Appendix: Robba rings for compact p-adic Lie groups

by Peter Schneider

23.6.2010

The Robba ring is a fundamental tool in p-adic differential equations and in p-adic Galois representations. It is defined as a ring of certain infinite Laurent series in one variable over a p-adic field K. Conceptually it is related to the cyclotomic $\mathbb{Z}_p$-extension of K whose Galois group is isomorphic to the additive group of p-adic integers $G = \mathbb{Z}_p$. In fact, the Robba ring can be understood in terms of the completed group ring $\mathbb{Z}_p[[G]]$ by a process of localization and completion.

Recent developments in the theory of p-adic Galois representations require the use of more general compact p-adic Lie groups G. In particular G might be nonabelian like $G = GL_n(\mathbb{Z}_p)$. It then becomes a natural question whether an analog of the Robba ring exists in this situation, possibly being constructed out of the completed group ring $\mathbb{Z}_p[[G]]$. But $\mathbb{Z}_p[[G]]$, in general, will be noncommutative. Hence localization becomes too difficult a process. The idea of these notes grew out of the technique of algebraic microlocalization.

In the first section we will adapt microlocalization to the framework of p-adic Banach algebras. This means that ring filtrations are replaced by multiplicative nonarchimedean norms. More importantly, for the application we have in mind, it is crucial to generalize the theory in such a way that the microlocalization can be preformed simultaneously with respect to finitely many such norms. Apparently this has not been done in the algebraic context of ring filtrations.

In the second section we apply this new technique to construct, under mild assumptions on G, various rings out of $\mathbb{Z}_p[[G]]$ culminating in a ring $R(G, K)$ which we call a Robba ring of G. Actually the ring $R(G, K)$ does depend, which we suppress in the notation, on the choice of a global coordinate system for the p-adic Lie group G. That such a phenomenon occurs in higher dimensions might not be too surprising.

In the third section we show that various variants of the Robba ring, which classically play an important role, also exist in our general setting.

The constructions in these notes were first presented at a workshop at Münster in 2005. Due to the lack of applications they were not published at the time. Given the progress made by G. Zabradi on structural properties of these rings it seemed appropriate to add these essentially unchanged notes as an appendix to his paper.

1 Generalized microlocalization of quasi-abelian normed algebras

Let K be a nonarchimedean complete field with absolute value $|\ |$. To motivate the construction in this section we consider the Tate algebra $K\langle T\rangle$ over K, i.e., the ring of power series $f(T) = \sum_{n\geq 0} \lambda_n T^n$ over K in one variable T which converge on the closed unit disk. Its natural norm is the Gauss norm given by

$$|f|_1 := \sup_{n \geq 0} |\lambda_n| \ .$$

But for any $0 < r \leq 1$ we also have the norm

$$|f|_r := \sup_{n \geq 0} |\lambda_n| r^n$$

on $K\langle T \rangle$. If some power of r lies in the value group $|K^\times|$ then the completion of $K\langle T \rangle$ with respect to the norm $| \ |_r$ is the algebra of analytic functions on the closed disk of radius r around the origin. On the other hand, if we first invert the variable T and then complete with respect to the norm $| \ |_1$ then we obtain the analytic functions on the unit circle. Microlocalization can be viewed as a generalization of this latter construction to certain noncommutative normed algebras. Finally, we may invert the variable T and then complete with respect to the norm $\max(| \ |_1, | \ |_r)$ in order to obtain the algebra of analytic functions on the closed annulus of inner and outer radius r and 1, respectively. The purpose of this section is to generalize the concept of microlocalization in such a way that we obtain a noncommutative analog of this third construction.

In fact, what we are going to do is a rather straightforward modification of the arguments and results in [Spr]. But since we work with normed algebras instead of filtered rings and since the paper [Spr] is partly obscured my confusing typographical errors we include, for the benefit of the reader, complete proofs.

Having specific applications in mind we do not strive for ultimate generality. It is clear that similar ideas will work in the more general context considered in [vdE].

We fix a (usually noncommutative) unital K-algebra A. A (nonarchimedean) norm $| \ |$ on A is called multiplicative if

$$|1| = 1 \qquad \text{and} \qquad |ab| = |a| \cdot |b| \quad \text{for any } a, b \in A.$$

Let $| \ |$ be such a multiplicative norm. The ring A of course then is an integral domain. For later reference we also recall the following triviality.

Remark 1.1. *If $|a_0 - b_0| < |b_0|$ and $|a_1 - b_1| < |b_1|$ then $|a_0 a_1 - b_0 b_1| < |b_0 b_1|$.*

Proof. We compute

$$\begin{aligned} |a_0 a_1 - b_0 b_1| &= |(a_0 - b_0) a_1 + b_0 (a_1 - b_1)| \\ &= \max(|a_0 - b_0| \cdot |a_1|, |b_0| \cdot |a_1 - b_1|) \\ &< |b_0| \cdot |b_1| = |b_0 b_1| \ . \end{aligned}$$

□

In this paper we are mostly interested in norms of the following much more restricted kind.

Definition 1.2. *The multiplicative norm $| \ |$ on A is called quasi-abelian if there is a constant $0 < \gamma < 1$ such that*

$$(qa) \qquad |ab - ba| \leq \gamma \cdot |ab| \qquad \text{for any } a, b \in A \ .$$

Throughout the paper we in fact fix a finite family of quasi-abelian norms $|\ |_1, \ldots, |\ |_m$ on A. Corresponding to each norm $|\ |_i$ we introduce the function

$$\Delta_i(x,y) := |s|_i^{-1} \cdot |t|_i^{-1} \cdot |at - sb|_i$$

on $A \setminus \{0\} \times A$ where $x = (s,a)$ and $y = (t,b)$. As a second input we fix a multiplicatively closed subset S of A (by convention this includes the requirement that $1 \in S$ but $0 \notin S$). The *saturation* S_i of S with respect to $|\ |_i$ is the set

$$S_i := \{a \in A : |at - s|_i < |s|_i \text{ for some } s, t \in S\}.$$

Note that this definition is symmetric in that, due to the condition (qa), we have

$$S_i = \{a \in A : |ta - s|_i < |s|_i \text{ for some } s, t \in S\}.$$

Lemma 1.3. *S_i is a multiplicatively closed subset containing S.*

Proof. Obviously $S \subseteq S_i$ and $0 \notin S_i$. Let $a, b \in S_i$ and $s, t, s', t' \in S$ such that

$$|at - s|_i < |s|_i \quad \text{and} \quad |bt' - s'|_i < |s'|_i\ .$$

By Remark 1.1 we then have $|atbt' - ss'|_i < |ss'|_i$. Because of

$$\begin{aligned} |abtt' - ss'|_i &= |abtt' - atbt' + atbt' - ss'|_i \\ &\leq \max(|a|_i |bt - tb|_i |t'|_i, |atbt' - ss'|_i) \end{aligned}$$

it therefore suffices, in order to obtain $ab \in S_i$, to check that

$$|a|_i |bt - tb|_i |t'|_i < |ss'|_i = |atbt'|_i$$

but which is a consequence of the condition (qa). □

The crucial consequence of the condition (qa) on which everything later on relies is the following "approximative" Ore condition.

Proposition 1.4. *For any $\epsilon > 0$ and any $(s,a) \in S \times A$ we have:*

i. There is a pair $(t,b) \in S \times A$ such that

$$|at - sb|_i \leq \epsilon |a|_i |t|_i \qquad \textit{(resp. } |ta - bs|_i \leq \epsilon |a|_i |t|_i\textit{)}$$

and $|s|_i |b|_i \leq |a|_i |t|_i$ for any $1 \leq i \leq m$; if $\epsilon < 1$ then $|s|_i |b|_i = |a|_i |t|_i$;

ii. if in i. we have $a \in S$ and $\epsilon < 1$ then $b \in S_1 \cap \ldots \cap S_m$.

Proof. i. Put $a_0 := a$ and $a_n := a_{n-1}s - sa_{n-1}$ for $n \geq 1$. By (qa) we have a constant $0 < \gamma < 1$ such that

$$|a_n|_i \leq \gamma |a_{n-1}|_i |s|_i \qquad \text{and hence} \qquad |a_n|_i \leq \gamma^n |s|_i^n |a|_i$$

for any $n \geq 0$ and any $1 \leq i \leq m$. We therefore find an $\ell \in \mathbb{N}$ such that $|a_\ell|_i \leq \epsilon |a|_i |s^\ell|_i$ for any $1 \leq i \leq m$. By induction with respect to n one checks that

$$as^n = \sum_{j=0}^{n-1} \binom{n}{j} s^{n-j} a_j + a_n \tag{1}$$

and

$$(2) \qquad a_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} s^j a s^{n-j} \ .$$

We put $t := s^\ell$ and $b := \sum_{j=0}^{\ell-1} \binom{\ell}{j} s^{\ell-j-1} a_j$ and obtain from (1) that $at = sb + a_\ell$ and hence

$$|at - sb|_i \leq \epsilon |a|_i |t|_i$$

for any $1 \leq i \leq m$. By (2) we have $|a_n|_i \leq |s|_i^n |a|_i$ for any $n \geq 0$ and therefore $|b|_i \leq |s^{\ell-1}|_i |a|_i = |s|_i^{-1} |t|_i |a|_i$. The stated identity in case $\epsilon < 1$ is obvious. The second half of the assertion is shown analogously. ii. This is clear. □

Corollary 1.5. *For any $\epsilon > 0$ and any $x \in S \times A$ there is a $\xi \in S \times A$ such that $\Delta_i(x, \xi) \leq \epsilon$ for any $1 \leq i \leq m$.*

We now introduce, for any $1 \leq i \leq m$, the function

$$d_i(x, y) := \inf_{\xi \in S_i \times A} \max(\Delta_i(x, \xi), \Delta_i(y, \xi))$$

on $(S_i \times A)^2$ as well as the function

$$d(x, y) := \max(d_1(x, y), \ldots, d_m(x, y))$$

on $(S \times A)^2$. Obviously we have $d(x, y) = d(y, x) \geq 0$. Furthermore, it follows from Cor. 1.5 that $d(x, x) = 0$.

Proposition 1.6. *For any $x, y, z \in S \times A$ we have*

$$d(x, z) \leq \max(d(x, y), d(y, z)) \ .$$

Proof. It certainly suffices to establish the inequality

$$d_i(x, z) \leq \max(d_i(x, y), d_i(y, z))$$

for each individual $1 \leq i \leq m$. Given any constant $\gamma_0 > \max(d_i(x, y), d_i(y, z))$ we have to show that $d_i(x, z) < \gamma_0$. Let $x = (s, a)$, $y = (t, b)$, and $z = (u, c)$. We find $\xi = (\sigma, \alpha)$ and $\eta = (\tau, \beta)$ in $S_i \times A$ such that $\Delta_i(x, \xi)$, $\Delta_i(y, \xi)$, $\Delta_i(y, \eta)$, and $\Delta_i(z, \eta)$ all are smaller than γ_0, i.e., such that

$$|a\sigma - s\alpha|_i < \gamma_0 |s|_i |\sigma|_i \ , \ |b\sigma - t\alpha|_i < \gamma_0 |t|_i |\sigma|_i \ ,$$
$$|b\tau - t\beta|_i < \gamma_0 |t|_i |\tau|_i \ , \ |c\tau - u\beta|_i < \gamma_0 |u|_i |\tau|_i \ .$$

We choose a $0 < \epsilon < 1$ such that $\epsilon |b|_i \leq \gamma_0 |t|_i$ and $\epsilon |c|_i \leq \gamma_0 |u|_i$. By Prop. 1.4.i applied to the multiplicative set S_i and $(\tau, \sigma) \in S_i \times A$ there is a pair $(v, d) \in S_i \times A$ such that

$$|\sigma v - \tau d|_i < \epsilon |\sigma|_i |v|_i < |\sigma|_i |v|_i = |\tau|_i |d|_i \ .$$

It follows that

$$\begin{aligned} |t\alpha v - t\beta d|_i &= |t\alpha v - b\sigma v + b\sigma v - b\tau d + b\tau d - t\beta d|_i \\ &< \max(\gamma_0 |t|_i |\sigma|_i |v|_i, \epsilon |b|_i |\sigma|_i |v|_i, \gamma_0 |t|_i |\tau|_i |d|_i) \\ &= \gamma_0 |t|_i |\sigma|_i |v|_i \ . \end{aligned}$$

By the multiplicativity of $|\ |_i$ we then must have

$$|\alpha v - \beta d|_i < \gamma_0 |\sigma|_i |v|_i \ .$$

We therefore obtain

$$\begin{aligned} |s\beta d - a\sigma v|_i &= |s\beta d - s\alpha v + s\alpha v - a\sigma v|_i \\ &< \gamma_0 |s|_i |\sigma|_i |v|_i = \gamma_0 |s|_i |\sigma v|_i \end{aligned}$$

and

$$\begin{aligned} |u\beta d - c\sigma v|_i &= |u\beta d - c\tau d + c\tau d - c\sigma v|_i \\ &< \max(\gamma_0 |u|_i |\tau|_i |d|_i, \epsilon |c|_i |\sigma|_i |v|_i) \\ &= \gamma_0 |u|_i |\sigma|_i |v|_i = \gamma_0 |u|_i |\sigma v|_i \ . \end{aligned}$$

This means that

$$d_i(x,z) \le \max(\Delta_i(x,(\sigma v, \beta d)), \Delta_i(z,(\sigma v, \beta d))) < \gamma_0 \ .$$

□

We obtain that d is a pseudometric on the set $S \times A$. The object of interest in this paper is the Hausdorff completion of the pseudometric space $(S \times A, d)$. But first we give a few explicit formulae for d.

Lemma 1.7. *For any $1 \le i \le m$ and any $a, b \in A$ and $s, t \in S_i$ we have:*

i. $d_i((s,a),(ts,ta)) = 0$;

ii. $d_i((s,a),(s,b)) = |s|_i^{-1} |a-b|_i$;

iii. $d_i((s,a),(1,0)) = |s|_i^{-1} |a|_i$;

iv. $d_i((s,a),(t,a)) \le |s|_i^{-1} |t|_i^{-1} |a|_i |s-t|_i$.

Proof. i. For any $\xi \in A \setminus \{0\} \times A$ we have

$$\Delta_i((ts,ta),\xi) = \Delta_i((s,a),\xi) \ .$$

Hence $d_i((s,a),(ts,ta)) = \inf_{\xi \in S_i \times A} \Delta_i((s,a),\xi)$ and the assertion follows from Cor. 1.5 applied to $S_i \times A$.

ii. For $\xi = (\sigma, \alpha) \in A \setminus \{0\} \times A$ we have

$$\begin{aligned} \max(\Delta_i((s,a),\xi), \Delta_i((s,b),\xi)) &= |s|_i^{-1} |\sigma|_i^{-1} \max(|a\sigma - s\alpha|_i, |b\sigma - s\alpha|_i) \\ &\ge |s|_i^{-1} |\sigma|_i^{-1} |a\sigma - b\sigma|_i = |s|_i^{-1} |a-b|_i \ . \end{aligned}$$

Hence $d_i((s,a),(s,b)) \ge |s|_i^{-1} |a-b|_i$. On the other hand we have

$$\begin{aligned} \Delta_i((s,b),\xi) &= |s|_i^{-1} |\sigma|_i^{-1} |b\sigma - s\alpha|_i \\ &= |s|_i^{-1} |\sigma|_i^{-1} |b\sigma - a\sigma + a\sigma - s\alpha|_i \\ &\le \max(|s|_i^{-1} |b-a|_i, \Delta_i((s,a),\xi)) \ . \end{aligned}$$

Using Cor. 1.5 (applied to $S_i \times A$) it follows that $d_i((s,a),(s,b)) \le |s|_i^{-1}|a-b|_i$.

iii. As a special case of ii. we have $d_i((s,a),(s,0)) = |s|_i^{-1}|a|_i$. We also have $d_i((1,0),(s,0)) = 0$ by i. It then follows from Prop. 1.6 that $d_i((s,a),(1,0)) = d_i((s,a),(s,0)) = |s|_i^{-1}|a|_i$.

iv. For $\xi = (\sigma,\alpha) \in A \setminus \{0\} \times A$ we have

$$\begin{aligned}\Delta_i((t,a),\xi) &= |t|_i^{-1}|\sigma|_i^{-1}|a\sigma - t\alpha|_i \\ &\le |t|_i^{-1}|\sigma|_i^{-1}\max(|a\sigma - s\alpha|_i, |s-t|_i|\alpha|_i) \ .\end{aligned}$$

By Prop. 1.4.i there is a $\xi \in S_i \times A$, for any $\epsilon > 0$, such that

$$|a\sigma - s\alpha|_i \le \epsilon|a|_i|\sigma|_i \qquad \text{and} \qquad |\alpha|_i \le |s|_i^{-1}|a|_i|\sigma|_i \ .$$

For such a ξ we have

$$\Delta_i((t,a),\xi) \le |t|_i^{-1}\max(\epsilon|a|_i, |s|_i^{-1}|a|_i|s-t|_i) \ .$$

It follows that

$$\begin{aligned}d_i((s,a),(t,a)) &\le \max(\Delta_i((s,a),\xi), \Delta_i((t,a),\xi)) \\ &= \max(|s|_i^{-1}|\sigma|_i^{-1}|a\sigma - s\alpha|_i, \Delta_i((t,a),\xi)) \\ &\le \max(\epsilon|s|_i^{-1}|a|_i, \epsilon|t|_i^{-1}|a|_i, |s|_i^{-1}|t|_i^{-1}|a|_i|s-t|_i)\end{aligned}$$

for any $\epsilon > 0$ and hence $d_i((s,a),(t,a)) \le |s|_i^{-1}|t|_i^{-1}|a|_i|s-t|_i$. □

Let now $\mathcal{C}(S \times A)$ denote the set of all Cauchy sequences $(x_j)_{j\in\mathbb{N}}$ (w.r.t. d) in $S \times A$. It contains $S \times A$ via the constant sequences. The pseudometric d extends to $\mathcal{C}(S \times A)$ by

$$d((x_j)_j,(y_j)_j) := \lim_{j\to\infty} d(x_j,y_j) \ .$$

We let $B := A\langle S; |\ |_1, \ldots, |\ |_m\rangle$ denote the quotient of $\mathcal{C}(S \times A)$ by the equivalence relation

$$(x_j)_j \sim (y_j)_j \quad \text{if} \quad d((x_j)_j,(y_j)_j) = 0 \ .$$

The pseudometric d passes to a metric on B which we again denote by d. The metric space (B,d) together with the obvious map $S \times A \longrightarrow B$ is the Hausdorff completion of the pseudometric space $(S \times A, d)$. We let $s^{-1}a \in B$ denote the image of $(s,a) \in S \times A$. Obviously

$$s^{-1}a = t^{-1}b \qquad \text{if and only if} \qquad d((s,a),(t,b)) = 0 \ .$$

By Lemma 1.7.ii the composed map

$$\begin{aligned}(A, \max(|\ |_1, \ldots, |\ |_m)) &\longrightarrow (B,d) \\ a &\longmapsto 1^{-1}a =: a\end{aligned}$$

is an isometry.

In the following we will show that B naturally is a K-Banach algebra. We begin with the scalar multiplication by K. On the set $S \times A$ we have a K-action given by

$$\lambda(s,a) := (s,\lambda a) \ .$$

It satisfies

$$d(\lambda x, \lambda y) = |\lambda| d(x,y)$$

and therefore extends to a K-action

$$\lambda(x_j)_j := (\lambda x_j)_j$$

on $\mathcal{C}(S \times A)$ which in turn descends to a K-action on B. The natural map $A \longrightarrow B$ is K-equivariant.

Next we construct the addition, and begin with the following immediate consequence of Lemma 1.7.ii.

Remark 1.8. *For any $s, t \in S$ and $a_1, a_2, b_1, b_2 \in A$ we have*

$$d((s, a_1 + a_2), (s, b_1 + b_2)) \le \max(d((s, a_1), (s, b_1)), d((s, a_2), (s, b_2))) \ .$$

Lemma 1.9. *Let $0 < \epsilon < 1$; for $s, t \in S$ and $a, b, c, d \in A$ such that*

$$d((s,a),(t,c)) \le \epsilon \quad \text{and} \quad d((s,b),(t,d)) \le \epsilon$$

we have

$$d((s, a+b), (t, c+d)) \le \epsilon \max_{1 \le i \le m} \max(1, |s|_i^{-1}|a|_i, |s|_i^{-1}|b|_i, |t|_i^{-1}|c|_i, |t|_i^{-1}|d|_i) \ .$$

Proof. Applying Prop. 1.4 to (t, s) we find $(u, v) \in S \times S_1 \cap \ldots \cap S_m$ such that

$$|us - vt|_i \le \epsilon |u|_i |s|_i < |u|_i |s|_i = |v|_i |t|_i$$

for any $1 \le i \le m$. We claim that

$$d_i((s,a),(us,vc)) \le \epsilon \max(1, |s|_i^{-1}|a|_i) \ .$$

For any $\xi = (\sigma, \alpha) \in S_i \times A$ we have

$$\begin{aligned} |vc\sigma - us\alpha|_i &= |v(c\sigma - t\alpha) + (vt - us)\alpha|_i \\ &\le \max(|t|_i^{-1}|u|_i|s|_i|c\sigma - t\alpha|_i, \epsilon|u|_i|s|_i|\alpha|_i) \end{aligned}$$

and hence

$$\begin{aligned} \Delta_i((us,vc),\xi) &\le \max(|t|_i^{-1}|\sigma|_i^{-1}|c\sigma - t\alpha|_i, \epsilon|\sigma|_i^{-1}|\alpha|_i) \\ &= \max(\Delta_i((t,c),\xi), \epsilon|\sigma|_i^{-1}|\alpha|_i) \ . \end{aligned}$$

Since $d((s,a),(t,c)) \le \epsilon$ we find a ξ with $\max(\Delta_i((s,a),\xi), \Delta_i((t,c),\xi)) \le \epsilon$. We conclude that

$$\begin{aligned} d_i((s,a),(us,vc)) &\le \max(\Delta_i((s,a),\xi), \Delta_i((us,vc),\xi)) \\ &\le \epsilon \max(1, |\sigma|_i^{-1}|\alpha|_i) \ . \end{aligned}$$

But $\Delta_i((s,a),\xi) \le \epsilon$ also implies that $|\sigma|_i^{-1}|\alpha|_i \le \max(|s|_i^{-1}|a|_i, \epsilon)$. Hence our claim (observe that in case $|s|_i^{-1}|a|_i < \epsilon < 1$ we have $\epsilon \max(1, |\sigma|_i^{-1}|\alpha|_i) = \epsilon = \epsilon \max(1, |s|_i^{-1}|a|_i)$. By the same reasoning we also have

$$d_i((s,b),(us,vd)) \le \epsilon \max(1, |s|_i^{-1}|b|_i) \ .$$

Applying Lemma 1.7.i to the element $v \in S_1 \cap \ldots \cap S_m$ we obtain

$$\begin{aligned} d_i((s, a+b), (t, c+d)) &= d_i((s, a+b), (vt, v(c+d))) \\ &\le \max(d_i((s, a+b), (us, v(c+d))), d_i((us, v(c+d)), (vt, v(c+d)))) \ . \end{aligned}$$

Using again Lemma 1.7.i and Remark 1.8 we have

$$\begin{aligned} d_i((s, a+b), (us, v(c+d))) &= d_i((us, u(a+b)), (us, v(c+d))) \\ &\le \max(d_i((us, ua), (us, vc)), d_i((us, ub), (us, vd))) \\ &= \max(d_i((s, a), (us, vc)), d_i((s, b), (us, vd))) \\ &\le \epsilon \max(1, |s|_i^{-1}|a|_i, |s|_i^{-1}|b|_i) \ . \end{aligned}$$

According to Lemma 1.7.iv we have

$$\begin{aligned} & d_i((us, v(c+d)), (vt, v(c+d))) \\ & \quad \le |u|_i^{-1}|s|_i^{-1}|v|_i^{-1}|t|_i^{-1}|v|_i|c+d|_i|us - vt|_i) \\ & \quad \le \epsilon |t|_i^{-1}|c+d|_i \ . \end{aligned}$$

It follows that

$$d_i((s, a+b), (t, c+d)) \le \epsilon \max(1, |s|_i^{-1}|a|_i, |s|_i^{-1}|b|_i, |t|_i^{-1}|c+d|_i) \ .$$

This finishes the proof. □

Corollary 1.10. *For $s, t \in S$ and $a, b, c, d \in A$ such that $s^{-1}a = t^{-1}c$ and $s^{-1}b = t^{-1}d$ we have $s^{-1}(a+b) = t^{-1}(c+d)$.*

Proof. Our assumption amounts to $d((s, a), (t, c)) = d((s, b), (t, d)) = 0$. The previous lemma then implies that $d((s, a+b), (t, c+d)) = 0$. □

At this point we introduce, for any $n \in \mathbb{N}$, the subset

$$\begin{aligned} P^{(n)} := \{(e_1, \ldots, e_n) \in B^n : & \text{ there are } a_1, \ldots, a_n \in A \text{ and } s \in S \text{ such} \\ & \text{that } e_j = s^{-1}a_j \text{ for any } 1 \le j \le n\} \end{aligned}$$

of B^n. Of course, $P^{(1)}$ is just the image of the natural map $S \times A \longrightarrow B$. By Lemma 1.7.iii the element

$$0 := s^{-1}0$$

is well defined in $P^{(1)}$ (i.e., is independent of the choice of $s \in S$). Cor. 1.10 says that the map

$$\begin{aligned} P^{(2)} &\longrightarrow B \\ (e, f) &\longmapsto e + f := s^{-1}(a+b) \quad \text{if } e = s^{-1}a, f = s^{-1}b \end{aligned}$$

is well defined. We obviously have:

1. $e + f = f + e$,

2. $e + 0 = e$,

3. $e + (-1)e = 0$,

4. $(e + f) + g = e + (f + g)$ for $(e, f, g) \in P^{(3)}$,

5. $\lambda(e + f) = \lambda e + \lambda f$ for $\lambda \in K$,

6. $(\lambda + \mu)e = \lambda e + \mu e$ for $\lambda, \mu \in K$.

For any $e \in B$ we put

$$|e| := d(e, 0) .$$

It follows from Lemma 1.7.ii and iii that for $(e, f) \in P^{(2)}$ we have

7. $d(e, f) = |e - f|$.

With this notation we may rephrase Lemma 1.9 as follows.

Corollary 1.11. *Let $0 < \epsilon < 1$; for any (e_1, f_1) and (e_2, f_2) in $P^{(2)}$ such that*

$$d(e_1, e_2) \leq \epsilon \quad \textit{and} \quad d(f_1, f_2) \leq \epsilon$$

we have

$$d(e_1 + f_1, e_2 + f_2) \leq \epsilon \max(1, |e_1|, |e_2|, |f_1|, |f_2|) .$$

This corollary shows that the map $+ : P^{(2)} \longrightarrow B$ is continuous and extends continuously to the closure of $P^{(2)}$ in B^2 (cf. [B-GT] II§3.6 Prop. 11 and IX§2.3).

Lemma 1.12. *$P^{(n)}$, for any $n \in \mathbb{N}$, is dense in B^n (for the product topology).*

Proof. We have to find, for any given $\epsilon > 0$ and $x_1, \ldots, x_n \in S \times A$, elements $a_1, \ldots, a_n \in A$ and $s \in S$ such that

$$d(x_j, (s, a_j)) < \epsilon \qquad \text{for any } 1 \leq j \leq n .$$

We may assume $n > 1$ and, by induction with respect to n, also that $x_1 = (t, b_1), \ldots, x_{n-1} = (t, b_{n-1})$. Let $x_n = (u, c)$ and choose $0 < \eta < 1$ such that $\eta \cdot \max\limits_{1 \leq i \leq m} |c|_i |u|_i^{-1} < \epsilon$. By Prop. 1.4.i there are elements $(\sigma, \alpha), (\tau, \beta) \in S \times A$ such that

$$\begin{aligned} |\sigma t - \alpha u|_i &\leq \eta |\sigma|_i |t|_i < |\sigma|_i |t|_i = |\alpha_i|_i |u|_i \\ |c\tau - u\beta|_i &\leq \eta |c|_i |\tau|_i \leq |c|_i |\tau|_i = |u|_i |\beta|_i \end{aligned}$$

for any $1 \leq i \leq m$. In particular

$$\begin{aligned} |\alpha c\tau - \sigma t\beta|_i &\leq \max(|\alpha|_i |c\tau - u\beta|_i, |\alpha u - \sigma t|_i |\beta|_i) \\ &\leq \eta \max(|\alpha|_i |c|_i |\tau|_i, |\beta|_i |\sigma|_i |t|_i) \\ &= \eta |u|_i^{-1} |c|_i |\tau|_i |\sigma|_i |t|_i . \end{aligned}$$

We put $y_j := (\sigma t, \sigma b_j)$ for $1 \leq j \leq n - 1$ and $y_n := (\sigma t, \alpha c)$. According to Lemma 1.7.i we have

$$d(x_j, y_j) = 0 \qquad \text{for any } 1 \leq j \leq n - 1 .$$

Furthermore

$$\begin{aligned} d_i(x_n, y_n) &\leq \max(\Delta_i(x_n, (\tau, \beta)), \Delta_i(y_n, (\tau, \beta))) \\ &= \max(|u|_i^{-1} |\tau|_i^{-1} |c\tau - u\beta|_i, |\sigma|_i^{-1} |t|_i^{-1} |\tau|_i^{-1} |\alpha c\tau - \sigma t\beta|_i) \\ &\leq \eta |c|_i |u|_i^{-1} < \epsilon . \end{aligned}$$

$\square$

It follows in particular that our map $+ : P^{(2)} \longrightarrow B$ extends continuously to a map

$$\begin{aligned} B \times B &\longrightarrow B \\ (e,f) &\longmapsto e+f \end{aligned}$$

which satisfies 1.-7. (for the associativity use Lemma 1.12 for $n = 3$). We see that $| \ |$ is a nonarchimedean norm which makes B into a K-Banach space. The natural map

$$(A, \max(| \ |_1, \ldots, | \ |_m)) \longrightarrow (B, | \ |)$$

is an isometry of normed K-vector spaces.

In order to construct the multiplication on B we proceed in a similar way.

Lemma 1.13. *Let $0 < \epsilon < 1$; for $s, t, u, v \in S$ and $a, b, c, d \in A$ such that*

$$d((s, at), (u, cv)) \leq \epsilon \quad \textit{and} \quad d((t, b), (v, d)) \leq \epsilon$$

we have

$$d((s, ab), (u, cd)) \leq \epsilon \max_{1 \leq i \leq m} \max(1, \frac{|at|_i}{|s|_i}, \frac{|at|_i}{|s|_i}\frac{|b|_i}{|t|_i}, \frac{|b|_i}{|t|_i}, \frac{|cv|_i}{|u|_i}, \frac{|cv|_i}{|u|_i}\frac{|d|_i}{|v|_i}, \frac{|d|_i}{|v|_i}) \ .$$

Proof. Let $0 \leq i \leq m$. We find $\xi = (\sigma, \alpha)$ and $\eta = (\tau, \beta)$ in $S_i \times A$ such that

$$\begin{aligned} |b\sigma - t\alpha|_i \leq \epsilon |\sigma|_i |t|_i \ , \ |d\sigma - v\alpha|_i \leq \epsilon |\sigma|_i |v|_i, \\ |at\tau - s\beta|_i \leq \epsilon |\tau|_i |s|_i \ , \ |cv\tau - u\beta|_i \leq \epsilon |\tau|_i |u|_i \ . \end{aligned}$$

In particular, we have $|\sigma|_i^{-1}|\alpha|_i \leq \max(|t|_i^{-1}|b|_i, \epsilon) \leq \max(|t|_i^{-1}|b|_i, 1)$ and also $|\sigma|_i^{-1}|\alpha|_i \leq \max(|v|_i^{-1}|d|_i, 1)$. Using Prop. 1.4.i (for S_i) we choose $(\rho, \kappa) \in S_i \times A$ such that

$$|\alpha\rho - \tau\kappa|_i \leq \epsilon |\alpha|_i |\rho|_i \ , \ |\tau|_i |\kappa|_i = |\alpha|_i |\rho|_i \ .$$

We then have

$$\begin{aligned} |ab\sigma\rho - s\beta\kappa|_i &= |a(b\sigma - t\alpha)\rho + at(\alpha\rho - \tau\kappa) + (at\tau - s\beta)\kappa|_i \\ &\leq \epsilon \max(|a|_i|\sigma|_i|t|_i|\rho|_i, |a|_i|t|_i|\alpha|_i|\rho|_i, |s|_i|\alpha|_i|\rho|_i) \end{aligned}$$

and hence

$$\begin{aligned} \Delta_i((s, ab), (\sigma\rho, \beta\kappa)) &\leq \epsilon \max(|s|_i^{-1}|at|_i, |s|_i^{-1}|at|_i|\sigma|_i^{-1}|\alpha|_i, |\sigma|_i^{-1}|\alpha|_i) \\ &\leq \epsilon \max(|s|_i^{-1}|at|_i, |s|_i^{-1}|at|_i|t|_i^{-1}|b|_i, |t|_i^{-1}|b|_i, 1) \ . \end{aligned}$$

Similarly

$$\begin{aligned} |cd\sigma\rho - u\beta\kappa|_i &= |c(d\sigma - v\alpha)\rho + cv(\alpha\rho - \tau\kappa) + (cv\tau - u\beta)\kappa|_i \\ &\leq \epsilon \max(|c|_i|\sigma|_i|v|_i|\rho|_i, |c|_i|v|_i|\alpha|_i|\rho|_i, |u|_i|\alpha|_i|\rho|_i) \end{aligned}$$

and

$$\begin{aligned} \Delta_i((u, cd), (\sigma\rho, \beta\kappa)) &\leq \epsilon \max(|u|_i^{-1}|cv|_i, |u|_i^{-1}|cv|_i|\sigma|_i^{-1}|\alpha|_i, |\sigma|_i^{-1}|\alpha|_i) \\ &\leq \epsilon \max(|u|_i^{-1}|cv|_i, |u|_i^{-1}|cv|_i|v|_i^{-1}|d|_i, |v|_i^{-1}|d|_i, 1) \ . \end{aligned}$$

We obtain

$$d_i((s, ab), (u, cd)) \leq \epsilon \max(1, \frac{|at|_i}{|s|_i}, \frac{|at|_i}{|s|_i}\frac{|b|_i}{|t|_i}, \frac{|b|_i}{|t|_i}, \frac{|cv|_i}{|u|_i}, \frac{|cv|_i}{|u|_i}\frac{|d|_i}{|v|_i}, \frac{|d|_i}{|v|_i}) \ .$$

□

Corollary 1.14. *For $s, t, u, v \in S$ and $a, b, c, d \in A$ such that $s^{-1}(at) = u^{-1}(cv)$ and $t^{-1}b = v^{-1}d$ we have $s^{-1}(ab) = u^{-1}(cd)$.*

This corollary says that on the subset

$$Q := \{(s^{-1}a, t^{-1}b) \in (P^{(1)})^2 : a \in At\}$$

of B^2 the map

$$\begin{aligned} Q &\longrightarrow B \\ (e, f) &\longmapsto e \cdot f := s^{-1}(ab) \quad \text{if } e = s^{-1}(at), f = t^{-1}b \end{aligned}$$

is well defined.

Corollary 1.15. *Let $0 < \epsilon < 1$; for any (e_1, f_1) and (e_2, f_2) in Q such that*

$$d(e_1, e_2) \leq \epsilon \quad \text{and} \quad d(f_1, f_2) \leq \epsilon$$

we have

$$d(e_1 \cdot f_1, e_2 \cdot f_2) \leq \epsilon \max(1, |e_1|, |e_1||f_1|, |f_1|, |e_2|, |e_2||f_2|, |f_2|) \ .$$

This corollary shows that the map $\cdot : Q \longrightarrow B$ is continuous and extends continuously to the closure of Q in B^2.

Lemma 1.16. *For any given $t \in S$ the set $\{s^{-1}(at) : a \in A, s \in S\}$ is dense in B.*

Proof. Let $(u, c) \in S \times A$ and let $\epsilon > 0$. By Prop. 1.4.i we find $(\sigma, \alpha) \in S \times A$ such that $|\sigma c - \alpha t|_i \leq \epsilon |\sigma|_i |c|_i$ for any $1 \leq i \leq m$. Using Lemma 1.7.i and ii we obtain

$$\begin{aligned} d(u^{-1}c, (\sigma u)^{-1}(\alpha t)) &= d((\sigma u)^{-1}(\sigma c), (\sigma u)^{-1}(\alpha t)) \\ &= \max_{1 \leq i \leq m} |u|_i^{-1} |\sigma|_i^{-1} |\sigma c - \alpha t|_i \\ &\leq \epsilon \max_{1 \leq i \leq m} |u|_i^{-1} |c|_i \ . \end{aligned}$$

□

This lemma implies that Q is dense in B^2. Hence by continuous extension we have a map

$$\begin{aligned} B \times B &\longrightarrow B \\ (e, f) &\longmapsto e \cdot f \ . \end{aligned}$$

Lemma 1.17. *For any $e, f, g \in B$ we have:*

i. $d(e \cdot f, e \cdot g) \leq |e| d(f, g)$ *and* $e \cdot (f + g) = e \cdot f + e \cdot g$;

ii. $d(e \cdot g, f \cdot g) \leq d(e, f)|g|$ *and* $(e + f) \cdot g = e \cdot g + f \cdot g$;

iii. $(e \cdot f) \cdot g = e \cdot (f \cdot g)$.

Proof. By continuity all three assertions need only to be checked on an appropriate dense subset of B^3.

i. As a consequence of Lemma 1.12 (for $n = 2$) and Lemma 1.16 the set

$$\{(s^{-1}(at), t^{-1}b, t^{-1}c) \in (P^{(1)})^3 : a, b, c \in A, s, t \in S\}$$

is dense in B^3. For a triple in this set the first inequality is immediate from Lemma 1.7.ii and iii and the second identity follows from the definitions.

ii. As a consequence of Lemma 1.16 the set $\{(s_1^{-1}(a't), s_2^{-1}(b't), t^{-1}c)\}$ is dense in B^3. In addition, the proof of Lemma 1.12 (for $n = 2$) shows that a pair $(s_1^{-1}(a't), s_2^{-1}(b't))$ can be approximated by a pair of the form $(s^{-1}(at), s^{-1}(bt))$. Hence the set

$$\{(s^{-1}(at), s^{-1}(bt), t^{-1}c) \in (P^{(1)})^3 : a, b, c \in A, s, t \in S\}$$

is dense in B^3. For a triple in this set the first inequality again is immediate from Lemma 1.7.ii and iii and the second identity again follows from the definitions.

iii. By Lemma 1.16 the set

$$\{(s^{-1}(at), t^{-1}(bu), u^{-1}c) \in (P^{(1)})^3 : a, b, c \in A, s, t, u \in S\}$$

is dense in B^3. On this subset the asserted associativity is immediate from the definition of the multiplication. □

We see that the multiplication $\cdot$ in B is distributive and associative. It is easy to see, using Lemma 1.7.i and the density of Q, that 1 is a unit element and that this multiplication is compatible with the scalar multiplication by K. Finally, $|1| = 1$ and as a consequence of Lemma 1.171.16.i we have $|e \cdot f| \leq |e||f|$. Hence we conclude that

B is a unital K-Banach algebra with submultiplicative norm $| \ |$

and that the natural map $(A, \max(| \ |_1, \dots, | \ |_m)) \longrightarrow (B, | \ |)$ is an isometric unital homomorphism of normed K-algebras. By construction we have $(s^{-1}1) \cdot s = s^{-1}s = 1$ and $s \cdot (s^{-1}a) = a$ for any $a \in A$ and $s \in S$. In particular, the elements of the multiplicative set S become invertible in the ring B.

Proposition 1.18 (Universal property)**.** *Let $(D, | \ |_D)$ be a unital K-Banach algebra and let $\phi : A \longrightarrow D$ be any unital homomorphism of K-algebras such that:*

(i) $\phi(s) \in D^\times$ for any $s \in S$;

(ii) there is a constant $\gamma > 0$ such that $|\phi(s)^{-1}\phi(a)|_D \leq \gamma \max_{1 \leq i \leq m} |s|_i^{-1}|a|_i$ for any $s \in S, a \in A$ (in particular, ϕ is continuous);

then there is a unique continuous unital homomorphism of K-Banach algebras

$$\phi_S : A\langle S; | \ |_1, \dots, | \ |_m\rangle \longrightarrow D$$

such that $\phi_S|A = \phi$. If $| \ |_D$ is submultiplicative and the constant in (*ii*) *can be chosen to be $\gamma = 1$ then ϕ_S is norm decreasing.*

Proof. The subset $P^{(1)} = \{s^{-1}a : a \in A, s \in S\}$ is dense in B. Because of $s \cdot (s^{-1}a) = a$ we have to have $\phi_S(s^{-1}a) = \phi(s)^{-1}\phi(a)$. Hence the uniqueness of ϕ_S is clear. To establish existence let $\gamma_0 > 0$ be a constant such that $|d_1 d_2|_D \le \gamma_0 |d_1|_D |d_2|_D$ for any $d_1, d_2 \in D$. We claim that there is a constant $\gamma_1 > 0$ such that

$$|\phi(s)^{-1}\phi(a) - \phi(t)^{-1}\phi(b)|_D \le \gamma_1 |s^{-1}a - t^{-1}b|$$

for any $(s,a),(t,b) \in S \times A$. Choose $\eta > 0$ such that $\eta|b| \le \eta|t||t^{-1}||b| \le |s^{-1}|^{-1}|s^{-1}a - t^{-1}b|$. Since $P^{(1)}$ is dense in B we find $(\sigma, \alpha) \in S \times A$ such that

$$|st^{-1} - \sigma^{-1}\alpha| \le \eta \ .$$

We then have

$$(\sigma s)^{-1}(\sigma a - \alpha b) - (\sigma s)^{-1}(\sigma s - \alpha t)t^{-1}b = s^{-1}a - t^{-1}b$$

and

$$|(\sigma s)^{-1}(\sigma s - \alpha t)t^{-1}b| \le |s^{-1}|\eta|b| \le |s^{-1}a - t^{-1}b| \ .$$

It follows that $|(\sigma s)^{-1}(\sigma a - \alpha b)| \le |s^{-1}a - t^{-1}b|$ and hence, by (ii), that

$$|\phi(\sigma s)^{-1}\phi(\sigma a - \alpha b)|_D \le \gamma |s^{-1}a - t^{-1}b| \ .$$

On the other hand

$$\begin{aligned}
&|\phi(\sigma s)^{-1}\phi(\sigma s - \alpha t)\phi(t)^{-1}\phi(b)|_D \\
&\quad \le \gamma_0 |\phi(\sigma s)^{-1}\phi(\sigma s - \alpha t)|_D |\phi(t)^{-1}\phi(b)|_D \\
&\quad \le \gamma_0\gamma^2 |(\sigma s)^{-1}(\sigma s - \alpha t)||t^{-1}b| \\
&\quad \le \gamma_0\gamma^2 |(\sigma s)^{-1}(\sigma s - \alpha t)t^{-1}||t||t^{-1}||b| \\
&\quad \le \gamma_0\gamma^2 |s^{-1}|\eta|t||t^{-1}||b| \\
&\quad \le \gamma_0\gamma^2 |s^{-1}a - t^{-1}b| \ .
\end{aligned}$$

Setting $\gamma_1 := \max(\gamma, \gamma_0\gamma^2)$ we finally obtain

$$\begin{aligned}
&|\phi(s)^{-1}\phi(a) - \phi(t)^{-1}\phi(b)|_D \\
&\quad = |\phi(\sigma s)^{-1}\phi(\sigma a - \alpha b) - \phi(\sigma s)^{-1}\phi(\sigma s - \alpha t)\phi(t)^{-1}\phi(b)|_D \\
&\quad \le \max(\gamma|s^{-1}a - t^{-1}b|, \gamma_0\gamma^2|s^{-1}a - t^{-1}b|) \\
&\quad = \gamma_1 |s^{-1}a - t^{-1}b| \ .
\end{aligned}$$

This proves our claim and implies that the map

$$\begin{aligned}
S \times A &\longrightarrow D \\
(s,a) &\longmapsto \phi(s)^{-1}\phi(a)
\end{aligned}$$

is uniformly continuous and therefore extends continuously to a map $\phi_S : B \longrightarrow D$. It is straightforward to check that ϕ_S is a unital homomorphism of K-algebras such that $\phi_S|A = \phi$. $\square$

Although the norm $|\ |$ on B is only submultiplicative it nevertheless is quasi-abelian in the following sense. We fix a constant $0 < \gamma < 1$ so that the condition (qa) is satisfies simultaneously for any $|\ |_i$.

Lemma 1.19. *$|u^{-1}s^{-1}ec - u^{-1}es^{-1}c| \le \gamma \cdot |u^{-1}s^{-1}ec| = \gamma \cdot |u^{-1}es^{-1}c|$ for any $u, s \in S$, $c \in A$, and $e \in B$.*

Proof. By Lemma 1.16 we may assume that $e = t^{-1}(bs)$ with $t \in S$ and $b \in A$. We then compute

$$\begin{aligned}
&|u^{-1}s^{-1}t^{-1}bsc - u^{-1}t^{-1}bc| \\
&\quad = |u^{-1}s^{-1}t^{-1}bsc - u^{-1}t^{-1}s^{-1}bsc + u^{-1}t^{-1}s^{-1}bsc - u^{-1}t^{-1}s^{-1}sbc| \\
&\quad \le \max(d((tsu, bsc), (stu, bsc)), d((stu, bsc), (stu, sbc))) \\
&\quad = \max_i \max(|tsu|_i^{-1}|stu|_i|bsc|_i|tsu - stu|_i, |stu|_i^{-1}|bsc - sbc|_i) \\
&\quad \le \gamma \cdot \max_i |stu|_i^{-1}|bsc|_i = \gamma \cdot \max_i |tu|_i^{-1}|bc|_i \\
&\quad = \gamma \cdot |u^{-1}t^{-1}bc|
\end{aligned}$$

using Lemma 1.7 in the fourth line. □

Lemma 1.20. *$|s^{-1}aec - s^{-1}eac| \le \gamma \cdot |s^{-1}aec| = \gamma \cdot |s^{-1}eac|$ for any $s \in S$, $a, c \in A$, and $e \in B$.*

Proof. By density we may assume that $e = t^{-1}b$ with $t \in S$ and $b \in A$. We then compute

$$\begin{aligned}
&|s^{-1}at^{-1}bc - s^{-1}t^{-1}bac| \\
&\quad = |s^{-1}at^{-1}bc - s^{-1}t^{-1}abc + s^{-1}t^{-1}abc - s^{-1}t^{-1}bac| \\
&\quad \le \max(|s^{-1}at^{-1}bc - s^{-1}t^{-1}abc|, |s^{-1}t^{-1}abc - s^{-1}t^{-1}bac|) \\
&\quad \le \max(\gamma \cdot |s^{-1}t^{-1}abc|, d((ts, abc), (ts, bac))) \\
&\quad = \max(\gamma \cdot |s^{-1}t^{-1}abc|, \max_i |ts|_i^{-1}|abc - bac|_i) \\
&\quad \le \gamma \cdot \max(|s^{-1}t^{-1}abc|, \max_i |ts|_i^{-1}|abc|_i) \\
&\quad = \gamma \cdot |s^{-1}t^{-1}abc| \\
&\quad = \gamma \cdot |s^{-1}at^{-1}bc|
\end{aligned}$$

using the previous lemma in the fourth and the last line. □

Proposition 1.21. *There is a constant $0 < \gamma < 1$ such that, for any elements $e_1, \ldots, e_n \in B$ and any permutation σ of $\{1, \ldots, n\}$, we have*

$$|e_1 \cdot \ldots \cdot e_n - e_{\sigma(1)} \cdot \ldots \cdot e_{\sigma(n)}| \le \gamma \cdot |e_1 \cdot \ldots \cdot e_n| = \gamma \cdot |e_{\sigma(1)} \cdot \ldots \cdot e_{\sigma(n)}| \ .$$

Proof. Since any permutation is a product of elementary transpositions it suffices to prove that

$$|ge_0e_1f - ge_1e_0f| \le \gamma \cdot |ge_0e_1f| = \gamma \cdot |ge_1e_0f|$$

for any $g, e_0, e_1, f \in B$. Since

$$\begin{aligned}
ge_0e_1f - ge_1e_0f = g(e_0e_1f) - (e_0e_1f)g + e_0e_1(fg) - e_1e_0(fg) \\
+ (e_1e_0f)g - g(e_1e_0f)
\end{aligned}$$

we may in fact assume that $g = 1$. By density it furthermore suffices to consider the case $f = u^{-1}c$ with $u \in S$ and $c \in A$. Using the identity

$$e_0e_1u^{-1}c - e_1e_0u^{-1}c = (e_0e_1)u^{-1}c - u^{-1}(e_0e_1)c + u^{-1}e_0e_1c - u^{-1}e_1e_0c \\ + u^{-1}(e_1e_0)c - e_1e_0u^{-1}c$$

we therefore are reduced to showing that

$$|u^{-1}e_0e_1c - u^{-1}e_1e_0c| \leq \gamma \cdot |u^{-1}e_0e_1c| = \gamma \cdot |u^{-1}e_1e_0c|$$

for any $u \in S$, $c \in A$, and $e_0, e_1 \in B$. But for $e_0 = s^{-1}a$ with $s \in S$ and $a \in A$ this is a consequence of Lemmas 1.19 and 1.20. The case of a general e_0 then follows by density. □

Remark 1.22. *For a single initial quasi-abelian norm on A (i.e., $m = 1$) the resulting norm $|\ |$ on B again is multiplicative (and quasi-abelian).*

Proof. We need to show that $|ef| = |e||f|$. By continuity it suffices to consider the case $e = s^{-1}a$ and $f = t^{-1}b$ with $s, t \in S$ and $a, b \in A$. But then, using Prop. 1.21 and Lemma 1.7.iii, we compute

$$|s^{-1}at^{-1}b| = |(ts)^{-1}ab| = |ts|_1^{-1}|ab|_1 = |s|_1^{-1}|a|_1|t|_1^{-1}|b|_1 = |s^{-1}a||t^{-1}b| \ .$$

□

2 Noncommutative annuli for uniform pro-p-groups

For the rest of this paper we assume that $\mathbb{Q}_p \subseteq K \subseteq \mathbb{C}_p$ is discretely valued. For simplicity we also assume that $p \neq 2$. We fix a uniform pro-p-group G as well as an ordered basis $h_1, \ldots, h_d$ of G. Then the map

$$\begin{aligned} \psi : \qquad \mathbb{Z}_p^d &\xrightarrow{\sim} G \\ (x_1, \ldots x_d) &\longmapsto h_1^{x_1} \cdot \ldots \cdot h_d^{x_d} \end{aligned}$$

is a bijective global chart for G as a locally $\mathbb{Q}_p$-analytic manifold (cf. [DDMS] §4.2). Using this chart we may identify the K-Fréchet spaces of locally analytic distributions

$$\psi^* : D(G, K) \xrightarrow{\cong} D(\mathbb{Z}_p^d, K)$$

As usual, we embed the group G into the algebra $D(G, K)$ via the Dirac distributions. We write $b_i := h_i - 1$ and $\mathbf{b}^\alpha := b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_d^{\alpha_d}$, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. Any distribution $\mu \in D(G, K)$ has a unique convergent expansion

$$\mu = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha$$

with $d_\alpha \in K$ such that, for any $0 < r < 1$, the set $\{|d_\alpha| r^\alpha\}_{\alpha \in \mathbb{N}_0^d}$ is bounded. Here we put $r^\alpha := r^{\alpha_1 + \ldots + \alpha_d}$. Conversely, any such series is convergent in $D(G, K)$. The Fréchet topology on $D(G, K)$ is defined by the family of norms

$$|\mu|_r := \sup_{\alpha \in \mathbb{N}_0^d} |d_\alpha| r^\alpha$$

for $0 < r < 1$. It is shown in [ST] Thm. 4.5 that, for any $1/p < r < 1$ such that $r \in p^{\mathbb{Q}}$, the norm $|\ |_r$ on $D(G,K)$ is multiplicative and quasi-abelian. Whereas $D(G,K)$ is a "noncommutative open unit disk" the completion $D_r(G,K)$ of $D(G,K)$ with respect to $|\ |_r$ is a "noncommutative closed disk of radius r". We now use the technique of the previous section to construct corresponding "noncommutative closed annuli" for any radii $1/p < r_0 \leq r < 1$ such that $r_0, r \in p^{\mathbb{Q}}$.

We take $A := D_r(G,K)$, the two norms $|\ |_{r_0}$ and $|\ |_r$, and the multiplicatively closed subset $S \subseteq D_r(G,K)$ generated by $b_1, \ldots, b_d$. We define

$$D_{[r_0,r]}(G,K) := A\langle S; |\ |_{r_0}, |\ |_r\rangle$$

and we let $|\ |_{r_0,r}$ denote its natural norm. Since $\max(|\ |_{r_0}, |\ |_r) = |\ |_r$ the canonical homomorphism of K-Banach algebras $D_r(G,K) \longrightarrow D_{[r_0,r]}(G,K)$ is an isometry for the natural norms.

Lemma 2.1. *For $1/p < r \leq r' \leq r'' < 1$, any integer $m \geq 0$, and any $\mu \in D(G,K)$ we have*

$$\frac{|\mu|_{r'}}{r'^m} \leq \max(\frac{|\mu|_r}{r^m}, \frac{|\mu|_{r''}}{r''^m}) \ .$$

Proof. Since exponential function and logarithm are monotonous functions it suffices to show that the function

$$t \longmapsto \log(\frac{|\mu|_{\exp(t)}}{\exp(t)^m})$$

on $(-\infty, 0)$ is convex. But it is a supremum

$$t \longmapsto \sup_\alpha(\log|d_\alpha| + (|\alpha| - m)t)$$

of affine functions and hence is visibly convex. $\square$

Let $1/p \leq r_0 < r \leq r' \leq r'' < r_1 \leq 1$ all contained in $p^{\mathbb{Q}}$ and consider the composed unital homomorphism of K-Banach algebras

$$D_{r''}(G,K) \longrightarrow D_{r'}(G,K) \longrightarrow D_{[r,r']}(G,K)$$

which is norm decreasing and makes the elements in S invertible. Let $\mu \in D_{r''}(G,K)$ and suppose that the monomial s in the $b_1, \ldots, b_d$ has m factors. Using Lemma 2.1 we compute

$$\begin{aligned} |s^{-1}\mu|_{r,r'} &= \max(|s|_r^{-1}|\mu|_r, |s|_{r'}^{-1}|\mu|_{r'}) = \max(\frac{|\mu|_r}{r^m}, \frac{|\mu|_{r'}}{r'^m}) \\ &\leq \max(\frac{|\mu|_r}{r^m}, \frac{|\mu|_{r''}}{r''^m}) = \max(|s|_r^{-1}|\mu|_r, |s|_{r''}^{-1}|\mu|_{r''}) \ . \end{aligned}$$

This shows that the assumptions of the universal property Prop. 1.18 are satisfied. The above composed homomorphism extends uniquely to a norm decreasing unital homomorphism of K-Banach algebras

$$D_{[r,r'']}(G,K) \longrightarrow D_{[r,r']}(G,K) \ .$$

We then may pass to the projective limit with respect to r'' and obtain the K-Fréchet algebra

$$D_{[r,r_1)}(G,K) := \varprojlim_{r \leq r'' < r_1} D_{[r,r'']}(G,K)$$

representing a “noncommutative half open annulus”.

A similar argument shows that the natural homomorphism $D_{r''}(G,K) \longrightarrow D_{[r',r'']}(G,K)$ extends uniquely to a norm decreasing unital homomorphism of K-Banach algebras

$$D_{[r,r'']}(G,K) \longrightarrow D_{[r',r'']}(G,K)\ .$$

Again we obtain a K-Fréchet algebra

$$D_{(r_0,r'']}(G,K) := \varprojlim_{r_0<r\leq r''} D_{[r,r'']}(G,K)\ .$$

It also follows that the Fréchet algebras $D_{[r,r_1)}(G,K)$ form an inductive system with respect to r. Especially in the case $r_1 = 1$ the inductive limit

$$R(G,K) := \varinjlim D_{[r,1)}(G,K)$$

is a locally convex unital K-algebra which we call the *Robba ring* of G. Although we suppress this in the notation this ring does depend on the initial choice of a basis $\{h_1,\ldots,h_d\}$ of G but not on its ordering. It is shown in [ST] in the discussion following Thm. 4.10 that, for $1/p \leq r < 1$, the norm $\|\ \|_r$ on $D(G,K)$ is completely independent of the choice of the ordered basis. It follows that each $D_{[r,r']}(G,K)$ together with its norm $\|\ \|_{r,r'}$ as well as the topological algebras $D_{[r,r_1)}(G,K)$, $D_{(r_0,r'']}(G,K)$, and $R(G,K)$ do not depend on the ordering of the chosen basis of G. In fact a little more is true. In the ring $\mathbb{Z}_p[[Z]]$ of formal power series in one variable over $\mathbb{Z}_p$ we have, for any $x \in \mathbb{Z}_p$, the identity

$$(3)\qquad (1+Z)^x - 1 = \sum_{i\geq 1}\binom{x}{i} Z^i = Z(x + Zf_x(Z)) \qquad \text{with } f_x \in \mathbb{Z}_p[[Z]].$$

If $x \in \mathbb{Z}_p^\times$ then $x + Zf_x(Z)$ is a unit in $\mathbb{Z}_p[[Z]]$. This shows that for $x \in \mathbb{Z}_p^\times$ we have $h_i^x - 1 \in b_i \cdot D(G,K)^\times$ for any $1 \leq i \leq d$. Applying the universal property Prop. 1.18 we conclude that replacing $\{h_1,\ldots,h_d\}$ by $\{h_1^{x_1},\ldots,h_d^{x_d}\}$ for some $x_1,\ldots,x_d \in \mathbb{Z}_p^\times$ does not change the Banach algebras $D_{[r,r']}(G,K)$ together with their norm $\|\ \|_{r,r'}$ and hence does not change the topological algebras $D_{[r,r_1)}(G,K)$, $D_{(r_0,r'']}(G,K)$, and $R(G,K)$ either.

In order to be able to work with these rings we will show that their elements can be viewed as Laurent series. For that we introduce the affinoid domain

$$X^d_{[r,r']} := \{(z_1,\ldots,z_d) \in \mathbb{C}_p^d : r \leq |z_1| = \ldots = |z_d| \leq r'\}\ .$$

The ring $\mathcal{O}_K(X^d_{[r,r']})$ of K-analytic functions on $X^d_{[r,r']}$ is the ring of all Laurent series

$$f(Z_1,\ldots,Z_d) = \sum_{\alpha\in\mathbb{Z}^d} d_\alpha \mathbf{Z}^\alpha$$

with $d_\alpha \in K$ and such that $\lim_{|\alpha|\to\infty} |d_\alpha|\rho^\alpha = 0$ for any $r \leq \rho \leq r'$. Here

$$\mathbf{Z}^\alpha := Z_1^{\alpha_1}\cdot\ldots\cdot Z_d^{\alpha_d},\ |\alpha| := |\alpha_1| + \ldots + |\alpha_d|,\ \text{and } \rho^\alpha := \rho^{\alpha_1+\ldots\alpha_d}$$

(with the abuse of notation that $|\alpha_i|$ exceptionally means the archimedean absolute value). Since $\rho^\alpha \leq \max(r^\alpha, r'^\alpha)$ for any $r \leq \rho \leq r'$ and any $\alpha \in \mathbb{Z}^d$ the latter condition on f is equivalent to

$$\lim_{|\alpha|\to\infty} |d_\alpha| r^\alpha = \lim_{|\alpha|\to\infty} |d_\alpha| r'^\alpha = 0\ .$$

The spectral norm on the affinoid algebra $\mathcal{O}_K(X^d_{[r,r']})$ is given by

$$\begin{aligned}|f|_{X^d_{[r,r']}} &= \sup_{r\le\rho\le r'}\max_{\alpha\in\mathbb{Z}^d}|d_\alpha|\rho^\alpha\\ &= \max(\max_{\alpha\in\mathbb{Z}^d}|d_\alpha|r^\alpha, \max_{\alpha\in\mathbb{Z}^d}|d_\alpha|r'^\alpha)\ .\end{aligned}$$

Setting $\mathbf{b}^\alpha := b_1^{\alpha_1}\cdot\ldots\cdot b_d^{\alpha_d}$ for any $\alpha = (\alpha_1,\ldots,\alpha_d)\in\mathbb{Z}^d$ we claim that $f(b_1,\ldots,b_d) := \sum_{\alpha\in\mathbb{Z}^d} d_\alpha\mathbf{b}^\alpha$ converges in $D_{[r,r']}(G,K)$. As a consequence of Prop. 1.21 and Lemma 1.7.iii we have

$$|\mathbf{b}^\alpha|_{r,r'} = \max(r^\alpha, r'^\alpha)$$

for any $\alpha\in\mathbb{Z}^d$. Hence

$$\lim_{|\alpha|\to\infty}|d_\alpha\mathbf{b}^\alpha|_{r,r'} = \lim_{|\alpha|\to\infty}\max(|d_\alpha|r^\alpha, |d_\alpha|r'^\alpha) = \max(\lim_{|\alpha|\to\infty}|d_\alpha|r^\alpha, \lim_{|\alpha|\to\infty}|d_\alpha|r'^\alpha) = 0\ .$$

Therefore

$$\begin{aligned}\mathcal{O}_K(X^d_{[r,r']}) &\longrightarrow D_{[r,r']}(G,K)\\ f &\longmapsto f(b_1,\ldots,b_d)\end{aligned}$$

is a well defined K-linear map. In order to investigate this map we introduce the filtration

$$F^iD_{[r,r']}(G,K) := \{e\in D_{[r,r']}(G,K) : |e|_{r,r'}\le|p|^i\}\qquad\text{for } i\in\mathbb{R}$$

on $D_{[r,r']}(G,K)$. Since K is discretely valued and $r,r'\in p^{\mathbb{Q}}$ this filtration is quasi-integral in the sense of [ST] §1. The corresponding graded ring $gr^\cdot D_{[r,r']}(G,K)$, by Prop. 1.21, is commutative. We let $\sigma(e)\in gr^\cdot D_{[r,r']}(G,K)$ denote the principal symbol of any element $e\in D_{[r,r']}(G,K)$.

Proposition 2.2. *i. $gr^\cdot D_{[r,r']}(G,K)$ is a free $gr^\cdot K$-module with basis $\{\sigma(\mathbf{b}^\alpha) : \alpha\in\mathbb{Z}^d\}$.*

ii. The map

$$\begin{aligned}\mathcal{O}_K(X^d_{[r,r']}) &\xrightarrow{\cong} D_{[r,r']}(G,K)\\ f &\longmapsto f(b_1,\ldots,b_d)\end{aligned}$$

is a K-linear isometric bijection.

Proof. Since $\{s^{-1}\mu : s\in S, \mu\in D_r(G,K)\}$ is dense in $D_{[r,r']}(G,K)$ every element in the graded ring $gr^\cdot D_{[r,r']}(G,K)$ is of the form $\sigma(s^{-1}\mu)$. Suppose that $\mu = \sum_{\alpha\in\mathbb{N}_0^d} d_\alpha\mathbf{b}^\alpha$ and $s = \mathbf{b}^\beta$ for some $\beta\in\mathbb{N}_0^d$. Then $s^{-1}\mu = \sum_{\alpha\in\mathbb{N}_0^d} d_\alpha\mathbf{b}^{-\beta}\mathbf{b}^\alpha$ and, using Lemma 1.7.iii we compute

$$\begin{aligned}|s^{-1}\mu|_{r,r'} &= \max(|s|_r^{-1}|\mu|_r, |s|_{r'}^{-1}|\mu|_{r'})\\ &= \max(\max_{\alpha\in\mathbb{N}_0^d}|d_\alpha|r^{\alpha-\beta}, \max_{\alpha\in\mathbb{N}_0^d}|d_\alpha|r'^{\alpha-\beta})\\ &= \max_{\alpha\in\mathbb{N}_0^d}|d_\alpha|\max(r^{\alpha-\beta}, r'^{\alpha-\beta})\\ &= \max_{\alpha\in\mathbb{N}_0^d}|d_\alpha||\mathbf{b}^{-\beta}\mathbf{b}^\alpha|_{r,r'}\ .\end{aligned}$$

It follows that $gr^{\cdot} D_{[r,r']}(G,K)$ as a $gr^{\cdot} K$-module is generated by the principal symbols $\sigma(\mathbf{b}^{-\beta}\mathbf{b}^{\alpha})$ with $\alpha, \beta \in \mathbb{N}_0^d$. But it also follows that, for a fixed $\beta \in \mathbb{N}_0^d$, the principal symbols $\sigma(\mathbf{b}^{-\beta}\mathbf{b}^{\alpha})$ with α running over $\mathbb{N}_0^d$ are linearly independent over $gr^{\cdot} K$. By Prop. 1.21 we may permute the factors in $\sigma(\mathbf{b}^{-\beta}\mathbf{b}^{\alpha})$ arbitrarily. Hence $gr^{\cdot} D_{[r,r']}(G,K)$ as a $gr^{\cdot} K$-module also is generated by the principal symbols $\sigma(\mathbf{b}^{\alpha})$ with α running over $\mathbb{Z}^d$. Since any given finite number of the latter can be written in the form $\sigma(\mathbf{b}^{-\beta}\mathbf{b}^{\alpha})$ with $\alpha \in \mathbb{N}_0^d$ and a joint $\beta \in \mathbb{N}_0^d$ we in fact obtain that $gr^{\cdot} D_{[r,r']}(G,K)$ is a free $gr^{\cdot} K$-module with basis $\{\sigma(\mathbf{b}^{\alpha}) : \alpha \in \mathbb{Z}^d\}$.

On the other hand, we of course have

$$\begin{aligned} f(b_1,\ldots,b_d) &\leq \max_{\alpha\in\mathbb{Z}^d} |d_\alpha||\mathbf{b}^\alpha|_{r,r'} \\ &= \max_{\alpha\in\mathbb{Z}^d} |d_\alpha| \max(r^\alpha, r'^\alpha) = \max(\max_{\alpha\in\mathbb{Z}^d} |d_\alpha| r^\alpha, \max_{\alpha\in\mathbb{Z}^d} |d_\alpha| r'^\alpha) \\ &= |f|_{X^d_{[r,r']}} \ . \end{aligned}$$

This means that if we introduce on $\mathcal{O}_K(X^d_{[r,r']})$ the filtration defined by the spectral norm then the asserted map respects the filtrations, and by the above reasoning it induces an isomorphism between the associated graded rings. Hence, by completeness of these filtrations, it is an isometric bijection. □

If G is commutative the map in Prop. 2.2, of course, is an isometric isomorphism of Banach algebras. But in general it is very far from being multiplicative.

One useful consequence of Prop. 2.2 is that we have

$$| \ |_{r,r'} = \max(| \ |_{r,r}, | \ |_{r',r'})$$

on $D_{[r,r']}(G,K)$.

The ring $\mathcal{O}_K(X^d_{[r,1)})$ of K-analytic functions on the rigid variety

$$X^d_{[r,1)} := \{(z_1,\ldots,z_d) \in \mathbb{C}_p^d : r \leq |z_1| = \ldots = |z_d| < 1\}$$

is the Fréchet algebra of all Laurent series

$$f(Z_1,\ldots,Z_d) = \sum_{\alpha\in\mathbb{Z}^d} d_\alpha \mathbf{Z}^\alpha$$

with $d_\alpha \in K$ and such that $\lim_{|\alpha|\to\infty} |d_\alpha|\rho^\alpha = 0$ for any $r \leq \rho < 1$. We also introduce the locally convex K-algebra

$$\mathcal{R}^d_K := \varinjlim_r \mathcal{O}_K(X^d_{[r,1)}) \ .$$

By limit arguments the map in Prop. 2.2 extends to K-linear topological isomorphisms

$$\mathcal{O}_K(X^d_{[r,1)}) \xrightarrow{\cong} D_{[r,1)}(G,K)$$

and

$$\mathcal{R}^d_K \xrightarrow{\cong} R(G,K) \ .$$

The coefficients $c_{\beta\gamma,\alpha} \in \mathbb{Q}_p$ in the expansions

$$\mathbf{b}^\beta \mathbf{b}^\gamma = \sum_{\alpha\in\mathbb{Z}^d} c_{\beta\gamma,\alpha} \mathbf{b}^\alpha \qquad \text{for any } \beta,\gamma \in \mathbb{Z}^d$$

can be viewed in any of the rings under consideration.

Lemma 2.3. $|c_{\beta\gamma,\alpha}| \le \min(1, p^{\alpha-\beta-\gamma})$ *for any* $\alpha \ne \beta+\gamma$, *and* $|c_{\beta\gamma,\beta+\gamma} - 1| < 1$.

Proof. By Prop. 1.21 the coefficients of the expansion

$$\mathbf{b}^\beta \mathbf{b}^\gamma - \mathbf{b}^{\beta+\gamma} = \sum_{\alpha \in \mathbb{Z}^d} c'_{\beta\gamma,\alpha} \mathbf{b}^\alpha$$

satisfy

$$|c'_{\beta\gamma,\alpha}| r^\alpha < |\mathbf{b}^{\beta+\gamma}|_{r,r} = r^{\beta+\gamma}$$

for any $1/p < r < 1$ in $p^{\mathbb{Q}}$. Hence $|c'_{\beta\gamma,\alpha}| < r^{\beta+\gamma-\alpha}$. By letting tend r to 1 and p^{-1}, respectively, we obtain $|c'_{\beta\gamma,\alpha}| \le \min(1, p^{\alpha-\beta-\gamma})$. For $\alpha = \beta+\gamma$ this means $|c'_{\beta\gamma,\beta+\gamma}| < 1$. □

3 Bounded rings

In $\mathcal{R}_K^d$ we have the subrings

$$\mathcal{R}_K^{d,b} := \{f = \sum_{\alpha \in \mathbb{Z}^d} d_\alpha \mathbf{Z}^\alpha \in \mathcal{R}_K^d : \|f\| := \sup_\alpha |d_\alpha| < \infty\}$$

and

$$\mathcal{R}_K^{d,int} := \{f \in \mathcal{R}_K^d : \|f\| \le 1\} \ .$$

It is well known that the norm $\| \ \|$ on $\mathcal{R}_K^{d,b}$ is multiplicative. We let $\mathcal{E}_K^d$ denote the completion of $\mathcal{R}_K^{d,b}$ with respect to $\| \ \|$ obtaining a K-Banach algebra.

Lemma 3.1. $\mathcal{E}_K^d$ *is the ring of all formal series* $\sum_{\alpha \in \mathbb{Z}^d} d_\alpha \mathbf{Z}^\alpha$ *such that* $\sup_\alpha |d_\alpha| < \infty$ *and* $\lim_{\sum \alpha_i \le m, |\alpha| \to \infty} |d_\alpha| = 0$ *for any* $m \in \mathbb{N}$.

Proof. Let $\widetilde{\mathcal{E}}$ denote the vector space of all these formal series in the assertion. It is easy to see that $\widetilde{\mathcal{E}}$ is complete with respect to $\| \ \|$. We also need the subspace $\widetilde{\mathcal{E}}_0 \subseteq \widetilde{\mathcal{E}}$ of all formal series $\sum_{\alpha \in \mathbb{Z}^d} d_\alpha \mathbf{Z}^\alpha$ such that, for any $m \in \mathbb{N}$, there are only finitely many α with $\sum_i \alpha_i \le m$ and $d_\alpha \ne 0$.

In a first step we consider an arbitrary element $\sum_{\alpha \in \mathbb{Z}^d} d_\alpha \mathbf{Z}^\alpha$ in $\mathcal{R}_K^d$. Then $\lim_{|\alpha| \to \infty} |d_\alpha| r^\alpha = 0$ for some $0 < r < 1$. It follows that

$$\lim_{\sum \alpha_i \le 0, |\alpha| \to \infty} |d_\alpha| \le \lim_{\sum \alpha_i \le 0, |\alpha| \to \infty} |d_\alpha| r^\alpha = 0$$

and

$$r^m \lim_{\sum \alpha_i = m, |\alpha| \to \infty} |d_\alpha| = \lim_{\sum \alpha_i = m, |\alpha| \to \infty} |d_\alpha| r^\alpha = 0 \qquad \text{for any } m \in \mathbb{N} \ .$$

Hence

$$\lim_{\sum \alpha_i \le m, |\alpha| \to \infty} |d_\alpha| = 0 \qquad \text{for any } m \in \mathbb{N} \ .$$

In particular, this shows that $\mathcal{R}_K^{d,b} \subseteq \widetilde{\mathcal{E}}$.

In a second step we suppose that $\sum_{\alpha \in \mathbb{Z}^d} d_\alpha \mathbf{Z}^\alpha$ lies in $\widetilde{\mathcal{E}}_0$. We claim that $\lim_{|\alpha| \to \infty} |d_\alpha| r^\alpha = 0$ for any $0 < r < 1$. Let $\epsilon > 0$. We have show that $|d_\alpha| r^\alpha < \epsilon$ for all but finitely many α. Choose $m \in \mathbb{N}$ in such a way that $(\sup_\alpha |d_\alpha|) r^m < \epsilon$. Then certainly $|d_\alpha| r^\alpha < \epsilon$ for any α such that

$\sum \alpha_i \geq m$. But by assumption there are only finitely many nonzero d_α with $\sum \alpha_i \leq m$. This establishes that $\widetilde{\mathcal{E}}_0 \subseteq \mathcal{R}_K^{d,b}$.

In a third step we argue that $\widetilde{\mathcal{E}}_0$ is dense in $\widetilde{\mathcal{E}}$. Let $f = \sum_{\alpha\in\mathbb{Z}^d} d_\alpha \mathbf{Z}^\alpha$ be an arbitrary element in $\widetilde{\mathcal{E}}$. For any $\epsilon > 0$ the sets

$$N_0(\epsilon) := \{\alpha : \sum \alpha_i \leq 0, |d_\alpha| \geq \epsilon\} \quad \text{and} \quad N_m(\epsilon) := \{\alpha : \sum \alpha_i = m, |d_\alpha| \geq \epsilon\} \text{ for } m \in \mathbb{N}$$

are finite. Hence $f_\epsilon := \sum_{\alpha\in N(\epsilon)} d_\alpha \mathbf{Z}^\alpha$ with $N(\epsilon) := \bigcup_{m\geq 0} N_m(\epsilon)$ lies in $\widetilde{\mathcal{E}}_0$ and $\|f - f_\epsilon\| < \epsilon$. $\square$

Let $R^b(G,K)$ and $R^{int}(G,K)$ denote the image in $R(G,K)$ of $\mathcal{R}_K^{d,b}$ and $\mathcal{R}_K^{d,int}$, respectively, under the above isomorphism. By transport of structure we view both subspaces as normed spaces with respect to $\| \ \|$.

Lemma 3.2. *For any $e \in R(G,K)$ we have*

$$\lim_{r<1, r\to 1} |e|_{r,r} = \begin{cases} \|e\| & \text{if } e \in R^b(G,K), \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Let $0 < r_0 < 1$ be such that $e \in D_{[r_0,1)}(G,K)$. We certainly may assume that $e \neq 0$. Then we may consider the function

$$\phi(\rho) := \log(|e|_{\exp(\rho),\exp(\rho)})$$

on $[\rho_0, 0)$ where $\rho_0 := \log(r_0)$. Let $e = \sum_{\alpha\in\mathbb{Z}^d} d_\alpha \mathbf{b}^\alpha$ and define $N := \{\alpha \in \mathbb{Z}^d : d_\alpha \neq 0\}$. Then

$$\phi(\rho) = \max_{\alpha\in N}(\log(|d_\alpha|) + (\sum_i \alpha_i)\rho) \ .$$

In particular, as a supremum of affine functions the function ϕ is convex on $[\rho_0, 0)$. If e is not in $R^b(G,K)$ then $\{\log(|d_\alpha|)\}_\alpha$ is unbounded which easily implies that $\lim_{\rho<0,\rho\to 0} \phi(\rho) = \infty$. On the other hand, if $e \in R^b(G,K)$ then ϕ extends by $\phi(0) := \log(\|e\|)$ to a convex function on $[\rho_0, 0]$, and we have

$$\log(|d_\alpha|) + (\sum_i \alpha_i)\rho \leq \phi(\rho) \leq \phi(0) + \frac{\phi(\rho_0) - \phi(0)}{\rho_0}\rho$$

for any $\alpha \in N$ and any $\rho \in [\rho_0, 0]$. Let

$$M := \{\beta \in N : |d_\beta| = \max_\alpha |d_\alpha| = \psi(1)\} \ .$$

We have $\sum_i \beta_i \geq \frac{\phi(\rho_0)-\phi(0)}{\rho_0}$ for any $\beta \in M$. Hence we may choose a $\gamma \in M$ in such a way that $\sum_i \gamma_i$ is minimal. Then

$$\log(|d_\beta|) + (\sum_i \beta_i)\rho \leq \log(|d_\gamma|) + (\sum_i \gamma_i)\rho$$

for any $\beta \in M$ and any $\rho \in [\rho_0, 0]$. On the other hand, if we put

$$c := \max\{\log(|d_\alpha|) : \alpha \in N \setminus M\}$$

then

$$\log(|d_\alpha|) + (\sum_i \alpha_i)\rho \le c + \frac{\phi(\rho_0) - c}{\rho_0}\rho$$

for any $\alpha \in N \setminus M$ and any $\rho \in [\rho_0, 0]$. Altogether we obtain

$$\phi(\rho) \le \max(c + \frac{\phi(\rho_0) - c}{\rho_0}\rho, \log(|d_\gamma|) + (\sum_i \gamma_i)\rho)$$

for any $\rho \in [\rho_0, 0]$. We certainly find a $\rho_0 < \rho_1 < 0$ such that

$$c + \frac{\phi(\rho_0) - c}{\rho_0}\rho \le \log(|d_\gamma|) + (\sum_i \gamma_i)\rho$$

for $\rho \in [\rho_1, 0]$. We conclude that on $[\rho_1, 0]$ the function ϕ coincides with the affine function $\log(|d_\gamma|) + (\sum_i \gamma_i)\rho$. □

Proposition 3.3. *$R^b(G,K)$ and $R^{int}(G,K)$ are subrings of $R(G,K)$; moreover, the norm $\| \ \|$ is multiplicative in this ring multiplication.*

Proof. For any $f_1 = \sum_{\beta \in \mathbb{Z}^d} c_\beta \mathbf{b}^\beta$ and $f_2 = \sum_{\gamma \in \mathbb{Z}^d} d_\gamma \mathbf{b}^\gamma$ in $R(G,K)$ we have

$$\begin{aligned} f_1 f_2 = (\sum_\beta c_\beta \mathbf{b}^\beta)(\sum_\gamma d_\gamma \mathbf{b}^\gamma) &= \sum_{\beta,\gamma} c_\beta d_\gamma \mathbf{b}^\beta \mathbf{b}^\gamma \\ = \sum_{\beta,\gamma} c_\beta d_\gamma \sum_\alpha c_{\beta\gamma,\alpha} \mathbf{b}^\alpha &= \sum_\alpha (\sum_{\beta,\gamma} c_\beta d_\gamma c_{\beta\gamma,\alpha}) \mathbf{b}^\alpha \ . \end{aligned}$$

If $f_1, f_2 \in R^b(G,K)$ we therefore, using Lemma 2.3, obtain

$$\|f_1 f_2\| = \sup_\alpha |\sum_{\beta,\gamma} c_\beta d_\gamma c_{\beta\gamma,\alpha}| \le \sup_\beta |c_\beta| \cdot \sup_\gamma |d_\gamma| = \|f_1\| \cdot \|f_2\| < \infty \ .$$

Hence $R^b(G,K)$ and $R^{int}(G,K)$ are subrings and $\| \ \|$ is submultiplicative. But $\| \ \|$, according to Lemma 3.2 and Lemma 1.22, is a limit of multiplicative norms and therefore is, in fact, multiplicative. □

As a consequence of this latter proposition we may complete the algebra $R^b(G,K)$ with respect to the norm $\| \ \|$ obtaining a K-Banach algebra $E(G,K)$ with multiplicative norm $\| \ \|$. Of course, as Banach spaces, we have our isometric isomorphism

$$\mathcal{E}_K^d \xrightarrow{\cong} E(G,K) \ .$$

It follows from Lemma 3.2 that the rings $R^b(G,K)$, $R^{int}(G,K)$, and $E(G,K)$ together with their norm $\| \ \|$ are independent of the ordering of our chosen basis $h_1, \ldots, h_d$ of G (because this is the case for the norms $| \ |_{r,r}$ as we have argued earlier).

The argument in the proof of Lemma 2.3 has another interesting consequence. To formulate it we introduce the following convention. Any of the rings $D_{[r_0,r]}(G,K)$, $D_{[r_0,1)}(G,K)$, $R(G,K)$, and $E(G,K)$ has its natural multiplication which in general is noncommutative and to which we some times refer as the intrinsic multiplication (always written as $(e,f) \longmapsto ef$). But using the bijection from Prop. 1.21 and its extensions these rings carry, by transport of structure, also a commutative multiplication which we write as $(e,f) \longmapsto e \circ f$. The notation e^{-1} always will refer to the inverse with respect to the intrinsic multiplication.

Lemma 3.4. *For any $e, f \in D_{[r_0,r]}(G,K)$ we have*

$$|ef - e \circ f|_{r_0,r} < |ef|_{r_0,r} = |e \circ f|_{r_0,r} \ .$$

Proof. Since $|\ |_{r_0,r} = \max(|\ |_{r_0,r_0}, |\ |_{r,r})$ it suffices to treat the case $r_0 = r$. Then the norm is multiplicative so that we have to show that

$$|ef - e \circ f|_{r,r} < |e|_{r,r}|f|_{r,r} \ .$$

Let $e = \sum_{\beta \in \mathbb{Z}^d} c_\beta \mathbf{b}^\beta$ and $f = \sum_{\gamma \in \mathbb{Z}^d} d_\gamma \mathbf{b}^\gamma$. We have

$$ef = \sum_\alpha (\sum_{\beta,\gamma} c_\beta d_\gamma c_{\beta\gamma,\alpha})\mathbf{b}^\alpha \quad \text{and} \quad e \circ f = \sum_\alpha (\sum_{\beta+\gamma=\alpha} c_\beta d_\gamma)\mathbf{b}^\alpha$$

and hence

$$|ef - e \circ f|_{r,r} = \max_\alpha |\sum_{\beta+\gamma\neq\alpha} c_\beta d_\gamma c_{\beta\gamma,\alpha} + \sum_{\beta+\gamma=\alpha} c_\beta d_\gamma (c_{\beta\gamma,\alpha} - 1)|r^\alpha \ .$$

From the proof of Lemma 2.3 we know that

$$|c_{\beta\gamma,\alpha}|r^\alpha < r^{\beta+\gamma} \quad \text{for } \beta + \gamma \neq \alpha \text{ and } \quad |c_{\beta\gamma,\beta+\gamma} - 1| < 1 \ .$$

We deduce that

$$\begin{aligned} |ef - e \circ f|_{r,r} &< \max_{\beta,\gamma} |c_\beta d_\gamma| r^{\beta+\gamma} \\ &\leq \max_\beta |c_\beta| r^\beta \cdot \max_\gamma |d_\gamma| r^\gamma \\ &= |e|_{r,r} \cdot |f|_{r,r} \ . \end{aligned}$$

□

Proposition 3.5. *Let D be any of the rings $D_{[r_0,r]}(G,K)$, $D_{[r_0,1)}(G,K)$, or $R(G,K)$; we then have:*

i. $e \in D$ is a unit with respect to the intrinsic multiplication if and only if it is a unit with respect to the commutative multiplication;

ii. any left or right unit in D is a unit.

Proof. By limit arguments it suffices to consider the case $D = D_{[r_0,r]}(G,K)$. Suppose first that e is a commutative unit, i.e., $e \circ f = 1$ for some $f \in D$. By Lemma 3.4 we then have $|ef - 1|_{r_0,r} < 1$ which implies, since we are in a Banach algebra, that ef is an intrinsic unit. Starting from $f \circ e = 1$ we similarly obtain that fe is an intrinsic unit. Hence e and f are intrinsic units. Now let, vice versa, e be a left intrinsic unit (the case of a right one being analogous), say, $ef = 1$ for some $f \in D$. By a totally analogous reasoning as above we obtain that $e \circ f = f \circ e$ and hence e and f are commutative units. But then we actually may apply the first part of the proof to conclude that e is an intrinsic unit. □

Proposition 3.6. *i. $R^b(G,K) \cap R(G,K)^\times = R^b(G,K)^\times$;*

ii. an element in $R^b(G,K)$ is a unit with respect to the intrinsic multiplication if and only if it is a unit with respect to the commutative multiplication.

Proof. i. Let $e \in R(G,K)^\times$ such that $\|e\| < \infty$. Suppose that $e \in D_{[r_0,1)}(G,K)^\times$. By Lemma 3.2 and its proof we know that $\lim_{r<1,r\to 1} |e|_{r,r} = \|e\|$ and that the function $\phi(\rho) := \log(|e|_{\exp(\rho),\exp(\rho)})$ is an affine function on $[\rho_1, 0]$ for ρ_1 sufficiently close to 1. Since the norms $|\ |_{\exp(\rho),\exp(\rho)}$, by Lemma 1.22, are multiplicative on $D_{[r_0,1)}(G,K)$ it follows that

$$\lim_{\rho<0,\rho\to 0} \log(|e^{-1}|_{\exp(\rho),\exp(\rho)}) = -\lim_{\rho<0,\rho\to 0} \phi(\rho) = -\log(\|e\|) < \infty\ .$$

Hence, again by Lemma 3.2, we have $e^{-1} \in R^b(G,K)$. ii. This follows from Prop. 3.5.i by applying the present assertion i. to G as well as the commutative group $\mathbb{Z}_p^d$. $\square$

Lemma 3.7. *We have $h_i^x - 1 \in R^{int}(G,K)^\times$ for any $x \in \mathbb{Z}_p \setminus \{0\}$ and any $1 \le i \le d$.*

Proof. Write $x = p^m y$ with $y \in \mathbb{Z}_p^\times$. We know from (3) that $(1+Z)^y - 1 \in Z \cdot \mathbb{Z}_p[[Z]]^\times$ and hence $(1+Z)^x - 1 \in [(1+Z)^{p^m} - 1] \cdot \mathbb{Z}_p[[Z]]^\times$. Moreover the leading coefficient of $(1+Z)^{p^m} - 1$ is equal to 1. Hence

$$h_i^x - 1 \in (h_i^{p^m} - 1) \cdot R^{int}(G,K)^\times \qquad \text{and} \qquad \|h_i^x - 1\| = 1\ . \tag{4}$$

By Prop. 3.6.i and the multiplicativity of $\|\ \|$ it now suffices to show that $h_i^{p^m} - 1$ is invertible in $R(G,K)$. But the polynomial $(1+Z)^{p^m} - 1$ has no zeros in an appropriate annulus $r_m \le |z| < 1$. This implies that $h_i^{p^m} - 1 \in D_{[r_0,1)}(G,K)^\times$. $\square$

The formula (4) implies that the rings $R^{int}(G,K) \subseteq R^b(G,K) \subseteq E(G,K)$ together with the norm $\|\ \|$ do not change if we replace the generating set $\{h_1, \ldots, h_d\}$ by $\{h_1^{x_1}, \ldots, h_d^{x_d}\}$ for some $x_1, \ldots, x_d \in \mathbb{Z}_p^\times$.

References

[B-GT] Bourbaki N.: *General Topology.* Berlin-Heidelberg-New York: Springer 1989

[DDMS] Dixon J.D., du Sautoy M.P.F., Mann A., Segal D.: *Analytic Pro-p-Groups.* Cambridge Univ. Press 1999

[ST] Schneider P., Teitelbaum J.: *Algebras of p-adic distributions and admissible representations.* Invent. math. 153, 145-196 (2003)

[Spr] Springer T.A.: *Microlocalisation algebrique.* In Sém. d'Algèbre Paul Dubreil et M.-P. Malliavin 1984, Lecture Notes Math. 1146, pp. 299-316. Berlin-Heidelberg-New York: Springer 1985

[vdE] van den Essen A.: *Algebraic Micro-Localization.* Communic. Alg. 14, 971-1000 (1986)