ is U-bi-invariant and $v \in E^U$. We also fix a basis $e_1, \ldots, e_m$ of the finite dimensional vector space E^U of U-fixed vectors. We then have

$$({}^g\psi)|G^0 * v \in g(E^U)$$

for any $g \in G$ and hence find uniquely determined $\varphi_1(g), \ldots, \varphi_m(g) \in \mathbb{C}$ such that

$$(+) \qquad ({}^g\psi)|G^0 * v = \varphi_1(g)ge_1 + \ldots + \varphi_m(g)ge_m \ .$$

This defines functions $\varphi_1, \ldots, \varphi_m : G/G^0 \longrightarrow \mathbb{C}$. We claim that each φ_i lies in $\mathcal{S}(G/G^0)$. Let $\ell_1, \ldots, \ell_m$ be U-invariant linear forms on E whose restrictions to E^U form a basis dual to $\{e_1, \ldots, e_m\}$. Then

$$\begin{aligned}\varphi_i(g) &= \ell_i({}^{g^{-1}}(({}^g\psi)|G^0) * v) = \ell_i((\psi^g)|G^0 * g^{-1}v) \\ &= \ell_i(\int\limits_{G^0} \psi(hg^{-1})hg^{-1}v dh) = \int\limits_{G^0} \psi(hg^{-1})\ell_i(hg^{-1}v)dh\end{aligned}$$

(the last identity is a consequence of [SSZ] App. Prop. 1). We have to show that, for any $g \in G$,

$$\varphi_i(g^{-1}z^{-1}) = \int\limits_{G^0} \psi(hgz)\ell_i(hgzv)dh = \omega(z) \int\limits_{G^0} \psi(hgz)\ell_i(hgv)dh$$

as a function of $z \in Z$ lies in $\mathcal{S}(Z)$. As a consequence of [Sil] 4.4.5 the matrix coefficient

$$\gamma_{\ell_i,gv}(h) := \begin{cases} \ell_i(h^{-1}gv) & \text{if } h \in G^0, \\ 0 & \text{if } h \in G \setminus G^0 \end{cases}$$

lies in $\mathcal{S}(G^0) \subseteq \mathcal{S}(G)$. Hence the convolution $\gamma_{\ell_i,gv} * \psi^{g^{-1}}$ lies in $\mathcal{S}(G)$ and its restriction to Z in $\mathcal{S}(Z)$. We compute

$$\begin{aligned}\omega(z)(\gamma_{\ell_i,gv} * \psi^{g^{-1}})(z) &= \omega(z) \int\limits_G \gamma_{\ell_i,gv}(h^{-1})\psi^{g^{-1}}(hz)dh \\ &= \omega(z) \int\limits_{G^0} \psi(hgz)\ell_i(hgv)dh \\ &= \varphi_i(g^{-1}z^{-1}) \ .\end{aligned}$$

Since ω is unitary it follows that $\varphi_i(g^{-1}.^{-1}) \in \mathcal{S}(Z)$. Having established this intermediate claim we now may form the element

$$\sum_{i=1}^m \varphi_i \otimes e_i \in \mathcal{S}(G/G^0) \otimes_{\mathbb{C}} E \ .$$

We will show that

$$J(\sum_{i=1}^m \varphi_i \otimes e_i) = \psi \otimes v$$

holds true. According to Prop. 2.3 we have a split monomorphism

$$\begin{array}{rcl}\mathcal{S}(G) \otimes_{\mathcal{S}(G^0)} E & \longrightarrow & \mathcal{S}(G)^r \\ \psi \otimes v & \longmapsto & (\psi * \gamma_{\tilde{v}_1,v}, \ldots, \psi * \gamma_{\tilde{v}_r,v})\end{array}$$

where $\widetilde{v}_1, \ldots, \widetilde{v}_r$ are appropriate smooth linear forms on E. Hence the above identity can be checked after composition with any map of the form

$$\begin{array}{rcl} \mathcal{S}(G) \otimes_{\mathcal{S}(G^0)} E & \longrightarrow & \mathcal{S}(G) \\ \psi \otimes v & \longmapsto & \psi * \gamma_{\tilde{v},v} \end{array}$$

where $\widetilde{v}$ is any smooth linear form on E. By making the subgroup U smaller if necessary we certainly may assume that $\widetilde{v}$ is U-invariant. The left hand side becomes

$$\begin{aligned} &\sum_{i=1}^{m} \sum_{g \in R} \mathrm{vol}(Z^0)^{-1} \cdot (\omega(\varphi_i^{g^{-1}}|Z) *_Z \epsilon_U^{g^{-1}}) * \gamma_{\tilde{v},ge_i} \\ &\qquad = \sum_{i=1}^{m} \sum_{g \in R} \mathrm{vol}(Z^0)^{-1} \cdot (\int_Z \omega(z)\varphi_i(zg)\epsilon_U^{(zg)^{-1}} dz) * \gamma_{\tilde{v},ge_i} \\ &\qquad = \sum_{i=1}^{m} \sum_{g \in R} \mathrm{vol}(Z^0)^{-1} \cdot \int_Z \omega(z)\varphi_i(zg)\epsilon_U^{(zg)^{-1}} * \gamma_{\tilde{v},ge_i} dz \\ &\qquad = \sum_{i=1}^{m} \sum_{g \in R} \mathrm{vol}(Z^0)^{-1} \cdot \int_Z \varphi_i(zg)\epsilon_U^{(zg)^{-1}} * \gamma_{\tilde{v},\omega(z)ge_i} dz \\ &\qquad = \sum_{i=1}^{m} \sum_{g \in R} \sum_{z \in Z/Z^0} \varphi_i(zg)\epsilon_U^{(zg)^{-1}} * \gamma_{\tilde{v},\omega(z)ge_i} \\ &\qquad = \sum_{i=1}^{m} \sum_{g \in G/G^0} \varphi_i(g)\epsilon_U^{g^{-1}} * \gamma_{\tilde{v},ge_i} \\ &\qquad = \sum_{i=1}^{m} \sum_{g \in G/G^0} \varphi_i(g)({}^{g^{-1}}\gamma_{\tilde{v},ge_i}) \\ &\qquad = \sum_{g \in G/G^0} {}^{g^{-1}}(\sum_{i=1}^{m} \varphi_i(g)\gamma_{\tilde{v},ge_i}) \qquad . \end{aligned}$$

Here the second identity follows from the separate continuity of the convolution $*$ in $\mathcal{S}(G)$. The second last identity comes form the left U-invariance of the function ${}^{g^{-1}}\gamma_{\tilde{v},ge_i}$ which is a consequence of the U-invariance of $\widetilde{v}$. Given a function $\phi \in \mathcal{S}(G)$ we write $\phi^0 \in \mathcal{S}(G)$ for the extension by zero of the restriction $\phi|G^0$. For any $g \in G$ we have

$${}^{g^{-1}}(({}^g\psi)^0)(h) = \begin{cases} \psi(h) & \text{if } h \in g^{-1}G^0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\psi = \sum_{g \in G/G^0} {}^{g^{-1}}(({}^g\psi)^0)$$

is a convergent expansion in $\mathcal{S}(G, U)$. Our right hand side therefore becomes

$$\begin{aligned} \psi * \gamma_{\tilde{v},v} &= (\sum_{g \in G/G^0} {}^{g^{-1}}(({}^g\psi)^0)) * \gamma_{\tilde{v},v} \\ &= \sum_{g \in G/G^0} {}^{g^{-1}}(({}^g\psi)^0 * \gamma_{\tilde{v},v}) \ . \end{aligned}$$

It remains to note that as a consequence of (+) we have

$$({}^g\psi)^0 * \gamma_{\tilde v,v} = \gamma_{\tilde v,({}^g\psi)^0 * v} = \sum_{i=1}^m \varphi_i(g)\gamma_{\tilde v,ge_i} \ .$$

In order to compute the composite

$$\mathcal{S}(G/G^0) \otimes_{\mathbb{C}} E \xrightarrow{J} \mathcal{S}(G) \otimes_{\mathcal{S}(G^0)} E \xrightarrow{I} C^\infty(X^1, E)$$

we need the Fourier isomorphism (cf. [Tre] Thm. 51.3)

$$\begin{array}{rcl} Fourier : \mathcal{S}(G/G^0) & \xrightarrow{\cong} & C^\infty(X^1) \\ \varphi & \longmapsto & \widehat{\varphi}(\chi) := \sum\limits_{g \in G/G^0} \varphi(g)\chi(g^{-1}) \ . \end{array}$$

Using a sufficiently small compact open subgroup U we have

$$\begin{aligned} I \circ J(\varphi \otimes v)(\chi) &= I(\varphi \circ (\epsilon_U \otimes v))(\chi) \\ &= I(\mathrm{vol}(Z^0)^{-1} \cdot \sum_{g \in R} (\omega(\varphi^{g^{-1}}|Z) *_Z \epsilon_U^{g^{-1}}) \otimes gv)(\chi) \\ &= \sum_{g \in R} \mathrm{vol}(Z^0)^{-1} \cdot (\omega(\varphi^{g^{-1}}|Z) *_Z \epsilon_U^{g^{-1}}) * const_{gv})(\chi) \\ &= \sum_{g \in R} \mathrm{vol}(Z^0)^{-1} \cdot (\chi(\omega(\varphi^{g^{-1}}|Z) *_Z \epsilon_U^{g^{-1}})) * gv \\ &= \sum_{g \in G/G^0} (\chi(\varphi(g)\epsilon_U^{g^{-1}})) * gv \\ &= \sum_{g \in G/G^0} \varphi(g)(\chi^g \epsilon_U) * v \\ &= (\sum_{g \in G/G^0} \varphi(g)\chi(g^{-1}))v \\ &= \widehat{\varphi}(\chi)v \\ &= (\widehat{\varphi} \cdot const_v)(\chi) \ . \end{aligned}$$

Here the fourth identity comes from the definition (∗). The fifth identity is based on the separate continuity of the $\mathcal{S}(G)$-action on E ([SSZ] App. Prop. 1) and is entirely analogous to the computation in the proof of Lemma 3.3. Hence we obtain the formula

$$I \circ J = Fourier \otimes \mathrm{id}_E \ .$$

Proposition 3.4: *Both maps, I and J, are bijective.*

Proof: In Lemma 3.3 we have seen that J is surjective, and the above discussion showed that the composite $I \circ J$ is bijective.

Corollary 3.5: *i. $C^\infty(X^1, E) \cong \mathcal{S}(G) \otimes_{\mathcal{S}(G^0)} E$ as $\mathcal{S}(G)$-modules;*

ii. $C^\infty(X^1, E)$ is finitely generated and projective as $\mathcal{S}(G)$-module;

iii. $C^\infty(X^1, E)$ is a projective generator of the category $\mathcal{M}^t(G)_E$.

Proof: i. This is the bijectivity of the map I in Prop. 3.4. ii. The properties in question are preserved under base extension. Hence, by the first assertion, it suffices to show that the $\mathcal{S}(G^0)$-module E is finitely generated and projective. The latter was done in Prop. 2.3. Moreover, since E is a direct sum of finitely many simple $\mathcal{S}(G^0)$-modules it must be finitely generated. iii. Since $C^\infty(X^1, E)$ is projective it is a generator if and only if any simple object in $\mathcal{M}^t(G)_E$ is a quotient of $C^\infty(X^1, E)$. By construction these simple objects are the twists E_χ for $\chi \in X^1$ for which we have the evaluation maps $C^\infty(X^1, E) \longrightarrow E_\chi$.

Let $D \subseteq Dis$ denote the X^1-orbit of the isomorphism class of E so that $\mathcal{M}^t(G)_E = \mathcal{M}^t(G)_D$.

Corollary 3.6: *i. The functor*

$$\begin{array}{rcl} \mathcal{M}^t(G)_E & \xrightarrow{\sim} & \mathcal{M}^r(\mathrm{End}_{\mathcal{S}(G)}(C^\infty(X^1, E))) \\ V & \longmapsto & \mathrm{Hom}_{\mathcal{S}(G)}(C^\infty(X^1, E), V) \end{array}$$

is an equivalence of categories;

ii. $\mathcal{Z}^t(G)_D \cong Z(\mathrm{End}_{\mathcal{S}(G)}(C^\infty(X^1, E)))$.

Proof: This is a standard consequence of $C^\infty(X^1, E)$ being a finitely generated projective generator of the abelian category $\mathcal{M}^t(G)_E$ which has arbitrary direct sums (cf. [Ber] Lemma 22 or [Pop] Cor. 3.7.4).

The action of $C^\infty(X^1)$ by multiplication on $C^\infty(X^1, E)$ defines a unital ring homomorphism

$$\begin{array}{rcl} \mu : C^\infty(X^1) & \longrightarrow & \mathrm{End}_{\mathcal{S}(G)}(C^\infty(X^1, E)) \\ f & \longmapsto & \mu_f \ . \end{array}$$

In X^1 we have the subgroup X^1_E of all χ such that $E_\chi \cong E$ as $\mathcal{S}(G)$-modules. This group X^1_E is finite since if E_χ and E are isomorphic they must have the same central character which implies that $\chi|Z = 1$. The action of X^1_E on X^1 being free the quotient X^1/X^1_E again is a C^∞-manifold. The ring $C^\infty(X^1/X^1_E)$ of C^∞-function on X^1/X^1_E coincides with the subring of those functions in $C^\infty(X^1)$ which are constant on the cosets of X^1_E.

Proposition 3.7: *The map μ restricts to an isomorphism*

$$C^\infty(X^1/X^1_E) \xrightarrow{\cong} Z(\mathrm{End}_{\mathcal{S}(G)}(C^\infty(X^1,E))) \ .$$

Proof: Using Cor. 3.5.i we have

$$\begin{aligned}\mathrm{End}_{\mathcal{S}(G)}(C^\infty(X^1,E)) &\cong \mathrm{Hom}_{\mathcal{S}(G)}(\mathcal{S}(G)\otimes_{\mathcal{S}(G^0)} E, C^\infty(X^1,E))\\ &\cong \mathrm{Hom}_{\mathcal{S}(G^0)}(E, C^\infty(X^1)\otimes_{\mathbb{C}} E)\\ &\cong C^\infty(X^1)\otimes_{\mathbb{C}} \mathrm{End}_{\mathcal{S}(G^0)}(E) \ .\end{aligned}$$

One checks that the map μ, under this identification, corresponds to the map

$$\begin{array}{ccc} C^\infty(X^1) & \longrightarrow & C^\infty(X^1)\otimes_{\mathbb{C}} \mathrm{End}_{\mathcal{S}(G^0)}(E)\\ f & \longmapsto & f\otimes \mathrm{id}_E \ . \end{array}$$

For any $\chi \in X^1_E$ we fix an isomorphism of $\mathcal{S}(G)$-modules $\alpha_\chi : E_\chi \xrightarrow{\cong} E$ and view it as an element in $\mathrm{End}_{\mathcal{S}(G^0)}(E)$. We claim that, as in [Ber] Prop. 28, the α_χ for $\chi \in X^1_E$ form a basis of the vector space $\mathrm{End}_{\mathcal{S}(G^0)}(E)$. For this we consider the action of G by conjugation on $\mathrm{End}_{\mathcal{S}(G^0)}(E)$. It is trivial on G^0 and on the center of G, hence factorizes through a finite quotient of G/G^0, and therefore is diagonalizable. A nonzero eigenvector with eigencharacter $\chi \in X^1$ is the same as a G-equivariant linear isomorphism $E_\chi \xrightarrow{\cong} E$. By [SSZ] App. Cor. 2 the latter automatically is $\mathcal{S}(G)$-equivariant and by Schur's lemma it then must be a scalar multiple of α_χ.

We write $A_\chi \in \mathrm{End}_{\mathcal{S}(G)}(C^\infty(X^1,E))$ for the endomorphism corresponding to $1\otimes\alpha_\chi \in C^\infty(X^1)\otimes_{\mathbb{C}} \mathrm{End}_{\mathcal{S}(G^0)}(E)$. For $f\otimes v \in C^\infty(X^1,E)$ and $f=\widehat{\varphi}$ we have

$$\begin{aligned}A_\chi(f\otimes v) &= A_\chi(I\circ J(\varphi\otimes v))\\ &= A_\chi \circ I(\varphi\circ(\epsilon_U\otimes v))\\ &= A_\chi\circ I\big(\mathrm{vol}(Z^0)^{-1}\cdot \sum_{g\in R}(\omega(\varphi^{g^{-1}}|Z) *_Z \epsilon_U^{g^{-1}})\otimes gv\big)\\ &= \mathrm{vol}(Z^0)^{-1}\cdot\sum_{g\in R} A_\chi\circ I\big((\omega(\varphi^{g^{-1}}|Z) *_Z \epsilon_U^{g^{-1}})\otimes gv\big)\\ &= \mathrm{vol}(Z^0)^{-1}\cdot\sum_{g\in R}(\omega(\varphi^{g^{-1}}|Z) *_Z \epsilon_U^{g^{-1}}) * (1\otimes\alpha_\chi(gv)) \ .\end{aligned}$$

For any $\chi' \in X^1$ we further compute

$$\begin{aligned}
A_\chi(f \otimes v)(\chi') &= \mathrm{vol}(Z^0)^{-1} \cdot \sum_{g \in R} [\chi'(\omega(\varphi^{g^{-1}}|Z) *_Z \epsilon_U^{g^{-1}})] * \alpha_\chi(gv) \\
&= \mathrm{vol}(Z^0)^{-1} \cdot \sum_{g \in R} [\chi'(\omega(\varphi^{g^{-1}}|Z) *_Z \epsilon_U^{g^{-1}})] * \chi^{-1}(g) g \alpha_\chi(v) \\
&= \mathrm{vol}(Z^0)^{-1} \cdot \sum_{g \in R} [\chi'(\omega((\chi^{-1}\varphi)^{g^{-1}}|Z) *_Z \epsilon_U^{g^{-1}})] * g \alpha_\chi(v) \\
&= \Big(\sum_{g \in G/G^0} \chi^{-1}(g)\varphi(g)\chi'(g^{-1}) \Big) \alpha_\chi(v) \\
&= \widehat{\varphi}(\chi\chi')\alpha_\chi(v) \\
&= [f(\chi.) \otimes \alpha_\chi(v)](\chi') \ .
\end{aligned}$$

Here the third identity comes from the fact that $\chi|Z = 1$ and the fourth identity is completely analogous to the corresponding identity in our earlier computation of the composite $I \circ J$. This establishes the explicit formula

$$A_\chi(f \otimes v) = f(\chi.) \otimes \alpha_\chi(v) \qquad \text{for any } f \otimes v \in C^\infty(X^1) \otimes_{\mathbb{C}} E \ .$$

In particular we have

$$A_\chi \circ \mu_f = \mu_{f(\chi.)} \circ A_\chi \qquad \text{for any } f \in C^\infty(X^1) \ .$$

From this one easily deduces, using that C^∞-functions on X^1 separate points, that

$$Z(\mathrm{End}_{\mathcal{S}(G)}(C^\infty(X^1, E))) = \{\mu_f : f \in C^\infty(X^1/X^1_E)\} \ .$$

4. The decomposition of $\mathcal{M}^t(G)$

The Harish Chandra spectrum $\Omega^t(G)$ of G is defined as follows. A discrete pair (L, τ) of G consists of a Levi subgroup L of G and an irreducible discrete series representation τ of L. The group G acts by conjugation on the set of (isomorphism classes of) discrete pairs and $\Omega^t(G)$ is defined to be the set of G-orbits of this action. By [Wal] III.4.1, for any tempered irreducible G-representation V there is up to conjugation a unique discrete pair (L, τ), called the discrete support of V, such that V is a subquotient of a G-representation parabolically induced from (L, τ). This gives a surjective map

$$\nu^t \ : \ \mathrm{Irr}^t(G) \longrightarrow \Omega^t(G)$$

which sends V to its discrete support. Given any discrete pair (L,τ) we have the map

$$\begin{array}{rcl} X^1_{nr}(L) & \longrightarrow & \Omega^t(G) \\ \chi & \longmapsto & G\text{-orbit of } [(L,\chi\tau)] \ . \end{array}$$

The images of these maps are called the (connected) components of $\Omega^t(G)$. They partition the Harish Chandra spectrum $\Omega^t(G)$. For any component Θ we consider the subcategory

$$\mathcal{M}^t(\Theta) := \mathcal{M}^t(G)_{(\nu^t)^{-1}(\Theta)} \ .$$

In this section we will establish the following main result.

Theorem 4.1: $\mathcal{M}^t(G) = \prod_\Theta \mathcal{M}^t(\Theta)$.

The proof requires various preparations. A component Θ will be called discrete, resp. induced, if for the discrete pairs (L,τ) in Θ we have $L = G$, resp. $L \neq G$. From Prop. 2.4 we know already that

$$\mathcal{M}^t(G) = \mathcal{M}^t(G)_{Ind} \times \mathcal{M}^t(G)_{Dis} \quad \text{and} \quad \mathcal{M}^t(G)_{Dis} = \prod_{\Theta \ discrete} \mathcal{M}^t(\Theta) \ .$$

The corresponding decomposition of any V in $\mathcal{M}^t(G)$ will be written as

$$V = V_{Ind} \oplus V_{Dis} \quad \text{and} \quad V_{Dis} = \bigoplus_{\Theta \ discrete} V_\Theta \ .$$

It remains to establish the decomposition

$$\mathcal{M}^t(G)_{Ind} = \prod_{\Theta \ induced} \mathcal{M}^t(\Theta) \ .$$

The subsequent arguments will rely crucially on the tempered parabolic induction and restriction functors

$$\mathrm{Ind}_P^G : \mathcal{M}^t(M) \longrightarrow \mathcal{M}^t(G) \quad \text{and} \quad r^t_{G,P} : \mathcal{M}^t(G) \longrightarrow \mathcal{M}^t(M)$$

constructed in the first part of this paper ([SZ]). Here $P \subseteq G$ is any parabolic subgroup with Levi component M. Let $\overline{P}$ denote the corresponding opposite parabolic subgroup. We recall that these functors are exact ([SZ] §2 and Prop. 5.1) and that in the sequence

$$\mathrm{Ind}_{\overline{P}}^G \ , \ r^t_{G,P} \ , \ \mathrm{Ind}_P^G \ , \ r^t_{G,\overline{P}}$$

each functor is right adjoint to the one immediately preceding it ([SZ] Prop. 3.1 and Thm. 5.5). Having a left as well as a right adjoint both functors commute with arbitrary direct sums and direct products. We also recall ([SZ] Cor. 5.3) that $r^t_{G,P}$, for an admissible tempered V, coincides with Waldspurger's construction V^w_P in [Wal] III.3.1.

Lemma 4.2: *For $P \neq G$ and any V in $\mathcal{M}^t(G)_{Dis}$ we have $r^t_{G,P}(V) = 0$.*

Proof: Since $r^t_{G,P}$ commutes with arbitrary direct sums we may assume that V lies in $\mathcal{M}^t(G)_D$ for some discrete X^1-orbit $D \subseteq Dis$. By Cor. 3.5.iii the category $\mathcal{M}^t(G)_D$ has the projective generator $C^\infty(X^1, E)$. Using again the commutation of $r^t_{G,P}$ with arbitrary direct sums as well as its exactness we are further reduced to showing that

$$r^t_{G,P}(C^\infty(X^1, E)) = 0 \ .$$

The evaluation maps induce an embedding

$$C^\infty(X^1, E) \hookrightarrow \prod_{\chi \in X^1}{}' E_\chi \ .$$

Using this time the commutation of $r^t_{G,P}$ with arbitrary direct products and its exactness it finally suffices to prove that $r^t_{G,P}(V) = 0$ for any irreducible discrete series representation V of G. This can be found in [Wal] III.3.2.

Lemma 4.3: *For $P \neq G$ and any V in $\mathcal{M}^t(M)$ the parabolically induced representation $\mathrm{Ind}^G_P(V)$ lies in $\mathcal{M}^t(G)_{Ind}$.*

Proof: Suppose that for some X^1-orbit $D \subseteq Dis$ the component $\mathrm{Ind}^G_P(V)_D$ in $\mathcal{M}^t(G)_D$ is nonzero. By adjunction we then obtain a nonzero homomorphism

$$r^t_{G,P}(\mathrm{Ind}^G_P(V)_D) \longrightarrow V \ .$$

But, according to the previous lemma, the left hand side vanishes which constitutes a contradiction.

There is the obvious map

$$\begin{array}{rrcl} \iota_M : & \Omega^t(M) & \longrightarrow & \Omega^t(G) \\ & M\text{-orbit of } (L,\tau) & \longmapsto & G\text{-orbit of } (L,\tau) \ . \end{array}$$

The image under ι_M of a connected component in $\Omega^t(M)$ is a connected component in $\Omega^t(G)$.

Proposition 4.4: *Let $\Theta_0 \subseteq \Omega^t(M)$ be a discrete component and put $\Theta := \iota_M(\Theta_0)$; the parabolic induction functor restricts to a functor*

$$\mathrm{Ind}_P^G : \mathcal{M}^t(\Theta_0) \longrightarrow \mathcal{M}^t(\Theta) .$$

Proof: We have to show that, for any V in $\mathcal{M}^t(\Theta_0)$, any simple subquotient of $\mathrm{Ind}_P^G(V)$ has discrete support in Θ. Note that the category $\mathcal{M}^t(\Theta)$ is closed under arbitrary direct sums. Since Ind_P^G is exact and commutes with arbitrary direct sums it therefore suffices to consider our projective generator $V = C^\infty(X_{nr}^1(M), E)$ of $\mathcal{M}^t(\Theta_0)$ from Cor. 3.5.iii. Let now W_0 be a fixed but arbitrary simple subquotient of $\mathrm{Ind}_P^G(V)$ and let the discrete pair (L, τ) represent the discrete support of W_0. Recall that this means that W_0 is a subquotient of $\mathrm{Ind}_Q^G(\tau)$ where $Q \subseteq G$ is some parabolic subgroup with Levi component L. With τ also $\mathrm{Ind}_Q^G(\tau)$ is preunitary. Hence W_0 even is a submodule of $\mathrm{Ind}_Q^G(\tau)$. It then follows by adjunction that τ is a quotient of $r_{G,Q}^t(W_0)$ and a fortiori, by the exactness of $r_{G,Q}^t$, a subquotient of $(r_{G,Q}^t(\mathrm{Ind}_P^G(V)))_{Dis}$. Using the embedding

$$V = C^\infty(X_{nr}^1(M), E) \hookrightarrow \prod_{\chi \in X_{nr}^1(M)}' E_\chi$$

given by the evaluation maps and the commutation of all functors involved with arbitrary direct products we obtain an embedding

$$(r_{G,Q}^t(\mathrm{Ind}_P^G(V)))_{Dis} \hookrightarrow \prod_{\chi \in X_{nr}^1(M)}' (r_{G,Q}^t(\mathrm{Ind}_P^G(E_\chi)))_{Dis} .$$

It follows that $(r_{G,Q}^t(\mathrm{Ind}_P^G(E_\chi)))_{Dis} \neq 0$ for at least one $\chi \in X_{nr}^1(M)$. By the geometric lemma for admissible tempered representations ([Wal] III.3.3) this latter module has a (finite) filtration whose graded pieces are of the form

$$(\mathrm{Ind}_{L \cap {}^sP^{s^{-1}}}^L(s^*(r_{M, M\cap^{s^{-1}}Q^s}^t(E_\chi))))_{Dis}$$

where s^* denotes conjugation by an appropriate element $s \in G$ such that ${}^sM^{s^{-1}} \cap Q = L \cap {}^sP^{s^{-1}}$. Since E_χ is discrete series Lemma 4.2 says that $r_{M, M\cap^{s^{-1}}Q^s}^t(E_\chi) = 0$ unless $M \subseteq {}^{s^{-1}}Q^s$. In addition, Lemma 4.3 says that $\mathrm{Ind}_{L \cap {}^sP^{s^{-1}}}^L(.)_{Dis} = 0$ unless $L \subseteq {}^sP^{s^{-1}}$. We consequently see that we must have ${}^sM^{s^{-1}} = L$ and that the simple constituents of the nonzero module $(r_{G,Q}^t(\mathrm{Ind}_P^G(E_\chi)))_{Dis}$ are of the form s^*E_χ. Hence $(r_{G,Q}^t(\mathrm{Ind}_P^G(E_\chi)))_{Dis}$, for any $\chi \in X_{nr}^1(M)$, lies in

$$\prod_s \mathcal{M}^t(s\Theta_0) .$$

Being cut out by a central idempotent by Lemma 2.1 this category is closed under arbitrary direct products. We conclude that $(r^t_{G,Q}(\mathrm{Ind}^G_P(V)))_{Dis}$ and hence also τ lie in $\prod_s \mathcal{M}^t(s\Theta_0)$. This implies that (L,τ) is conjugate in G to some (M,E_χ), and this means that the G-orbit of (L,τ) lies in Θ.

We now begin the proof of Theorem 4.1. Let V be any module in $\mathcal{M}^t(G)$. We have to show that V can be decomposed into a direct sum of objects in the subcategories $\mathcal{M}^t(\Theta)$. It is a completely formal fact ([Ber] Lemma 28) that this decomposition property is inherited by submodules. Hence it suffices to embed our given module V into another one which decomposes as wanted. This we will do as follows (again in complete analogy with [Ber]).

We fix finitely many pairs $M_i \subseteq P_i$ of parabolic subgroups P_i with Levi factor M_i which form a system of representatives for the G-conjugacy classes of all such pairs. The projection maps $r^t_{G,P_i}(V) \to r^t_{G,P_i}(V)_{Dis}$ induce, by adjunction, a homomorphism

$$\alpha_V : V \longrightarrow \bigoplus_i \mathrm{Ind}^G_{P_i}(r^t_{G,P_i}(V)_{Dis}) \ .$$

We claim that this map is injective. Let V_0 be its kernel. We then have the commutative diagram

$$\begin{array}{ccc} V_0 & \xrightarrow{\alpha_{V_0}} & \bigoplus_i \mathrm{Ind}^G_{P_i}(r^t_{G,P_i}(V_0)_{Dis}) \\ \downarrow \subseteq & & \downarrow \\ V & \xrightarrow{\alpha_V} & \bigoplus_i \mathrm{Ind}^G_{P_i}(r^t_{G,P_i}(V)_{Dis}) \end{array}$$

in which the vertical arrows are injective. Hence $\alpha_{V_0} = 0$. By adjunction this implies that $r^t_{G,P_i}(V_0)_{Dis} = 0$ for any i. Suppose that V_0 is nonzero and let W_0 be one of its simple subquotients. We find a pair $M_{i_0} \subseteq P_{i_0}$ and an irreducible discrete series representation τ of M_{i_0} such that W_0 is a submodule of $\mathrm{Ind}^G_{P_{i_0}}(\tau)$. It follows by adjunction that $r^t_{G,P_{i_0}}(W_0)_{Dis}$ and a fortiori, by the exactness of $r^t_{G,P_{i_0}}$, that $r^t_{G,P_{i_0}}(V_0)_{Dis}$ is nonzero which is a contradiction.

Having established our claim we may assume that V is of the form $V = \mathrm{Ind}^G_P(V_1)$ where V_1 lies in $\mathcal{M}^t(M)_{Dis}$. Since the functor Ind^G_P commutes with arbitrary direct sums we may further reduce to the case that V_1 lies in $\mathcal{M}^t(\Theta_0)$ for some discrete component $\Theta_0 \subseteq \Omega^t(M)$. But then we know from Prop. 4.4 that V even is contained in a single $\mathcal{M}^t(\Theta)$.

5. Projective generators

In this section we fix a induced component Θ in the Harish Chandra spectrum $\Omega^t(G)$. Our goal is to construct an explicit projective generator for the category $\mathcal{M}^t(\Theta)$. We choose a parabolic subgroup P with Levi factor M and a discrete component $\Theta_0 \subseteq \Omega^t(M)$ such that

$$\iota_M(\Theta_0) = \Theta\ .$$

According to Cor. 3.5 the $\mathcal{S}(M)$-module $C^\infty(X^1_{nr}(M), E)$, for an appropriate irreducible discrete series representation E of M, is finitely generated and projective in $\mathcal{M}^t(M)$ and is a generator of $\mathcal{M}^t(\Theta_0)$. We put

$$I_P(\Theta_0) := \mathrm{Ind}_P^G(C^\infty(X^1_{nr}(M), E))$$

which according to Prop. 4.4 lies in $\mathcal{M}^t(\Theta)$.

Proposition 5.1: *$I_P(\Theta_0)$ is finitely generated and projective in $\mathcal{M}^t(G)$.*

Proof: By [SZ] Cor. 5.6 and Cor. 5.7 the functor Ind_P^G respects projective objects as well as objects of finite presentation. In addition we only need to observe that any finitely generated projective module is of finite presentation.

The functor

$$\mathrm{Ind}_P^G : \mathcal{M}^t(\Theta_0) \longrightarrow \mathcal{M}^t(\Theta)$$

(cf. Prop. 4.4) has the right adjoint functor

$$\begin{array}{rcl} r_{\Theta,\Theta_0;\overline{P}} : \mathcal{M}^t(\Theta) & \longrightarrow & \mathcal{M}^t(\Theta_0) \\ V & \longmapsto & r^t_{G,\overline{P}}(V)_{\Theta_0}\ . \end{array}$$

Proposition 5.2: *The functor $r_{\Theta,\Theta_0;Q}$, for any parabolic subgroup Q with Levi factor M, is faithful.*

Proof: In a first step we convince ourselves that $r_{\Theta,\Theta_0;Q}(V) \neq 0$ for any simple object V in $\mathcal{M}^t(\Theta)$. By definition there is a parabolic subgroup P' with Levi factor M and a $\chi \in X^1_{nr}(M)$ such that V is a subquotient of $\mathrm{Ind}_{P'}^G(E_\chi)$. But according to [Sil] 5.4.4.1(1) the two $\mathcal{S}(G)$-modules $\mathrm{Ind}_{P'}^G(E_\chi)$ and $\mathrm{Ind}_Q^G(E_\chi)$ have the same simple constituents. Hence V is a subquotient and then, by unitarity, even a submodule of $\mathrm{Ind}_Q^G(E_\chi)$. It follows by adjunction that E_χ is a quotient of $r^t_{G,Q}(V)$. Since E_χ lies in $\mathcal{M}^t(\Theta_0)$ this means that $r_{\Theta,\Theta_0;Q}(V) \neq 0$.

Since any nonzero finitely generated $\mathcal{S}(G)$-module has a simple quotient it follows immediately from the first step and the exactness of $r_{\Theta,\Theta_0;Q}$ that $r_{\Theta,\Theta_0;Q}(V) \neq 0$ for any nonzero finitely generated module V in $\mathcal{M}^t(\Theta)$. Clearly, applying the exactness of $r_{\Theta,\Theta_0;Q}$ to the inclusion of any finitely generated submodule, we in fact obtain $r_{\Theta,\Theta_0;Q}(V) \neq 0$ for any nonzero V in $\mathcal{M}^t(\Theta)$.

Recall that faithfulness means injectivity on Hom-spaces. Let therefore A be a homomorphism in $\mathcal{M}^t(\Theta)$ such that $r_{\Theta,\Theta_0;Q}(A) = 0$. Using once more the exactness of $r_{\Theta,\Theta_0;Q}$ we get $r_{\Theta,\Theta_0;Q}(\mathrm{im}(A)) = \mathrm{im}(r_{\Theta,\Theta_0;Q}(A)) = 0$ and consequently $\mathrm{im}(A) = 0$. This means of course that A was the zero map.

Corollary 5.3: *$I_P(\Theta_0)$ is a projective generator of $\mathcal{M}^t(\Theta)$.*

Proof: It remains to note that any functor which has a faithful right adjoint respects generators.

Corollary 5.4: *For any V in $\mathcal{M}^t(\Theta)$ and any parabolic subgroup Q with Levi factor M the map*

$$V \longrightarrow \mathrm{Ind}_Q^G(r_{G,Q}^t(V)_{\Theta_0})$$

corresponding under adjunction to the projection map $r_{G,Q}^t(V) \twoheadrightarrow r_{G,Q}^t(V)_{\Theta_0}$ is injective.

Proof: Similarly as in the proof of Thm. 4.1 we let V_0 denote the kernel of the map in question and we consider the commutative diagram

$$\begin{array}{ccc} V_0 & \longrightarrow & \mathrm{Ind}_Q^G(r_{G,Q}^t(V_0)_{\Theta_0}) \\ \subseteq \downarrow & & \downarrow \\ V & \longrightarrow & \mathrm{Ind}_Q^G(r_{G,Q}^t(V)_{\Theta_0}) \ . \end{array}$$

Since the vertical arrows are injective the upper horizontal arrow has to be the zero map. By adjunction this means that $r_{G,Q}^t(V_0)_{\Theta_0} = 0$. Hence Prop. 5.2 implies that $V_0 = 0$.

6. The center of $\mathcal{M}^t(G)$

Corresponding to the decomposition in Thm. 4.1 we have a decomposition

$$\mathcal{Z}^t(G) = \prod_{\Theta} \mathcal{Z}^t(G)_\Theta$$

of the center $\mathcal{Z}^t(G)$ into a direct product of ideals where Θ runs over all connected components of the Harish Chandra spectrum $\Omega^t(G)$. The ideal $\mathcal{Z}^t(G)_\Theta$

is generated by the idempotent Z_Θ which projects each V in $\mathcal{M}^t(G)$ onto its component V_Θ in $\mathcal{M}^t(G)_\Theta$.

If the component Θ is discrete then we have computed $\mathcal{Z}^t(G)_\Theta$ in Cor. 3.6.ii and Prop. 3.7 to be isomorphic to

$$\mathcal{Z}^t(G)_\Theta \cong C^\infty(X^1_{nr}(G)/X^1_{nr}(G)_E)$$

where E is an irreducible discrete series representation of G whose isomorphism class lies in $(\nu^t)^{-1}(\Theta)$ and where $X^1_{nr}(G)_E = \{\chi \in X^1_{nr}(G) : E_\chi \cong E\}$. Note that the map $\chi \longmapsto \nu^t([E_\chi])$ induces a bijection

$$X^1_{nr}(G)/X^1_{nr}(G)_E \xrightarrow{\simeq} \Theta$$

which we use to equip Θ with the structure of a C^∞-manifold. This manifold structure is easily seen to be independent of the choice of E. Hence our above computation reads more intrinsically

$$\mathcal{Z}^t(G)_\Theta \cong C^\infty(\Theta) \ .$$

We consider now an induced component Θ. As before let P be a parabolic subgroup with Levi factor M and let E be an irreducible discrete series representation of M such that Θ contains the G-orbit of (M, E). Let Θ_0 denote the component in $\Omega^t(M)$ which contains the M-orbit of (M, E). Then $\Theta = \iota_M(\Theta_0)$. As in the proof of Cor. 3.6 we deduce from Prop. 5.1 and Cor. 5.3 an isomorphism

$$\mathcal{Z}^t(G)_\Theta \cong Z(\mathrm{End}_{\mathcal{S}(G)}(\mathrm{Ind}^G_P(C^\infty(X^1_{nr}(M), E)))) \ .$$

Inducing the multiplication by functions gives the unital ring monomorphism

$$\begin{array}{rcl} C^\infty(X^1_{nr}(M)) & \longrightarrow & \mathrm{End}_{\mathcal{S}(G)}(\mathrm{Ind}^G_P(C^\infty(X^1_{nr}(M), E))) \\ f & \longmapsto & \mathrm{Ind}(\mu_f) \end{array}$$

which allows us to view $\mathrm{Ind}^G_P(C^\infty(X^1_{nr}(M), E))$ as a unital module over the commutative ring $C^\infty(X^1_{nr}(M))$.

Lemma 6.1: *Fix a good maximal compact subgroup K of G; for any open subgroup $U \subseteq K$ the U-fixed vectors $\mathrm{Ind}^G_P(C^\infty(X^1_{nr}(M), E))^U$ as a $C^\infty(X^1_{nr}(M))$-module are finitely generated and free.*

Proof: Using the Iwasawa decomposition $G = KP$ we obtain the K-equivariant isomorphisms

$$\begin{aligned} \mathrm{Ind}^G_P(C^\infty(X^1_{nr}(M), E)) &\cong \mathrm{Ind}^K_{P\cap K}(C^\infty(X^1_{nr}(M), E)) \\ &\cong C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} \mathrm{Ind}^K_{P\cap K}(E) \ . \end{aligned}$$

It remains to note that $\mathrm{Ind}_{P\cap K}^{K}(E)^U \cong \mathrm{Ind}_P^G(E)^U$ is a finite dimensional vector space since the smooth representation $\mathrm{Ind}_P^G(E)$ is admissible.

Let now $z \in \mathcal{Z}^t(G)_\Theta$ and let z', resp. z_χ for any $\chi \in X_{nr}^1(M)$, denote the $\mathcal{S}(G)$-module endomorphism induced by z on $\mathrm{Ind}_P^G(C^\infty(X_{nr}^1(M), E))$, resp. on $\mathrm{Ind}_P^G(E_\chi)$. By naturality z' is an endomorphism of $C^\infty(X_{nr}^1(M))$-modules as well. On the other hand we write $\mathbb{C}_\chi$ for the one dimensional $C^\infty(X_{nr}^1(M))$-module corresponding to the character of evaluation in χ. We then have the formulas

$$\mathrm{Ind}_P^G(C^\infty(X_{nr}^1(M), E)) \otimes_{C^\infty(X_{nr}^1(M))} \mathbb{C}_\chi = \mathrm{Ind}_P^G(E_\chi)$$

and

$$z' \otimes_{C^\infty(X_{nr}^1(M))} \mathbb{C}_\chi = z_\chi \ .$$

Lemma 6.2: *There is a function $f \in C^\infty(X_{nr}^1(M))$ such that $z' = \mathrm{Ind}(\mu_f)$.*

Proof: By [Wal] IV.2.2(i) (or [Sil] 2.5.9 and 4.4.6) and [Sil] 5.3.1.3 the subset Ξ of all characters $\chi \in X_{nr}^1(M)$ such that $\mathrm{Ind}_P^G(E_\chi)$ is irreducible is dense in $X_{nr}^1(M)$. By Schur's lemma $z_\chi \in \mathbb{C}$ is a scalar for any $\chi \in \Xi$. Let now U be an open subgroup of G as in Lemma 6.1. Then the endomorphism z' restricted to the finitely generated and free $C^\infty(X_{nr}^1(M))$-module $\mathrm{Ind}_P^G(C^\infty(X_{nr}^1(M), E))^U$ is given by a square matrix with entries in $C^\infty(X_{nr}^1(M))$. Over Ξ this square matrix specializes to a diagonal matrix with identical diagonal entries. By the density of Ξ the same then must hold true over all of $X_{nr}^1(M)$ which means that z' on the U-fixed vectors is multiplication by a function in $C^\infty(X_{nr}^1(M))$. Since we can make U arbitrarily small this remains true for z'.

Since $z' = \mathrm{Ind}(\mu_f)$ lies in the center of $\mathrm{End}_{\mathcal{S}(G)}(\mathrm{Ind}_P^G(C^\infty(X_{nr}^1(M), E)))$ the endomorphism μ_f must lie in the center of $\mathrm{End}_{\mathcal{S}(M)}(C^\infty(X_{nr}^1(M), E))$. We therefore know from Prop. 3.7 that f is a function on $X_{nr}^1(M)/X_{nr}^1(M)_E$ where, similarly as before, we put

$$X_{nr}^1(M)_E = \{\chi \in X_{nr}^1(M) : E_\chi \cong E\} \ .$$

More generally we obtain the following. We have

$$z_\chi = \text{ multiplication by } f(\chi)$$

for $\chi \in X_{nr}^1(M)$. Therefore, whenever $\mathrm{Ind}_P^G(E_\chi)$ and $\mathrm{Ind}_P^G(E_{\chi'})$, for any two $\chi, \chi' \in X_{nr}^1(M)$, have a common irreducible constituent we must have $f(\chi) = f(\chi')$. According to [Sil] 5.4.4.1(1) this occurs when the discrete pairs (M, E_χ)

and $(M, E_{\chi'})$ lie in the same G-orbit. Hence the function f factorizes over the surjection

$$\begin{array}{rcl} X^1_{nr}(M) & \longrightarrow\!\!\!\!\rightarrow & \Theta \\ \chi & \longmapsto & G\text{-orbit of } [(M, E_\chi)] \ . \end{array}$$

Lemma 6.3: *If $f \in C^\infty(X^1_{nr}(M))$ is the pullback of a function on Θ then* $\mathrm{Ind}(\mu_f)$ *lies in the center of* $\mathrm{End}_{\mathcal{S}(G)}(\mathrm{Ind}_P^G(C^\infty(X^1_{nr}(M), E)))$.

Proof: As before we abbreviate $I_P(\Theta_0) := \mathrm{Ind}_P^G(C^\infty(X^1_{nr}(M), E))$. By Cor. 3.6.ii and Prop. 3.7 the function f defines an element in the center of $\mathcal{M}^t(\Theta_0)$ which we denote by z_f. It acts on any E_χ by multiplication by $f(\chi)$. We now consider the diagram

$$\begin{array}{ccc} I_P(\Theta_0) & \longrightarrow & \mathrm{Ind}_P^G(r^t_{G,P}(I_P(\Theta_0))_{\Theta_0}) \\ \Big\downarrow{\scriptstyle \mathrm{Ind}(\mu_f)} & & \Big\downarrow{\scriptstyle \mathrm{Ind}_P^G(z_f)} \\ I_P(\Theta_0) & \longrightarrow & \mathrm{Ind}_P^G(r^t_{G,P}(I_P(\Theta_0))_{\Theta_0}) \end{array}$$

where the horizontal arrows represent the map corresponding under adjunction to the projection map $r^t_{G,P}(I_P(\Theta_0)) \longrightarrow\!\!\!\!\rightarrow r^t_{G,P}(I_P(\Theta_0))_{\Theta_0}$. According to Cor. 5.4 they are injective. It suffices to show that this diagram is commutative: any $\mathcal{S}(G)$-module endomorphism of $I_P(\Theta_0)$ extends by functoriality to an $\mathcal{S}(G)$-module endomorphism of $\mathrm{Ind}_P^G(r^t_{G,P}(I_P(\Theta_0))_{\Theta_0})$ which commutes with $\mathrm{Ind}_P^G(z_f)$ since z_f is central. Hence $\mathrm{Ind}(\mu_f)$ must be central provided the diagram commutes.

We let $\Xi \subseteq X^1_{nr}(M)$ denote the subset of all characters such that $s^*E_\chi \not\cong E_\chi$ for any nontrivial $s \in N_G(M)/M$ where $N_G(M)$ denotes the normalizer of M in G. By [Sil] 5.3.1.3 the subset Ξ is dense in $X^1_{nr}(M)$. Hence the product of evaluation maps

$$C^\infty(X^1_{nr}(M), E) \longrightarrow \prod_{\chi \in \Xi}{}' E_\chi$$

is injective and induces to an injective map

$$I_P(\Theta_0) \longrightarrow \prod_{\chi \in \Xi}{}' \mathrm{Ind}_P^G(E_\chi) \ .$$

Since all functors under consideration commute with arbitrary direct products it suffices therefore to establish the commutativity of the corresponding diagrams

$$\begin{array}{ccc} \mathrm{Ind}_P^G(E_\chi) & \longrightarrow & \mathrm{Ind}_P^G(r^t_{G,P}(\mathrm{Ind}_P^G(E_\chi))_{\Theta_0}) \\ \Big\downarrow{\scriptstyle f(\chi)\cdot} & & \Big\downarrow{\scriptstyle \mathrm{Ind}_P^G(z_f)} \\ \mathrm{Ind}_P^G(E_\chi) & \longrightarrow & \mathrm{Ind}_P^G(r^t_{G,P}(\mathrm{Ind}_P^G(E_\chi))_{\Theta_0}) \end{array}$$

for any $\chi \in \Xi$. But according to [Wal] III.7.3 we have

$$r^t_{G,P}(\mathrm{Ind}^G_P(E_\chi))_{\Theta_0} \cong \bigoplus_s s^* E_\chi$$

for any $\chi \in \Xi$ where s runs over all elements in $N_G(M)/M$ such that $s\Theta_0 = \Theta_0$. For these s the discrete pairs (M, E_χ) and (M, s^*E_χ) define the same point in Θ so that z_f, by our assumption on the function f, acts on the whole direct sum by multiplication by $f(\chi)$ as well.

These lemmas show that the center $\mathcal{Z}^t(G)_\Theta$ is isomorphic to the ring of all C^∞-functions on $X^1_{nr}(M)$ which are pullbacks from Θ. The finite group

$$W(\Theta_0) := \{s \in N_G(M)/M : s\Theta_0 = \Theta_0\}$$

acts on $X^1_{nr}(M)/X^1_{nr}(M)_E$ in such a way that $s^*E_\chi \cong E_{s^{-1}(\chi)}$. This action can also be described as being the conjugation action twisted by the cocycle $s \longmapsto s(1)$ where $1 \in X^1_{nr}(M)$ denotes the trivial character. Our above map $X^1_{nr}(M) \twoheadrightarrow \Theta$ descends to a bijection

$$W(\Theta_0)\backslash(X^1_{nr}(M)/X^1_{nr}(M)_E) \xrightarrow{\simeq} \Theta$$

which allows to equip Θ with the structure of an orbifold. We leave it to the reader to verify that this orbifold structure does not depend on the choice of (M, E). Our result now can be formulated as follows.

Proposition 6.4: $\mathcal{Z}^t(G)_\Theta \cong C^\infty(\Theta)$.

As the disjoint union of its components Θ the whole Harish Chandra spectrum $\Omega^t(G)$ naturally carries the structure of an orbifold (with infinitely many connected components). We obtain the following explicit computation of the center of the category $\mathcal{M}^t(G)$.

Theorem 6.5: $\mathcal{Z}^t(G) \cong C^\infty(\Omega^t(G))$.

7. The geometric lemma

A very important tool in the parabolic induction and restriction formalism for smooth representations is the geometric lemma which computes the composite $r_{G,P'} \circ \mathrm{Ind}^G_P$ in a way which is analogous to the Mackey formula in finite group representation theory. At this point we finally need to establish an analog of this geometric lemma for our functors for $\mathcal{S}$-modules.

Throughout this section we fix a minimal Levi subgroup $M_0 \subseteq G$. Let $W^G := N_G(M_0)/M_0$ denote the corresponding Weyl group. We also fix two parabolic subgroups $P, P' \subseteq G$ which contain M_0. Then P, resp. P', has a unique Levi subgroup M, resp. M', which contains M_0. We let N and N' denote the unipotent radicals of P and P', respectively. The usual geometric lemma starts from the following filtration of the functor

$$\mathrm{Ind}_P^G : \mathcal{M}(M) \longrightarrow \mathcal{M}(G) \ .$$

By the Bruhat decomposition the group G is the disjoint union of double cosets

$$G = \dot{\bigcup_{w \in W(M'|G|M)}} P'wP$$

where $W(M'|G|M) \subseteq W^G$ is an appropriate set of representatives for the double cosets in $W^{M'} \backslash W^G / W^M$ (if P and P' both are standard one can take the representatives of minimal length). In addition there is a total order $\leq$ on $W(M'|G|M)$ (in the standard case one can take any refinement of the Bruhat order) such that the

$$G_{\geq w} := \bigcup_{v \geq w} P'vP$$

form a decreasing filtration of G by open subsets and the

$$Fil^w \mathrm{Ind}_P^G(.) := \{F \in \mathrm{Ind}_P^G(.) : F \text{ has support in } G_{\geq w}\}$$

form a decreasing filtration of Ind_P^G viewed as a functor into $\mathcal{M}(P')$. Since the formation of the usual Jacquet functor $r_{G,P'}$ is exact and only uses the P'-action (and not the full G-action) we obtain, by setting

$$Fil^w r_{G,P'}(\mathrm{Ind}_P^G(.)) := (Fil^w \mathrm{Ind}_P^G(.))_{N'} \ ,$$

a decreasing filtration of the composed functor $r_{G,P'} \circ \mathrm{Ind}_P^G$. The geometric lemma compares the associated graded functor

$$gr^w(r_{G,P'} \circ \mathrm{Ind}_P^G) : \mathcal{M}(M) \longrightarrow \mathcal{M}(M')$$

with the composed functor

$$\begin{aligned} \mathrm{Ind}_{M' \cap \dot{w} P \dot{w}^{-1}}^{M'} \circ \dot{w}_* \circ r_{M, M \cap \dot{w}^{-1} P' \dot{w}} : \mathcal{M}(M) &\longrightarrow \mathcal{M}(M \cap \dot{w}^{-1} M' \dot{w}) \\ &\longrightarrow \mathcal{M}(M' \cap \dot{w} M \dot{w}^{-1}) \longrightarrow \mathcal{M}(M') \ . \end{aligned}$$

Here we have fixed once and for all a representative $\dot{w} \in N_G(M_0)$ of $w \in W^G$. This comparison is done via the map

$$\begin{array}{rcl} p_{\dot{w}} : Fil^w \mathrm{Ind}_P^G(.) & \longrightarrow & \mathrm{Ind}_{M' \cap \dot{w}P\dot{w}^{-1}}^{M'}(\dot{w}_*(r_{M,M\cap \dot{w}^{-1}P'\dot{w}}(.))) \\ F & \longmapsto & p_{\dot{w}}(F)(m') := \delta_{P'}^{-1/2}(m') \cdot \int\limits_{N'/N'\cap \dot{w}P\dot{w}^{-1}} \overline{F(m'n'\dot{w})} dn' \end{array}$$

where $x \longmapsto \overline{x}$ denotes the projection to the Jacquet module $(.)_{M\cap \dot{w}^{-1}N'\dot{w}}$ (note that $M\cap \dot{w}^{-1}N'\dot{w}$ is the unipotent radical of the parabolic subgroup $M\cap \dot{w}^{-1}P'\dot{w}$ of M). The function F has values in a representation space X where N acts trivially, and $\overline{F}$ has values in $\overline{X}$ where also $M \cap \dot{w}^{-1}N'\dot{w}$ acts trivially, hence $\overline{F(m'n'\dot{w}g)} = \overline{F(m'n'\dot{w})}$ for all $g \in (M \cap \dot{w}^{-1}N'\dot{w})N$ so that the integral is well defined.

Geometric Lemma 7.1: *The map $p_{\dot{w}}$ induces a natural isomorphism*

$$gr^w(r_{G,P'} \circ \mathrm{Ind}_P^G) \xrightarrow{\simeq} \mathrm{Ind}_{M'\cap \dot{w}P\dot{w}^{-1}}^{M'} \circ \dot{w}_* \circ r_{M,M\cap \dot{w}^{-1}P'\dot{w}} \ .$$

Proof: [BZ] 2.12 and 6.4.

The difficulty with a direct generalization of this result to our context of $\mathcal{S}$-modules lies in the fact that our functor $r^t_{G,P'}$ does depend on the full $\mathcal{S}(G)$-module structure (something like an algebra $\mathcal{S}(P')$ was not even defined). The passage from the filtration $Fil^w \mathrm{Ind}_P^G$ to a filtration of the functor $r^t_{G,P'} \circ \mathrm{Ind}_P^G$ therefore will necessarily be more involved. In fact we will need the full force of the theory which we have developed. We begin by using the natural splitting

$$r^t_{G,P'} \circ \mathrm{Ind}_P^G \longrightarrow r_{G,P'} \circ \mathrm{Ind}_P^G$$

provided by [SZ] Prop. 5.2 and here denoted simply by σ in order to define

$$Fil^w r^t_{G,P'}(\mathrm{Ind}_P^G(.)) := \sigma^{-1}(Fil^w r_{G,P'}(\mathrm{Ind}_P^G(.))) \ .$$

We also introduce the image $\widetilde{Fil}^w r^t_{G,P'}(\mathrm{Ind}_P^G(.))$ of $Fil^w r_{G,P'}(\mathrm{Ind}_P^G(.))$ under the natural surjection

$$r_{G,P'} \circ \mathrm{Ind}_P^G \longrightarrow r^t_{G,P'} \circ \mathrm{Ind}_P^G$$

denoted by π in the following. Of course, these at first are only filtrations by $\mathcal{H}(M')$-submodules satisfying

$$Fil^w r^t_{G,P'}(\mathrm{Ind}_P^G(.)) \subseteq \widetilde{Fil}^w r^t_{G,P'}(\mathrm{Ind}_P^G(.)) \ .$$

Proposition 7.2: *$Fil^w r^t_{G,P'}(\mathrm{Ind}^G_P(Y))$, for any Y in $\mathcal{M}^t(M)$, is an $\mathcal{S}(M')$-submodule of $r^t_{G,P'}(\mathrm{Ind}^G_P(Y))$, and we have*

$$Fil^w r^t_{G,P'}(\mathrm{Ind}^G_P(.)) = \widetilde{Fil}^w r^t_{G,P'}(\mathrm{Ind}^G_P(.)) \ .$$

Proof: *Step 1:* Suppose that Y is admissible. Then also $r^t_{G,P'}(\mathrm{Ind}^G_P(Y))$ is admissible by [SZ] Cor. 5.3. According to [SSZ] App. Cor. 2 any $\mathcal{H}(M')$-submodule of an admissible $\mathcal{S}(M')$-module is already an $\mathcal{S}(M')$-submodule. Moreover, if $Z_{M'}$ denotes the maximal split torus in the center of M' then $r_{G,P'}(\mathrm{Ind}^G_P(Y))$ decomposes into generalized eigenspaces with respect to the action of $Z_{M'}$. Obviously the filtration Fil^w is compatible with this decomposition. The image of σ coincides with the sum of those eigenspaces for which the corresponding eigencharacter of $Z_{M'}$ is unitary.

Step 2: Let $\{Y_i\}_{i\in I}$ be a family in $\mathcal{M}^t(M)$. If any Y_i satisfies our assertion then so, too, does their direct sum $\oplus_{i\in I} Y_i$. This is immediate from the fact that all functors commute with arbitrary direct sums.

Step 3: Suppose that there is a monomorphism of $\mathcal{S}(M)$-modules

$$Y \longrightarrow {\prod_{i\in I}}' Y_i$$

into the nondegenerate direct product of a family of admissible $\mathcal{S}(M)$-modules Y_i. Since all functors involved are exact and respect nondegenerate direct products we obtain monomorphisms of $\mathcal{H}(M')$-modules

$$r_{G,P'}(\mathrm{Ind}^G_P(Y)) \longrightarrow {\prod_{i\in I}}' r_{G,P'}(\mathrm{Ind}^G_P(Y_i))$$

(this one obviously respects the filtration) and

$$\begin{array}{c} \mathrm{Ind}^{M'}_{M'\cap \dot{w}P\dot{w}^{-1}}(\dot{w}_*(r_{M,M\cap \dot{w}^{-1}P'\dot{w}}(Y))) \\ \big\downarrow \\ {\prod_{i\in I}}' \ \mathrm{Ind}^{M'}_{M'\cap \dot{w}P\dot{w}^{-1}}(\dot{w}_*(r_{M,M\cap \dot{w}^{-1}P'\dot{w}}(Y_i))) \ . \end{array}$$

The Geometric Lemma 7.1 then implies that the former map on the graded objects

$$gr^w r_{G,P'}(\mathrm{Ind}^G_P(Y)) \longrightarrow {\prod_{i\in I}}' gr^w r_{G,P'}(\mathrm{Ind}^G_P(Y_i))$$

still is injective. It follows that the diagram

$$\begin{array}{ccc} Fil^w r_{G,P'}(\mathrm{Ind}_P^G(Y)) & \xrightarrow{\subseteq} & r_{G,P'}(\mathrm{Ind}_P^G(Y)) \\ \downarrow & & \downarrow \\ \prod'_{i\in I} Fil^w r_{G,P'}(\mathrm{Ind}_P^G(Y_i)) & \xrightarrow{\subseteq} & \prod'_{i\in I} r_{G,P'}(\mathrm{Ind}_P^G(Y_i)) \end{array}$$

is cartesian. Hence also the diagram

$$\begin{array}{ccc} Fil^w r^t_{G,P'}(\mathrm{Ind}_P^G(Y)) & \xrightarrow{\subseteq} & r^t_{G,P'}(\mathrm{Ind}_P^G(Y)) \\ \downarrow & & \downarrow \\ \prod'_{i\in I} Fil^w r^t_{G,P'}(\mathrm{Ind}_P^G(Y_i)) & \xrightarrow{\subseteq} & \prod'_{i\in I} r^t_{G,P'}(\mathrm{Ind}_P^G(Y_i)) \end{array}$$

is cartesian. It therefore follows from the first step applied to the Y_i that $Fil^w r^t_{G,P'}(\mathrm{Ind}_P^G(Y))$ is an $\mathcal{S}(M')$-submodule of $r^t_{G,P'}(\mathrm{Ind}_P^G(Y))$ and that

$$\widetilde{Fil}^w r^t_{G,P'}(\mathrm{Ind}_P^G(Y)) \subseteq Fil^w r^t_{G,P'}(\mathrm{Ind}_P^G(Y)) \ .$$

Step 4: Let Y be a quotient of Z in $\mathcal{M}^t(M)$. If Z satisfies our assertion then so, too, does Y. The map $Fil^w \mathrm{Ind}_P^G(Z) \twoheadrightarrow Fil^w \mathrm{Ind}_P^G(Y)$ and hence, by the exactness of the functor $r_{G,P'}$, also the map

$$Fil^w r_{G,P'}(\mathrm{Ind}_P^G(Z)) \twoheadrightarrow Fil^w r_{G,P'}(\mathrm{Ind}_P^G(Y))$$

are surjective. On the other hand, by assumption we have

$$Fil^w r_{G,P'}(\mathrm{Ind}_P^G(Z)) = Fil^w \mathrm{im}(\sigma) \oplus Fil^w \ker(\pi)$$

where on the right hand side we have the direct sum of the filtrations induced on the subspaces $\mathrm{im}(\sigma)$ and $\ker(\pi)$. The above surjectivity therefore implies the corresponding decomposition

$$Fil^w r_{G,P'}(\mathrm{Ind}_P^G(Y)) = Fil^w \mathrm{im}(\sigma) \oplus Fil^w \ker(\pi)$$

and, in particular, the surjectivity of the map

$$Fil^w r^t_{G,P'}(\mathrm{Ind}_P^G(Z)) \twoheadrightarrow Fil^w r^t_{G,P'}(\mathrm{Ind}_P^G(Y)) \ .$$

Step 5: Let Y in $\mathcal{M}^t(M)$ be arbitrary. By Step 2 we may assume that Y lies in one component category (cf. Thm. 4.1). By Step 4 we may further assume that Y is of the form $\oplus_{i\in I}\Pi$ where Π is a fixed projective generator of this component category. Step 2 again reduces us to the case $Y = \Pi$. It remains to observe that the specific projective generator which we have constructed in Cor. 5.3 satisfies the assumption in Step 3 (induce the evaluation embedding $C^\infty(X^1, E) \hookrightarrow \prod_{\chi\in X^1}{}' E_\chi$ to M).

Hence we have the associated graded functor

$$gr^w(r^t_{G,P'} \circ \mathrm{Ind}^G_P) : \mathcal{M}^t(M) \longrightarrow \mathcal{M}^t(M') \ .$$

Moreover, the second part of Prop. 7.2 means that

$$Fil^w r_{G,P'}(\mathrm{Ind}^G_P(.)) = Fil^w \mathrm{im}(\sigma) \oplus Fil^w \ker(\pi)$$

where on the right hand side we have the direct sum of the filtrations induced on the subspaces $\mathrm{im}(\sigma)$ and $\ker(\pi)$. Hence

$$gr^w r_{G,P'}(\mathrm{Ind}^G_P(.)) = gr^w \mathrm{im}(\sigma) \oplus gr^w \ker(\pi) \ . \tag{1}$$

On the other hand there is the natural decomposition

$$\begin{aligned} &\mathrm{Ind}^{M'}_{M'\cap\dot{w}P\dot{w}^{-1}}(\dot{w}_*(r_{M,M\cap\dot{w}^{-1}P'\dot{w}}(.))) \\ &\qquad = \mathrm{Ind}^{M'}_{M'\cap\dot{w}P\dot{w}^{-1}}(\dot{w}_*(\mathrm{im}(\sigma))) \oplus \mathrm{Ind}^{M'}_{M'\cap\dot{w}P\dot{w}^{-1}}(\dot{w}_*(\ker(\pi))) \end{aligned} \tag{2}$$

where σ and π denote here our section and corresponding projection for the Jacquet functor $r_{M,M\cap\dot{w}^{-1}P'\dot{w}}$.

Lemma 7.3: *The map $p_{\dot{w}}$ transforms the decomposition* (1) *into the decomposition* (2)*.*

Proof: This is reduced, by the technique in the proof of Prop. 7.2, to the case of an admissible Y in $\mathcal{M}^t(M)$. In this case the claim follows from a comparison of the eigencharacters of $Z_{M'}$ on both sides (compare the proof of [Wal] III.3.3).

We now introduce the composed map

$$\begin{aligned} p^t_{\dot{w}} : gr^w(r^t_{G,P'} \circ \mathrm{Ind}^G_P)(.) &\xrightarrow{\sigma} gr^w(r_{G,P'} \circ \mathrm{Ind}^G_P)(.) \\ &\xrightarrow{p_{\dot{w}}} \mathrm{Ind}^{M'}_{M'\cap\dot{w}P\dot{w}^{-1}}(\dot{w}_*(r_{M,M\cap\dot{w}^{-1}P'\dot{w}}(.))) \\ &\longrightarrow \mathrm{Ind}^{M'}_{M'\cap\dot{w}P\dot{w}^{-1}}(\dot{w}_*(r^t_{M,M\cap\dot{w}^{-1}P'\dot{w}}(.))) \end{aligned}$$

which obviously is a $\mathcal{H}(M')$-equivariant natural transformation.

Theorem 7.4: *The map $p^t_{\dot w}$ induces a natural isomorphism*

$$gr^w(r^t_{G,P'} \circ \mathrm{Ind}^G_P) \xrightarrow{\simeq} \mathrm{Ind}^{M'}_{M'\cap\dot wP\dot w^{-1}} \circ \dot w_* \circ r^t_{M,M\cap\dot w^{-1}P'\dot w}$$

of functors from $\mathcal{M}^t(M)$ to $\mathcal{M}^t(M')$.

Proof: As a consequence of the Geometric Lemma 7.1 and Lemma 7.3 the map in question is bijective. It remains to show that it is $\mathcal{S}(M')$-equivariant which by proceeding one more time as in the proof of Prop. 7.2 is reduced to an admissible Y in $\mathcal{M}^t(M)$. In this case it is a direct consequence of [Wal] III.3.3 and [SSZ] App. Cor. 2.

8. The Plancherel isomorphism

For objects we write the decomposition in Thm. 4.1 as

$$V = \bigoplus_{\Theta} V_\Theta \ .$$

It can be applied, in particular, to $\mathcal{S}(G)$ as a left $\mathcal{S}(G)$-module giving the decomposition into right ideals

$$\mathcal{S}(G) = \bigoplus_{\Theta} \mathcal{S}(G)_\Theta \ .$$

Using the central idempotents which define these decompositions one easily checks that:

– $\mathcal{S}(G)_\Theta$ is a 2-sided ideal in $\mathcal{S}(G)$;

– $\mathcal{S}(G)_{\Theta'} * V_\Theta = 0$ whenever $\Theta' \neq \Theta$.

The purpose of the Plancherel isomorphism is an explicit computation of these 2-sided ideals in spectral terms. For any nondegenerate $\mathcal{S}(G)$-module V the space $\mathrm{End}_{\mathbb{C}}(V)$ of $\mathbb{C}$-linear endomorphisms of V naturally is, by composition with the $\mathcal{S}(G)$-action on the source and on the target, an $(\mathcal{S}(G), \mathcal{S}(G))$-bimodule. An endomorphism in the nondegenerate part of this bimodule will be called smooth. In other words, an endomorphism $A : V \longrightarrow V$ is smooth if there is a compact open subgroup $U \subseteq G$ such that

$$A(\epsilon_U * v) = \epsilon_U * A(v) = A(v) \qquad \text{for any } v \in V \ .$$

We let $\mathrm{End}^\infty_{\mathbb{C}}(V)$ denote the (nonunital) subring of all smooth endomorphisms.

We now fix again a component Θ of the Harish Chandra spectrum $\Omega^t(G)$. But we consider at first an arbitrary finitely generated projective generator Π of the component category $\mathcal{M}^t(\Theta)$. Let

$$\Lambda := \mathrm{End}_{\mathcal{S}(G)}(\Pi) \ .$$

Then (cf. [Pop] Cor. 3.7.4) the functors

$$\begin{array}{ccc} \mathcal{M}^t(\Theta) & \xrightarrow{\sim} & \mathcal{M}^r(\Lambda) \\ V & \longmapsto & \mathrm{Hom}_{\mathcal{S}(G)}(\Pi, V) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{M}^r(\Lambda) & \xrightarrow{\sim} & \mathcal{M}^t(\Theta) \\ \mathcal{X} & \longmapsto & \mathcal{X} \otimes_\Lambda \Pi \end{array}$$

are quasi-inverse equivalences of categories. The $\mathcal{S}(G)$-action on Π obviously defines a homomorphism of rings

$$\mathcal{S}(G) \longrightarrow \mathrm{End}^\infty_\Lambda(\Pi) := \mathrm{End}^\infty_{\mathbb{C}}(\Pi) \cap \mathrm{End}_\Lambda(\Pi)$$

which factorizes through $\mathcal{S}(G)_\Theta$.

Proposition 8.1: $\mathcal{S}(G)_\Theta \xrightarrow{\cong} \mathrm{End}^\infty_\Lambda(\Pi)$.

Proof: Let $U \subseteq G$ be an arbitrary compact open subgroup. By the above equivalences of categories we have the natural isomorphism

$$(\mathcal{S}(G) * \epsilon_U)_\Theta \cong \mathrm{Hom}_{\mathcal{S}(G)}(\Pi, (\mathcal{S}(G) * \epsilon_U)_\Theta) \otimes_\Lambda \Pi\ .$$

Since the evaluation in ϵ_U gives an isomorphism $\mathrm{Hom}_{\mathcal{S}(G)}(\mathcal{S}(G) * \epsilon_U, V) \cong \epsilon_U * V$ which is exact in V we see that $\mathcal{S}(G) * \epsilon_U$ is a finitely generated projective object in $\mathcal{M}^t(G)$. Hence $(\mathcal{S}(G) * \epsilon_U)_\Theta$ is finitely generated projective in $\mathcal{M}^t(\Theta)$ and $\mathcal{Y} := \mathrm{Hom}_{\mathcal{S}(G)}(\Pi, (\mathcal{S}(G) * \epsilon_U)_\Theta)$ is finitely generated projective in $\mathcal{M}^r(\Lambda)$. It follows in particular that $\mathcal{Y}$ is a reflexive Λ-module, i.e., setting $\mathcal{Y}^\sharp := \mathrm{Hom}_\Lambda(\mathcal{Y}, \Lambda)$ the natural map $\mathcal{Y} \xrightarrow{\cong} \mathcal{Y}^{\sharp\sharp}$ is an isomorphism. We therefore obtain

$$(\mathcal{S}(G) * \epsilon_U)_\Theta \cong \mathcal{Y} \otimes_\Lambda \Pi \cong \mathcal{Y}^{\sharp\sharp} \otimes_\Lambda \Pi \cong \mathrm{Hom}_\Lambda(\mathcal{Y}^\sharp, \Pi)\ .$$

On the other hand we have

$$\begin{aligned} \mathcal{Y}^\sharp &= \mathrm{Hom}_\Lambda(\mathcal{Y}, \Lambda) \\ &= \mathrm{Hom}_\Lambda(\mathrm{Hom}_{\mathcal{S}(G)}(\Pi, (\mathcal{S}(G) * \epsilon_U)_\Theta), \mathrm{Hom}_{\mathcal{S}(G)}(\Pi, \Pi)) \\ &\cong \mathrm{Hom}_{\mathcal{S}(G)}((\mathcal{S}(G) * \epsilon_U)_\Theta, \Pi) \\ &= \mathrm{Hom}_{\mathcal{S}(G)}(\mathcal{S}(G) * \epsilon_U, \Pi) \\ &\cong \epsilon_U * \Pi\ . \end{aligned}$$

We leave it to the reader to check that the resulting isomorphism

$$(\mathcal{S}(G) * \epsilon_U)_\Theta \cong \mathrm{Hom}_\Lambda(\mathcal{Y}^\sharp, \Pi) \cong \mathrm{Hom}_\Lambda(\epsilon_U * \Pi, \Pi)$$

is the one given by the action of $\mathcal{S}(G) * \epsilon_U$. Our assertion follows from this by passing to the limit with respect to a sequence of shrinking subgroups U, where on the right hand side we use the maps $f \in \mathrm{Hom}_\Lambda(\epsilon_U * \Pi, \Pi) \mapsto f \circ \epsilon_U \in \mathrm{Hom}_\Lambda(\epsilon_{U'} * \Pi, \Pi)$ if $U' \subseteq U$.

Remark: The limit argument in the last step of the previous proof becomes more transparent if one notes that $\epsilon_U * \Pi$ is finitely generated and projective as a (left) Λ-module. This directly implies that any map in $\mathrm{Hom}_\Lambda(\epsilon_U * \Pi, \Pi)$ is smooth as an endomorphism of Π. By the first formula in the previous proof we find elements $\pi_1, \ldots, \pi_m \in \epsilon_U * \Pi$ and $q_1, \ldots, q_m \in \mathrm{Hom}_{\mathcal{S}(G)}(\Pi, (\mathcal{S}(G) * \epsilon_U)_\Theta)$ such that $\sum_{i=1}^m q_i(\pi_i) = \epsilon_{U,\Theta} :=$ Θ-component of ϵ_U. Then the Λ-module maps

$$\begin{array}{rcl} \epsilon_U * \Pi & \longrightarrow & \Lambda^m \\ \pi & \longmapsto & (q_1(.)\pi, \ldots, q_m(.)\pi) \end{array}$$

and

$$\begin{array}{rcl} \Lambda^m & \longrightarrow & \epsilon_U * \Pi \\ (\lambda_1, \ldots, \lambda_m) & \longmapsto & \sum_{i=1}^m \lambda_i(\pi_i) \end{array}$$

exhibit $\epsilon_U * \Pi$ as a a direct summand of a finitely generated free Λ-module.

For the projective generators Π which we have constructed in this paper we will be able to compute the ring $\mathrm{End}_\Lambda^\infty(\Pi)$ explicitly. Let therefore $P \subseteq G$ be a parabolic subgroup with Levi factor M and let $\Theta_0 \subseteq \Omega^t(M)$ be a discrete component such that $\iota_M(\Theta_0) = \Theta$. For an appropriate irreducible discrete series representation E of M the $\mathcal{S}(M)$-module

$$\Pi_0 := C^\infty(X^1_{nr}(M), E)$$

is a projective generator of $\mathcal{M}^t(\Theta_0)$ and

$$\Pi := \mathrm{Ind}_P^G(\Pi_0) \ ,$$

by Prop. 5.1 and Cor. 5.3, is a finitely generated projective generator of $\mathcal{M}^t(\Theta)$. Induction of operators defines a unital ring homomorphism

$$\Lambda_0 := \mathrm{End}_{\mathcal{S}(M)}(\Pi_0) \longrightarrow \Lambda := \mathrm{End}_{\mathcal{S}(G)}(\Pi) \ .$$

In a first step we will compute the ring $\mathrm{End}_{\Lambda_0}^\infty(\Pi)$ where, as a general piece of notation, we put

$$\mathrm{End}_{\Lambda'}^\infty(\Pi) := \mathrm{End}_{\mathbb{C}}^\infty(\Pi) \cap \mathrm{End}_{\Lambda'}(\Pi)$$

for any ring Λ' mapping to Λ. Let us recall from the proof of Prop. 3.7 what we know about the ring Λ_0. First of all there is the injective unital ring homomorphism

$$\begin{array}{rcl} \mu : C^\infty(X^1_{nr}(M)) & \longrightarrow & \Lambda_0 \\ f & \longmapsto & \mu_f \ . \end{array}$$

Secondly, for any $\chi \in X^1_{nr}(M)_E := \{\chi \in X^1_{nr}(M) : E_\chi \cong E\}$ we have a specific automorphism $A_\chi \in \Lambda_0^\times$ (uniquely determined up to a scalar). They satisfy:

1) There is an $\alpha_\chi \in \mathrm{End}_{\mathcal{S}(M^0)}(E)^\times$ such that

$$A_\chi(f \otimes v) = f(\chi.) \otimes \alpha_\chi(v) \qquad \text{for any } f \otimes v \in C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} E = \Pi_0 \ .$$

2) $A_\chi \circ \mu_f = \mu_{f(\chi.)} \circ A_\chi$.

3) The set $\{A_\chi\}_\chi$ is a basis of Λ_0 as a (left) $C^\infty(X^1_{nr}(M))$-module (via μ).

4) $A_\chi \circ A_{\chi'} = c(\chi, \chi') \cdot A_{\chi\chi'}$ for some constant $c(\chi, \chi') \in \mathbb{C}^\times$.

Fixing a good maximal compact subgroup $K \subseteq G$ we may identify

$$\begin{aligned} \Pi &= \mathrm{Ind}_P^G(C^\infty(X^1_{nr}(M), E)) \\ &= \mathrm{Ind}_{P\cap K}^K(C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} E) \\ &= C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} \mathrm{Ind}_{P\cap K}^K(E) \ . \end{aligned}$$

The Λ_0-action on Π (via the induction of operators) is given on the last term as follows:

– The subring $C^\infty(X^1_{nr}(M))$ acts in the obvious way through multiplication on the first factor.

– $\mathrm{Ind}_P^G(A_\chi)(f \otimes F) = f(\chi.) \otimes \mathrm{Ind}_{P\cap K}^K(\alpha_\chi)(F)$.

Moreover this identification leads to another identification

$$\begin{aligned} \mathrm{End}^\infty_{C^\infty(X^1_{nr}(M))}(\Pi) &= \mathrm{End}^\infty_{C^\infty(X^1_{nr}(M))}(C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} \mathrm{Ind}_{P\cap K}^K(E)) \\ &= C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} \mathrm{End}^\infty_{\mathbb{C}}(\mathrm{Ind}_{P\cap K}^K(E)) \ . \end{aligned}$$

Since $\mathrm{Ind}_P^G(A_\chi)$ is an automorphism conjugation by $\mathrm{Ind}_P^G(A_\chi)$ on $\mathrm{End}^\infty_{\mathbb{C}}(\Pi)$ is well defined. Because of 4) this conjugation actually defines an action of the group $X^1_{nr}(M)_E$ on $\mathrm{End}^\infty_{\mathbb{C}}(\Pi)$. We obviously have

$$\mathrm{End}^\infty_{\Lambda_0}(\Pi) = \mathrm{End}^\infty_{C^\infty(X^1_{nr}(M))}(\Pi) \cap \mathrm{End}^\infty_{\mathbb{C}}(\Pi)^{X^1_{nr}(M)_E}$$

where the superscript $X^1_{nr}(M)_E$ on the right hand side indicates, as usual, the subspace of $X^1_{nr}(M)_E$-fixed vectors. As a consequence of 2) the action of $X^1_{nr}(M)_E$ on $\mathrm{End}^\infty_{\mathbb{C}}(\Pi)$ in fact respects the subspace $\mathrm{End}^\infty_{C^\infty(X^1_{nr}(M))}(\Pi)$ so that we may simply write

$$\begin{aligned} \mathrm{End}^\infty_{\Lambda_0}(\Pi) &= \mathrm{End}^\infty_{C^\infty(X^1_{nr}(M))}(\Pi)^{X^1_{nr}(M)_E} \\ &= [C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} \mathrm{End}^\infty_{\mathbb{C}}(\mathrm{Ind}_{P\cap K}^K(E))]^{X^1_{nr}(M)_E} \ . \end{aligned}$$

Note that the action of $\chi \in X^1_{nr}(M)_E$ on $C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} \mathrm{End}^\infty_{\mathbb{C}}(\mathrm{Ind}^K_{P\cap K}(E))$ is given by

$$f \otimes B \longmapsto f(\chi.) \otimes \mathrm{Ind}^K_{P\cap K}(\alpha_\chi) \circ B \circ \mathrm{Ind}^K_{P\cap K}(\alpha_\chi)^{-1} \ .$$

In the second step we want to understand Λ as a (left) Λ_0-module. By adjunction we have

$$\Lambda = \mathrm{End}_{\mathcal{S}(G)}(\mathrm{Ind}^G_P(\Pi_0)) = \mathrm{Hom}_{\mathcal{S}(M)}(r^t_{G,P} \circ \mathrm{Ind}^G_P(\Pi_0), \Pi_0) \ .$$

The filtration on $r^t_{G,P} \circ \mathrm{Ind}^G_P(\Pi_0)$ constructed in the last section, by Thm. 7.4, satisfies

$$gr^w(r^t_{G,P} \circ \mathrm{Ind}^G_P(\Pi_0)) \cong \mathrm{Ind}^M_{M\cap \dot{w}P\dot{w}^{-1}}(\dot{w}_*(r^t_{M,M\cap\dot{w}^{-1}P\dot{w}}(\Pi_0))) \ .$$

By Lemma 4.2 the right hand side is zero if $M \not\subseteq \dot{w}^{-1}P\dot{w}$ which is equivalent to $M \neq \dot{w}^{-1}M\dot{w}$. So we are left with representatives $\dot{w} \in N_G(M_0) \cap N_G(M)$. We recall that w runs over the representatives of minimal length for the double cosets in $W^M \setminus W^G/W^M$ where the Weyl groups are formed with respect to a fixed choice of minimal Levi subgroup $M_0 \subseteq M$. One checks that the obvious map

$$[N_G(M_0) \cap N_G(M)]/N_M(M_0) \xrightarrow{\cong} N_G(M)/M$$

is an isomorphism of groups. Hence the associated graded object of our filtration on $r^t_{G,P} \circ \mathrm{Ind}^G_P(\Pi_0)$ is

$$\bigoplus_{w\in N_G(M)/M} \dot{w}_*\Pi_0 \ .$$

Since all filtration steps are projective $\mathcal{S}(M)$-modules this filtration in fact splits and therefore induces a filtration $Fil^w\Lambda$ on Λ by left Λ_0-modules such that

$$gr^w\Lambda \cong \mathrm{Hom}_{\mathcal{S}(M)}(\dot{w}_*\Pi_0, \Pi_0) \ .$$

Obviously $\mathrm{Hom}_{\mathcal{S}(M)}(\dot{w}_*\Pi_0, \Pi_0) = 0$ if $w\Theta_0 \neq \Theta_0$. The filtration $Fil^w\Lambda$ therefore can be viewed as indexed by $w \in W(\Theta_0)$. We now introduce the set

$$\Gamma_E := \{(\chi, w) \in X^1_{nr}(M) \times W(\Theta_0) : \dot{w}_*E \cong E_\chi\} \ .$$

One easily checks that Γ_E in fact is a subgroup of the semidirect product of $X^1_{nr}(M)$ by $W(\Theta_0)$ (with respect to the natural action of the latter on the former). For any $(\chi, w) \in \Gamma_E$ we fix an isomorphism $\alpha_{\chi,w} \in \mathrm{Hom}_{\mathcal{S}(M)}(\dot{w}_*E, E_\chi)$ and obtain the isomorphism of $\mathcal{S}(M)$-modules

$$\begin{array}{rcl} A_{\chi,w} : \dot{w}_*(\mathcal{S}(M) \otimes_{\mathcal{S}(M^0)} E) & \xrightarrow{\cong} & \mathcal{S}(M) \otimes_{\mathcal{S}(M^0)} E \\ \phi \otimes v & \longmapsto & {}^{\dot{w}}\phi^{\dot{w}^{-1}} \otimes \alpha_{\chi,w}(v) \ . \end{array}$$

Using the isomorphism I in Cor. 3.5.i this leads to the $\mathcal{S}(M)$-equivariant isomorphism

$$\begin{array}{rcl} A_{\chi,w} : \dot{w}_*\Pi_0 = \dot{w}_*(C^\infty(X^1_{nr}(M), E)) & \xrightarrow{\cong} & C^\infty(X^1_{nr}(M), E) = \Pi_0 \\ \widetilde{f} & \longmapsto & [\chi' \mapsto \alpha_{\chi,w}(\widetilde{f}((\chi^{-1}\chi')(w.w^{-1})))]. \end{array}$$

Since $A_{\chi,1} = A_{\chi^{-1}}$ and since $A_{\chi,1} \circ A_{\chi',w}$ coincides with $A_{\chi\chi',w}$ up to a nonzero constant it follows from 3) that the $A_{\chi,w}$, for any given $w \in W(\Theta_0)$, form a basis of $\mathrm{Hom}_{\mathcal{S}(M)}(\dot{w}_*\Pi_0, \Pi_0)$ as a (left) $C^\infty(X^1_{nr}(M))$-module. We emphasize that $\dot{w}_*\Pi_0$ and Π_0 have the same underlying $C^\infty(X^1_{nr}(M))$-module (cf. the paragraph before Prop. 3.7); on $\dot{w}_*\Pi_0$ only the $\mathcal{S}(M)$-action is changed by the conjugation by $\dot{w}^{-1}$ on M.

To pass from $gr^w\Lambda$ to Λ we need the theory of intertwining operators. This requires the introduction of poles as we will describe subsequently. Let $\mathcal{O} := \mathcal{O}(X_{nr}(M))$ denote the subring in $C^\infty(X^1_{nr}(M))$ of algebraic functions on the complex algebraic torus $X_{nr}(M)$ of unramified characters of M, and let $\mathcal{K} := \mathcal{K}(X_{nr}(M))$ be the field of fractions of $\mathcal{O}$. Since $\mathcal{O}$ acts on Π by $\mathcal{S}(G)$-equivariant endomorphisms the scalar extension $\mathcal{K} \otimes_{\mathcal{O}} \Pi$ carries a natural nondegenerate $\mathcal{S}(G)$-action through the second factor.

Proposition 8.2: *For any $w \in W(\Theta_0)$ there is an $\mathcal{S}(G)$-equivariant and $\mathcal{K} \otimes_{\mathcal{O}} C^\infty(X^1_{nr}(M))$-linear isomorphism*

$$J_w : \mathcal{K} \otimes_{\mathcal{O}} \Pi = \mathcal{K} \otimes_{\mathcal{O}} \mathrm{Ind}^G_P(\Pi_0) \xrightarrow{\cong} \mathcal{K} \otimes_{\mathcal{O}} \mathrm{Ind}^G_{\dot{w}^{-1}P\dot{w}}(\Pi_0) = \mathcal{K} \otimes_{\mathcal{O}} \mathrm{Ind}^G_P(\dot{w}_*\Pi_0)$$

which is natural in E and such that the diagram

$$\begin{array}{ccccc} r^t_{G,P}(\Pi) & \xrightarrow{r^t_{G,P}(J_w)} & r^t_{G,P}(\mathcal{K} \otimes_{\mathcal{O}} \mathrm{Ind}^G_P(\dot{w}_*\Pi_0)) & \xrightarrow{=} & r^t_{G,P}(\mathrm{Ind}^G_P(\mathcal{K} \otimes_{\mathcal{O}} \dot{w}_*\Pi_0)) \\ \uparrow \subseteq & & & & \downarrow adjunction \\ Fil^w r^t_{G,P}(\Pi) & \xrightarrow{p^t_{\dot{w}}} & \dot{w}_*\Pi_0 & \xrightarrow{\subseteq} & \mathcal{K} \otimes_{\mathcal{O}} \dot{w}_*\Pi_0 \end{array}$$

is commutative.

Proof: This is a streamlined version of the corresponding considerations in [Wal]. We first recall that, by Lemma 7.3 and Thm. 7.4, we have the commutative diagram

$$\begin{array}{ccc} Fil^w \mathrm{Ind}^G_P(\Pi_0) & \longrightarrow & Fil^w r^t_{G,P}(\Pi) \\ & {}_{p_{\dot{w}}}\searrow \quad \swarrow_{p^t_{\dot{w}}} & \\ & \dot{w}_*\Pi_0 & \end{array}$$

with the canonical projection map in the top row and with

$$p_{\dot{w}}(F) = \int\limits_{N/N\cap\dot{w}P\dot{w}^{-1}} F(n\dot{w})dn = \int\limits_{N/N\cap\dot{w}N\dot{w}^{-1}} F(n\dot{w})dn$$

(we refer to [BZ] Prop. 6.1(b) for the identity $N\cap\dot{w}P\dot{w}^{-1} = N\cap\dot{w}N\dot{w}^{-1}$). All three maps in the diagram visibly are $C^\infty(X^1_{nr}(M))$-linear.

As before let $\Xi \subseteq X^1_{nr}(M)$ denote the subset of all characters such that $\dot{w}_*E_\chi \not\cong E_\chi$ for any nontrivial $w \in N_G(M)/M$. It is shown in [Wal] IV.2.2 that, for $\chi \in \Xi$, we have the well defined G-equivariant isomorphism

$$J^\chi_w : \mathrm{Ind}^G_P(E_\chi) \xrightarrow{\cong} \mathrm{Ind}^G_P((\dot{w}_*E_\chi)^{\approx}) = \mathrm{Ind}^G_P(\dot{w}_*E_\chi)$$

given by regular extension of absolutely convergent integrals

$$F \longmapsto [g \mapsto (\ell \mapsto \int\limits_{N/N\cap\dot{w}N\dot{w}^{-1}} \ell(F(gn\dot{w}))dn)]\ .$$

Since both sides are admissible tempered G-representations J^χ_w even is an isomorphism of $\mathcal{S}(G)$-modules. Obviously the composed map

$$r^t_{G,P}(\mathrm{Ind}^G_P(E_\chi)) \xrightarrow{r^t_{G,P}(J^\chi_w)} r^t_{G,P}(\mathrm{Ind}^G_P(\dot{w}_*E_\chi)) \xrightarrow{adjunction} \dot{w}_*E_\chi$$

restricts to the map $p^t_{\dot{w}}$ on $Fil^w r^t_{G,P}(\mathrm{Ind}^G_P(E_\chi))$. Most importantly, [Wal] IV.1.1 says that if we identify $\mathrm{Ind}^G_P(E_\chi)$ via restriction of functions with $\mathrm{Ind}^K_{P\cap K}(E)$ then, for any $F \in \mathrm{Ind}^K_{P\cap K}(E)$, there are elements $f_0, f_1, \ldots, f_r \in \mathcal{O}$ with $f_0 \neq 0$ and $F_1, \ldots, F_r \in \mathrm{Ind}^K_{P\cap K}(\dot{w}_*E)$ such that we have

$$(1) \qquad f_0(\chi)\cdot J^\chi_w(F) = \sum_{i=1}^r f_i(\chi)\cdot F_i \qquad \text{for any } \chi \in \Xi\,.$$

Hence we define J_w to be the map

$$\begin{array}{c} \Pi = C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} \mathrm{Ind}^K_{P\cap K}(E) \\ \Big\downarrow {\scriptstyle f\otimes F \mapsto \sum_{i=1}^r \frac{1}{f_0}\otimes f f_i \otimes F_i} \\ \mathcal{K}\otimes_{\mathcal{O}} C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} \mathrm{Ind}^K_{P\cap K}(\dot{w}_*E) = \mathcal{K}\otimes_{\mathcal{O}} \mathrm{Ind}^G_P(\dot{w}_*\Pi_0)\ . \end{array}$$

For convenience we assume from now on that the representatives $\dot{w}$ are chosen to lie in K. That J_w is well defined, is a linear map, and is $\mathcal{S}(G)$-equivariant is a consequence of the following general argument. In all three cases we need to show that specific elements H in the target of J_w vanish. We write $H = \frac{1}{f_0} \otimes F_0$ with $0 \neq f_0 \in \mathcal{O}$ and $F_0 \in \mathrm{Ind}_P^G(\dot{w}_*\Pi_0)$. The complement $D(f_0)$ of the zero set of f_0 is a Zariski open subset of $X_{nr}(M)$ and $\mathcal{O}(D(f_0)) = \mathcal{O}_{f_0}$. Since $C^\infty(X_{nr}^1(M))$ is flat over $\mathcal{O}$ (see the Appendix) the subring $\mathcal{O}$ contains no zero divisors of $C^\infty(X_{nr}^1(M))$ and we have

$$H \in \mathcal{O}_{f_0} \otimes_{\mathcal{O}} \mathrm{Ind}_P^G(\dot{w}_*\Pi_0) \subseteq \mathcal{K} \otimes_{\mathcal{O}} \mathrm{Ind}_P^G(\dot{w}_*\Pi_0) \ .$$

For $\chi \in D(f_0) \cap X_{nr}^1(M)$ the map $\mathrm{Ind}_P^G(\dot{w}_*\Pi_0) \longrightarrow \mathrm{Ind}_P^G(\dot{w}_*E_\chi)$ of evaluation in χ, which we write as $F \mapsto F(\chi)$, extends in an obvious way to a map

$$\mathcal{O}_{f_0} \otimes_{\mathcal{O}} \mathrm{Ind}_P^G(\dot{w}_*\Pi_0) \longrightarrow \mathrm{Ind}_P^G(\dot{w}_*E_\chi) \ .$$

It sends H to $H(\chi) := f_0(\chi)^{-1} \cdot F_0(\chi)$. Since $D(f_0) \cap \Xi$ is dense in $X_{nr}^1(M)$ any element in $\mathrm{Ind}_P^G(\dot{w}_*\Pi_0)$ and a fortiori our H is determined by its evaluations in all $\chi \in D(f_0) \cap \Xi$. But due to the equation (1) the specific nature of H together with the fact that any J_w^χ is a well defined $\mathcal{S}(G)$-equivariant homomorphism implies that $H(\chi) = 0$.

An entire analogous reasoning, but this time applied to specific elements in $\mathcal{K} \otimes_{\mathcal{O}} \dot{w}_*\Pi_0$ establishes the commutativity of the asserted diagram.

By construction the map J_w is $C^\infty(X_{nr}^1(M))$-linear. It remains to check that the extended $\mathcal{K} \otimes_{\mathcal{O}} C^\infty(X_{nr}^1(M))$-linear map

$$J_w : \mathcal{K} \otimes_{\mathcal{O}} C^\infty(X_{nr}^1(M)) \otimes_{\mathbb{C}} \mathrm{Ind}_{P\cap K}^K(E) \longrightarrow \mathcal{K} \otimes_{\mathcal{O}} C^\infty(X_{nr}^1(M)) \otimes_{\mathbb{C}} \mathrm{Ind}_{P\cap K}^K(\dot{w}_*E)$$

is an isomorphism. By G-equivariance it suffices to establish the bijectivity of the restricted maps

(2)
$$\begin{array}{c} \mathcal{K} \otimes_{\mathcal{O}} C^\infty(X_{nr}^1(M)) \otimes_{\mathbb{C}} \mathrm{Ind}_{P\cap K}^K(E)^U \\ \Big\downarrow {\scriptstyle J_w} \\ \mathcal{K} \otimes_{\mathcal{O}} C^\infty(X_{nr}^1(M)) \otimes_{\mathbb{C}} \mathrm{Ind}_{P\cap K}^K(\dot{w}_*E)^U \end{array}$$

for any compact open subgroup $U \subseteq K$. Since $E \cong \dot{w}_*E$ as a $P \cap K$-representation we have

$$\dim_{\mathbb{C}} \mathrm{Ind}_{P\cap K}^K(E)^U = \dim_{\mathbb{C}} \mathrm{Ind}_{P\cap K}^K(\dot{w}_*E)^U < \infty \ .$$

Hence (2) is a linear map between finitely generated free $\mathcal{K} \otimes_{\mathcal{O}} C^\infty(X_{nr}^1(M))$-modules of the same rank. It therefore descends to a map

(3) $$C^\infty(X_{nr}^1(M))_{f_0} \otimes_{\mathbb{C}} \mathrm{Ind}_{P\cap K}^K(E)^U \longrightarrow C^\infty(X_{nr}^1(M))_{f_0} \otimes_{\mathbb{C}} \mathrm{Ind}_{P\cap K}^K(\dot{w}_*E)^U$$

where $C^\infty(X^1_{nr}(M))_{f_0}$ denotes the localization of the ring $C^\infty(X^1_{nr}(M))$ in an appropriate nonzero element $f_0 \in \mathcal{O}$ (which depends on U). Of course, in this we always can replace a given f_0 by a multiple of it. We will make use of this in the following way. The subset $\mathcal{U} \subseteq X_{nr}(M)$ of all characters such that $\dot{w}_* E_\chi \not\cong E_\chi$ for any nontrivial $w \in N_G(M)/M$ is Zariski open. We always choose f_0 in such a way that $D(f_0)$ is contained in $\mathcal{U}$. This has the effect that the maximal ideal space of the ring $C^\infty(X^1_{nr}(M))_{f_0}$ which is $X^1_{nr}(M) \cap D(f_0)$ (see below) is contained in Ξ. By the Nakayama lemma it suffices, for (3) to be an isomorphism, that it is an isomorphism after reduction modulo each maximal ideal of $C^\infty(X^1_{nr}(M))_{f_0}$. But since we are in Ξ the latter gives the isomorphisms J^χ_w restricted to the U-invariant vectors.

In the above proof we have used the following fact of which we include a proof for the convenience of the reader.

Remark 8.3: *The maximal ideal space of* $C^\infty(X^1_{nr}(M))$ *is* $X^1_{nr}(M)$.

Proof: By Gelfand's theorem $X^1_{nr}(M)$ is the maximal ideal space of the Banach algebra $C(X^1_{nr}(M))$ of all continuous functions on $X^1_{nr}(M)$. This bijection is realized by sending a point $\chi \in X^1_{nr}(M)$ to the maximal ideal $\mathfrak{m}_\chi := \{f \in C(X^1_{nr}(M)) : f(\chi) = 0\}$. Sending χ to $\mathfrak{m}_\chi \cap C^\infty(X^1_{nr}(M))$ then is an injection of $X^1_{nr}(M)$ into the maximal ideal space of $C^\infty(X^1_{nr}(M))$. Let now $\mathfrak{m}$ be an arbitrary maximal ideal in $C^\infty(X^1_{nr}(M))$. Since

$$C(X^1_{nr}(M))^\times \cap C^\infty(X^1_{nr}(M)) = C^\infty(X^1_{nr}(M))^\times$$

and since the units $C(X^1_{nr}(M))^\times$ form an open subset in $C(X^1_{nr}(M))$ we see that the closure $\overline{\mathfrak{m}}$ of $\mathfrak{m}$ in $C(X^1_{nr}(M))$ does not contain any units. Since, by the Stone-Weierstrass theorem, $C^\infty(X^1_{nr}(M))$ is dense in $C(X^1_{nr}(M))$ it follows that $\overline{\mathfrak{m}}$ is a proper ideal in $C(X^1_{nr}(M))$. Hence there is a $\chi \in X^1_{nr}(M)$ such that $\overline{\mathfrak{m}} \subseteq \mathfrak{m}_\chi$ which implies $\mathfrak{m} = \mathfrak{m}_\chi \cap C^\infty(X^1_{nr}(M))$.

For any $(\chi, w) \in \Gamma_E$ we now consider the composite $\mathcal{S}(G)$-equivariant isomorphism

$$\widetilde{A}_{\chi,w} := (\mathrm{id}_\mathcal{K} \otimes \mathrm{Ind}^G_P(A_{\chi,w})) \circ J_w : \mathcal{K} \otimes_\mathcal{O} \Pi \longrightarrow \mathcal{K} \otimes_\mathcal{O} \Pi$$

It satisfies the commutative diagram:

$$\begin{array}{ccccc}
r^t_{G,P}(\Pi) & & \xrightarrow{\ r^t_{G,P}(\widetilde{A}_{\chi,w})\ } & & r^t_{G,P}(\mathcal{K} \otimes_\mathcal{O} \Pi) \\
\subseteq \big\uparrow & & & & \big\downarrow \mathit{adjunction} \\
Fil^w r^t_{G,P}(\Pi) & \longrightarrow & \dot{w}_* \Pi_0 & \xrightarrow{\ A_{\chi,w}\ } & \mathcal{K} \otimes_\mathcal{O} \Pi_0
\end{array}$$

To formulate its linearity property with respect to $\mathcal{K}$ we introduce the automorphisms of rings

$$\begin{array}{rcl} \iota_{(\chi,w)} : \mathcal{R} & \xrightarrow{\cong} & \mathcal{R} \\ f & \longmapsto & [\chi' \mapsto f((\chi^{-1}\chi')(w.w^{-1}))] \ . \end{array}$$

for any of the rings $\mathcal{R} = C^\infty(X^1_{nr}(M))$, $\mathcal{O}$, $\mathcal{K}$, or $\mathcal{K} \otimes_{\mathcal{O}} C^\infty(X^1_{nr}(M))$. This defines a (left) action of the group Γ_E on $\mathcal{R}$. For any (left) $\mathcal{R}$-module Y we let $\iota^*_{(\chi,w)}Y$ denote its pullback along $\iota_{(\chi,w)}$. For any $C^\infty(X^1_{nr}(M))$-bimodule Y we introduce the sub-bimodule

$$Y_{(\chi,w)} := \{y \in Y : yf = \iota_{(\chi,w)}(f)y \ \text{ for any } f \in C^\infty(X^1_{nr}(M))\} \ .$$

One checks that

$$A_{\chi,w} \in \mathrm{Hom}_{(\mathcal{S}(M),C^\infty(X^1_{nr}(M)))}(\dot{w}_*\Pi_0, \iota^*_{(\chi,w)}\Pi_0) = \mathrm{Hom}_{\mathcal{S}(M)}(\dot{w}_*\Pi_0, \Pi_0)_{(\chi,w)} \ .$$

It follows that

$$\widetilde{A}_{\chi,w} \in \mathrm{Hom}_{(\mathcal{S}(G),\mathcal{K}\otimes_{\mathcal{O}} C^\infty(X^1_{nr}(M)))}(\mathcal{K} \otimes_{\mathcal{O}} \Pi, \iota^*_{(\chi,w)}(\mathcal{K} \otimes_{\mathcal{O}} \Pi)) \ .$$

Remark 8.4: *The map*

$$\begin{array}{rcl} \mathcal{K} \otimes_{\mathcal{O}} \Lambda & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{S}(G)}(\Pi, \mathcal{K} \otimes_{\mathcal{O}} \Pi) \\ k \otimes B & \longmapsto & [x \mapsto k \otimes B(x)] \end{array}$$

is an isomorphism of $(\mathcal{K} \otimes_{\mathcal{O}} C^\infty(X^1_{nr}(M)), C^\infty(X^1_{nr}(M))$*-bimodules.*

Proof: We note that the left, resp. right, $C^\infty(X^1_{nr}(M))$-module structure on $\Lambda = \mathrm{End}_{\mathcal{S}(G)}(\Pi)$ comes from the $C^\infty(X^1_{nr}(M))$-action on Π as the second, resp. first, entry. This makes clear the asserted linearity properties. The map in question is the inductive limit of the corresponding maps

$$\mathcal{K} \otimes_{\mathcal{O}} \mathrm{End}_{\mathcal{S}(G,U)}(\Pi^U) \longrightarrow \mathrm{Hom}_{\mathcal{S}(G,U)}(\Pi^U, \mathcal{K} \otimes_{\mathcal{O}} \Pi^U)$$

with U varying over the compact open subgroups of G. As a consequence of Prop. 5.1 the unital $\mathcal{S}(G,U)$-module Π^U is finitely generated and projective, provided U is sufficiently small, in which case the isomorphism is clear.

Remark 8.5: *We have*

$$(\mathcal{K} \otimes_{\mathcal{O}} \Lambda)_{(\chi,w)} \xrightarrow{\cong} \mathrm{Hom}_{(\mathcal{S}(G),\mathcal{K}\otimes_{\mathcal{O}} C^\infty(X^1_{nr}(M)))}(\mathcal{K} \otimes_{\mathcal{O}} \Pi, \iota^*_{(\chi,w)}(\mathcal{K} \otimes_{\mathcal{O}} \Pi)) \ .$$

We see that $\widetilde{A}_{\chi,w}$ can naturally be viewed as an element in $(\mathcal{K} \otimes_{\mathcal{O}} \Lambda)_{(\chi,w)}$.

Proposition 8.6: *i.* $\mathcal{K} \otimes_{\mathcal{O}} \Lambda = \bigoplus_{(\chi,w)\in\Gamma_E} (\mathcal{K} \otimes_{\mathcal{O}} \Lambda)_{(\chi,w)}$*;*

ii. $(\mathcal{K} \otimes_{\mathcal{O}} \Lambda)_{(\chi,w)} \cong (\mathcal{K} \otimes_{\mathcal{O}} gr^w \Lambda)_{(\chi,w)} \cong (\mathcal{K} \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{S}(M)}(\dot{w}_*\Pi_0, \Pi_0))_{(\chi,w)}$*;*

iii. $\mathcal{K} \otimes_{\mathcal{O}} \Lambda$ *is a free* $\mathcal{K} \otimes_{\mathcal{O}} C^\infty(X^1_{nr}(M))$*-module with basis* $\{\widetilde{A}_{\chi,w}\}_{(\chi,w)\in\Gamma_E}$*.*

Proof: This is immediate from the two facts that $\widetilde{A}_{\chi,w}$ maps to $A_{\chi,w}$ and that $A_{\chi,w}$ is a basis of $\mathrm{Hom}_{\mathcal{S}(M)}(\dot{w}_*\Pi_0, \Pi_0)_{(\chi,w)}$ as a (left) $C^\infty(X^1_{nr}(M))$-module.

Corollary 8.7: $\mathrm{End}^\infty_\Lambda(\Pi)$ *consists of all* $B \in \mathrm{End}^\infty_{C^\infty(X^1_{nr}(M))}(\Pi)$ *such that*

$$\widetilde{A}_{\chi,w} \circ B = B \circ \widetilde{A}_{\chi,w} \qquad \text{for any } (\chi, w) \in \Gamma_E$$

viewed in $\mathcal{K} \otimes_{\mathcal{O}} \mathrm{End}_{\mathbb{C}}(\Pi)$*.*

Proof: Since Π is free as a $C^\infty(X^1_{nr}(M))$-module and $\mathcal{O}$ contains no zero divisors of $C^\infty(X^1_{nr}(M))$ it follows that Π is $\mathcal{O}$-torsion free. This implies that

$$\mathrm{End}_{\mathbb{C}}(\Pi) \subseteq \mathcal{K} \otimes_{\mathcal{O}} \mathrm{End}_{\mathbb{C}}(\Pi)$$

and hence that commutation of elements in $\mathrm{End}^\infty_{C^\infty(X^1_{nr}(M))}(\Pi)$ with elements in Λ can be tested in $\mathcal{K} \otimes_{\mathcal{O}} \mathrm{End}_{\mathbb{C}}(\Pi)$.

Corollary 8.8: *For any two elements* $\gamma, \gamma' \in \Gamma_E$ *there is a function* $k_{\gamma,\gamma'} \in (\mathcal{K} \otimes_{\mathcal{O}} C^\infty(X^1_{nr}(M)))^\times$ *such that*

$$\widetilde{A}_\gamma \circ \widetilde{A}_{\gamma'} = k_{\gamma,\gamma'} \cdot \widetilde{A}_{\gamma\gamma'} \ .$$

Proof: Both, $\widetilde{A}_\gamma \circ \widetilde{A}_{\gamma'}$ and $\widetilde{A}_{\gamma\gamma'}$ are bases of the free rank one $\mathcal{K} \otimes_{\mathcal{O}} C^\infty(X^1_{nr}(M))$-module $(\mathcal{K} \otimes_{\mathcal{O}} \Lambda)_{\gamma\gamma'}$.

We of course have the inverse isomorphisms $\widetilde{A}_\gamma^{-1}$ as well. As a consequence of the last corollary the group Γ_E acts on $\mathrm{End}_{\mathcal{K}\otimes_{\mathcal{O}} C^\infty(X^1_{nr}(M))}(\mathcal{K} \otimes_{\mathcal{O}} \Pi)$ by

$$(\gamma, B) \longmapsto \widetilde{A}_\gamma^{-1} \circ B \circ \widetilde{A}_\gamma \ .$$

We also note that this action, in contrast to the individual operators J_w and $\widetilde{A}_\gamma$, does not depend on the particular choice of the representatives $\dot{w}$. Therefore Prop. 8.1 and Cor. 8.7 can be reformulated as follows where, as usual, the superscript Γ_E indicates the formation of the Γ_E-fixed elements.

Theorem 8.9: *We have*

$$\mathcal{S}(G)_\Theta \xrightarrow{\cong} \mathrm{End}^\infty_\Lambda(\Pi) = \mathrm{End}^\infty_{C^\infty(X^1_{nr}(M))}(\Pi) \cap \mathrm{End}_{\mathcal{K}\otimes_{\mathcal{O}} C^\infty(X^1_{nr}(M))}(\mathcal{K} \otimes_{\mathcal{O}} \Pi)^{\Gamma_E} \ .$$

By analytic considerations ([Wal] V.3.1) it can be shown that $\mathrm{End}^{\infty}_{C^\infty(X^1_{nr}(M))}(\Pi)$ in fact is Γ_E-invariant in $\mathrm{End}_{\mathcal{K}\otimes_{\mathcal{O}} C^\infty(X^1_{nr}(M))}(\mathcal{K}\otimes_{\mathcal{O}}\Pi)$. Hence we may rewrite the above Plancherel isomorphism as

$$\mathcal{S}(G)_\Theta \xrightarrow{\cong} \mathrm{End}^{\infty}_{C^\infty(X^1_{nr}(M))}(\Pi)^{\Gamma_E} = [C^\infty(X^1_{nr}(M)) \otimes_{\mathbb{C}} \mathrm{End}^{\infty}_{\mathbb{C}}(\mathrm{Ind}^K_{P\cap K}(E))]^{\Gamma_E} \ .$$

In this form the right hand side can be viewed as a space of operator valued C^∞-functions on $X^1_{nr}(M)$. Observe that the group Γ_E acts on $X^1_{nr}(M)$ via

$$((\chi, w), \chi') \longmapsto \chi'(w^{-1}.w)\chi \ .$$

The corresponding quotient space $\Gamma_E \backslash X^1_{nr}(M)$ coincides with our earlier space $W(\Theta_0)\big\backslash(X^1_{nr}(M)/X^1_{nr}(M)_E) \xrightarrow{\simeq} \Theta$. Hence our description in Prop. 6.4 of the center of $\mathcal{M}^t(\Theta)$ is directly compatible with the above Plancherel isomorphism.

Appendix: Flatness of the ring of C^∞-functions

Let X be a C^∞-manifold which we assume to be paracompact and hence metrizable. Let $\mathcal{E}$ denote the sheaf of $\mathbb{C}$-valued C^∞-functions on X. For any $\mathcal{E}(X)$-module M we have the $\mathcal{E}$-module presheaf $\mathcal{M}$ on X defined by

$$\mathcal{M}(U) := M \otimes_{\mathcal{E}(X)} \mathcal{E}(U) \qquad \text{for } U \subseteq X \text{ open} \ .$$

Lemma A.1:

If the $\mathcal{E}(X)$-module M is finitely presented then $\mathcal{M}$ is a sheaf.

Proof: We write M as the cokernel

$$\mathcal{E}(X)^m \xrightarrow{\alpha_X} \mathcal{E}(X)^n \longrightarrow M \longrightarrow 0$$

of a homomorphism α_X between finitely generated free $\mathcal{E}(X)$-modules. The map α_X extends in an obvious way to a homomorphism $\alpha : \mathcal{E}^m \longrightarrow \mathcal{E}^n$ of $\mathcal{E}$-module sheaves. Define the $\mathcal{E}$-module sheaf $\mathcal{M} := \mathrm{coker}(\alpha)$. Since any $\mathcal{E}$-module sheaf $\mathcal{N}$ is soft ([GR] A.4.2 Satz 5) the section functor $\mathcal{N} \longmapsto \mathcal{N}(U)$, for any fixed open subset $U \subseteq X$, is exact ([GR] A.4.4). Hence in the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{E}(U)^m & \xrightarrow{\alpha_U} & \mathcal{E}(U)^n & \longrightarrow & \mathcal{M}(U) & \longrightarrow & 0 \\
\uparrow \cong & & \uparrow \cong & & \uparrow & & \\
\mathcal{E}(X)^m \otimes_{\mathcal{E}(X)} \mathcal{E}(U) & \longrightarrow & \mathcal{E}(X)^n \otimes_{\mathcal{E}(X)} \mathcal{E}(U) & \longrightarrow & M \otimes_{\mathcal{E}(X)} \mathcal{E}(U) & \longrightarrow & 0
\end{array}$$

the upper row is exact. The lower row is exact, too, by the right exactness of the tensor product. It follows that the broken vertical arrow is an isomorphism.

Now let $X = (S^1)^d$ be the d-dimensional compact real torus. By $\mathcal{O}(X)$ we denote the subring of complex Laurent polynomials in $\mathcal{E}(X)$.

Proposition A.2:

$\mathcal{E}(X)$ is flat over $\mathcal{O}(X)$.

Proof: It suffices to show that, for any ideal $I \subseteq \mathcal{O}(X)$, the map

$$M := I \otimes_{\mathcal{O}(X)} \mathcal{E}(X) \longrightarrow \mathcal{O}(X) \otimes_{\mathcal{O}(X)} \mathcal{E}(X) = \mathcal{E}(X)$$

is injective. Since $\mathcal{O}(X)$ is noetherian I is a finitely presented $\mathcal{O}(X)$-module so that M is a finitely presented $\mathcal{E}(X)$-module. If $\mathcal{M}$ denotes the sheaf for M as constructed above it suffices to show that the induced map $\mathcal{M} \longrightarrow \mathcal{E}$ of $\mathcal{E}$-module sheaves is injective. But this can be checked stalkwise. Let $x \in X$ be a point. The map $\mathcal{M}_x \longrightarrow \mathcal{E}_x$ between the stalks in x is the map

$$I \otimes_{\mathcal{O}(X)} \mathcal{E}_x \longrightarrow \mathcal{E}_x \ .$$

Let $\mathcal{O}_x$ denote the localization of $\mathcal{O}(X)$ in x. Since $\mathcal{O}_x$ is flat over $\mathcal{O}(X)$ the map

$$I \otimes_{\mathcal{O}(X)} \mathcal{O}_x \longrightarrow \mathcal{O}_x$$

is injective. Finally, according to [Mal] Ex. III.4.11 and Cor. VI.1.12, the ring $\mathcal{E}_x$ is flat over $\mathcal{O}_x$.

9. Some homological algebra

As was shown in [SSZ] App. Cor. 2 the full subcategory $\mathcal{M}^t_{adm}(G)$ of $\mathcal{M}^t(G)$ consisting of all objects which as smooth G-representations are admissible also can be viewed, via the forgetful functor, as a full subcategory of $\mathcal{M}(G)$. This means that

$$\mathrm{Hom}_{\mathcal{S}(G)}(V_1, V_2) = \mathrm{Hom}_{\mathcal{H}(G)}(V_1, V_2)$$

for any two V_1 and V_2 in $\mathcal{M}^t_{adm}(G)$. In this section, which is somewhat independent of the previous sections, we will investigate the question whether even

$$\mathrm{Ext}^*_{\mathcal{S}(G)}(V_1, V_2) = \mathrm{Ext}^*_{\mathcal{H}(G)}(V_1, V_2)$$

holds true. Note that both abelian categories, $\mathcal{M}^t(G)$ and $\mathcal{M}(G)$, have enough projective and injective objects so that the Ext-functors can be defined and computed in the usual multitude of ways. For simplicity we will always assume in this section that the center of G is finite.

We fix a V_1 in $\mathcal{M}^t_{adm}(G)$ which we assume for now to be of finite length. By [SS] Cor. II.3.2 we find a projective resolution

$$\ldots \longrightarrow Y_m \longrightarrow \ldots \longrightarrow Y_0 \longrightarrow V_1 \longrightarrow 0$$

of V_1 in $\mathcal{M}(G)$ where any of the projective $\mathcal{H}(G)$-modules Y_m is a finite direct sum of projective $\mathcal{H}(G)$-modules of the form

$$\text{c-Ind}_U^G(H)$$

with H a finite dimensional smooth representation of a compact open subgroup $U \subseteq G$ and where c-Ind denotes compact induction.

Lemma 9.1: *i.* $\mathcal{S}(G) \otimes_{\mathcal{H}(G)} V_1 \xrightarrow{\cong} V_1$*;*

ii. $\mathcal{S}(G) \otimes_{\mathcal{H}(G)} Y_. \longrightarrow V_1$ *is a projective resolution of* V_1 *in* $\mathcal{M}^t(G)$*.*

Proof: We have to show that the complex

$$\ldots \longrightarrow \mathcal{S}(G) \otimes_{\mathcal{H}(G)} Y_m \longrightarrow \ldots \longrightarrow \mathcal{S}(G) \otimes_{\mathcal{H}(G)} Y_0 \longrightarrow V_1 \longrightarrow 0$$

is exact. By [Mey] Thm. 28 (compare the proof of Thm. 39) its topological counterpart

$$\ldots \longrightarrow \mathcal{S}(G) \widehat{\otimes}_{\mathcal{H}(G)} Y_m \longrightarrow \ldots \longrightarrow \mathcal{S}(G) \widehat{\otimes}_{\mathcal{H}(G)} Y_0 \longrightarrow V_1 \longrightarrow 0$$

is known to be exact. Here $\widehat{\otimes}_{\mathcal{H}(G)}$ stands for the complete bornological tensor product. The spaces Y_m are given the finest bornology and $\mathcal{S}(G)$ carries the natural bornology coming from its structure as a topological algebra. We therefore are reduced to showing that the natural map

$$\mathcal{S}(G) \otimes_{\mathcal{H}(G)} Y_m \longrightarrow \mathcal{S}(G) \widehat{\otimes}_{\mathcal{H}(G)} Y_m$$

is bijective. By the particular structure of Y_m we are further reduced to considering the map

$$\mathcal{S}(G) \otimes_{\mathcal{H}(G)} \text{c-Ind}_U^G(H) \longrightarrow \mathcal{S}(G) \widehat{\otimes}_{\mathcal{H}(G)} \text{c-Ind}_U^G(H) \ .$$

Let $U_0 \subseteq U$ be an open normal subgroup which acts trivially on H. We then have

$$\text{c-Ind}_U^G(H) = (\mathcal{H}(G) * \epsilon_{U_0} \otimes_{\mathbb{C}} H)^{U/U_0}$$

where the U/U_0-fixed vectors are formed with respect to the diagonal action (with the right translation action on the first factor). Obviously the right hand side is the image of an $\mathcal{H}(G)$-equivariant projector on $\mathcal{H}(G) * \epsilon_{U_0} \otimes_{\mathbb{C}} H$. We therefore may compute

$$\begin{aligned}\mathcal{S}(G)\,\widehat{\otimes}_{\mathcal{H}(G)}\,\text{c-Ind}_U^G(H) &= \mathcal{S}(G)\,\widehat{\otimes}_{\mathcal{H}(G)}\,(\mathcal{H}(G) * \epsilon_{U_0} \otimes_{\mathbb{C}} H)^{U/U_0} \\ &= (\mathcal{S}(G)\,\widehat{\otimes}_{\mathcal{H}(G)}\,(\mathcal{H}(G) * \epsilon_{U_0} \otimes_{\mathbb{C}} H))^{U/U_0} \\ &= ((\mathcal{S}(G)\,\widehat{\otimes}_{\mathcal{H}(G)}\,\mathcal{H}(G) * \epsilon_{U_0}) \otimes_{\mathbb{C}} H)^{U/U_0} \\ &= (\mathcal{S}(G) * \epsilon_{U_0} \otimes_{\mathbb{C}} H)^{U/U_0}\ .\end{aligned}$$

But an exactly analogous computation also gives

$$\mathcal{S}(G) \otimes_{\mathcal{H}(G)} \text{c-Ind}_U^G(H) = (\mathcal{S}(G) * \epsilon_{U_0} \otimes_{\mathbb{C}} H)^{U/U_0}\ .$$

Proposition 9.2: *For any V in $\mathcal{M}^t_{adm}(G)$ we have $\mathcal{S}(G) \otimes_{\mathcal{H}(G)} V \xrightarrow{\cong} V$.*

Proof: The $\mathcal{H}(G)$-module V is the filtered inductive limit of its finitely generated $\mathcal{H}(G)$-submodules. On the one hand, these submodules being admissible and finitely generated are of finite length ([BeD] Rem. 3.12). On the other hand, by [SSZ] App. Cor. 2 they are in fact $\mathcal{S}(G)$-submodules. Hence the assertion follows from Lemma 9.1.i by a limit argument.

Proposition 9.3: *For any V_1 in $\mathcal{M}^t_{adm}(G)$ and any V_2 in $\mathcal{M}^t(G)$ we have*

$$\text{Ext}^*_{\mathcal{S}(G)}(V_1, V_2) = \text{Ext}^*_{\mathcal{H}(G)}(V_1, V_2)\ .$$

Proof: Suppose first that V_1 is of finite length. using Lemma 9.1 we compute

$$\begin{aligned}\text{Ext}^*_{\mathcal{H}(G)}(V_1, V_2) &= h^*(\text{Hom}_{\mathcal{H}(G)}(Y., V_2)) \\ &= h^*(\text{Hom}_{\mathcal{S}(G)}(\mathcal{S}(G) \otimes_{\mathcal{H}(G)} Y., V_2)) \\ &= \text{Ext}^*_{\mathcal{S}(G)}(V_1, V_2)\ .\end{aligned}$$

In the general case we write V_1 as in the proof of Prop. 9.2 as a filtered inductive limit of finite length $\mathcal{S}(G)$-modules X_α and use for both sides of the assertion the corresponding spectral sequence ([Jen] Thm. 4.2)

$$E_2^{r,s} = \varprojlim_{\alpha}{}^{(r)}\, \text{Ext}^s_{\ldots}(X_\alpha, V_2) \Longrightarrow \text{Ext}^{r+s}_{\ldots}(V_1, V_2)\ .$$

The E_2-terms are the same by the first step. Hence the abutments coincide as well.

The case of an infinite center can be dealt with in a completely analogous way by working in categories of modules on which the action of the connected center of G is fixed.

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