Smooth representations and Hecke modules in characteristic p

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To the memory of Robert Steinberg

Abstract

Let G be a p-adic Lie group and $I \subseteq G$ be a compact open subgroup which is a torsionfree pro-p-group. Working over a coefficient field k of characteristic p we introduce a differential graded Hecke algebra for the pair (G, I) and show that the derived category of smooth representations of G in k-vector spaces is naturally equivalent to the derived category of differential graded modules over this Hecke DGA.

1 Background and motivation

Let G be a d-dimensional p-adic Lie group, and let k be any field. We denote by $\mathrm{Mod}_k(G)$ the category of smooth G-representations in k-vector spaces. It obviously has arbitrary direct sums.

We fix a compact open subgroup $I \subseteq G$. In $\mathrm{Mod}_k(G)$ we then have the representation

$$\mathrm{ind}_I^G(1) := \text{all } k\text{-valued functions with finite support on } G/I$$

with G acting by left translations. Being generated by a single element, which is the characteristic function of the trivial coset, $\mathrm{ind}_I^G(1)$ is a compact object in $\mathrm{Mod}_k(G)$. It generates the full subcategory $\mathrm{Mod}_k^I(G)$ of all representations V in $\mathrm{Mod}_k(G)$ which are generated by their I-fixed vectors V^I. In general $\mathrm{Mod}_k^I(G)$ is not an abelian category. The Hecke algebra of I by definition is the endomorphism ring

$$\mathcal{H}_I := \mathrm{End}_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1))^{\mathrm{op}} \ .$$

We let $\mathrm{Mod}(\mathcal{H}_I)$ denote the category of left unital $\mathcal{H}_I$-modules. There is the pair of adjoint functors

$$\begin{aligned} H^0 : \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Mod}(\mathcal{H}_I) \\ V &\longmapsto V^I = \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), V) \ , \end{aligned}$$

and

$$\begin{aligned} T_0 : \mathrm{Mod}(\mathcal{H}_I) &\longrightarrow \mathrm{Mod}_k^I(G) \subseteq \mathrm{Mod}_k(G) \\ M &\longmapsto \mathrm{ind}_I^G(1) \otimes_{\mathcal{H}_I} M \ . \end{aligned}$$

If the characteristic of k does not divide the pro-order of I then the functor H^0 is exact. Then $\mathrm{ind}_I^G(1)$ is a projective compact object in $\mathrm{Mod}_k(G)$. Since it does not generate the full category $\mathrm{Mod}_k(G)$ one cannot apply the Gabriel-Popescu theorem (cf. [KS] Thm. 8.5.8) to the functor H^0. Nevertheless, in this case, one might hope for a close relation between the categories $\mathrm{Mod}_k^I(G)$ and $\mathrm{Mod}(\mathcal{H}_I)$. This indeed happens, for example, for a connected reductive group G and its Iwahori subgroup I and the field $k = \mathbb{C}$ (cf. [Ber] Cor. 3.9(ii)). In addition, in this situation the algebra $\mathcal{H}_I$ turns out to be a generalized affine Hecke algebra so that its structure is explicitly known. Therefore in characteristic zero Hecke algebras have become one of the most important tools in the investigation of smooth G-representations.

In this light it is a pressing question to better understand the relation between the two categories $\mathrm{Mod}_k(G)$ and $\mathrm{Mod}(\mathcal{H}_I)$ in the opposite situation where k has characteristic p. Since p always will divide the pro-order of I the functor H^0 certainly is no longer exact. Both functors H^0 and T_0 now have a very complicated behaviour and little is known ([Koz], [Oll], [OS]). This suggests that one should work in a derived framework which takes into account the higher cohomology of I.

This paper will demonstrate that, by doing this not in a naive way but in an appropriate differential graded context, the situation does improve drastically. We will show the somewhat surprising result that the object $\mathrm{ind}_I^G(1)$ becomes a compact generator of the full derived category of G provided I is a torsionfree pro-p-group.

The main result of this paper was proved already in 2007 but remained unpublished. At the time we gave a somewhat ad hoc proof. Although the main arguments remain unchanged we now, by appealing to a general theorem of Keller, have arranged them in a way which makes the reasoning more transparent. In the context of the search for a p-adic local Langlands program there is increasing interest in studying derived situations (cf. [Ha]). We also have now ([OS]) the first examples of explicit computations of the cohomology groups $H^i(I, \mathrm{ind}_I^G(1))$. I hope that these are sufficient reasons to finally publish the paper. I thank W. Soergel for a very inspiring discussion about injective resolutions and DGA's.

2 The unbounded derived category of G

We assume from now on throughout the paper that the field k has characteristic p and that I is a torsionfree pro-p-group. Let us first of all collect a few properties of the abelian category $\mathrm{Mod}_k(G)$.

Lemma 1. *i. $\mathrm{Mod}_k(G)$ is (AB5), i.e., it has arbitrary colimits and filtered colimits are exact.*

ii. $\mathrm{Mod}_k(G)$ is (AB3), i.e., it has arbitrary limits.*

iii. $\mathrm{Mod}_k(G)$ has enough injective objects.

iv. $\mathrm{Mod}_k(G)$ is a Grothendieck category, i.e., it satisfies (AB5) and has a generator.

v. $V^I \neq 0$ for any nonzero V in $\mathrm{Mod}_k(G)$.

Proof. i. This is obvious. ii. Take the subspace of smooth vectors in the limit of k-vector spaces. iii. This is shown in [Vig] I.5.9. Alternatively it is a consequence of iv. (cf. [KS] Thm. 9.6.2). v. Since I is pro-p where p is the characteristic of k, the only irreducible smooth representation of I is the trivial one.

iv. Because of i. it remains to exhibit a generator of $\mathrm{Mod}_k(G)$. We define

$$Y := \bigoplus_J \mathrm{ind}_J^G(1)$$

where J runs over all open subgroups in G. For any V in $\mathrm{Mod}_k(G)$ we have

$$\mathrm{Hom}_{\mathrm{Mod}_k(G)}(Y, V) = \prod_J V^J \ .$$

Since $V = \bigcup_J V^J$ we easily deduce that Y is a generator of $\mathrm{Mod}_k(G)$. □

As usual, let $D(G) := D(\mathrm{Mod}_k(G))$ be the derived category of unbounded complexes in $\mathrm{Mod}_k(G)$.

Remark 2. *$D(G)$ has arbitrary direct sums, which can be computed as direct sums of complexes.*

Proof. See the first paragraph in [KS] §14.3. □

According to [Laz] V.2.2.8 and [Ser] the group I has cohomological dimension d. This means that the higher derived functors of the left exact functor

$$\begin{aligned} \mathrm{Mod}_k(I) &\longrightarrow \mathrm{Vec}_k \\ E &\longmapsto E^I \end{aligned}$$

into the category Vec_k of k-vector spaces are zero in degrees $> d$. On the other hand the restriction functor

$$\begin{aligned} \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Mod}_k(I) \\ V &\longmapsto V|I \end{aligned}$$

is exact and respects injective objects. The latter is a consequence of the fact that compact induction

$$\begin{aligned} \mathrm{Mod}_k(I) &\longrightarrow \mathrm{Mod}_k(G) \\ E &\longmapsto \mathrm{ind}_I^G(E) \end{aligned}$$

is an exact left adjoint (compare [Vig] I.5.7). Hence the higher derived functors of the composed functor

$$\begin{aligned} H^0(I, .) : \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Vec}_k \\ V &\longmapsto V^I \end{aligned}$$

are given by $V \longmapsto H^i(I, V|I)$ and vanish in degrees $> d$. It follows that the total right derived functor

$$RH^0(I, .) : D(G) \longrightarrow D(\mathrm{Vec}_k)$$

between the corresponding (unbounded) derived categories exists ([Har] I.5.3).

To compute $RH^0(I,.)$ we use the formalism of K-injective complexes as developed in [Spa]. We let $C(\mathrm{Mod}_k(G))$ and $K(\mathrm{Mod}_k(G))$ denote the category of unbounded complexes in $\mathrm{Mod}_k(G)$ with chain maps and homotopy classes of chain maps, respectively, as morphisms. The K-injective complexes form a full triangulated subcategory $K_{inj}(\mathrm{Mod}_k(G))$ of $K(\mathrm{Mod}_k(G))$. Exactly in the same way as [Spa] Prop. 3.11 one can show that any complex in $C(\mathrm{Mod}_k(G))$ has a right K-injective resolution (recall from Lemma 1.ii that the category $\mathrm{Mod}_k(G)$ has inverse limits). Alternatively one may apply [Se] Thm. 3.13 or [KS] Thm. 14.3.1 based upon Lemma 1.iv. The existence of K-injective resolutions means that the natural functor

$$K_{inj}(\mathrm{Mod}_k(G)) \xrightarrow{\simeq} D(G)$$

is an equivalence of triangulated categories. We fix a quasi-inverse $\mathbf{i}$ of this functor. Then the derived functor $RH^0(I,.)$ is naturally isomorphic to the composed functor

$$D(G) \xrightarrow{\mathbf{i}} K_{inj}(\mathrm{Mod}_k(G)) \longrightarrow K(\mathrm{Vec}_k) \longrightarrow D(\mathrm{Vec}_k)$$

with the middle arrow given by

$$V^\bullet \mapsto \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), V^\bullet) \ .$$

Explanation: Let $V^\bullet$ be a complex in $C(\mathrm{Mod}_k(G))$. To compute $RH^0(I,.)$ according to [Har] one chooses a quasi-isomorphism $V^\bullet \xrightarrow{\sim} C^\bullet$ into a complex consisting of objects which are acyclic for the functor $H^0(I,.)$. On the other hand let $V^\bullet \xrightarrow{\sim} A^\bullet$ be a quasi-isomorphism into a K-injective complex. By [Spa] Prop. 1.5.(c) we then have, in $K(\mathrm{Mod}_k(G))$, a unique commutative diagram:

$$\begin{array}{ccc} & & C^\bullet \\ & \nearrow & \downarrow \\ V^\bullet & & \\ & \searrow & \downarrow \\ & & A^\bullet \end{array}$$

We claim that the induced map

$$(C^\bullet)^I \xrightarrow{\sim} (A^\bullet)^I$$

is a quasi-isomorphism. Choose quasi-isomorphisms

$$A^\bullet \xrightarrow{\sim} \tilde{C}^\bullet \xrightarrow{\sim} \tilde{A}^\bullet$$

where $\tilde{C}^\bullet$ consists of $H^0(I,.)$-acyclic objects and $\tilde{A}^\bullet$ is K-injective. By [Spa] Prop. 1.5.(b) the composite is an isomorphism in $K(\mathrm{Mod}_k(G))$ and hence induces a quasi-isomorphism $(A^\bullet)^I \xrightarrow{\sim} (\tilde{A}^\bullet)^I$. But by [Har] Thm. I 5.1 and Cor. I.5.3.(γ) the composite $C^\bullet \xrightarrow{\sim} A^\bullet \xrightarrow{\sim} \tilde{C}^\bullet$ also induces a quasi-isomorphism $(C^\bullet)^I \xrightarrow{\sim} (\tilde{C}^\bullet)^I$.

Lemma 3. *The (hyper)cohomology functor $H^\ell(I,.)$, for any $\ell \in \mathbb{Z}$, commutes with arbitrary direct sums in $D(G)$.*

Proof. First of all we observe that the cohomology functor $H^\ell(I,.)$ commutes with arbitrary direct sums in $\mathrm{Mod}_k(G)$ ([S-CG] I.2.2 Prop. 8). This, in particular, implies that arbitrary direct sums of $H^0(I,.)$-acyclic objects in $\mathrm{Mod}_k(G)$ again are $H^0(I,.)$-acyclic. Let now $(V_j^\bullet)_{j\in J}$ be a family of objects in $D(G)$, where we view each $V_j^\bullet$ as an actual complex. Then, according to Remark 2 the direct sum of the $V_j^\bullet$ in $D(G)$ is represented by the direct sum complex $\oplus_j V_j^\bullet$. Now we choose quasi-isomorphisms $V_j^\bullet \xrightarrow{\sim} C_j^\bullet$ in $C(\mathrm{Mod}_k(G))$ where all representations C_j^m are $H^0(I,.)$-acyclic. By the preliminary observation the direct sum map $\oplus_j V_j^\bullet \xrightarrow{\sim} C^\bullet := \oplus_j C_j^\bullet$ again is a quasi-isomorphism where all terms of the target complex are $H^0(I,.)$-acyclic. We therefore obtain

$$H^\ell(I, \oplus_j V_j^\bullet) = h^\ell((C^\bullet)^I) = \oplus_j h^\ell((C_j^\bullet)^I) = \oplus_j H^\ell(I, V_j^\bullet) \ .$$

□

As usual, we view $\mathrm{Mod}_k(G)$ as the full subcategory of those complexes in $D(G)$ which have zero terms outside of degree zero.

Lemma 4. *$ind_I^G(1)$ is a compact object in $D(G)$.*

Proof. We have to show that the functor $\mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(1),.)$ commutes with arbitrary direct sums in $D(G)$. For any $V^\bullet$ in $D(G)$ we compute

(1) $\quad \mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(1), V^\bullet) = \mathrm{Hom}_{K(\mathrm{Mod}_k(G))}(\mathrm{ind}_I^G(1), \mathbf{i}(V^\bullet)) = h^0(\mathbf{i}(V^\bullet)^I) = H^0(I, V^\bullet) \ ,$

where the first identity uses [Spa] Prop. 1.5.(b). The claim therefore follows from Lemma 3. □

Proposition 5. *Let $E^\bullet$ in $D(I)$; then $E^\bullet = 0$ if and only if $H^j(I, E^\bullet) = 0$ for any $j \in \mathbb{Z}$.*

Proof. The completed group ring $\Omega := \varprojlim_N k[I/N]$ of I over k, where N runs over all open normal subgroups of I, is a pseudocompact local ring (cf. [Sch] §19). If $\mathfrak{m} \subseteq \Omega$ denotes the maximal ideal then $\Omega/\mathfrak{m} = k$. Since Ω is noetherian ([Laz] V.2.2.4 for $k = \mathbb{F}_p$ and [Sch] Thm. 33.4 together with [B-AC] Chap. IX §2.3 Prop. 5 in general) its pseudocompact topology coincides with the $\mathfrak{m}$-adic topology ([Sch] Lemma 19.8). This implies that:

- $\Omega/\mathfrak{m}^j$ lies in $\mathrm{Mod}_k(I)$ for any $j \in \mathbb{N}$.
- For any E in $\mathrm{Mod}_k(I)$ we have

$$E = \bigcup_{j \in \mathbb{N}} E^{\mathfrak{m}^j = 0} \qquad \text{where} \quad E^{\mathfrak{m}^j=0} := \{v \in E : \mathfrak{m}^j v = 0\}.$$

Because of

$$E^{\mathfrak{m}^j=0} = \mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, E)$$

we need to consider the left exact functors $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j,.)$ on $\mathrm{Mod}_k(I)$. Their right derived functors of course are $\mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j,.)$. In particular

$$\mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m},.) = H^i(I,.) \ .$$

For any $j \in \mathbb{N}$ we have the short exact sequence

$$0 \longrightarrow \mathfrak{m}^j/\mathfrak{m}^{j+1} \longrightarrow \Omega/\mathfrak{m}^{j+1} \longrightarrow \Omega/\mathfrak{m}^j \longrightarrow 0$$

in $\mathrm{Mod}_k(I)$. Moreover, $\mathfrak{m}^j/\mathfrak{m}^{j+1} \cong k^{n(j)}$ for some $n(j) \geq 0$ since Ω is noetherian. The associated long exact Ext-sequence therefore reads

$$\ldots \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j,.) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^{j+1},.) \longrightarrow H^i(I,.)^{n(j)} \longrightarrow \ldots$$

By induction with respect to j we deduce that:

– Each functor $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j,.)$ has cohomological dimension $\leq d$.

– Each $H^0(I,.)$-acyclic object in $\mathrm{Mod}_k(I)$ is $\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j,.)$-acyclic for any $j \geq 1$.

It follows that the total right derived functors $R\mathrm{Hom}_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j,.)$ on $D(I)$ exist. More explicitly, let $E^\bullet$ be any complex in $D(I)$ and choose a quasi-isomorphism $E^\bullet \xrightarrow{\sim} C^\bullet$ into a complex consisting of $H^0(I,.)$-acyclic objects. It then follows that we have the short exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, C^\bullet) \longrightarrow \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^{j+1}, C^\bullet) \longrightarrow \big((C^\bullet)^I\big)^{n(j)} \longrightarrow 0 \ .$$

Suppose now that $RH^0(I,E^\bullet) = 0$. This means that the complex $(C^\bullet)^I$ is exact. By induction with respect to j we obtain the exactness of the complex

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}_k(I)}(\Omega/\mathfrak{m}^j, C^\bullet) = (C^\bullet)^{\mathfrak{m}^j=0}$$

for any $j \in \mathbb{N}$. Hence $C^\bullet$ and $E^\bullet$ are exact. □

Proposition 6. *$\mathrm{ind}_I^G(1)$ is a generator of the triangulated category $D(G)$ in the sense that any strictly full triangulated subcategory of $D(G)$ closed under all direct sums which contains $\mathrm{ind}_I^G(1)$ coincides with $D(G)$.*

Proof. By (1) we have

$$\mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(1)[j], V^\bullet) = \mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(1), V^\bullet[-j]) = H^0(I, V^\bullet[-j]) = H^{-j}(I, V^\bullet)$$

for any $V^\bullet$ in $D(G)$. Hence Prop. 5 implies that the family of shifts $\{\mathrm{ind}_I^G(1)[j]\}_{j\in\mathbb{Z}}$ is a generating set of $D(G)$ in the sense of [Ne2] Def. 8.1.1. On the other hand, by Lemma 4, each shift $\mathrm{ind}_I^G(1)[j]$ is a compact object. In the language of [Ne2] this means that $\{\mathrm{ind}_I^G(1)[j]\}_{j\in\mathbb{Z}}$ is an $\aleph_0$-perfect class consisting of $\aleph_0$-small objects (loc. cit. Remark 4.2.6 and Def. 4.2.7). According to [Ne2] Lemma 4.2.1 the class $\{\mathrm{ind}_I^G(1)[j]\}_{j\in\mathbb{Z}}$ then is β-perfect for any infinite cardinal β. Hence [Ne2] Thm. 8.3.3 applies and shows (see the explanations in 3.2.6-8) that any strictly full triangulated subcategory of $D(G)$ closed under all direct sums which contains $\mathrm{ind}_I^G(1)$, and therefore the whole class $\{\mathrm{ind}_I^G(1)[j]\}_{j\in\mathbb{Z}}$, coincides with $D(G)$. □

3 The Hecke DGA

In order to also "derive" the picture on the Hecke algebra side we fix an injective resolution $\mathrm{ind}_I^G(1) \xrightarrow{\sim} \mathcal{I}^\bullet$ in $C(\mathrm{Mod}_k(G))$ and introduce the differential graded algebra

$$\mathcal{H}_I^\bullet := \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}}$$

over k. We recall that

$$\mathcal{H}_I^n = \prod_{q\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathcal{I}^q, \mathcal{I}^{q+n})$$

with differential

$$(da)_q(x) = d(a_q(x)) - (-1)^n a_{q+1}(dx)$$

for $a = (a_q) \in \mathcal{H}_I^n$ and multiplication

$$(ba)_q := (-1)^{mn} a_{q+m} \circ b_q$$

for $a = (a_q) \in \mathcal{H}_I^n$ and $b = (b_q) \in \mathcal{H}_I^m$. The cohomology of $\mathcal{H}_I^\bullet$ is given by

$$h^*(\mathcal{H}_I^\bullet) = \mathrm{Ext}^*_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), \mathrm{ind}_I^G(1))$$

(compare [Har] I§6). In particular

$$h^0(\mathcal{H}_I^\bullet) = \mathcal{H}_I \ .$$

Remark 7. *$h^*(\mathcal{H}_I^\bullet) = H^*(I, ind_I^G(1))$ and, in particular, $h^i(\mathcal{H}_I^\bullet) = 0$ for $i > d$.*

Proof. We compute

$$\begin{aligned} h^*(\mathcal{H}_I^\bullet) &= \mathrm{Ext}^*_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), \mathrm{ind}_I^G(1)) = h^*(\mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), \mathcal{I}^\bullet)) \\ &= h^*((\mathcal{I}^\bullet)^I) = H^*(I, \mathrm{ind}_I^G(1)) \ . \end{aligned}$$

□

Let $D(\mathcal{H}_I^\bullet)$ be the derived category of differential graded left $\mathcal{H}_I^\bullet$-modules. Note that $\mathcal{H}_I^\bullet$ is a compact generator of $D(\mathcal{H}_I^\bullet)$ ([Ke2] §2.5). It is well known that $\mathcal{H}_I^\bullet$ and $D(\mathcal{H}_I^\bullet)$ do not depend, up to quasi-isomorphism and equivalence, respectively, on the choice of the injective resolution $\mathcal{I}^\bullet$. For the convenience of the reader we briefly recall the argument. Let $\mathrm{ind}_I^G(1) \xrightarrow{\sim} \mathcal{J}^\bullet$ be a second injective resolution in $C(\mathrm{Mod}_k(G))$, and let $\mathcal{I}^\bullet \xrightarrow{f} \mathcal{J}^\bullet$ be a homotopy equivalence inducing the identity on $\mathrm{ind}_I^G(1)$ with homotopy inverse g. We form the differential graded algebra

$$\mathcal{A}^\bullet := \{(a,b) \in \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{J}^\bullet)^{\mathrm{op}} \times \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}} : a \circ f = f \circ b\}$$

(w.r.t. componentwise multiplication) and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{A}^\bullet & \xrightarrow{\mathrm{pr}_2} & \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}} \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle b \mapsto f \circ b} \\ \mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{J}^\bullet)^{\mathrm{op}} & \xrightarrow{a \mapsto a \circ f} & \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet, \mathcal{J}^\bullet). \end{array}$$

Obviously, the maps pr_i are homomorphisms of differential graded algebras (and the bottom horizontal and right perpendicular arrows are homotopy equivalences of complexes). By direct inspection one checks that the pr_i, in fact, are quasi-isomorphisms. Hence the differential graded algebras $\mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}}$ and $\mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{J}^\bullet)^{\mathrm{op}}$ are naturally quasi-isomorphic to each other. Moreover, by appealing to [BL] Thm. 10.12.5.1, we see that the functors

$$D(\mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet)^{\mathrm{op}}) \xrightarrow[(\mathrm{pr}_2)_*]{\sim} D(\mathcal{A}^\bullet) \xleftarrow[(\mathrm{pr}_1)_*]{\sim} D(\mathrm{End}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{J}^\bullet)^{\mathrm{op}})$$

are equivalences of triangulated categories.

There is the following pair of adjoint functors

$$H : D(G) \longrightarrow D(\mathcal{H}_I^\bullet) \qquad \text{and} \qquad T : D(\mathcal{H}_I^\bullet) \longrightarrow D(G) \ .$$

For any K-injective complex $V^\bullet$ in $\mathrm{Mod}_k(G)$ the natural chain map

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet, V^\bullet) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathrm{ind}_I^G(1), V^\bullet)$$

is a quasi-isomorphism. But the left hand term in a natural way is a differential graded left $\mathcal{H}_I^\bullet$-module. In fact we have the functor

$$\begin{aligned} K_{inj}(\mathrm{Mod}_k(G)) &\longrightarrow K(\mathcal{H}_I^\bullet) \\ V^\bullet &\longmapsto \mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}(\mathcal{I}^\bullet, V^\bullet) \end{aligned}$$

into the homotopy category $K(\mathcal{H}_I^\bullet)$ of differential graded left $\mathcal{H}_I^\bullet$-modules which allows us to define the composed functor

$$H : D(G) \xrightarrow{\mathbf{i}} K_{inj}(\mathrm{Mod}_k(G)) \longrightarrow K(\mathcal{H}_I^\bullet) \longrightarrow D(\mathcal{H}_I^\bullet) \ .$$

The diagram

(2)
$$\begin{array}{ccc} D(G) & \xrightarrow{H} & D(\mathcal{H}_I^\bullet) \\ & {\scriptstyle RH^0(I,.)} \searrow & \downarrow {\scriptstyle \text{forget}} \\ & & D(\mathrm{Vec}_k) \end{array}$$

then is commutative up to natural isomorphism.

For the functor T in the opposite direction we first note that $\mathcal{I}^\bullet$ is naturally a differential graded right $\mathcal{H}_I^\bullet$-module so that we can form the graded tensor product $\mathcal{I}^\bullet \otimes_{\mathcal{H}_I^\bullet} M^\bullet$ with any differential graded left $\mathcal{H}_I^\bullet$-module $M^\bullet$. This tensor product is naturally a complex in $C(\mathrm{Mod}_k(G))$. We now define T to be the composite

$$T : D(\mathcal{H}_I^\bullet) \xrightarrow{\mathbf{p}} K_{pro,\mathcal{H}_I^\bullet} \xrightarrow{\mathcal{I}^\bullet \otimes_{\mathcal{H}_I^\bullet}} K(\mathrm{Mod}_k(G)) \longrightarrow D(G) \ .$$

Here $K_{pro,\mathcal{H}_I^\bullet}$ denotes the full triangulated subcategory of $K(\mathcal{H}_I^\bullet)$ consisting of K-projective modules and $\mathbf{p}$ is a quasi-inverse of the equivalence of triangulated categories $K_{pro,\mathcal{H}_I^\bullet} \xrightarrow{\simeq} D(\mathcal{H}_I^\bullet)$ (compare [BL] 10.12.2.9).

The usual standard computation shows that T is left adjoint to H.

4 The main theorem

We need one more property of the derived category $D(G)$.

Lemma 8. *The triangulated category $D(G)$ is algebraic.*

Proof. The composite functor

$$D(G) \xrightarrow{\mathbf{i}} K_{inj}(\mathrm{Mod}_k(G)) \xrightarrow{\subseteq} K(\mathrm{Mod}_k(G))$$

is a fully faithful exact functor between triangulated categories. Hence the assertion follows from [Kra] §7.5 Lemma. □

In view of Lemmas 4 and 8 and Prop. 6 all assumptions of Keller's theorem ([Ke1] Thm. 4.3, [Ke2] Thm. 3.3.a; compare also [BvB] Thm. 3.1.7) are satisfied and we obtain our main result.

Theorem 9. *The functor H is an equivalence between triangulated categories*

$$D(G) \xrightarrow{\sim} D(\mathcal{H}_I^\bullet) \ .$$

Of course, it follows formally that the adjoint functor T is a left inverse of H.

Remark 10. *The full subcategory $D(G)^c$ of all compact objects in $D(G)$ is the smallest strictly full triangulated subcategory closed under direct summands which contains $ind_I^G(1)$.*

Proof. In view of Lemma 4 and Prop. 6 this follows from [Ne1] Lemma 2.2. □

The subcategory $D(G)^c$ should be viewed as the analog of the subcategory of perfect complexes in the derived category of a ring (cf. [Ke2] §1.4 Lemma).

Another important subcategory of $D(G)$ is the bounded derived category $D^b(G) := D^b(\mathrm{Mod}_k(G))$. Correspondingly we have the full subcategory $D^b(\mathcal{H}_I^\bullet)$ of all differential graded modules $M^\bullet$ in $D(\mathcal{H}_I^\bullet)$ such that $h^j(M^\bullet) = 0$ for all but finitely many $j \in \mathbb{Z}$. Since I has finite cohomological dimension the commutative diagram (2) shows that H restricts to a fully faithful functor

$$D^b(G) \longrightarrow D^b(\mathcal{H}_I^\bullet) \ .$$

On the other hand the behaviour of the functor T is controlled by an Eilenberg-Moore spectral sequence

$$E_2^{r,s} = \mathrm{Tor}_{-r}^{h^*(\mathcal{H}_I^\bullet)}(\mathrm{ind}_I^G(1), h^*(M^\bullet))^s \Longrightarrow h^{r+s}(T(M^\bullet))$$

([May] Thm. 4.1). This suggests that except in very special cases the functor T will not preserve the bounded subcategories.

5 Complements

5.1 The top cohomology

A first step in the investigation of the DGA $\mathcal{H}_I^\bullet$ might be the computation of its cohomology algebra $h^*(\mathcal{H}_I^\bullet)$. By Remark 7 the latter is concentrated in degrees 0 to d. Of course the usual Hecke algebra $\mathcal{H}_I = h^0(\mathcal{H}_I^\bullet)$ is a subalgebra of $h^*(\mathcal{H}_I^\bullet)$. We determine here the top cohomology $h^d(\mathcal{H}_I^\bullet)$ as a right $\mathcal{H}_I$-module.

Using the I-equivariant linear map

$$\begin{aligned} \pi_I : \mathrm{ind}_I^G(1) &\longrightarrow \mathrm{ind}_I^G(1)^I = \mathcal{H}_I \\ \phi &\longmapsto \big[h \mapsto \sum_{g \in I/I\cap hIh^{-1}} \phi(gh)\big] \end{aligned}$$

we obtain the map

$$\pi_I^* : h^*(\mathcal{H}_I^\bullet) = H^*(I, \mathrm{ind}_I^G(1)) \xrightarrow{H^*(I,\pi_I)} H^*(I, \mathcal{H}_I) = H^*(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I \ .$$

The last equality in this chain comes from the universal coefficient theorem which is applicable since I as a Poincaré group ([Laz] V.2.5.8) has finite cohomology $H^*(I, \mathbb{F}_p)$. Of course, as a ring $\mathcal{H}_I$ is a right module over itself. For our purposes we have to consider a modification of this module structure which is specific to characteristic p.

As a k-vector space $\mathrm{ind}_I^G(1)^I = \mathcal{H}_I$ has the basis $\{\chi_{IxI}\}_{x\in I\backslash G/I}$ consisting of the characteristic functions of the double cosets IxI. If we denote the multiplication in the algebra $\mathcal{H}_I$, as usual, by the symbol "$*$" for convolution then in this basis it is given by the formula

$$\chi_{IxI} * \chi_{IhI} = \sum_{y\in I\backslash G/I} c_{x,y;h}\chi_{IyI}$$

where the coefficients are

$$c_{x,y;h} = (\chi_{IxI} * \chi_{IhI})(y) = \sum_{y\in G/I} \chi_{IxI}(g)\chi_{IhI}(g^{-1}y) = |IxI \cap yIh^{-1}I/I| \cdot 1_k$$

with 1_k denoting the unit element in the field k. Of course, for fixed x and h we have $c_{x,y;h} = 0$ for all but finitely many $y \in I\backslash G/I$. But $IxI \cap yIh^{-1}I \neq \emptyset$ implies $IxI \subseteq IyIh^{-1}I$; by compactness the latter is a finite union of double cosets. Hence also for fixed y and h we have $c_{x,y;h} \neq 0$ for at most finitely many $x \in I\backslash G/I$. It follows that by combining the transpose of these coefficient matrices with the anti-automorphism

$$\begin{aligned} \mathcal{H}_I &\longrightarrow \mathcal{H}_I \\ \chi &\longmapsto \chi^*(g) := \chi(g^{-1}) \end{aligned}$$

we obtain through the formula

$$\chi_{IxI} *_\tau \chi_{IhI} := \sum_{y\in I\backslash G/I} c_{y,x;h^{-1}}\chi_{IyI}$$

a new right action of $\mathcal{H}_I$ on itself. We denote this new module by $\mathcal{H}_I^\tau$.

Comment: We compute

$$\begin{aligned} |IyI/I| \cdot c_{x,y;h} &= |IyI/I| \cdot (\chi_{IxI} * \chi_{IhI})(y) \\ &= \sum_{z\in G/I} \chi_{IyI}(z)\big(\chi_{IxI} * \chi^*_{Ih^{-1}I}\big)(z) \\ &= \big(\chi_{IyI} * (\chi_{IxI} * \chi^*_{Ih^{-1}I})^*\big)(1) \\ &= \big((\chi_{IyI} * \chi_{Ih^{-1}I}) * \chi^*_{IxI}\big)(1) \\ &= \sum_{z\in G/I} (\chi_{IyI} * \chi_{Ih^{-1}I})(z)\chi_{IxI}(z) \\ &= |IxI/I| \cdot (\chi_{IyI} * \chi_{Ih^{-1}I})(x) \\ &= |IxI/I| \cdot c_{y,x;h^{-1}} \ . \end{aligned}$$

This, of course, is valid with integral coefficients (instead of k). Moreover $|IxI/I|$ is always a power of p. It follows that over any field of characteristic different from p one has $\mathcal{H}_I^\tau \cong \mathcal{H}_I$. It also follows that $c_{x,y;h} = c_{y,x;h^{-1}}$ whenever both are nonzero.

It is straightforward to check that

$$\pi_I(\phi) *_\tau \chi_{IhI} = \pi_I(\phi * \chi_{IhI})$$

holds true for any $\phi \in \mathrm{ind}_I^G(1)$ and any $h \in G$. Hence

$$\pi_I : \mathrm{ind}_I^G(1) \longrightarrow \mathcal{H}_I^\tau \qquad \text{and} \qquad \pi_I^* : h^*(\mathcal{H}_I^\bullet) \longrightarrow H^*(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I^\tau$$

are maps of right $\mathcal{H}_I$-modules.

Proposition 11. *The map π_I^d is an isomorphism*

$$h^d(\mathcal{H}_I^\bullet) \xrightarrow{\cong} H^d(I, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{H}_I^\tau$$

of right $\mathcal{H}_I$-modules. By fixing a basis of the one dimensional $\mathbb{F}_p$-vector space $H^d(I, \mathbb{F}_p)$ we therefore obtain $h^d(\mathcal{H}_I^\bullet) \cong \mathcal{H}_I^\tau$ as right $\mathcal{H}_I$-modules.

Proof. It remains to show that π_I^d is bijective. We have the I-equivariant decomposition

$$\mathrm{ind}_I^G(1) = \bigoplus_{x \in I\backslash G/I} \mathrm{ind}_{I\cap xIx^{-1}}^I(1) \ .$$

The map π_I restricts to

$$\begin{aligned} \pi_I : \mathrm{ind}_{I\cap xIx^{-1}}^I(1) &\longrightarrow k \cdot \chi_{IxI} \subseteq \mathcal{H}_I \\ \phi &\longmapsto \big(\sum_{y \in I/I\cap xIx^{-1}} \phi(y) \big) \cdot \chi_{IxI} \ . \end{aligned}$$

Since $H^*(I, .)$ commutes with arbitrary direct sums it therefore suffices to show that the map

$$H^d(I, \phi \mapsto \sum_{y \in I/I\cap xIx^{-1}} \phi(y)) : H^d(I, \mathrm{ind}_{I\cap xIx^{-1}}^I(1_{\mathbb{F}_p})) \longrightarrow H^d(I, \mathbb{F}_p)$$

is bijective. Using Shapiro's lemma this latter map identifies (cf. [S-CG] Chap. I §2.5) with the corestriction map

$$\mathrm{Cor} : H^d(I \cap xIx^{-1}, \mathbb{F}_p) \longrightarrow H^d(I, \mathbb{F}_p)$$

which for Poincaré groups of dimension d is an isomorphism of one dimensional vector spaces ([S-CG] (4) on p. 37). □

5.2 The easiest example

As an example, we will make explicit the case where $G = I = \mathbb{Z}_p$ is the additive group of p-adic integers, which we nevertheless write multiplicatively with unit element e. In order to distinguish it from the unit element $1 \in k$ we will denote the multiplicative unit in $\mathbb{Z}_p$ by γ. Let Ω denote the completed group ring of $\mathbb{Z}_p$ over k. We have:

a) The category $\mathrm{Mod}_k(G)$ coincides with the category of torsion Ω-modules.

b) Sending $\gamma - 1$ to t defines an isomorphism of k-algebras $\Omega \cong k[[t]]$ between Ω and the formal power series ring in one variable t over k.

For any V in $\mathrm{Mod}_k(G)$ we have the smooth G-representation $C^\infty(G, V)$ of all V-valued locally constant functions on G where $g \in G$ acts on $f \in C^\infty(G, V)$ by ${}^g f(h) := g(f(g^{-1}h))$. One easily checks:

c) $C^\infty(G, V) = C^\infty(G, k) \otimes_k V$ with the diagonal G-action on the right hand side.

d) The map $\mathrm{Hom}_{\mathrm{Mod}_k(G)}(W, C^\infty(G, V)) \xrightarrow{\cong} \mathrm{Hom}_k(W, V)$ sending F to $[w \mapsto F(w)(e)]$ is an isomorphism for any W in $\mathrm{Mod}_k(G)$. It follows that $C^\infty(G, V)$ is an injective object in $\mathrm{Mod}_k(G)$.

e) The short exact sequence

$$0 \longrightarrow V \longrightarrow C^\infty(G,k) \otimes_k V \xrightarrow{\gamma_* - 1 \otimes \mathrm{id}} C^\infty(G,k) \otimes_k V \longrightarrow 0 \ , \tag{3}$$

where $\gamma_*(\phi)(h) = \phi(h\gamma)$, is an injective resolution of V in $\mathrm{Mod}_k(G)$.

f) For any $g \in G$ define the map $F_g : C^\infty(G,k) \to C^\infty(G,k)$ by $F_g(\phi)(h) := \phi(hg)$. In particular, $F_\gamma = \gamma_*$. Sending g to F_g defines an isomorphism of k-algebras

$$\Omega \xrightarrow{\cong} \mathrm{End}_{\mathrm{Mod}_k(G)}(C^\infty(G,k)) \ .$$

Obviously $\mathrm{ind}_I^G(1) = k$ is the trivial G-representation. By (3) we may take for $\mathcal{I}^\bullet$ the injective resolution

$$C^\infty(G,k) \xrightarrow{\gamma_* - 1} C^\infty(G,k) \longrightarrow 0 \longrightarrow \ldots$$

Using (f) we deduce that $\mathcal{H}_I^\bullet$ is

$$\ldots \longrightarrow \mathcal{H}_I^{-1} = \Omega \xrightarrow{d^{-1}} \mathcal{H}_I^0 = \Omega \times \Omega \xrightarrow{d^0} \mathcal{H}_I^1 = \Omega \longrightarrow \ldots$$

with

$$d^{-1}a = ((\gamma - 1)a, (\gamma - 1)a) \qquad \text{and} \qquad d^0(a,b) = (\gamma - 1)(a - b)$$

and multiplication

$$(a_{-1}, (a_0, b_0), a_1) \cdot (a'_{-1}, (a'_0, b'_0), a'_1) \\ = (a'_0 a_{-1} + a'_{-1} b_0, (a'_0 a_0 - a'_{-1} a_1, b'_0 b_0 - a'_1 a_{-1}), a'_1 a_0 + b'_0 a_1) \ .$$

Using b) we then identify $\mathcal{H}_I^\bullet$ with the upper row in the commutative diagram

$$\begin{array}{ccccc} k[[t]] & \xrightarrow{a \mapsto (ta, ta)} & k[[t]] \times k[[t]] & \xrightarrow{(a,b) \mapsto t(a-b)} & k[[t]] \\ \uparrow & & \uparrow {\scriptstyle a \mapsto (a,a)} & & \uparrow {\scriptstyle \subseteq} \\ 0 & \longrightarrow & k & \xrightarrow{0} & k. \end{array}$$

We view the bottom row as the differential graded algebra of dual numbers $k[\epsilon]/(\epsilon^2)$ in degrees 0 and 1 with the zero differential. It is easy to check that the vertical arrows in the above diagram constitute a quasi-isomorphism of differential graded algebras. In particular, this says that $\mathcal{H}_I^\bullet$ is quasi-isomorphic to its cohomology algebra with zero differential (ϵ corresponds to the projection map $G = \mathbb{Z}_p \to \mathbb{F}_p \subseteq k$ as a generator of $H^1(G,k) = \mathrm{Hom}^{cont}(\mathbb{Z}_p, k)$). According to our Thm. 9 we therefore obtain that H composed with the pullback along the above quasi-isomorphism is an equivalence of triangulated categories

$$D(\mathbb{Z}_p) \xrightarrow{\sim} D(k[\epsilon]/(\epsilon^2)) \ . \tag{4}$$

We finish by determining this functor explicitly. Let V be an object in $\mathrm{Mod}_k(G)$. Using the injective resolution (3) we can represent $H(V)$ by the complex

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}_k(G)}([C^\infty(G,k) \xrightarrow{\gamma_* - 1} C^\infty(G,k)], [C^\infty(G,k) \otimes_k V \xrightarrow{\gamma_* - 1 \otimes \mathrm{id}} C^\infty(G,k) \otimes_k V]) \ .$$

Furthermore, using the identifications in c) and d) this latter complex can be computed to be the complex

$$\mathrm{Hom}_k(C^\infty(G,k),V) \xrightarrow{d^{-1}} \mathrm{Hom}_k(C^\infty(G,k),V) \times \mathrm{Hom}_k(C^\infty(G,k),V) \xrightarrow{d^0} \mathrm{Hom}_k(C^\infty(G,k),V)$$

in degrees -1, 0, and 1 with the differentials

$$\begin{aligned} d^{-1}f &= (f \circ (\gamma_* - 1), f \circ (\gamma_* - 1) + (\gamma - 1) \circ f \circ \gamma_*) \quad \text{and} \\ d^0(f_0, f_1) &= (\gamma - 1) \circ f_0 \circ \gamma_* + (f_0 - f_1) \circ (\gamma_* - 1) \ . \end{aligned}$$

Let $\delta_e \in \mathrm{Hom}_k(C^\infty(G,k),k)$ denote the "Dirac distribution" $\delta_e(\phi) := \phi(e)$ in the unit element. The diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \mathrm{Hom}_k(C^\infty(G,k),V) \\ \downarrow & & \downarrow d^{-1} \\ V & \xrightarrow{v \mapsto (\delta_e(.)v, \delta_e(.)\gamma(v))} & \mathrm{Hom}_k(C^\infty(G,k),V) \times \mathrm{Hom}_k(C^\infty(G,k),V) \\ {\scriptstyle \gamma - 1}\downarrow & & \downarrow d^0 \\ V & \xrightarrow{v \mapsto \delta_e(.)v} & \mathrm{Hom}_k(C^\infty(G,k),V) \end{array}$$

is commutative. We claim that the horizontal arrows form a quasi-isomorphism $\alpha^\bullet$. In order to define a map in the opposite direction we let $\phi_1 \in C^\infty(G,k)$ denote the constant function with value 1. Using that $\gamma_*(\phi_1) = \phi_1$ one checks that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_k(C^\infty(G,k),V) & \longrightarrow & 0 \\ {\scriptstyle d^{-1}}\downarrow & & \downarrow \\ \mathrm{Hom}_k(C^\infty(G,k),V) \times \mathrm{Hom}_k(C^\infty(G,k),V) & \xrightarrow{(f_0,f_1) \mapsto f_0(\phi_1)} & V \\ {\scriptstyle d^0}\downarrow & & \downarrow {\scriptstyle \gamma - 1} \\ \mathrm{Hom}_k(C^\infty(G,k),V) & \xrightarrow{f \mapsto f(\phi_1)} & V \end{array}$$

is commutative. Hence the horizontal arrows define a homomorphism of complexes $\beta^\bullet$ such that $\beta^\bullet \circ \alpha^\bullet = \mathrm{id}$. Applying $\mathrm{Hom}_k(.,V)$ to our injective resolution of k we obtain the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_k(C^\infty(G,k),V) \xrightarrow{f \mapsto f \circ (\gamma_* - 1)} \mathrm{Hom}_k(C^\infty(G,k),V) \xrightarrow{\beta^1} V \longrightarrow 0 \ .$$

This implies that d^{-1} is injective and that $\mathrm{im}(d^0) \supseteq \ker(\beta^1)$. The former says that the cohomology in degree -1 is zero. Because of

$$\mathrm{Hom}_k(C^\infty(G,k),V) = \ker(\beta^1) \oplus \mathrm{im}(\alpha^1) \tag{5}$$

the latter shows the surjectivity of $h^1(\alpha^\bullet)$. Hence $h^1(\alpha^\bullet)$ is bijective. A pair (f_0, f_1) represents a class in $\ker(h^0(\beta^\bullet))$ if and only if $d^0(f_0, f_1) = 0$ and $\beta^0(f_0, f_1) = 0$. The first condition implies that

$$f_1 \circ (\gamma_* - 1) = (\gamma - 1) \circ f_0 \circ \gamma_* + f_0 \circ (\gamma_* - 1) \ .$$

By (5) the second condition says that we may write $f_0 = \delta_e(.)v + f\circ(\gamma_* - 1)$ for $v := f_0(\phi_1) \in V$ and some $f \in \mathrm{Hom}_k(C^\infty(G,k), V)$. Inserting this into the above equation we obtain

$$f_1 \circ (\gamma_* - 1) = \delta_e(.)(\gamma(v) - v) + (\gamma \circ f \circ \gamma_* - f) \circ (\gamma_* - 1) \ .$$

It follows that

$$\gamma(v) = v \qquad \text{and} \qquad f_1 = (\gamma \circ f \circ \gamma_* - f) \ .$$

Using this last identity one checks that $(f_0, f_1) = d^{-1}f + (\delta_e(.)v, 0)$. But we have $0 = d^0(\delta_e(.)v, 0) = \delta_e(\gamma_*.)(\gamma - 1)(v) + \delta_e((\gamma_* - 1).)v = \delta_e((\gamma_* - 1).)v$, which implies that $v = 0$. We conclude that $h^0(\beta^\bullet)$ is injective and hence bijective and that therefore $h^0(\alpha^\bullet)$ is bijective.

A differential graded $k[\epsilon]/(\epsilon^2)$-module is the same as a graded k-vector space with two anti-commuting differentials ϵ and d of degree 1. Given the smooth G-representation V we form the graded $k[\epsilon]/(\epsilon^2)$-module $k[\epsilon]/(\epsilon^2) \otimes_k V$ (sitting in degrees 0 and 1) and equip it with the differential $d_V(v_0 + v_1\epsilon) := (\gamma - 1)(v_0)\epsilon$. The above computations together with the fact that ϵ corresponds to the identity in $\mathcal{H}_I^1 = \mathrm{Hom}_{\mathrm{Mod}_k(G)}(\mathcal{I}^0, \mathcal{I}^1) = \mathrm{End}_{\mathrm{Mod}_k(G)}(C^\infty(G,k))$ proves the following

Proposition 12. *The equivalence* (4) *sends V in $Mod_k(G)$ to the differential graded module* $(k[\epsilon]/(\epsilon^2) \otimes_k V, d_V)$.

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