

# On the codimension of modules over skew power series rings with applications to Iwasawa algebras

by Peter Schneider and Otmar Venjakob

## Introduction

The purpose of this paper is twofold. First of all we will set up a general notion of skew power series rings. In a skew power series ring in one variable  $t$  over a ring  $R$  of coefficients the way the variable  $t$  commutes with a coefficient  $a \in R$  should be subject to a relation of the form

$$ta = \sigma(a)t + \delta(a)$$

where  $\sigma$  is an automorphism of the ring  $R$  and  $\delta$  is a certain kind of derivation on  $R$ . This is modelled on the well known notion of skew polynomial rings (cf. [MCR] 1.2). But it is clear that in the context of power series a convergence issue will arise. It therefore seems quite natural to work in the context of topological rings. Having our applications in mind we will place ourselves in the context of pseudocompact rings. In the first section we will see that to have a well defined ring structure on skew power series subjected to the above relation essentially comes down to a certain nilpotency condition on the derivation  $\delta$ . We also will construct a natural filtration on the skew power series ring such that the associated graded ring is a skew polynomial ring (in fact with zero derivation). This provides a basic tool to establish ring theoretic properties of skew power series rings and extends work of the second author in [Ve2].

Suppose now that  $S$  is a noetherian skew power series ring over the noetherian ring  $R$ . Our second theme in sections 2 and 3 is the investigation of the class of  $S$ -modules which are finitely generated as  $R$ -modules. In Thm. 2.2 we establish a short exact sequence for any such module  $M$  which presents  $M$  as the cokernel of a twisted endomorphism of the  $S$ -module  $S \otimes_R M$ . This fact will be explored in two ways. First, through a careful analysis of the associated long exact Ext-sequence we will show in Thm. 3.1 that the groups  $\text{Ext}_S^*(M, S)$  and  $\text{Ext}_R^{*-1}(M, R)$  are (up to a twist by  $\sigma$ ) naturally isomorphic. In the case where the rings  $R$  and  $S$  are Auslander regular this means that the geometric intuition that the codimension of  $M$  as an  $S$ -module is one higher than the codimension of  $M$  as an  $R$ -module is correct. Second, under an additional condition on the form of the automorphism  $\sigma$  it allows to show that the class of  $M$  in the Grothendieck group of modules of codimension at most one less vanishes. In fact, this can be formulated (Prop. 3.4) as the vanishing of a certain natural map on higher  $K$ -theory, which is in the spirit of the Gersten conjecture for commutative regular local rings (cf. [Ger]).

In section 4 we show that all these results apply to Iwasawa algebras  $\Lambda(G)$  of  $p$ -adic Lie groups  $G$ . The investigation of these rings, which play an increasingly important role in number theory, certainly was our original motivation.

In [CFKSV] a Iwasawa main conjecture for elliptic curves  $E$  without complex multiplication is formulated, claiming a deep relationship between a certain torsion  $\Lambda(G)$ -module (the Selmer group of  $E$ ) and the values of the Hasse-Weil  $L$ -function of  $E$ . Here  $p$  is a prime number at which  $E$  has good ordinary reduction and  $G$  is the Galois group of the extension of the field of rational numbers  $\mathbb{Q}$  which arises by adjoining the  $p$ -power division points of  $E$  and thus, by a theorem of Serre, is an open subgroup of  $GL_2(\mathbb{Z}_p)$ . This  $G$  possesses a natural quotient isomorphic to the additive group of  $p$ -adic integers  $\mathbb{Z}_p$ , the Galois group of the so called cyclotomic  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}$ . We denote the corresponding normal subgroup in  $G$  by  $H$ . It then is easy to see that  $\Lambda(G)$  is isomorphic to a skew power series ring over  $\Lambda(H)$ . Those torsion  $\Lambda(G)$ -modules which are finitely generated over  $\Lambda(H)$  play a central role in the formulation of the main conjecture. On the other hand, in [CSS] a structure theorem for finitely generated torsion  $\Lambda(G)$ -modules up to pseudo-null modules (see section 4 for the definition) was proven. In this context our results imply that any pseudo-null  $\Lambda(G)$ -module which is finitely generated as a  $\Lambda(H)$ -module has zero class in the Grothendieck group of all finitely generated torsion  $\Lambda(G)$ -modules.

In the final section we briefly discuss how our results extend to skew Laurent series rings.

We remark that everything that follows holds true in an analogous way, and with much simpler proofs, for skew polynomial rings. But whereas the analog of our short exact sequence in Thm. 2.2 is contained in [MCR] 7.5.2 its application to the study of Ext-groups in section 3 seems to have gone unnoticed in the literature. So the analog of our Thm. 3.1 constitutes a generalization of Rees' lemma in [Ree]. In fact, we later will use a result from [LVO] (their Thm. III.3.4.6) which is based on such a generalization but which is stated in loc. cit. with an incomplete proof.

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## 0. Notations and reminders

For any (unital) ring  $R$  we let  $\mathfrak{M}(R)$  denote the abelian category of left (unital)  $R$ -modules. If  $R$  is left noetherian then the finitely generated left  $R$ -modules form a full abelian subcategory  $\mathfrak{M}_{fg}(R)$  of  $\mathfrak{M}(R)$ ; we let  $G(R)$  be the Grothendieck group of the category  $\mathfrak{M}_{fg}(R)$  with respect to short exact sequences. By  $\text{Jac}(R)$  we denote the Jacobson radical of  $R$ .

A left pseudocompact ring  $R$  is a (unital) topological ring which is Hausdorff and complete and which has a fundamental system  $(L_i)_{i \in I}$  of open neighbourhoods of zero consisting of left ideals  $L_i \subseteq R$  such that  $R/L_i$  is of finite length as an  $R$ -module. A pseudocompact left  $R$ -module is a (unital) topological  $R$ -module which is Hausdorff and complete and which has a fundamental system  $(M_i)_{i \in I}$

of open neighbourhoods of zero consisting of submodules  $M_i \subseteq M$  such that the  $R$ -module  $M/M_i$  is of finite length. We let  $\mathfrak{PM}(R)$  denote the category of pseudocompact left  $R$ -modules with continuous  $R$ -linear maps. The category  $\mathfrak{PM}(R)$  is abelian with exact projective limits and the forgetful functor  $\mathfrak{PM}(R) \rightarrow \mathfrak{M}(R)$  is faithful and exact and commutes with projective limits ([Ga1] IV.3 Thm. 3, [vGB] Prop. 3.3). This implies easily that any finitely generated submodule of a pseudocompact module is closed and is pseudocompact in the subspace topology. An arbitrary direct product of pseudocompact modules is pseudocompact. A pseudocompact  $R$ -module  $M$  will be called topologically free if it is topologically isomorphic to a pseudocompact module of the form  $\prod_{i \in I} R$ ; the set of elements of  $M$  corresponding to the elements  $(\dots, 0, 1, 0, \dots)$ , with the 1 in the  $i$ -th place, then is called a pseudobasis of  $M$ . Topologically free pseudocompact modules are projective objects in the category  $\mathfrak{PM}(R)$ ; one easily concludes that  $\mathfrak{PM}(R)$  has enough projective objects ([Bru] Cor. 1.3 and Lemma 1.6). In particular, for any two pseudocompact  $R$ -modules  $M$  and  $N$ , Ext-functors  $\mathcal{E}xt_R^*(M, N)$  may be defined by left deriving the functor  $\text{Hom}_{\mathfrak{PM}(R)}(\cdot, N)$ . Suppose that  $R$  is left noetherian. Then any finitely generated left  $R$ -module carries a unique pseudocompact topology. Hence we have a natural fully faithful and exact embedding  $\mathfrak{M}_{fg}(R) \rightarrow \mathfrak{PM}(R)$ . Given a finitely generated  $R$ -module  $M$  and an arbitrary pseudocompact  $R$ -module  $N$  one easily shows, by using a resolution of  $M$  by finitely generated projective  $R$ -modules, that one has

$$\mathcal{E}xt_R^*(M, N) = \text{Ext}_R^*(M, N) ,$$

where the right hand side denotes the usual Ext-groups of any two modules in  $\mathfrak{M}(R)$ .

There are obvious “right” versions of everything above. If we are dealing with these we usually will indicate this by adding a superscript “r” to the notation; e.g.,  $\mathfrak{M}^r(R)$  denotes the category of right  $R$ -modules. By an ideal we always mean a 2-sided ideal, and a ring will be called noetherian if it is left and right noetherian.

*Remark 0.1:* Suppose that  $R$  is left noetherian; we then have:

- i. If  $R$  is left pseudocompact then
  - the ring  $R/\text{Jac}(R)$  is left artinian, and
  - the pseudocompact topology on  $R$  is the  $\text{Jac}(R)$ -adic topology;
- ii. if there is an ideal  $I \subseteq R$  such that  $R/I$  is left artinian and  $R$  is  $I$ -adically complete then  $R$  is left pseudocompact in the  $I$ -adic topology.

Proof: i. See [vGB] Cor. 3.14. ii. Each  $I^n/I^{n+1}$  is a finitely generated left module over the left artinian ring  $R/I$  and hence is of finite length. It follows inductively that each  $R/I^n$  is a left  $R$ -module of finite length.

*Remark 0.2:* Suppose the left pseudocompact ring  $R$  is noetherian; then  $R$  also is right pseudocompact (in the same topology).

Proof: This follows from Remark 0.1 since the noetherian and left artinian ring  $R/I$  necessarily is right artinian as well ([CH] p.55).

Suppose that  $R \longrightarrow S$  is a continuous (unital) homomorphism of noetherian pseudocompact rings. For any pseudocompact left  $R$ -module  $M$  we define its (completed) *base change* to  $S$  by

$$S \widehat{\otimes}_R M := \varprojlim_{i,L} S/\text{Jac}(S)^i \otimes_R M/L$$

where  $i$  runs over  $\mathbb{N}$  and  $L$  runs over all open submodules of  $M$ . Since  $M/L$  is of finite length over  $R$  we find an integer  $j \geq 0$  such that  $\text{Jac}(R)^j M \subseteq L$ . By possibly increasing  $j$  we have

$$S/\text{Jac}(S)^i \otimes_R M/L = S/\text{Jac}(S)^i \otimes_{R/\text{Jac}(R)^j} M/L .$$

Because  $M/L$  is finitely generated over the artinian ring  $R/\text{Jac}(R)^j$  it follows that  $S/\text{Jac}(S)^i \otimes_{R/\text{Jac}(R)^j} M/L$  is finitely generated over the artinian ring  $S/\text{Jac}(S)^i$  and hence is of finite length over  $S$ . Therefore  $S \widehat{\otimes}_R M$  is a projective limit of finite length  $S$ -modules and as such is pseudocompact. We have constructed in this way a functor

$$S \widehat{\otimes}_R . : \mathfrak{PM}(R) \longrightarrow \mathfrak{PM}(S) .$$

It is right exact since projective limits in  $\mathfrak{PM}(S)$  are exact. If  $M$  is finitely generated then, by looking at a finite presentation of  $M$  over  $R$ , it is rather obvious that one has

$$S \widehat{\otimes}_R M = S \otimes_R M .$$

*Remark 0.3:* If  $S$  is pseudocompact and topologically free as a right  $R$ -module then the functor  $S \widehat{\otimes}_R .$  is exact.

Proof: As a topological abelian group  $S \widehat{\otimes}_R M$  only depends on the structure of  $S$  as a pseudocompact right  $R$ -module. Under our additional assumption we may write  $S$  as the projective limit  $S = \varprojlim S_i$  of right factor  $R$ -modules  $S_i$  of  $S$  which are finitely generated and free over  $R$ . By the argument in the proof of [Bru] Lemma A.4 the natural map

$$S \widehat{\otimes}_R M \longrightarrow \varprojlim (S_i \otimes_R M)$$

is bijective. Since  $S_i$  is free over  $R$  and since the transition maps in the projective system  $\{S_i \otimes_R M\}_i$  are surjective (for any  $M$ ) the right hand side of the above bijection is an exact functor in  $M$ .

## 1. Skew power series rings

In order to motivate our later construction of skew power series rings we first recall briefly the well known notion of a skew polynomial ring (cf. [MCR] 1.2). Let  $R$  be an arbitrary unital ring. We suppose given an automorphism  $\sigma$  of the ring  $R$  as well as a left  $\sigma$ -derivation  $\delta : R \rightarrow R$  which is an additive map satisfying

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b) \quad \text{for any } a, b \in R .$$

The left  $R$ -module of all (left) polynomials  $a_0 + a_1t + \dots + a_mt^m$  over  $R$  in the variable  $t$  carries a unique unital ring structure which extends the  $R$ -module structure and satisfies

$$ta = \sigma(a)t + \delta(a) \quad \text{for any } a \in R .$$

This ring is denoted by  $R[t; \sigma, \delta]$ . Put  $\sigma' := \sigma^{-1}$  and  $\delta' := -\delta\sigma^{-1}$ . Then  $\delta'$  is a right  $\sigma'$ -derivation, i.e., an additive map satisfying

$$\delta'(ab) = \delta'(a)\sigma'(b) + a\delta'(b) \quad \text{for any } a, b \in R .$$

The skew polynomial ring  $R[t; \sigma, \delta]$  can alternatively be described as the right  $R$ -module of all (right) polynomials  $a_0 + ta_1 + \dots + t^ma_m$  over  $R$  with the multiplication determined by the relation

$$at = t\sigma'(a) + \delta'(a) \quad \text{for any } a \in R .$$

To make this explicit we let  $M_{k,l}(Y, Z)$ , for any integers  $k, l \geq 0$ , denote the sum of all noncommutative monomials in two variables  $Y, Z$  with  $k$  factors  $Y$  and  $l$  factors  $Z$ . In  $R[t; \sigma, \delta]$  one then has the formulas

$$(1) \quad \sum_{i \geq 0} t^i a_i = \sum_{j \geq 0} \left( \sum_{i \geq j} M_{i-j, j}(\delta, \sigma)(a_i) \right) t^j$$

and

$$(2) \quad \sum_{j \geq 0} a_j t^j = \sum_{i \geq 0} t^i \left( \sum_{j \geq i} M_{j-i, i}(\delta', \sigma')(a_j) \right) .$$

Furthermore, the multiplication is explicitly given by

$$(3) \quad \left( \sum_{i \geq 0} t^i a_i \right) \left( \sum_{k \geq 0} t^k b_k \right) = \sum_{m \geq 0} t^m \left( \sum_{n=0}^m \sum_{k \geq n} M_{k-n, n}(\delta', \sigma')(a_{m-n}) b_k \right)$$

and

$$(4) \quad \left( \sum_{j \geq 0} a_j t^j \right) \left( \sum_{l \geq 0} b_l t^l \right) = \sum_{m \geq 0} \left( \sum_{n=0}^m \sum_{j \geq n} a_j M_{j-n, n}(\delta, \sigma)(b_{m-n}) \right) t^m ,$$

respectively.

After this reminder we always in this section let  $R$  be a noetherian pseudocompact ring,  $\sigma$  a topological automorphism of  $R$ , and  $\delta : R \rightarrow R$  a continuous left  $\sigma$ -derivation. We define  $S$  to be the left  $R$ -module of all (left) formal power series  $\sum_{i \geq 0} a_i t^i$  over  $R$  in the variable  $t$  and view it as a pseudocompact  $R$ -module with respect to the obvious direct product topology. Then a formal power series  $x = \sum_{i \geq 0} a_i t^i$  can be considered as an expansion of the element  $x$  into a convergent sum. The skew polynomial ring  $R[t; \sigma, \delta]$  is a dense submodule of  $S$ . To extend its ring structure by continuity to  $S$  we need to ensure that the sums which form the coefficients on the right hand sides of the formulas (1) - (4) and which then are infinite converge. We let  $\mathcal{A}_k$  denote the set of all noncommutative monomials in three variables  $Y, Z$ , and  $Z'$  with exactly  $k$  factors  $Y$ , and we put  $\mathcal{A}_{\geq l} := \bigcup_{k \geq l} \mathcal{A}_k$ .

**Definition:** A left or right  $\sigma$ -derivation  $\tilde{\delta} : R \rightarrow R$  is called  $\sigma$ -nilpotent if for any  $n \geq 1$  there is an  $m \geq 1$  such that

$$M(\tilde{\delta}, \sigma, \sigma^{-1})(R) \subseteq \text{Jac}(R)^n \quad \text{for any } M \in \mathcal{A}_{\geq m} .$$

*Remark 1.1:* i. If  $\tilde{\delta}$  and  $\sigma$  commute then  $M(\tilde{\delta}, \sigma, \sigma^{-1})(R) = \tilde{\delta}^k(R)$  for  $M \in \mathcal{A}_k$  and hence  $\tilde{\delta}$  is  $\sigma$ -nilpotent if and only if it is topologically nilpotent.

ii. If  $\tilde{\delta}$  satisfies  $\tilde{\delta}(R) \subseteq \text{Jac}(R)$  and  $\tilde{\delta}(\text{Jac}(R)) \subseteq \text{Jac}(R)^2$  then  $\tilde{\delta}(\text{Jac}(R)^n) \subseteq \text{Jac}(R)^{n+1}$  holds true for any  $n \geq 0$  and, because of  $\sigma^{\pm 1}(\text{Jac}(R)) = \text{Jac}(R)$ , it follows that  $\tilde{\delta}$  is  $\sigma$ -nilpotent.

We assume from now on that  $\delta$  is  $\sigma$ -nilpotent. As before we put  $\sigma' := \sigma^{-1}$  and  $\delta' := -\delta\sigma^{-1}$ . Clearly  $\delta'$  is  $\sigma'$ -nilpotent. One easily checks that by reading (4) as a definition we obtain a continuous multiplication on  $S$  with respect to which  $S$  is a left pseudocompact ring (compare the reasoning in [Ga2] 0.4). Since also the formulas (1) - (3) hold now more generally in  $S$  we see that  $S$  is right pseudocompact as well and that  $\{t^i : i \geq 0\}$  is a pseudobasis of  $S$  as a left and as a right  $R$ -module. We call  $R[[t; \sigma, \delta]] := S$  a (formal) *skew power series ring* over  $R$ .

*Remark 1.2:* If  $\sigma$  and  $\delta$  commute with each other then the map

$$\hat{\sigma} : \begin{array}{ccc} S & \longrightarrow & S \\ \sum_i t^i a_i & \longmapsto & \sum_i t^i \sigma(a_i) \end{array}$$

is a topological automorphism of the pseudocompact ring  $S$ .

Proof: The map obviously is additive, bijective, and a homeomorphism. It remains to check its multiplicativity on elements of the form  $x = t^i a$  and  $y = t^j b$  with  $a, b \in R$ . By a straightforward induction it in fact suffices to consider the case where  $j = 1$  and  $b = 1$ . We compute

$$\widehat{\sigma}(t^i a t) = \widehat{\sigma}(t^i (t \sigma^{-1}(a) - \delta(\sigma^{-1}(a)))) = t^{i+1} a - t^i \delta(a) = t^i \sigma(a) t = \widehat{\sigma}(t^i a) \widehat{\sigma}(t) .$$

In order to transfer properties of the ring  $R$  to the ring  $S$  we introduce a certain filtration on  $R$ . For  $k \in \mathbb{N}$  let  $P_k$  denote the set of all sequences  $\underline{m} = (m_1, \dots, m_r)$  of natural numbers  $m_i > 0$  and varying length  $r$  such that  $m_1 + \dots + m_r = k$ . We define  $\Delta_0 := \{1_R\}$  and

$$\Delta_k := \sum_{\underline{m} \in P_k} \sum_{M_i \in \mathcal{A}_{m_i}} M_1(\delta, \sigma, \sigma^{-1})(R) \cdot \dots \cdot M_r(\delta, \sigma, \sigma^{-1})(R)$$

for  $k \geq 1$ . The following properties are easily verified by induction:

- (a)  $\Delta_k \cdot \Delta_l \subseteq \Delta_{k+l}$ ,
- (b)  $\sigma(\Delta_k) = \Delta_k$ ,
- (c)  $\delta(\Delta_k) \subseteq \Delta_{k+1}$ ,  $\delta(R\Delta_k) \subseteq R\Delta_{k+1}$ ,  $\delta(\Delta_k R) \subseteq \Delta_{k+1}R$ ,
- (d)  $R\Delta_{k+1} \subseteq R\Delta_k$ ,
- (e)  $R\Delta_k = \Delta_k R$ .

In particular, the  $I_k := R\Delta_k$ , for  $k \geq 0$ , form a descending series of 2-sided ideals in  $R$  with  $I_k \cdot I_l \subseteq I_{k+l}$ . The associated graded ring is denoted, as usual, by  $gr_I R$ . The automorphism  $\sigma$  induces an automorphism  $\bar{\sigma}$  of  $gr_I R$ ; the derivation  $\delta$  induces a  $\bar{\sigma}$ -derivation  $\bar{\delta}$  of degree 1 on  $gr_I R$ .

*Remark 1.3:* i. If  $\delta$  is  $\sigma$ -nilpotent with  $\delta(R) \subseteq \text{Jac}(R)$  then we have  $\bigcap I_k = 0$ ;  
ii. if  $\bigcap I_k = 0$  then  $\delta$  is  $\sigma$ -nilpotent and  $R = \varprojlim R/I_k$ .

Proof: i. This is obvious. ii. First of all we note that  $M(\delta, \sigma, \sigma^{-1})(R) \subseteq \Delta_k \subseteq I_k$  for  $M \in \mathcal{A}_k$ . On the other hand, since  $R$  is noetherian the ideals  $I_k$  are closed in  $R$ . Since  $R/\text{Jac}(R)$  is artinian it therefore follows from [Ga1] IV.3. Prop. 11 that for any  $n \geq 1$  there is a  $k \geq 1$  such that  $I_k \subseteq \text{Jac}(R)^n$ . Moreover, [Ga1] IV.3 Prop. 10 says that the canonical map  $R \longrightarrow \varprojlim R/I_k$  is an isomorphism of pseudocompact rings.

The filtration  $I_k$  of  $R$  (with  $I_k := R$  for  $k < 0$ ) induces a filtration  $J_k$  by

$$J_k := \prod_{i \geq 0} I_{k-i} t^i .$$

Indeed we have the following.

**Lemma 1.4:** *i. Each  $J_k$  is a closed 2-sided ideal in  $S$ ;*

*ii.  $J_k \cdot J_l \subseteq J_{k+l}$ ;*

*iii. if  $\bigcap I_k = 0$  then  $S = \varprojlim S/J_k$ ;*

*iv. for the associated graded ring  $gr_J S$  we have  $gr_J S \cong (gr_I R)[\bar{t}; \bar{\sigma}, \bar{\delta}]$  where  $\bar{t}$  denotes the principal symbol of  $t$  in  $gr_J S$ .*

Proof: i. and ii. are direct consequences of the above properties of  $I_k$  and the earlier formula (4). iii. If  $\bigcap I_k = 0$  then also  $\bigcap J_k = 0$  and we may apply again [Ga1] IV.3 Prop. 10. iv. This is straightforward.

**Lemma 1.5:** *Suppose that  $\delta$  is  $\sigma$ -nilpotent with  $\delta(R) \subseteq \text{Jac}(R)$  and that  $gr_I R$  is noetherian, resp. Auslander regular; then  $S$  is noetherian, resp. Auslander regular.*

Proof: (The notion of Auslander regularity will be recalled in section 3.) By Remark 1.3.i and Lemma 1.4.iii the ring  $S$  is complete with respect to the filtration  $J_k$ . Moreover, it is well known that the skew polynomial ring  $gr_J S \cong gr_I R[\bar{t}; \bar{\sigma}]$  over the noetherian, resp. Auslander regular, ring  $gr_I R$  is noetherian, resp. Auslander regular, as well (cf. [MCR] 1.2.9 and [LVO] III.3.4.6). Finally, it is a general fact that a complete filtered ring is noetherian, resp. Auslander regular, if its associated graded ring has this property (cf. [LVO] I.1.2.3 and III.2.2.5).

Since in our main application in section 4 the maps  $\sigma$  and  $\delta$  commute with each other we want to mention an example where this is not the case. Let  $R = R_0[[X]]$  be the commutative formal power series ring in one variable  $X$  over the ring  $R_0 = \mathbb{Z}_p$  or  $R_0 = \mathbb{F}_p$ . Its Jacobson radical  $\text{Jac}(R)$  is the ideal generated by  $p$  and  $X$ . We fix a unit  $u \in R^\times$  and let  $\sigma$  denote the unique continuous automorphism of  $R$  such that  $\sigma(X) = uX$  and  $\sigma|_{R_0} = id$ . We also fix an element  $F \in \text{Jac}(R)^2$ . There is a unique continuous  $\sigma$ -derivation  $\delta$  on  $R$  such that  $\delta(X) = F$  and  $\delta|_{R_0} = 0$ . It satisfies  $\delta(\text{Jac}(R)^j) \subseteq \text{Jac}(R)^{j+1}$  for any  $j \geq 0$  and therefore is  $\sigma$ -nilpotent. Hence the corresponding skew power series ring  $R[[t; \sigma, \delta]]$  is defined. Obviously,  $\sigma$  and  $\delta$  do not commute with each other in general. To be completely explicit let  $R_0 = \mathbb{F}_p$ ,  $F := X^p$ , and  $u \in \mathbb{F}_p \setminus \{0, 1\}$ . Then  $I_k = RX^{pk}$ ; hence  $gr_I R$  is the polynomial ring in one variable over  $\mathbb{F}_p[X]/\mathbb{F}_p[X]X^p$ .

In our main application later the  $\sigma$ -nilpotence of  $\delta$  will be a consequence of the subsequent lemma. There the situation will be such that  $R$  is contained in another noetherian pseudocompact ring  $S'$  in such a way that:

–  $S'$  is pseudocompact and topologically free as a left as well as a right  $R$ -module;



– there is an element  $t \in S'$  such that the powers  $\{t^i : i \geq 0\}$  form a pseudobasis of  $S'$  as a left and as a right  $R$ -module and such that

$$ta = \sigma(a)t + \delta(a) \quad \text{for any } a \in R .$$

**Lemma 1.6:** *Suppose that we are in the above situation (in particular,  $S'$  is noetherian) and that  $\sigma$  and  $\delta$  commute with each other; we then have:*

i.  $\delta^n(a) = \sum_{i=0}^n \binom{n}{i} (-1)^i t^{n-i} \sigma^i(a) t^i$  for any  $a \in R$  and any integer  $n \geq 0$ ;

ii. for any open left ideal  $L \subseteq R$  there is an  $n \geq 0$  such that  $\delta^j(R) \subseteq L$  for any  $j \geq n$ .

Proof: i. This is proved by an easy induction starting from the defining equation for a  $\sigma$ -derivation.

ii. The topology of  $R$  is induced by the topology of  $S'$ . Since  $S'$  is noetherian we therefore find a  $k \geq 0$  such that  $\text{Jac}(S')^k \cap R \subseteq L$ . Since  $\text{Jac}(S')^k$  is open in  $S'$  all but finitely many members of the pseudobasis  $\{t^i : i \geq 0\}$  must lie in  $\text{Jac}(S')^k$ . Hence there is an  $m \geq 0$  such that  $S' t^m S' \cap R \subseteq \text{Jac}(S')^k \cap R \subseteq L$ . But it is an immediate consequence of the formula in i. that  $\delta^{2m}(R) \subseteq S' t^m S' \cap R$ .

We see that in the situation of the lemma  $\delta$  necessarily is topologically nilpotent, hence  $\sigma$ -nilpotent by Remark 1.1.i, and that therefore  $S'$  is isomorphic to the skew power series ring  $R[[t; \sigma, \delta]]$ .

## 2. A short exact sequence

For the rest of this paper we fix a skew power series ring  $S = R[[t; \sigma, \delta]]$  over a noetherian pseudocompact ring  $R$  as constructed in the previous section. In particular,  $\delta$  is  $\sigma$ -nilpotent. We also assume that  $S$  is noetherian.

**Lemma 2.1:** *For any module  $M$  in  $\mathfrak{M}_{fg}(S)$  we have  $\bigcap_{k \geq 0} t^k M = \{0\}$ .*

Proof: By [vGB] Cor. 3.14 the pseudocompact topology on  $M$  is the  $\text{Jac}(S)$ -adic one. It follows that  $\bigcap_{k \geq 0} \text{Jac}(S)^k M = \{0\}$ . Since  $\text{Jac}(S)^k$  is open in  $S$  all but finitely many members of the pseudobasis  $\{t^i : i \geq 0\}$  lie in  $\text{Jac}(S)^k$ . Hence  $\bigcap_{k \geq 0} t^k M \subseteq \bigcap_{k \geq 0} \text{Jac}(S)^k M = \{0\}$ .

We claim that the forgetful functor induces an exact and faithful functor between abelian categories

$$\mathfrak{P}\mathfrak{M}(S) \longrightarrow \mathfrak{P}\mathfrak{M}(R) .$$

Let  $M$  be a pseudocompact left  $S$ -module and let  $M_0 \subseteq M$  be an open submodule. Then  $M/M_0$  is of finite length as  $S$ -module. Hence we have  $\text{Jac}(S)^i M \subseteq M_0$  for some  $i > 0$  so that in fact  $M/M_0$  is an  $S/\text{Jac}(S)^i$ -module of finite length. But

by Remark 0.1.i, since  $S$  is assumed to be noetherian, the  $R$ -module  $S/\text{Jac}(S)^i$  and consequently the  $R$ -module  $M/M_0$  is of finite length. It follows that  $M$  is pseudocompact as an  $R$ -module. Clearly, the base extension functor  $S \widehat{\otimes}_R \cdot$  is left adjoint to the above functor and hence preserves projective objects.

For any left  $R$ -module  $M$  the twisted left  $R$ -module  ${}^\sigma M$  has the same underlying additive group as  $M$  but with  $R$  acting through the automorphism  $\sigma^{-1}$ . If  $M$  is pseudocompact then  ${}^\sigma M$  obviously is pseudocompact as well.

We also introduce the abelian category  $\mathfrak{M}_R(S)$  of those left  $S$ -modules which are finitely generated as  $R$ -modules.

**Theorem 2.2:** *For any module  $M$  in  $\mathfrak{M}_R(S)$  the sequence*

$$0 \longrightarrow S \otimes_R {}^\sigma M \xrightarrow{\kappa} S \otimes_R M \xrightarrow{\mu} M \longrightarrow 0$$

with  $\kappa(x \otimes m) := xt \otimes m - x \otimes tm$  and  $\mu(x \otimes m) := xm$  is an exact sequence of finitely generated  $S$ -modules.

Proof: Obviously all  $S$ -modules in the sequence under consideration are finitely generated. To see that  $\kappa$  is well defined fix an  $a \in R$ . Then

$$\begin{aligned} \kappa(xa \otimes m) &= xat \otimes m - xa \otimes tm \\ &= xat \otimes m - x \otimes atm \\ &= xt\sigma^{-1}(a) \otimes m - x \otimes t\sigma^{-1}(a)m \\ &= xt \otimes \sigma^{-1}(a)m - x \otimes t\sigma^{-1}(a)m \\ &= \kappa(x \otimes \sigma^{-1}(a)m) . \end{aligned}$$

Clearly,  $\kappa$  and  $\mu$  are  $S$ -module maps such that  $\mu \circ \kappa = 0$ , and  $\mu$  is surjective. It remains to show that  $\kappa$  is injective and that the kernel of  $\mu$  is contained in the image of  $\kappa$ .

It is immediate that the kernel of  $\mu$  is generated additively by the elements of the form

$$x \otimes m - 1 \otimes xm \quad \text{for } x \in S \text{ and } m \in M .$$

As a finitely generated module  $S \otimes_R M$  carries a natural pseudocompact topology with respect to which any submodule, being finitely generated as well, is closed. It therefore follows from the power series expansion of  $x$  that  $\ker(\mu)$  as a  $S$ -module is generated by the elements

$$t^i \otimes m - 1 \otimes t^i m \quad \text{for } i \geq 0 \text{ and } m \in M .$$

But

$$t^i \otimes m - 1 \otimes t^i m = \sum_{j=0}^{i-1} t^j (t \otimes t^{i-j-1} m - 1 \otimes tt^{i-j-1} m) .$$

Hence in fact the elements  $t \otimes m - 1 \otimes tm = \kappa(1 \otimes m) \in \text{im}(\kappa)$  generate  $\ker(\mu)$ .

Finally, to establish the injectivity of  $\kappa$  it is convenient to make the identifications

$$\begin{aligned} S \otimes_R {}^\sigma M &\xrightarrow{\cong} \prod_{i \geq 0} M & \text{and} & & S \otimes_R M &\xrightarrow{\cong} \prod_{i \geq 0} M \\ (\sum_{i \geq 0} t^i a_i) \otimes m &\longmapsto (\sigma^{-1}(a_i)m)_i & & & (\sum_{i \geq 0} t^i a_i) \otimes m &\longmapsto (a_i m)_i \end{aligned}$$

which are possible since  $M$  is finitely generated over  $R$ . A straightforward computation shows that the map  $\kappa$  under this identification is given by  $(m_i)_i \mapsto (m_{i-1} - tm_i)_i$  (where  $m_{-1} := 0$ ). An element in the kernel of  $\kappa$  therefore corresponds to a tuple  $(m_i)_i$  such that  $m_i = tm_{i+1}$  for any  $i \geq -1$ . Hence all  $m_i$  lie in the intersection  $\bigcap_{k \geq 0} t^k M$  which by Lemma 2.1 is equal to zero.

**Corollary 2.3:** *i. Suppose that  $\sigma$  is of the form  $\sigma(\cdot) = \gamma \cdot \gamma^{-1}$  for some unit  $\gamma \in S$ . Then for any  $M$  in  $\mathfrak{M}_R(S)$  the map  ${}^\sigma M \xrightarrow{\gamma} M$  is an isomorphism of  $R$ -modules. By Thm. 2.2 we therefore have a short exact sequence of finitely generated  $S$ -modules of the form*

$$0 \longrightarrow S \otimes_R M \longrightarrow S \otimes_R M \longrightarrow M \longrightarrow 0 .$$

*In particular, the class of  $M$  in the Grothendieck group  $G(S)$  is zero.*

*ii. For any module  $M$  in  $\mathfrak{M}_R^r(S)$  there is a corresponding exact sequence*

$$0 \longrightarrow M^{\sigma^{-1}} \otimes_R S \xrightarrow{\kappa^r} M \otimes_R S \xrightarrow{\mu^r} M \longrightarrow 0$$

*with  $\kappa^r(m \otimes x) := m \otimes tx - mt \otimes x$  and  $\mu^r(m \otimes x) := mx$ .*

### 3. Comparing codimensions

In this section we want to compare the Ext-groups  $\text{Ext}_R^*(M, R)$  and  $\text{Ext}_S^*(M, S)$  for any module  $M$  in  $\mathfrak{M}_R(S)$ . Since the rings  $R$  and  $S$  are noetherian these Ext-groups certainly are finitely generated right  $R$ - and  $S$ -modules, respectively. We will show that the right  $R$ -module structure on  $\text{Ext}_R^*(M, R)^\sigma$  can be extended in a natural way to a right  $S$ -module structure. In fact we start more generally with an arbitrary finitely generated left  $S$ -module  $M$  (which therefore is pseudocompact as  $S$ - and as  $R$ -module) and construct first a natural right  $S$ -module structure on  $\text{Hom}_{\mathfrak{P}\mathfrak{M}(R)}(M, R)^\sigma$ . The latter is a right  $R$ -module via

$$f^a(m) := f(m)\sigma^{-1}(a)$$

for any  $f \in \text{Hom}_{\mathfrak{P}\mathfrak{M}(R)}(M, R)$ ,  $a \in R$ , and  $m \in M$ . We now define

$$f^t(m) := \sigma^{-1}(f(tm) - \delta(f(m))) .$$

Obviously,  $f^t : M \rightarrow R$  is additive and continuous. the computation

$$\begin{aligned}
f^t(bm) &= \sigma^{-1}(f(tbm) - \delta(f(bm))) \\
&= \sigma^{-1}(f(\sigma(b)tm + \delta(b)m) - \delta(bf(m))) \\
&= \sigma^{-1}(\sigma(b)f(tm) + \delta(b)f(m) - \delta(b)f(m) - \sigma(b)\delta(f(m))) \\
&= b\sigma^{-1}(f(tm) - \delta(f(m)))
\end{aligned}$$

for any  $b \in R$  shows that  $f^t$  also is  $R$ -linear. Hence  $f^t \in \text{Hom}_{\mathfrak{PM}(R)}(M, R)$ . We next check that

$$\begin{aligned}
[(f^{\sigma(a)})^t + f^{\delta(a)}](m) &= \sigma^{-1}(f(tm)a - \delta(f(m)a)) + f(m)\sigma^{-1}(\delta(a)) \\
&= \sigma^{-1}(f(tm)a - \delta(f(m)a) - \sigma(f(m))\delta(a)) + f(m)\sigma^{-1}(\delta(a)) \\
&= \sigma^{-1}(f(tm) - \delta(f(m)))\sigma^{-1}(a) \\
&= [(f^t)^a](m) .
\end{aligned}$$

This shows that, if we let  $S_0 \subseteq S$  denote the *skew polynomial* subring of all finite expressions  $\sum_{i=0}^n t^i a_i$ , then by the above definitions  $S_0$  acts from the right on  $\text{Hom}_{\mathfrak{PM}(R)}(M, R)^\sigma$ . Moreover, this action is functorial with respect to maps in  $\mathfrak{PM}(S)$ . To see that it extends, by continuity, to an  $S$ -action we first establish the following assertion: For any  $j \geq 0$  there is a  $k(j) \geq 0$  such that

$$f^{t^k}(M) \subseteq \text{Jac}(R)^j \quad \text{for any } k \geq k(j).$$

For  $k \geq 0$  and  $0 \leq i \leq k$  we define inductively explicit noncommutative polynomials  $B_{i,k}(Y, Z, Z')$  by

$$B_{0,k} := 1, \quad B_{k+1,k+1} := Z^k Y Z'^k B_{k,k}$$

and

$$B_{i,k} := B_{i,k-1} + Z^{k-1} Y Z'^{k-1} B_{i-1,k-1} \quad \text{for } 0 < i < k .$$

We note that each  $B_{i,k}$  is a sum of monomials in  $\mathcal{A}_i$ . An easy induction then establishes the explicit formula

$$(+) \quad (f^{t^k})(m) = \sigma^{-k} \left( \sum_{i=0}^k (-1)^i B_{i,k}(\delta, \sigma, \sigma^{-1})(f(t^{k-i}m)) \right)$$

for any  $k \geq 0$ . By the  $\sigma$ -nilpotence of  $\delta$  there is an  $r \geq 1$  such that

$$(a) \quad B(\delta, \sigma, \sigma^{-1})(R) \subseteq \text{Jac}(R)^j$$

for any finite sum  $B$  of monomials in  $\mathcal{A}_{\geq r}$ . Since  $\delta$  is continuous we find a sequence of integers  $\ell_r \geq \dots \geq \ell_2 \geq \ell_1 := j$  such that

$$(b) \quad \delta^0(\text{Jac}(R)^{\ell_\rho}), \dots, \delta^{r-1}(\text{Jac}(R)^{\ell_\rho}) \subseteq \text{Jac}(R)^{\ell_{\rho-1}} \quad \text{for any } 2 \leq \rho \leq r .$$

Because  $f$  is continuous and  $M$  is finitely generated there is, by Remark 0.1.i, an  $n' \geq 0$  such that

$$f(\text{Jac}(S)^{n'} M) \subseteq \text{Jac}(R)^{\ell_r} .$$

From the proof of Lemma 2.1 we know that

$$t^n \subseteq \text{Jac}(S)^{n'}$$

for some  $n \geq 0$ . Hence

$$f(t^{i'} M) \subseteq \text{Jac}(R)^{\ell_r} \quad \text{for any } i' \geq n .$$

Together with (b) and the fact that  $\sigma$  is an automorphism of  $R$  this shows that

$$(c) \quad B(\delta, \sigma, \sigma^{-1})(f(t^{i'} M)) \subseteq \text{Jac}(R)^j \quad \text{for any } i' \geq n$$

whenever  $B$  is a finite sum of monomials in  $\mathcal{A}_0 \cup \dots \cup \mathcal{A}_{r-1}$ . We now choose  $k \geq k(j) := r + n$ . For any index  $i$  in the sum on the right hand side of the formula (+) we then have  $i \geq r$  or  $i < r, k - i \geq n$  which means that the corresponding summand  $B_{i,k}(\delta, \sigma, \sigma^{-1})(f(t^{k-i} m))$  lies in  $\text{Jac}(R)^j$  either by (a) or by (c). Hence

$$f^{t^k}(M) \subseteq \sigma^{-k}(\text{Jac}(R)^j) = \text{Jac}(R)^j .$$

Consider now an arbitrary element  $x = \sum_{i \geq 0} t^i a_i$  in  $S$ . Since  $R$  is complete the above assertion implies that the series

$$(f^x)(m) := \sum_{i \geq 0} ((f^{t^i})^{a_i})(m)$$

converges in  $R$ . It is clear that the resulting map  $f^x : M \rightarrow R$  is  $R$ -linear and continuous. It is straightforward to check that this indeed defines a right  $S$ -action on  $\text{Hom}_{\mathfrak{P}\mathfrak{M}(R)}(M, R)^\sigma$  which is functorial with respect to maps in  $\mathfrak{P}\mathfrak{M}(S)$ .

Let now  $P. \rightarrow M$  be a projective resolution of  $M$  in  $\mathfrak{M}_{fg}(S)$ . Since  $S$  is topologically free as a left  $R$ -module this also is a projective resolution of  $M$  in  $\mathfrak{P}\mathfrak{M}(R)$ . It follows that the terms in the complex  $\text{Hom}_{\mathfrak{P}\mathfrak{M}(R)}(P., R)^\sigma$  and hence the Ext-groups  $\mathcal{E}xt_R^*(M, R)^\sigma$  carry a natural right  $S$ -module structure.

Let us fix a module  $M$  in  $\mathfrak{M}_R(S)$ . Then, by the above discussion, the groups

$$\text{Ext}_R^j(M, R)^\sigma = \mathcal{E}xt_R^j(M, R)^\sigma$$

are right  $S$ -modules in the category  $\mathfrak{M}_R^r(S)$ . Applying the right module version of Thm. 2.2 to them we obtain the short exact sequences of right  $S$ -modules

$$0 \rightarrow \text{Ext}_R^j(M, R) \otimes_R S \xrightarrow{\kappa^r} \text{Ext}_R^j(M, R)^\sigma \otimes_R S \xrightarrow{\mu^r} \text{Ext}_R^j(M, R)^\sigma \rightarrow 0 .$$

On the other hand we may pass from the short exact sequence of Thm. 2.2 to the associated long exact Ext-sequence and obtain the exact sequence of right  $S$ -modules

$$\begin{aligned} \text{Ext}_S^j(S \otimes_R M, S) &\longrightarrow \text{Ext}_S^j(S \otimes_R {}^\sigma M, S) \longrightarrow \text{Ext}_S^{j+1}(M, S) \\ &\longrightarrow \text{Ext}_S^{j+1}(S \otimes_R M, S) \longrightarrow \text{Ext}_S^{j+1}(S \otimes_R {}^\sigma M, S) \end{aligned}$$

where the first and the last arrow are given by  $\text{Ext}_S^*(\kappa, S)$ . We claim that this latter map can naturally be identified with the map  $\kappa^r$  for the right  $S$ -module  $\text{Ext}_R^*(M, R)^\sigma$ . Hence the second exact sequence gives rise to the short exact sequence

$$0 \longrightarrow \text{Ext}_S^j(S \otimes_R M, S) \longrightarrow \text{Ext}_S^j(S \otimes_R {}^\sigma M, S) \longrightarrow \text{Ext}_S^{j+1}(M, S) \longrightarrow 0$$

which moreover is naturally isomorphic to the first exact sequence above. This establishes the following fact.

**Theorem 3.1:** *For any module  $M$  in  $\mathfrak{M}_R(S)$  we have  $\text{Hom}_S(M, S) = 0$  and  $\text{Ext}_S^j(M, S) = \text{Ext}_R^{j-1}(M, R)^\sigma$ , as right  $S$ -modules, for any  $j \geq 1$ .*

Our claim is the consequence of a series of steps. First of all, for any left  $R$ -module  $N$  we have the natural homomorphism of right  $S$ -modules

$$\begin{aligned} \text{Hom}_R(N, R) \otimes_R S &\longrightarrow \text{Hom}_R(N, S) = \text{Hom}_S(S \otimes_R N, S) \\ f \otimes x &\longmapsto [n \mapsto f(n)x] . \end{aligned}$$

It is an isomorphism if  $N$  is finitely generated. Since over a noetherian ring arbitrary direct products of (faithfully) flat modules are (faithfully) flat and since  $S$  is topologically free over  $R$  from the left as well as right it follows that  $S$  is faithfully flat as a left as well as a right  $R$ -module. Applying the above isomorphism to a resolution of the  $R$ -module  $M$  by finitely generated projective  $R$ -modules therefore leads to a natural  $S$ -linear isomorphism

$$\text{Ext}_R^*(M, R) \otimes_R S = \text{Ext}_S^*(S \otimes_R M, S) .$$

We therefore have to show the commutativity of the diagram

$$\begin{array}{ccc} \text{Ext}_R^j(M, R) \otimes_R S & \xrightarrow{\kappa^r} & \text{Ext}_R^j(M, R)^\sigma \otimes_R S \\ \downarrow = & & \downarrow = \\ & & \text{Ext}_R^j({}^\sigma M, {}^\sigma R)^\sigma \otimes_R S \\ & & \downarrow \text{Ext}_R^j({}^\sigma M, \sigma) \otimes id \\ & & \text{Ext}_R^j({}^\sigma M, R) \otimes_R S \\ & & \downarrow = \\ \text{Ext}_S^j(S \otimes_R M, S) & \xrightarrow{\text{Ext}_S^j(\kappa, S)} & \text{Ext}_S^j(S \otimes_R {}^\sigma M, S) . \end{array}$$

Since all relevant modules are finitely generated this diagram is equivalent to the diagram

$$\begin{array}{ccc}
\mathcal{E}xt_R^j(M, R) \otimes_R S & \xrightarrow{\kappa^r} & \mathcal{E}xt_R^j(M, R)^\sigma \otimes_R S \\
\downarrow = & & \downarrow = \\
\mathcal{E}xt_R^j(M, R) \otimes_R S & & \mathcal{E}xt_R^j(\sigma M, \sigma R)^\sigma \otimes_R S \\
\downarrow = & & \downarrow \mathcal{E}xt_R^j(\sigma M, \sigma) \otimes id \\
\mathcal{E}xt_R^j(M, R) \otimes_R S & & \mathcal{E}xt_R^j(\sigma M, R) \otimes_R S \\
\downarrow = & & \downarrow = \\
\mathcal{E}xt_S^j(S \otimes_R M, S) & \xrightarrow{\mathcal{E}xt_S^j(\kappa, S)} & \mathcal{E}xt_S^j(S \otimes_R \sigma M, S) .
\end{array}$$

The vertical identifications now can be viewed as being induced (through a projective resolution of  $M$  in  $\mathfrak{PM}(R)$ ) by the natural maps

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathfrak{PM}(R)}(N, R) \otimes_R S & \longrightarrow & \mathrm{Hom}_{\mathfrak{PM}(R)}(N, S) = \mathrm{Hom}_{\mathfrak{PM}(S)}(S \widehat{\otimes}_R N, S) \\
f \otimes x & \longmapsto & [n \mapsto f(n)x]
\end{array}$$

for any  $N$  in  $\mathfrak{PM}(R)$ . Since the whole diagram is functorial in the pseudocompact  $S$ -modules  $M$  we may reduce the proof of its commutativity, by using a projective resolution of  $M$  in  $\mathfrak{M}_{fg}(S)$ , to the case  $j = 0$ , i.e., to the commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathfrak{PM}(R)}(N, R) \otimes_R S & \xrightarrow{\kappa^r} & \mathrm{Hom}_{\mathfrak{PM}(R)}(N, R)^\sigma \otimes_R S \\
\downarrow & & \downarrow = \\
\mathrm{Hom}_{\mathfrak{PM}(R)}(N, R) \otimes_R S & & \mathrm{Hom}_{\mathfrak{PM}(R)}(\sigma N, \sigma R)^\sigma \otimes_R S \\
\downarrow & & \downarrow \mathrm{Hom}_{\mathfrak{PM}(R)}(\sigma N, \sigma) \otimes id \\
\mathrm{Hom}_{\mathfrak{PM}(R)}(N, R) \otimes_R S & & \mathrm{Hom}_{\mathfrak{PM}(R)}(\sigma N, R) \otimes_R S \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathfrak{PM}(S)}(S \widehat{\otimes}_R N, S) & \xrightarrow{\mathrm{Hom}_{\mathfrak{PM}(S)}(\kappa, S)} & \mathrm{Hom}_{\mathfrak{PM}(S)}(S \widehat{\otimes}_R \sigma N, S)
\end{array}$$

for any module  $N$  in  $\mathfrak{M}_{fg}(S)$ . This is a straightforward explicit computation which we leave to the reader. This concludes the proof of our claim and consequently the proof of Thm. 3.1.

To make use of Thm. 3.1 we need a reasonable theory of codimension. We therefore assume now in addition that

$R$  and  $S$  are (left and right) Auslander regular.

This means (for  $R$ ) that  $R$  is (left and right) regular, i.e., that any finitely generated (left or right)  $R$ -module has a finite resolution by finitely generated projective  $R$ -modules. Defining, for any finitely generated left or right  $R$ -module  $N$ , its *grade* or *codimension* by

$$j_R(N) := \min\{i \geq 0 : \text{Ext}_R^i(N, R) \neq 0\}$$

the Auslander regularity condition requires in addition that for any such  $N$  and any  $k \geq 0$  we have

$$j_R(N') \geq k \quad \text{for any } R\text{-submodule } N' \subseteq \text{Ext}_R^k(N, R) .$$

A standard method to introduce on the category  $\mathfrak{M}_{fg}(R)$  a codimension filtration (cf. [Ste] Chap. V and [CSS]§1) is to define, for any  $j \geq 0$ , the full subcategory

$$\begin{aligned} \mathfrak{M}^j(R) := & \text{ all modules } M \text{ in } \mathfrak{M}_{fg}(R) \text{ such that} \\ & \text{Ext}_R^i(M', R) = 0 \\ & \text{for any } i < j \text{ and any submodule } M' \subseteq M . \end{aligned}$$

It has the following properties:

- In any short exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  in  $\mathfrak{M}(R)$  the module  $M$  lies in  $\mathfrak{M}^j(R)$  if and only if  $M'$  and  $M''$  lie in  $\mathfrak{M}^j(R)$ ;
- any module in  $\mathfrak{M}_{fg}(R)$  has a unique largest submodule contained in  $\mathfrak{M}^j(R)$ .

We also recall (cf. [CSS]§1 and [Ve1]) that, because we are assuming  $R$  to be Auslander regular, we in fact have the simplified characterization:

- A module  $M$  in  $\mathfrak{M}_{fg}(R)$  lies in  $\mathfrak{M}^j(R)$  if and only if  $\text{Ext}_R^i(M, R) = 0$  for any  $i < j$ .

Note that, for any nonzero module  $M$  in  $\mathfrak{M}_{fg}(R)$ , its grade  $j_R(M)$  is the smallest non-negative integer such that  $M$  lies in  $\mathfrak{M}^{j_R(M)}(R)$ .

Let  $G_*(R)^{(j)}$  denote the Quillen  $K$ -groups (cf. [Qui]) of the abelian category  $\mathfrak{M}^j(R)$ . In particular,  $G_0(R)^{(j)}$  is the Grothendieck group of the category  $\mathfrak{M}^j(R)$  (w.r.t. short exact sequences).

The category  $\mathfrak{M}_R(S)$  can be viewed as an abelian subcategory of both categories  $\mathfrak{M}_{fg}(S)$  and  $\mathfrak{M}_{fg}(R)$ . Hence it becomes a natural problem to compare the codimensions as  $R$ - and  $S$ -module, respectively, of any  $M$  in  $\mathfrak{M}_R(S)$ .

**Corollary 3.2:** *i.  $j_S(M) = j_R(M) + 1$  for any module  $M$  in  $\mathfrak{M}_R(S)$ ;*

*ii.  $\mathfrak{M}^j(R) \cap \mathfrak{M}_R(S) = \mathfrak{M}^{j+1}(S) \cap \mathfrak{M}_R(S)$  for any  $j \geq 0$ .*



Proof: Apply Thm. 3.1.

The second problem which we want to address is the nature of the natural maps

$$G_*(S)^{(j+1)} \longrightarrow G_*(S)^{(j)}$$

induced by the corresponding inclusion of categories. As background for this one should keep in mind that the Gersten conjecture ([Ger] Prop. 7.7 and [Qui] §7 Prop. 5.6) is the claim that in the case of a commutative regular local noetherian ring (instead of  $S$ ) these maps are the zero maps; note that in this case the grade of a module as defined above is equal to the codimension of the support of the module (cf. [Ste] Prop. VII.6.8(ii)). We introduce the Quillen  $K$ -groups  $G_*^R(S)^{(j)}$  of the category  $\mathfrak{M}^j(R) \cap \mathfrak{M}_R(S)$ .

**Lemma 3.3:** *For any module  $N$  in  $\mathfrak{M}_{fg}(R)$  we have*

$$j_R(N) = j_S(S \otimes_R N) .$$

Proof: As discussed already after Thm. 3.1 we have that  $S$  is faithfully flat over  $R$  and that

$$\mathrm{Ext}_S^*(S \otimes_R N, S) = \mathrm{Ext}_R^*(N, R) \otimes_R S .$$

**Proposition 3.4:** *If the automorphism  $\sigma$  of  $R$  is of the form  $\sigma(\cdot) = \gamma \cdot \gamma^{-1}$  for some unit  $\gamma \in S^\times$  then the map*

$$G_*^R(S)^{(j)} \longrightarrow G_*(S)^{(j)}$$

*induced by the inclusion  $\mathfrak{M}^j(R) \cap \mathfrak{M}_R(S) \subseteq \mathfrak{M}^j(S)$  is the zero map for any  $j \geq 0$ .*

Proof: Suppose that the module  $M$  lies in  $\mathfrak{M}^j(R) \cap \mathfrak{M}_R(S)$ . Then by Cor. 3.2 and Lemma 3.3 all three  $S$ -modules which constitute the exact sequence

$$0 \longrightarrow S \otimes_R M \longrightarrow S \otimes_R M \longrightarrow M \longrightarrow 0 .$$

of Cor. 2.3.i lie in the abelian category  $\mathfrak{M}^j(S)$ . We therefore may view this short exact sequence as an exact sequence of natural transformations between the exact functors from the category  $\mathfrak{M}^j(R) \cap \mathfrak{M}_R(S)$  into the category  $\mathfrak{M}^j(S)$  given by inclusion and  $S \otimes_R \cdot$ , respectively. In this situation a fundamental theorem of Quillen asserts ([Qui] §3 Cor. 1) that the map induced on  $K$ -groups by the third functor is the difference of the maps induced by the first two functors. But these first two functors coincide. Hence the inclusion functor induces the zero map.

Since by Cor. 3.2 we have the commutative diagram

$$\begin{array}{ccc}
 & G_*^R(S)^{(j)} & \\
 \swarrow & & \searrow 0 \\
 G_*(S)^{(j+1)} & \longrightarrow & G_*(S)^{(j)}
 \end{array}$$

Prop. 3.4 may be viewed as a partial answer to our problem. But it is not clear whether this vanishing result can be considered as evidence for some version of Gersten's conjecture for certain noncommutative rings.

#### 4. Application to Iwasawa algebras

Let  $G$  be a compact  $p$ -adic Lie group. Its so called Iwasawa algebra

$$\Lambda(G) := \varprojlim \mathbf{Z}_p[G/U]$$

is the projective limit of the group rings  $\mathbf{Z}_p[G/U]$  where  $U$  ranges over the open normal subgroups in  $G$ . It is known to be (left and right) noetherian ([Laz] (V.2.2.4)). An important feature of  $\Lambda(G)$  is that it carries a natural compact Hausdorff topology. A fundamental system of open neighbourhoods of zero in  $\Lambda(G)$  is given by the submodules  $p^n\Lambda(G) + I(U)$  with  $n \in \mathbb{N}$  and  $U$  as before, where  $I(U)$  denotes the kernel of the projection map  $\Lambda(G) \rightarrow \Lambda(G/U)$  (cf. [NSW] Prop. (5.2.17) or [Laz] (II.2.2.2)). It follows that  $\Lambda(G)$  is left and right pseudocompact.

We remark that if  $G$  is pro- $p$  then  $\Lambda(G)$  is local, and if in addition  $G$  has no element of order  $p$  then  $\Lambda(G)$  is a regular local integral domain ([Neu]).

We now assume that there is a closed normal subgroup  $H \subseteq G$  such that  $G/H \cong \mathbf{Z}_p$ . Then with the pair of pseudocompact rings  $R := \Lambda(H) \subseteq \Lambda(G) =: S$  we are in the situation considered in this paper. This is seen by noting that  $G$  necessarily is the semidirect product of  $H$  and a subgroup  $\Gamma \cong \mathbf{Z}_p$ . We pick once and for all a topological generator  $\gamma$  of  $\Gamma$  and define the element  $t := \gamma - 1$  in  $\Lambda(\Gamma) \subseteq \Lambda(G)$ . It is well known that  $\Lambda(\Gamma)$  is the formal power series ring in the variable  $t$  over  $\mathbf{Z}_p$ . Hence (use Lemma 1.6) the Iwasawa algebra  $S = \Lambda(G)$  is the skew power series ring  $\Lambda(H)[[t; \sigma, \delta]]$  where  $\sigma(\cdot) := \gamma \cdot \gamma^{-1}$  and  $\delta := \sigma - id$  (cf. [Ve3] Example 5.1).

We recall (cf. [CSS]§1 and [Ve1]) that  $G$  always contains an open subgroup  $G'$  such that  $\Lambda(G')$  is left and right Auslander regular. Since

$$\mathrm{Ext}_{\Lambda(G)}^i(M, \Lambda(G)) = \mathrm{Ext}_{\Lambda(G')}^i(M, \Lambda(G'))$$

for any  $i \geq 0$  and any module  $M$  in  $\mathfrak{M}_{fg}(\Lambda(G))$  (cf. [Jan] Lemma 2.3) the codimension theory of section 3 continues to hold for general  $\Lambda(G)$  even if it is not Auslander regular.

Hence our main results Theorems 2.2 and 3.1, Cor. 3.2, and Prop. 3.4 hold true for the pair  $\Lambda(H) \subseteq \Lambda(G)$ . In Iwasawa theory the modules in  $\mathfrak{M}^2(\Lambda(G))$  are called pseudo-null modules. We conclude by pointing out that, as a special case of our results, any pseudo-null  $\Lambda(G)$ -module which is finitely generated as a  $\Lambda(H)$ -module has zero class in the Grothendieck group  $G_0(\Lambda(G))^{(1)}$  of all finitely generated torsion  $\Lambda(G)$ -modules ([CSS] p.4 and Lemma 2.4).

## 5. Skew Laurent series rings

Let  $R$  be again a pseudocompact ring and  $\sigma$  a topological automorphism of  $R$ . The left  $R$ -module  $T$  of all (left) formal Laurent series  $\sum_{i \gg -\infty} a_i t^i$  over  $R$  carries a unique unital ring structure which extends the  $R$ -module structure and satisfies  $ta = \sigma(a)t$ . The ring  $R[[t, t^{-1}; \sigma]] := T$  is called a skew Laurent series ring. It can alternatively be described by right Laurent series, it contains  $S := R[[t; \sigma]]$  as a subring, and it also arises as the localization  $S_t$  of  $S$  with respect to the Ore set  $\{1, t, t^2, \dots\}$ . In particular,  $T$  is a flat (left and right)  $S$ -module. In the following we assume that  $R$  is noetherian. Then  $S$ , by [MCR] 1.4.5, and  $T$ , as a localization of a noetherian ring, both are noetherian as well.

Similarly as before we let  $\mathfrak{M}_R(T)$  denote the abelian category of left  $T$ -modules which are finitely generated as  $R$ -modules. Any such module  $M$  lies in  $\mathfrak{M}_R(S)$  and satisfies  $T \otimes_S M = M$ . From this observation it is immediate that, for any  $M$  in  $\mathfrak{M}_R(T)$ , we obtain an analogue of Thm. 2.2 in the form of an exact sequence of (left)  $T$ -modules

$$0 \longrightarrow T \otimes_R M \longrightarrow T \otimes_S M \longrightarrow M \longrightarrow 0 .$$

Indeed, consider  $M$  as an  $S$ -module, apply  $T \otimes_S \cdot$  to the exact sequence resulting from that proposition and observe that an analog of Cor. 2.3.i applies because  $\sigma(\cdot) = t \cdot t^{-1}$ .

Furthermore, for  $M$  in  $\mathfrak{M}_R(T)$ , multiplication by  $t$  on the right  $S$ -module  $\text{Ext}_R^j(M, R)^\sigma$  is easily checked to be bijective. Hence the  $S$ -module structure extends uniquely to a right  $T$ -module structure on  $\text{Ext}_R^j(M, R)^\sigma$ . Using the natural isomorphism  $\text{Ext}_T^j(M, T) \cong \text{Ext}_S^j(M, S) \otimes_S T$  of right  $T$ -modules we obtain, again by base change, the analogue of Thm. 3.1:

$$\text{Ext}_T^j(M, T) \cong \text{Ext}_R^{j-1}(M, R)^\sigma$$

as right  $T$ -modules.

Now assume that  $R$  in addition is Auslander regular. By [LVO] III.3.4.6(3) the ring  $S$  then also is Auslander regular. The increasing filtration  $(St^{-j})_{j \in \mathbf{Z}}$  on  $T$  is complete with associated graded ring isomorphic to  $R[t, t^{-1}; \sigma]$ . It therefore follows from [LVO] III.3.4.6(2) and III.2.2.5 that  $T$  is Auslander regular. We also note that  $T$  is faithfully flat over  $R$ . Thus the analogs of Cor. 3.2 and Prop. 3.4 (the assumption there is satisfied in the present situation) hold for the ring extension  $T$  over  $R$  by exactly the same arguments.

Finally, we point out that all the results of this section also hold for skew Laurent polynomial rings  $R[t, t^{-1}; \sigma]$  over arbitrary (noetherian, respectively Auslander regular) rings  $R$  with ring automorphism  $\sigma$ . We omit the proofs since they are analogous but simpler.

## References

- [Bru] Brumer A.: Pseudocompact Algebras, Profinite Groups and Class Formations. *J. Algebra* 4, 442-470 (1966)
- [CH] Chatters A.W., Hajarnavis C.R.: Rings with chain conditions. Boston-London-Melbourne: Pitman 1980
- [CFKSV] Coates J., Fukaya T., Kato K., Sujatha R., Venjakob O.: The  $GL_2$  main conjecture for elliptic curves without complex multiplication. To appear in *Publ. Math. IHES*
- [CSS] Coates J., Schneider P., Sujatha R.: Modules over Iwasawa algebras. *J. Inst. Math. Jussieu* 2, 73–108 (2003)
- [Ga1] Gabriel P.: Des Catégories Abéliennes. *Bull. Soc. math. France* 90, 323-448 (1962)
- [Ga2] Gabriel P.: Etude infinitesimale des schemas en groupes. In *Schémas en Groupe I (SGA3), Exp. VII B*. *Lecture Notes Math.* 151, pp. 4746-562. Berlin-Heidelberg-New York: Springer 1970
- [Ger] Gersten S.M.: Some exact sequences in the higher  $K$ -theory of rings. In *Algebraic  $K$ -theory I*. *Lecture Notes Math.* 341, pp. 211-243. Berlin-Heidelberg-New York: Springer 1973
- [Jan] Jannsen U.: Iwasawa modules up to isomorphism. *Adv. Studies Pure Math.* 17, 171-207 (1989)
- [Laz] Lazard M.: Groupes analytiques  $p$ -adique. *Publ. Math. IHES* 26, 389-603 (1965)
- [LVO] Li Huishi, van Oystaeyen F.: Zariskian Filtrations. Dordrecht: Kluwer 1996
- [MCR] McConnell J.C., Robson J.C.: Noncommutative noetherian rings. Chichester: Wiley 1987

- [NSW] Neukirch J., Schmidt A., Wingberg K.: Cohomology of Number Fields. Grundlehren math. Wiss. 323. Berlin-Heidelberg-New York: Springer 2000
- [Neu] Neumann A.: Completed group algebras without zero divisors. Arch. Math. 51, 496-499 (1988)
- [Qui] Quillen D.: Higher algebraic  $K$ -theory I. In Algebraic  $K$ -theory I. Lecture Notes Math. 341, pp. 85-139. Berlin-Heidelberg-New York: Springer 1973
- [Ree] Rees D.: A theorem of homological algebra. Proc. Cambridge Philos. Soc. 52, 605-610 (1956)
- [Ste] Stenström B.: Rings of Quotients. Berlin-Heidelberg-New York: Springer 1975
- [vGB] van Gastel M., van den Bergh M.: Graded Modules of Gelfand-Kirillov Dimension One over Three-Dimensional Artin-Schelter Regular Algebras. J. Algebra 196, 251-282 (1997)
- [Ve1] Venjakob O.: On the structure theory of the Iwasawa algebra of a  $p$ -adic Lie group. J. Eur. Math. Soc. 4, 271-311 (2002)
- [Ve2] Venjakob O.: A non-commutative Weierstrass preparation theorem and applications to Iwasawa theory. J. reine angew. Math. 559, 153-191 (2003)
- [Ve3] Venjakob O.: Characteristic Elements in Noncommutative Iwasawa Theory. Habilitationsschrift, Heidelberg 2003

Universität Münster, Mathematisches Institut, Einsteinstr. 62, 48149 Münster, Germany

pschnei@math.uni-muenster.de

<http://www.uni-muenster.de/math/u/schneider/>

Universität Heidelberg, Mathematisches Institut, Im Neuenheimer Feld 288, 69120 Heidelberg, Germany

otmar@mathi.uni-heidelberg.de

<http://www.mathi.uni-heidelberg.de/~otmar/>