ZELEVINSKY OPERATIONS FOR MULTISEGMENTS AND A PARTIAL ORDER ON PARTITIONS

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Abstract. The motivation for this article comes from the representation theory of general linear groups $GL_n(F)$, where F is a p-adic local field. In a natural way it leads to a partial order on the set of partitions of $n \geq 1$. By way of an example we showed in Schneider and Zink (2016) that this partial order is strictly contained in the well known dominance order. Our aim in the present paper is to describe it explicitly in terms of partitions alone, without any reference to representations. It is not too complicated to see that our partial order contains the refinement partial order. Our key result is that the additional relations are generated by the following "basic" relations: Two partitions $\mathcal{P} = (\ell_1, ..., \ell_s)$ and $\mathcal{P}' = (\ell'_1, ..., \ell'_s)$ satisfy $\mathcal{P} \leq \mathcal{P}'$ if there exist permutations σ and π of $(1, \ldots, s)$ such that ℓ'_i and $\ell'_{\pi(i)}$ have the same parity and $\ell_{\sigma(i)} = \frac{\ell'_i + \ell'_{\pi(i)}}{2}$ for all $i = 1, ...s$. The input from representation theory, coming from the work of Zelevinsky (1980), consists of certain operations on the set of so called multisegments. We prove our result by a careful analysis of the commutation relations between these "Zelevinsky operations".

1. Introduction

The motivation for this article comes from the representation theory of general linear groups $GL_n(F)$, where F is a p-adic local field. In a natural way we were led to an explicit relation R on the set $\mathcal{D}_n^+$ of partitions of $n \geq 1$, which we will write as $\mathcal{P} \leq \mathcal{P}'$. It satisfies:

(i) $R \subseteq R_{dom}$, where R_{dom} is the dominance order relation written as $\mathcal{P} \leq_{dom} \mathcal{P}'$;
(ii) R is not transitive, such that $\mathcal{P} \leq \mathcal{P}' \leq \mathcal{P}''$ need not to imply $\mathcal{P} \leq \mathcal{P}''$.

Therefore the problem was to determine the transitive closure $R^+ \subseteq R_{dom}$ which surprisingly turned out to be strictly smaller than R_{dom}. This was indicated only by Example 2.29 in [SZ] in the case $n = 6$. Our aim in the present paper is to give a complete and explicit description of R^+ in general.

The difficulty comes from the fact that the relation R is given not directly in terms of partitions but in terms of a representation theoretic nature. The origin of R can be sketched as follows: We consider the set $Irr(G)$ of all isomorphism classes of irreducible smooth representations of a p-adic group G, which equivalently may be understood as simple modules over the Hecke algebra $\mathcal{H}(G)$. The *Jacobson topology* on $Irr(G)$ is given by the closed subsets $\mathcal{V}(J)$, for 2-sided ideals $J \subset \mathcal{H}(G)$, consisting of all simple $\mathcal{H}(G)$-modules which are annihilated by J. With respect to this Jacobson topology we have in $Irr(G)$ the dense subspace $Irr^t(G)$ of tempered classes. Let

$$Irr^t(G) = \dot{\bigcup}_{\Theta} Irr^t_{\Theta}$$

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denote its decomposition into connected components. The origin of our basic relation R is the relation

f:closure-rel (1) $$\Theta' \leq \Theta \quad \text{if and only if} \quad Irr^t_{\Theta'} \cap \overline{Irr^t_{\Theta}} \neq \emptyset \ ,$$

where the closure is formed in $Irr(G)$. We now specialize to the case $G_n = GL_n(F)$ and recall the transition from tempered components to partitions in more detail.

By the work of Bernstein and Zelevinsky irreducible representations, tempered representations, and tempered components Θ can be classified in terms of so called multisegments as follows. Let $\mathcal{C}$ be the set of irreducible cuspidal representations of all G_n and let $[\mathcal{C}] := \mathcal{C}/\sim$ be the set of inertial classes $[\sigma]$ where $\sigma \sim \sigma'$ are inertially equivalent if $\sigma' = \sigma \otimes |.|^s$ for some $s \in \mathbb{C}$ (here $|.| := |det|_F$). A *segment* of length ℓ is a sequence $\Delta = (\sigma, \sigma|.|, ..., \sigma|.|^{\ell-1})$ "supported" by cuspidal representations. Note that all constituents of Δ are in the same inertial class $[\sigma]$ which is uniquely determined by Δ.

Each $\sigma \in \mathcal{C}$ has the degree $d = d(\sigma)$, if it is a representation of G_d, as well as a unique polar decomposition

$$\sigma = \sigma_0 \otimes |.|^r \ ,$$

where the unitary representation σ_0 and the real number $r = r(\sigma)$ are uniquely determined. This induces a map

2 (2) $$\Delta \longmapsto r(\Delta) := (r, r+1, ..., r+\ell-1)$$

assigning segments which are supported by real numbers to our original segments. The segment Δ of length ℓ is called centered if $r(\Delta) = \big(\frac{1-\ell}{2}, ..., \frac{\ell-1}{2}\big)$. More generally one has to consider *multisegments*

$$a = (\Delta_1, ..., \Delta_r) = \sum_{\Delta} m_\Delta(a) \cdot \Delta \ ,$$

which we may consider either as multisets of segments or as effective divisors supported on the set of all segments. We call a a centered multisegment if all constituents Δ_i are centered.

The work of Bernstein and Zelevinsky establishes bijections

- $\Delta \leftrightarrow L(\Delta)$ between centered segments of degree $d(\Delta) := \ell(\Delta) \cdot d(\sigma) = n$ and all discrete series representations of G_n, and
- $a \leftrightarrow L(a)$ between multisegments of degree $d(a) := \sum_i d(\Delta_i) = n$ and all irreducible representations of G_n; moreover, $L(a)$ is tempered if and only if a is centered.

We use here the notation of Rodier ([Rod] §4); it corresponds to $L(a) = \langle a \rangle^t$ in [Zel].

A partition is a finite nonincreasing sequence $\mathcal{P} = (\ell_1 \geq \ell_2 \geq \ldots \geq \ell_s)$ of positive integers; more precisely, $\mathcal{P}$ is a partition of $|\mathcal{P}| := \sum_i \ell_i$. We put $s(\mathcal{P}) := s$. We will consider (finitely supported) *partition valued functions* on $[\mathcal{C}]$, writing them as divisors $\mathcal{P} = \sum_{[\sigma]} \mathcal{P}_{[\sigma]}[\sigma]$ over $[\mathcal{C}]$ where the coefficients $\mathcal{P}_{[\sigma]}$ are partitions. The degree of such a function $\mathcal{P}$ is defined as $d(\mathcal{P}) := \sum_{[\sigma]} |\mathcal{P}_{[\sigma]}| \cdot d(\sigma)$. To a multisegment $a = (\Delta_1, ..., \Delta_r)$ we assign the partition valued function

$$\mathcal{P}(a) := \sum_{[\sigma]} \mathcal{P}^a_{[\sigma]}[\sigma]$$

where $\mathcal{P}^a_{[\sigma]}$ is the partition which collects the lengths of all segments Δ in a which have their support in the inertial class $[\sigma]$. Thus we have $d(\mathcal{P}(a)) = d(a)$. Furthermore, with any partition valued function $\mathcal{P}$ we associate

$$Irr^t_{\Theta_{\mathcal{P}}} := \{L(a) \mid a \text{ is a centered multisegment of degree } n \text{ such that } \mathcal{P}(a) = \mathcal{P}\}.$$

Then $Irr^t_{\Theta_\mathcal{P}}$ is a connected component of $Irr^t(G)$ and

3 (3)
$$\{\mathcal{P} : d(\mathcal{P}) = n\} \xrightarrow{\simeq} \{\text{tempered components } \Theta \text{ of } G_n\}$$
$$\mathcal{P} \longmapsto \Theta_\mathcal{P}$$

is a bijection (cf. [SZS] §2). Sometimes we will also consider the larger set

$$Irr_{\Theta_\mathcal{P}} := \{L(a) \mid a \text{ is a multisegment of degree } n \text{ such that } \mathcal{P}(a) = \mathcal{P}\}$$

consisting of all irreducible representations $L(a)$ such that the underlying segments are of prescribed size but need not be centered. Then from

$$Irr(G_n) = \dot{\bigcup}_\mathcal{P} Irr_{\Theta_\mathcal{P}} \supset Irr^t(G_n) = \dot{\bigcup}_\mathcal{P} Irr^t_{\Theta_\mathcal{P}}$$

(cf. [SZS] §2), where the right side is dense in the left side with respect to the Jacobson topology ([SZ] Remark 1.6), one may conclude that

$$\overline{Irr_{\Theta_\mathcal{P}}} = \overline{Irr^t_{\Theta_\mathcal{P}}} \qquad \text{for all } \mathcal{P}.$$

Via (3) we may transport our relation (1) to the set of partition valued functions by defining

4 (4)
$$\mathcal{P} \leq \mathcal{P}' \quad \text{if and only if} \quad \Theta_{\mathcal{P}'} \leq \Theta_\mathcal{P} \ .$$

Note that we reverse here the order to make $\mathcal{P} \leq \mathcal{P}'$ compatible with the dominance order for partitions later on (see Remark 3.3). Examining the Jacobson topology we will prove in section 2 the following result.

Proposition 1.1. *$\mathcal{P} \leq \mathcal{P}'$ is equivalent to finding multisegments a and b such that*

- *$\mathcal{P}(a) = \mathcal{P}$ and $\mathcal{P}(b) = \mathcal{P}'$,*
- *b is centered, and*
- *$b \leq_Z a$, which means that the centered multisegment b is obtained from the multisegment a by performing a sequence of elementary Zelevinsky operations.*

Here we will not explain the Zelevinsky operations for multisegments ([Zel] 7.1; see sections 2 and 3), but we mention that applying such an operation to a multisegment a will not change the cuspidal support of a which is the set of all σ occurring there, including multiplicities. Therefore we can split our conditions into a set of subconditions each of them referring to multisegments which are supported in a single inertial class $[\sigma]$. More formally this says that

5 (5)
$$\sum_{[\sigma]} \mathcal{P}_{[\sigma]}[\sigma] \leq \sum_{[\sigma]} \mathcal{P}'_{[\sigma]}[\sigma] \qquad \text{if and only if} \qquad \mathcal{P}_{[\sigma]}[\sigma] \leq \mathcal{P}'_{[\sigma]}[\sigma] \text{ for any } [\sigma].$$

Comment. This corresponds to the fact that for partition valued functions $\mathcal{P}$, $\mathcal{P}'$ which have disjoint support (i.e., $\mathcal{P}_{[\sigma]} = \emptyset$ or $\mathcal{P}'_{[\sigma]} = \emptyset$ for all $[\sigma]$) we obtain direct product decompositions

$$Irr_{\Theta_{\mathcal{P}+\mathcal{P}'}} \leftrightarrow Irr_{\Theta_\mathcal{P}} \times Irr_{\Theta_{\mathcal{P}'}} \qquad \text{and} \qquad Irr^t_{\Theta_{\mathcal{P}+\mathcal{P}'}} \leftrightarrow Irr^t_{\Theta_\mathcal{P}} \times Irr^t_{\Theta_{\mathcal{P}'}} \ ,$$

where an element $(L(a), L(b))$ from the right hand side is mapped to the normalized parabolic induction $i_{G_{d+d'}, P_{d+d'}}(L(a) \otimes L(b)) = L(a+b)$ on the left. From [Zel] Prop. 8.5. we see that in this situation the induction is always irreducible.

This will bring us down to considerations similar as in [SZ], where we have considered representations with Iwahori fixed vector corresponding to multisegments a which are supported in the single inertial class [1] of unramified characters and will lead us from partition valued functions to partitions (see Prop. 1.3 below).

Finally to get rid of representation theory altogether we have to pass from multisegments which are supported by cuspidal representations to multisegments which are supported by real numbers.

A (real) number supported multisegment ${}_na = ({}_n\Delta_1, ..., {}_n\Delta_r)$ is by definition an effective divisor over the set of number supported segments ${}_n\Delta = (r, r+1, ..., r+\ell-1)$. For each length ℓ we let ${}_n\Delta(\ell) = (\frac{1-\ell}{2}, ..., \frac{\ell-1}{2})$ denote the unique number supported segment of length ℓ which is centered. Then we have:

- The map (2) from segments to number supported segments naturally induces a map $a \mapsto r(a)$ from multisegments to number supported multisegments.
- There is a well defined map ${}_na \mapsto \mathcal{P}^{\,{}_na}$ from number supported multisegments onto the set of partitions such that $\mathcal{P}^{\,{}_na}$ collects the lengths of all segments (including multiplicities) which occur in ${}_na$. And via $\ell \mapsto {}_n\Delta(\ell)$ this map has a section

$$\mathcal{P} = (\ell_1 \geq \cdots \geq \ell_s) \longmapsto {}_nb(\mathcal{P}) = ({}_n\Delta(\ell_1), ..., {}_n\Delta(\ell_s))$$

  from partitions to number supported multisegments which are centered.

Our central definition now is the following.

def **Definition 1.2.** *The basic relation $\mathcal{P} \leq \mathcal{P}'$ for partitions holds if there is a number supported multisegment ${}_na$ such that*

- ${}_nb(\mathcal{P}') \leq_Z {}_na$ *meaning that the centered multisegment ${}_nb(\mathcal{P}')$ is obtained from the multisegment ${}_na$ by performing a sequence of elementary Zelevinsky operations (cf. section 3.2), and*
- $\mathcal{P}^{\,{}_na} = \mathcal{P}$.

It is justified by the following result proved in section 2.

Proposition 1.3. *For two partition valued functions $\mathcal{P} = \sum_{[\sigma]} \mathcal{P}_{[\sigma]}[\sigma]$ and $\mathcal{P}' = \sum_{[\sigma]} \mathcal{P}'_{[\sigma]}[\sigma]$ we have*

$$\mathcal{P} \leq \mathcal{P}' \qquad \textit{if and only if} \qquad \mathcal{P}_{[\sigma]} \leq \mathcal{P}'_{[\sigma]} \textit{ for any } [\sigma].$$

In this way we have reformulated the relation $\Theta_{\mathcal{P}'} \leq \Theta_{\mathcal{P}}$ (cf. (1)) in terms of basic relations for partitions, which are the main subject of this paper and which we will study from section 3 on. Thus our multisegments a will be always number supported and we will denote by $a \mapsto \mathcal{P}(a)$ the map onto the set of partitions. We will see that the Zelevinsky operations for multisegments have natural counterparts for number supported multisegments. And again these operations are preserving the support which are now the underlying numbers. Thus the condition ${}_nb \leq_Z {}_na$ from our Def. 1.2 implies that we need to work only with multisegments ${}_na$ which are supported in $\frac{1}{2}\mathbb{Z}$ because this is true for the centered multisegments ${}_nb$.

In the preparational section 3 we will translate some of the basic notions from [Zel] to the language of number supported segments/multisegments. It will be convenient to work in the free abelian group $\mathcal{D}$ on the set $\mathcal{S}$ of all segments. Then we may interpret multisegments as the semigroup $\mathcal{D}^+$ of effective divisors and Zelevinsky operations as translation operators on $\mathcal{D}$.

The aim of this paper is a direct characterization of the basic relation $\mathcal{P} \leq \mathcal{P}'$ **only** in terms of partitions avoiding multisegments and Zelevinsky operations. What is easy to see is property (i) above and also the following property:

(iii) $R_{ref} \subset R$, where R_{ref} denotes the refinement order relation written as $\mathcal{P} \leq_{ref} \mathcal{P}'$.

The latter means that $\mathcal{P}$ is obtained from $\mathcal{P}'$ by a sequence of cutting constituents $\ell'_i = \ell_i(\mathcal{P}')$ into parts. On the level of multisegments it amounts to applying Zelevinsky operations of so called type 0. It is obvious that a refinement $\mathcal{P} \leq_{ref} \mathcal{P}'$ will increase the number $s(\mathcal{P}) > s(\mathcal{P}')$ of parts of the partition. As a first substantial step, in section 4, in particular Prop. 4.4, we study the opposite situation, namely basic relations $\mathcal{P} \leq \mathcal{P}'$ such that $s(\mathcal{P}) = s(\mathcal{P}')$; on the level of multisegments this means to apply Zelevinsky operations of so called type 1.

(iv) For two partitions $\mathcal{P} = (\ell_1 \geq ... \geq \ell_s)$ and $\mathcal{P}' = (\ell'_1 \geq ... \geq \ell'_s)$ we have $\mathcal{P} \leq \mathcal{P}'$ if and only if there exist permutations σ and π of $(1, \dots, s)$ such that ℓ'_i and $\ell'_{\pi(i)}$ have the same parity and $\ell_{\sigma(i)} = \frac{\ell'_i + \ell'_{\pi(i)}}{2}$ for all $i = 1, ...s$.

These latter basic relations will be called of type 1. Our main result now is the following (cf. Thm. 6.1.)

Theorem 1.4. *The relation $\mathcal{P} \leq \mathcal{P}'$ is basic if and only if there is a partition $\mathcal{P}''$ such that $\mathcal{P} \leq_{ref} \mathcal{P}''$ and $\mathcal{P}'' \leq \mathcal{P}'$ is a basic relation of type 1.*

Then of course the transitive closure R^+ is obtained via sequences consisting of, possibly both, refinement relations as in (iii) and relations as in (iv). The proper inclusion $R^+ \subset R_{dom}$ is an easy corollary. Moreover, using [Bry], we will consider covers $\mathcal{P} <_{dom} \mathcal{P}'$ for the dominance order, which means it is impossible to refine it as $\mathcal{P} <_{dom} \mathcal{P}'' <_{dom} \mathcal{P}'$ and we will see that covers for the dominance order usually are not contained in R^+ (Prop. 6.3).

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sec:Jacobson

2. Jacobson topology for general linear groups

As we have sketched already, the tempered components Θ for the general linear groups G_n can be parameterized in terms of partition valued functions $\mathcal{P} = \sum_{[\sigma]} \mathcal{P}_{[\sigma]}[\sigma]$ which are given on the set of inertial classes $[\sigma]$ of cuspidal representations. For an irreducible representation $L(b)$ of G_n we want to express the fact that $L(b) \in Irr^t(\Theta_{\mathcal{P}'}) \bigcap \overline{Irr^t(\Theta_{\mathcal{P}})}$ in terms of the multisegment b. Obviously we have that $L(b) \in Irr^t(\Theta'_{\mathcal{P}})$ if and only if b is a centered multisegment such that $\mathcal{P}(b) = \mathcal{P}'$.

For a multisegment $a = (\Delta_1, ..., \Delta_r)$ we let $\Pi^a := i_{G_n,P_a}(L(\Delta_1) \otimes \cdots \otimes L(\Delta_r))$ where i_{G_n,P_a} is the normalized parabolic induction functor from the standard parabolic subgroup P_a whose diagonal block sizes are the degrees of the segments in the support of a.

SZ **Lemma 2.1.** *The Jacobson closure $\overline{Irr^t(\Theta_{\mathcal{P}})} = \overline{Irr(\Theta_{\mathcal{P}})}$ contains all irreducible subquotients of Π^a, in particular therefore $L(a)$, for all multisegments a such that $\mathcal{P}(a) = \mathcal{P}$.*

Proof. This is a straightforward generalization of [SZ] Lemmas 2.20 and 2.21. By [Ber] Cor. 3.9 there are arbitrarily small compact open subgroups $J \subseteq G_n$ such that the characteristic function e_J of J is a special idempotent in $\mathcal{H}(G)$ in the sense of [SZ] §1. The statements and proofs of [SZ] Lemmas 2.20 and 2.21 remain correct if we use such a J, instead of the Iwahori subgroup, with the property that the representation $\mathrm{Ind}_P^G(\tau)$ is generated by its J-fixed vectors. □

Note that in the above lemma a may vary over all multisegments such that $\mathcal{P}(a) = \mathcal{P}$ which means we may shift the underlying segments by unramified twists. Fixing $\mathcal{P}$ only amounts to fixing the lengths of segments which are supported in a certain inertial class.

We also need the following result, which is Theorem 7.1 in [Zel]. But first we need to recall what elementary Zelevinsky operations for multisegments are. Let Δ_1, Δ_2 be two linked segments, i.e., such that $\Delta_1 \not\subseteq \Delta_2$, $\Delta_2 \not\subseteq \Delta_1$ and $\Delta_1 \cup \Delta_2$ is again a segment. Then either $\Delta_1 \cap \Delta_2$ is also a segment or $\Delta_1 \cap \Delta_2 = \emptyset$, in which case $\Delta_1 \cup \Delta_2$ is a segment of length $\ell_1 + \ell_2$. An *elementary Zelevinsky operation* on $a = (\Delta_1, ..., \Delta_r)$ means to form a new multisegment a' by replacing a pair Δ_i, Δ_j of linked segments from a by the pair $\Delta_i \cup \Delta_j$, $\Delta_i \cap \Delta_j$. For two multisegments a and b we write $b \leq_Z a$ if b arises from $a = (\Delta_1, ..., \Delta_r)$ via a sequence of elementary Zelevinsky operations.

Zel **Proposition 2.2.** *Let a and b be any two multisegments. Then the irreducible representation $L(b)$ is a subquotient of Π^a if and only if $b \leq_Z a$. (Note here that the order of multisegments in a plays no role. A different order will change the parabolically induced representation Π^a but not the set of its irreducible subquotients by [Zel] Thm. 1.9.)*

The segment $\Delta = (\sigma, \ldots, \sigma|.|^{\ell-1})$ gives rise to the effective divisor $\nu(\Delta) := \sigma + \ldots + \sigma|.|^{\ell-1}$ on the set $\mathcal{C}$. For a multisegment $a = \sum_\Delta m_\Delta \cdot \Delta$ we then define the effective divisor $\nu(a) := \sum_\Delta m_\Delta \cdot \nu(\Delta)$ on $\mathcal{C}$. Here we note that $\nu(a)$ will be considered as the cuspidal support of the representation $L(a)$ which is the conjugacy class of pairs

$$(M_{\nu(a)}, \sigma^{\nu(a)}) := \left(\times_{\sigma\in\nu(a)} G_{d(\sigma)}, \otimes_{\sigma\in\nu(a)}\sigma\right)$$

consisting of a block-diagonal Levi subgroup and a corresponding cuspidal representation where one has to take into account the multiplicities.

nu **Lemma 2.3.** *For any $L(b) \in \overline{Irr(\Theta_{\mathcal{P}})}$ there is an $L(a) \in Irr(\Theta_{\mathcal{P}})$ such that $\nu(a) = \nu(b)$.*

Proof. Taking [Zel] 4.4(b) into account this is a straightforward generalization of the argument in the proof of [SZ] Prop. 2.23 (which proves the assertion in the case where the cuspidal supports are unramified characters). It is based on the observation that the cuspidal support map ν satisfies

$$\nu\left(\overline{Irr(\Theta_{\mathcal{P}})}\right) \subseteq \overline{\nu(Irr(\Theta_{\mathcal{P}}))} = \nu(Irr(\Theta_{\mathcal{P}})) \ .$$

The left hand inclusion comes from the continuity of ν. It has been proved in [SZ] Cor. 2.8 for representations with Iwahori fixed vector. The proof carries over to our more general situation if we replace the Iwahori subgroup J by an appropriate smaller subgroup. For the right hand equality one has to show that the image $\nu(Irr(\Theta_{\mathcal{P}}))$ is a closed subset of the Bernstein spectrum (w.r.t. the Zariski topology). Again in the proof of [SZ] Prop. 2.23 this has been done for representations with Iwahori fixed vector where $\mathcal{P} = \mathcal{P}_{[1]}[1]$ is supported by unramified characters. The argument easily generalizes to cases $\mathcal{P} = \mathcal{P}_{[\sigma]}[\sigma]$ with cuspidal support in a single inertial class, and then, taking [Zel] 4.4(b) into account, also to $\mathcal{P}$ with larger support. □

Putting these facts together we will prove the following result.

Prop1 **Proposition 2.4.** *$\mathcal{P} \leq \mathcal{P}'$ is equivalent to finding multisegments a and b such that*

- *$\mathcal{P}(a) = \mathcal{P}$ and $\mathcal{P}(b) = \mathcal{P}'$,*
- *b is centered, and*
- *$b \leq_Z a$, which means that the centered multisegment b is obtained from the multisegment a by performing a sequence of elementary Zelevinsky operations.*

Proof. Assume we have found multisegments a and b with the asserted properties. Then by the first two conditions we have $L(b) \in Irr^t(\Theta_{\mathcal{P}'})$, and the remaining condition tells us that $L(b)$ is a subquotient of Π^a where $\mathcal{P}(a) = \mathcal{P}$. Using Lemma 2.1 we deduce that $L(b) \in \overline{Irr_{\Theta_{\mathcal{P}}}}$ so that the intersection is non-empty.

Conversely assume that $L(b) \in Irr^t(\Theta_{\mathcal{P}'}) \cap \overline{Irr(\Theta_{\mathcal{P}})}$. Then clearly b is centered and $\mathcal{P}(b) = \mathcal{P}'$. By Lemma 2.3 the condition that $L(b) \in \overline{Irr(\Theta_{\mathcal{P}})}$ implies that $\nu(b) = \nu(a)$ for some $L(a) \in Irr(\Theta_{\mathcal{P}})$. Hence we have

$$\nu(b) = \nu(a) \quad \text{for some } a \text{ such that } \mathcal{P}(a) = \mathcal{P}.$$

Let now $A = \sum_\sigma m_\sigma \cdot \sigma$ be any fixed effective divisor on $\mathcal{C}$. Then according to [Zel] Lemma 9.10 the set $\nu^{-1}(A) := \{a : \nu(a) = A\}$ has a unique minimal element a_0 such that $\nu(a) = A$ implies $a_0 \leq_Z a$.

We apply this observation to $A := \nu(b) = \nu(a)$ as above, hence $a, b \in \nu^{-1}(A)$. But b is a centered multisegment and as such obviously does not admit any Zelevinsky operation. It follows that b is the unique minimal element in $\nu^{-1}(A)$, and therefore $b \leq_Z a$. □

The final aim of this section is to reduce the relation $\mathcal{P} \leq \mathcal{P}'$ for partition valued functions to the basic relation for partitions as defined in Def. 1.2.

Prop2 **Proposition 2.5.** *For two partition valued functions* $\mathcal{P} = \sum_{[\sigma]} \mathcal{P}_{[\sigma]}[\sigma]$ *and* $\mathcal{P}' = \sum_{[\sigma]} \mathcal{P}'_{[\sigma]}[\sigma]$ *we have*

$$\mathcal{P} \leq \mathcal{P}' \qquad \text{if and only if} \qquad \mathcal{P}_{[\sigma]} \leq \mathcal{P}'_{[\sigma]} \text{ for any } [\sigma].$$

Proof. As explained for (5) it is already clear that $\mathcal{P} \leq \mathcal{P}'$ if and only if $\mathcal{P}_{[\sigma]}[\sigma] \leq \mathcal{P}'_{[\sigma]}[\sigma]$ for all inertial classes $[\sigma]$. Thus we may restrict to partition valued functions which are supported by a single inertial class $[\sigma]$ and prove

$$\mathcal{P}_{[\sigma]}[\sigma] \leq \mathcal{P}'_{[\sigma]}[\sigma] \quad \text{ if and only if } \quad \mathcal{P}_{[\sigma]} \leq \mathcal{P}'_{[\sigma]}\ .$$

First we assume that $\mathcal{P}_{[\sigma]}[\sigma] \leq \mathcal{P}'_{[\sigma]}[\sigma]$. According to Prop. 2.4 this means the existence of an a and a centered b such that

$$\mathcal{P}(b) = \mathcal{P}'_{[\sigma]}[\sigma],\ \mathcal{P}(a) = \mathcal{P}_{[\sigma]}[\sigma],\ \text{and } b \leq_Z a\ .$$

By applying the map r we deduce for the number supported multisegments $r(a)$ and $r(b)$ that $\mathcal{P}^{r(b)} = \mathcal{P}'_{[\sigma]}$, $\mathcal{P}^{r(a)} = \mathcal{P}_{[\sigma]}$, and $r(b) \leq_Z r(a)$. Moreover, with b also the number supported multisegment $r(b)$ will be centered. Therefore we deduce the basic relation $\mathcal{P}_{[\sigma]} \leq \mathcal{P}'_{[\sigma]}$ for partitions.

Conversely we now assume a basic relation $\mathcal{P}_{[\sigma]} \leq \mathcal{P}'_{[\sigma]}$ for partitions. This means that there exists a number supported multisegment ${}_na$ such that ${}_nb \leq_Z {}_na$ and $\mathcal{P}^{{}_na} = \mathcal{P}_{[\sigma]}$ with ${}_nb$ being the unique centered number supported multisegment which is determined by the partition $\mathcal{P}'_{[\sigma]}$. We fix a preunitary representative $\sigma_0 \in [\sigma]$ and consider the section

$$\begin{aligned} S : \mathbb{R} &\longrightarrow [\sigma] \\ r &\longmapsto \sigma_0 \otimes |.|^r \end{aligned}$$

inducing sections ${}_n\Delta \to S({}_n\Delta)$ and ${}_na \to S({}_na)$ from the set of number supported segments/multisegments to the set of segments/multisegments which are supported in the inertial class $[\sigma]$. Because of $\mathcal{P}^{{}_na} = \mathcal{P}_{[\sigma]}$ the multisegment $a := S({}_na)$ satisfies $\mathcal{P}(a) = \mathcal{P}_{[\sigma]}[\sigma]$. Similarly $b = S({}_nb)$ is a centered multisegment such that $\mathcal{P}(b) = \mathcal{P}'_{[\sigma]}[\sigma]$. But the section S respects Zelevinsky operations. Hence ${}_nb \leq_Z {}_na$ implies $b \leq_Z a$ and therefore $\mathcal{P}_{[\sigma]}[\sigma] \leq \mathcal{P}'_{[\sigma]}[\sigma]$. □

3. Preparations

3.1. **Partitions.** A partition is a finite nonincreasing sequence $\mathcal{P} = (\ell_1 \geq \ell_2 \geq \ldots \geq \ell_s)$ of natural numbers; we put $s(\mathcal{P}) := s$. Sometimes it will be convenient, though, to prolong such a sequence to the right by adding zeros. The set of all partitions carries two well known partial orders.

– The refinement order $\mathcal{P} \leq_{ref} \mathcal{P}'$ is the transitive hull of the relations where a single entry m of $\mathcal{P}'$ is replaced by the entries ℓ and ℓ' in $\mathcal{P}$ if $m = \ell + \ell'$.
– The dominance order is defined by $\mathcal{P} = (\ell_1 \geq \ell_2 \geq \ldots \geq \ell_s) \leq_{dom} (m_1 \geq m_2 \geq \ldots \geq m_t)$ if $\sum_{j=1}^{i} \ell_i \leq \sum_{j=1}^{i} m_i$ for any $i \geq 1$.

It is not difficult to see that the dominance order is finer than the refinement order. Our aim in this paper is to understand a third partial order $\leq$, to be defined in section 3.4, which sits in between these two. To make this precise we have still to transport the definition of elementary Zelevinsky operations from representation supported (as in section (2)) to number supported multisegments.

We also will use the point of view of effective divisors on the set of natural numbers $\mathbb{N}$. Let $\mathcal{D}_{\mathbb{N}}$ denote the free abelian group on the set $\mathbb{N}$. Its elements will be called virtual partitions and will be written as $\mathcal{P} = \sum_{\ell \geq 1} a_\ell[\ell]$. We define the subsemigroup $\mathcal{D}_{\mathbb{N}}^{+} := \{\sum_{\ell \geq 1} a_\ell[\ell] \in \mathcal{D}_{\mathbb{N}} : a_\ell \geq 0 \text{ for any } \ell \geq 1\}$. A partition $\mathcal{P} = (\ell_1 \geq \ell_2 \geq \ldots \geq \ell_s)$ can and will be identified with the element $\sum_{\ell \geq 1} a_\ell[\ell] \in \mathcal{D}_{\mathbb{N}}^{+} \setminus \{0\}$ where the multiplicity a_ℓ is the number of occurrences of the number ℓ in $\mathcal{P}$.

3.2. **Multisegments and Zelevinsky operations.** A segment $\Delta = (m, m+1, \ldots, m+\ell-1)$ is a sequence where $m \in \frac{1}{2}\mathbb{Z}$ is a half-integer and $\ell \in \mathbb{N}$ is a natural number called the length $\ell(\Delta)$ of the segment; define $m(\Delta) := m$ and $u(\Delta) := m + \ell - 1$. We let $\mathcal{D}$ denote the free abelian group on the set $\mathcal{S}$ of all segments. Its elements will simply be called divisors and will be written as $a = \sum_\Delta a_\Delta \Delta$. The support of a is the set $\{\Delta \in \mathcal{S} : a_\Delta \neq 0\}$. The divisor a is called effective if all $a_\Delta \geq 0$. The subsemigroup of effective divisors will be denoted by $\mathcal{D}^+$. A multisegment is a nonzero effective divisor.

Following Zelevinsky we introduce a certain partial order $\leq_Z$ on $\mathcal{D}^+$. For this we first define specific divisors as follows:

1) Let $E_1, E_2 \in \mathcal{S}$ be such that $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 \in \mathcal{S}$. Note that the segments $E_1, E_2, E_1 \cup E_2$ are pairwise different. We put $Z^0_{\{E_1,E_2\}} := (E_1 \cup E_2) - E_1 - E_2 \in \mathcal{D}$.
2) Let $\Delta_1, \Delta_2 \in \mathcal{S}$ be two linked segments, i.e., such that $\Delta_1 \not\subseteq \Delta_2$, $\Delta_2 \not\subseteq \Delta_1$, $\Delta_1 \cap \Delta_2 \in \mathcal{S}$. Then necessarily $\Delta_1 \cup \Delta_2 \in \mathcal{S}$, and the segments Δ_1, Δ_2, $\Delta_1 \cap \Delta_2$, and $\Delta_1 \cup \Delta_2$ are pairwise different. We put $Z^1_{\{\Delta_1,\Delta_2\}} := (\Delta_1 \cup \Delta_2) + (\Delta_1 \cap \Delta_2) - \Delta_1 - \Delta_2 \in \mathcal{D}$ and $A^1_{\{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\}} := \Delta_1 + \Delta_2 - (\Delta_1 \cup \Delta_2) - (\Delta_1 \cap \Delta_2) = -Z^1_{\{\Delta_1,\Delta_2\}}$.

Each of these divisors $D = Z^0_{\{E_1,E_2\}}$, $Z^1_{\{\Delta_1,\Delta_2\}}$, and $A^1_{\{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\}}$ gives rise to the corresponding translation operator $a \mapsto D + a$ on $\mathcal{D}$, where the cases $D = Z$ reflect the elementary Zelevinsky operations from section 2. In the following we will sometimes call operators of the form $Z^0_{\{E_1,E_2\}}$, resp. $Z^1_{\{\Delta_1,\Delta_2\}}$, resp. $A^1_{\{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\}}$, operators of type Z^0, resp. Z^1, resp. A^1. Of course, they do not respect $\mathcal{D}^+$. We therefore introduce the following language.

Definition 3.1. *Let $D_1, \ldots, D_s$ be a sequence of divisors. We say that the composite translation operator $D_1 + (D_2 + \ldots (D_s + .))$ is defined in a multisegment a if $a_s := D_s + a$, $a_{s-1} := D_{s-1} + a_s, \ldots, a_1 := D_1 + a_2$ all are multisegments.*

For any two multisegments a, a' we define

$$a \leq_Z a' \quad \text{if there is a sequence of operators of types } Z^0 \text{ and } Z^1$$
$$\text{whose composite } D \text{ is defined in } a' \text{ with } a = D + a'.$$

again by following Zelevinsky.

A basic technical problem, we will have to deal with later on, is the fact that if, for two divisors D_1 and D_2, the composite $D_1 + (D_2 + .)$ is defined in a multisegment a then the composite $D_2 + (D_1 + .)$ has no reason to be defined in a as well. Here we only give a trivial criterion. For any divisor $a = \sum_\Delta a_\Delta \Delta$ we define the divisor $a^{<0} := \sum_{a_\Delta < 0} a_\Delta \Delta$.

ivial-commute **Remark 3.2.** *Let D_1 and D_2 be any divisors and a any multisegment such that $a + D_1^{<0} + D_2^{<0}$ is a multisegment. Then $D_1 + (D_2 + .)$ and $D_2 + (D_1 + .)$ both are defined in a and $D_1 + (D_2 + a) = D_2 + (D_1 + a)$.*

tiseg-to-part

3.3. **From multisegments to partitions.** There is the obvious surjective homomorphism

f:partition (6)
$$\mathcal{P}: \quad \mathcal{D} \longrightarrow \mathcal{D}_{\mathbb{N}}$$
$$\sum_\Delta a_\Delta \Delta \longmapsto \sum_\Delta a_\Delta [\ell(\Delta)] \ .$$

It maps multisegments to partitions. The following observation is an exercise.

Z-dom **Remark 3.3.** *For any two multisegments a and a' the relation $a' \leq_Z a$ implies $\mathcal{P}(a) \leq_{dom} \mathcal{P}(a')$.*

The map $\mathcal{P}$, in fact, has a natural section. We first need more notations. For a segment Δ we let Δ^-, resp. Δ^+, denote the subsegment consisting of all entries ≤ 0, resp. ≥ 0. Obviously one has $\Delta = \Delta^- \cup \Delta^+$. Next we introduce the "reflection" map[1]

$$r: \quad \mathcal{S} \longrightarrow \mathcal{S}$$
$$(m, \ldots, m + \ell - 1) \longmapsto (-m - \ell + 1, \ldots, -m) \ .$$

It satisfies $r(\Delta^+) = r(\Delta)^-$ and $r(\Delta^-) = r(\Delta)^+$. We extend the maps $\Delta \longmapsto \Delta^-, \Delta^+, r(\Delta)$ additively to maps $a \longmapsto a^-, a^+, r(a)$ on $\mathcal{D}$.

Definition 3.4. *A multisegment $a = \sum_\Delta a_\Delta \Delta$ is called centered if $r(\Delta) = \Delta$ whenever $a_\Delta \neq 0$.*

It is clear that for any natural number ℓ there is a unique centered segment $\Delta(\ell) := (-\frac{\ell-1}{2}, \ldots, \frac{\ell-1}{2})$ of length ℓ. We therefore have the injective homomorphism

$$b: \quad \mathcal{D}_{\mathbb{N}} \longrightarrow \mathcal{D}$$
$$\sum_{\ell \geq 1} a_\ell [\ell] \longmapsto \sum_{\ell \geq 1} a_\ell \Delta(\ell) \ .$$

It satisfies $\mathcal{P} \circ b = \mathrm{id}$ and maps partitions to centered multisegments. As a consequence $a \mapsto b(\mathcal{P}(a))$ is a projection map from $\mathcal{D}^+ \setminus \{0\}$ onto the submonoid of centered multisegments. It linearly extends the map $\Delta \mapsto b(\mathcal{P}(\Delta)) = \Delta(\ell(\Delta))$ shifting Δ into centered position.

---

[1]The earlier map r on multisegments will not occur anymore.

subsec:new-po

3.4. **A new partial order on partitions.** As already formulated earlier in Def. 1.2, our basic definition in this paper is the following.

def:basic

Definition 3.5. *We say that the basic relation $\mathcal{P} \leq \mathcal{P}'$ holds between two partitions $\mathcal{P}$ and $\mathcal{P}'$ if there is a multisegment a such that $\mathcal{P} = \mathcal{P}(a)$ and $a \geq_Z b(\mathcal{P}')$.*

It is immediate from Remark 3.3 that a basic relation $\mathcal{P} \leq \mathcal{P}'$ implies that $\mathcal{P} \leq_{dom} \mathcal{P}'$. Hence the transitive hull of the basic relations is a partial order $\leq$ on the set of all partitions which is coarser than the dominance order. In fact, it is shown in [SZ] Example 2.29 that the dominance order is strictly finer than the order $\leq$. On the other hand [SZ] Cor. 2.25 says that $\leq$ is finer than the refinement order.

Z0-chain

Remark 3.6. *For two partitions $\mathcal{P}$ and $\mathcal{P}'$ the following properties are equivalent:*

a) $\mathcal{P} \leq_{ref} \mathcal{P}'$;
b) *$\mathcal{P} \leq \mathcal{P}'$ is a basic relation corresponding to $a \geq_Z b(\mathcal{P}')$ for a sequence entirely of operators of type Z^0;*
c) *there are multisegments a and a' such that $a \geq_Z a'$ for a sequence entirely of operators of type Z^0 and such that $\mathcal{P} = \mathcal{P}(a)$ and $\mathcal{P}' = \mathcal{P}(a')$.*

Proof. The implications $b) \Longrightarrow c) \Longrightarrow a)$ are obvious. We therefore assume that $\mathcal{P} \leq_{ref} \mathcal{P}'$ and need to prove b). We first consider the "elementary" case where a single entry $m = \ell + \ell'$ of $\mathcal{P}'$ is replaced by the entries ℓ and ℓ' in $\mathcal{P}$. Let a' be any multisegment such that $\mathcal{P}' = \mathcal{P}(a')$. Then a' must contain a segment Δ of length m. We write $\Delta = E \cup E'$ with segments E and E' of length ℓ and ℓ', respectively. Then $a' = Z^0_{\{E,E'\}} + a$ for some segment a such that $\mathcal{P} = \mathcal{P}(a)$. In particular, we have $a \geq_Z a'$. In the general case we still find, by transitivity, given any a' such that $\mathcal{P}' = \mathcal{P}(a')$ an a such that $\mathcal{P} = \mathcal{P}(a)$ and $a \geq_Z a'$ for a sequence entirely of operators of type Z^0. We finally apply this observation to $a' = b(\mathcal{P}')$. ☐

Later on in section 4 we will analyze the case when a basic relation is realized by a chain of operators of type Z^1.

c:precentered

3.5. **Precentered multisegments.** When studying the partial order $\leq$ the following concept will be very useful.

f:precentered

Definition 3.7. *A multisegment a is called precentered if a^+ and a have the same number of constituents and $r(a^+) = a^-$ (or equivalently if a^- and a have the same number of constituents and $r(a^-) = a^+$).*

crit-precent

Lemma 3.8. *A multisegment a is precentered if and only if there is a centered multisegment b with the same number of constituents as a such that $a^+ = b^+$ and $a^- = b^-$. Moreover, b is uniquely determined by a.*

Proof. Given $a = \sum a_\Delta \Delta$ we consider $b := \sum a_\Delta(\Delta^+ \cup r(\Delta^+))$, which is centered with the property $a^+ = b^+$. On the other hand $b^- = \sum a_\Delta r(\Delta^+) = r(a^+) = a^-$, because a is precentered. The uniqueness of b is obvious. Conversely if $a^+ = b^+$ and $a^- = b^-$, where b is centered, then $r(a^+) = r(b^+) = b^- = a^-$, hence a is precentered. ☐

This lemma allows us to introduce, for any given centered multisegment b, the set $B(b)$ of all precentered multisegments whose corresponding (as in Lemma 3.8) centered multisegment is b. This set $B(b)$ can be described as follows.

pi-map **Lemma 3.9.** *Let $b = (\Delta_1, \dots, \Delta_s)$ with $\Delta_i = \Delta(\ell_i)$ be a centered multisegment for which we have fixed an enumeration. Then the map*

$$\mathfrak{S}(\ell_1, \dots, \ell_s) \longrightarrow B(b)$$
$$\pi \longmapsto b_\pi := \sum_{\nu=1}^{s} (\Delta_\nu^+ \cup \Delta_{\pi(\nu)}^-)$$

is a well defined surjective map, where $\mathfrak{S}(\ell_1, \dots, \ell_s)$ denotes the subgroup of the symmetric group $\mathfrak{S}_s$ which consists of all permutations with the property that the integers $\ell_i + \ell_{\pi(i)}$, for any $1 \le i \le s$ are even.

Proof. Obviously we have $b_\pi^+ = b^+$ and $b_\pi^- = b^-$, hence $b_\pi \in B(b)$. On the other hand consider any $a \in B(b)$ which means $a^+ = b^+$ and $a^- = b^-$. So we may write $a = \sum_i \Delta_i'$ such that $\Delta_i'^+ = \Delta_i^+$. Then $a^- = b^-$ rewrites as $\sum_i \Delta_i'^- = \sum \Delta_i^-$ which implies $\Delta_i'^- = \Delta_{\pi(i)}^-$, hence $\Delta_i' = \Delta_i^+ \cup \Delta_{\pi(i)}^-$ for a certain permutation $\pi \in \mathfrak{S}_s$. Moreover, if the entries of Δ_i' are half-integers, resp. integers, then the entries of Δ_i^+ and $\Delta_{\pi(i)}^-$ and hence the entries of Δ_i and $\Delta_{\pi(i)}$ are half-integers, resp. integers, as well. This implies that each integer $\ell_i + \ell_{\pi(i)}$ is even and therefore that $\pi \in \mathfrak{S}(\ell_1, \dots, \ell_s)$. □

We see that the fixed enumeration $b = (\Delta_1, \dots, \Delta_s)$ induces a fixed enumeration $a = (\Delta_1', \dots, \Delta_s')$ for all multisegments $a \in B(b)$ which is given as $\Delta_i' = \Delta_i^+ \cup \Delta_{\pi(i)}^-$ for a certain permutation π and the notation $a = b_\pi$ refers to that enumeration. But note that different permutations π can produce the same b_π because the constituents of b may arise with multiplicities. We are going to see now that the set $B(b)$ of precentered multisegments of type b is stable under applying translation operators of type Z^1 and A^1 provided these operators are defined for a certain b_π.

In the case of an operator $A^1_{\{\Delta_1', \Delta_2'\}}$ of type A^1 it will be convenient to use the following notation:

$$\Delta_1' = C^- \cup \Delta_2' \cup C^+$$

where the segments C^-, resp. C^+, have entries which are strictly smaller, resp. strictly larger, than the entries of Δ_2'. Then

$$A^1_{\{\Delta_1', \Delta_2'\}} = (C^- \cup \Delta_2') + (\Delta_2' \cup C^+) - (C^- \cup \Delta_2' \cup C^+) - \Delta_2' \ .$$

respect-B **Lemma 3.10.** *If $b_\pi = (\Delta_1', \dots, \Delta_s') \in B(b)$ is given, and if we assume that*

$$Z^1_{\{\Delta_i', \Delta_j'\}} = \Delta_i' \cup \Delta_j' + \Delta_i' \cap \Delta_j' - \Delta_i' - \Delta_j'$$

is an operator of type Z^1 which is defined in b_π, then we have

$$Z^1_{\{\Delta_i', \Delta_j'\}} + b_\pi = b_{\pi(i,j)} \in B(b) \ .$$

Similarly if

$$A^1_{\{\Delta_i', \Delta_j'\}} = C^- \cup \Delta_j' + \Delta_j' \cup C^+ - \Delta_i' - \Delta_j'$$

is an operator of type A^1 which is defined in b_π, then we have

$$A^1_{\{\Delta_i', \Delta_j'\}} + b_\pi = b_{\pi(i,j)} \in B(b) \ .$$

Proof. First we consider $b'' = \sum_\nu \Delta''_\nu := A^1_{\{\Delta'_i,\Delta'_j\}} + b_\pi$ where the pair

$$(\Delta'_i, \Delta'_j) = (C^- \cup \Delta'_j \cup C^+, \Delta'_j) \text{ is replaced by the pair } (\Delta''_i, \Delta''_j) := (C^- \cup \Delta'_j, \Delta'_j \cup C^+).$$

Since b_π is precentered, the entries of C^-, resp. C^+, are strictly negative, resp. strictly positive. For b'' this implies:

f:plus-minus
$$\begin{aligned} \Delta''^{-}_i &= (C^- \cup \Delta'_j)^- &= \Delta'^{-}_i \\ \Delta''^{+}_i &= (C^- \cup \Delta'_j)^+ &= \Delta'^{+}_j \\ \Delta''^{+}_j &= (\Delta'_j \cup C^+)^+ &= \Delta'^{+}_i \\ \Delta''^{-}_j &= (\Delta'_j \cup C^+)^- &= \Delta'^{-}_j \end{aligned} \tag{7}$$

Thus we see that

$$\Delta''_i = \Delta''^{+}_i \cup \Delta''^{-}_i = \Delta'^{+}_j \cup \Delta'^{-}_i = \Delta^+_j \cup \Delta^-_{\pi(i)} \ ,$$

where the last equality uses the definition of b_π in Lemma 3.9. We may rewrite this as

f:1
$$\Delta''_i = \Delta^+_j \cup \Delta^-_{\pi(i,j)(j)} \ . \tag{8}$$

And similarly the identities (7) also imply that

f:2
$$\Delta''_j = \Delta^+_i \cup \Delta^-_{\pi(i,j)(i)} \ . \tag{9}$$

Finally if $\nu \notin \{i,j\}$ then

f:3
$$\Delta''_\nu = \Delta'_\nu = \Delta^+_\nu \cup \Delta^-_{\pi(\nu)} = \Delta^+_\nu \cup \Delta^-_{\pi(i,j)(\nu)} \tag{10}$$

and therefore

$$b'' = \sum_\nu \Delta''_\nu = \sum_\nu \Delta^+_\nu \cup \Delta^-_{\pi(i,j)(\nu)} = b_{\pi(i,j)} \ .$$

The equalities (8), (9), and (10) also show that, for any $1 \le \nu \le s$, the segments Δ_ν and $\Delta_{\pi(ij)(\nu)}$ both have the same kind of entries - half-integers or integers. This implies that $\pi(ij)$, indeed, lies in $\mathfrak{S}(\ell_1, \ldots, \ell_s)$.

Next we consider $b'' = \sum_\nu \Delta''_\nu := Z^1_{\{\Delta'_i,\Delta'_j\}} + b_\pi$ where the pair

$$(\Delta'_i, \Delta'_j) \text{ is replaced by the pair } (\Delta''_i, \Delta''_j) := (\Delta'_i \cup \Delta'_j, \Delta'_i \cap \Delta'_j).$$

Since $\Delta'_i \not\subseteq \Delta'_j$, $\Delta'_j \not\subseteq \Delta'_i$, and $\Delta'_i \cap \Delta'_j$ is a segment and since b_π is precentered we obtain exactly the same relations as in (7):

$$\begin{aligned} (\Delta''_i)^- &= \Delta'^{-}_i \\ (\Delta''_i)^+ &= \Delta'^{+}_j \\ (\Delta''_j)^- &= \Delta'^{-}_j \\ (\Delta''_j)^+ &= \Delta'^{+}_i \end{aligned}$$

Then precisely the same computation as before gives us $b'' = b_{\pi(i,j)}$ also in this second case. $\square$

sec:Z1-chain

4. Chains of operators of type Z^1

In this section we analyze basic relations which are realized by a chain of operators of type Z^1.

no-Z1 **Lemma 4.1.** *Let b' be a precentered multisegment in which no operator of type Z^1 is defined; then b' must be centered.*

Proof. We enumerate the multisegment $b' = \sum_i \Delta'_i$ in such a way that $i < j$ implies

$$\text{either } u(\Delta'_i) > u(\Delta'_j) \text{ or } u(\Delta'_i) = u(\Delta'_j) \text{ and } m(\Delta'_i) \leq m(\Delta'_j).$$

Our assumption on b' implies that $m(\Delta'_i) \leq m(\Delta'_j)$ whenever $u(\Delta'_i) > u(\Delta'_j)$. Thus for $i < j$ we will always have $u(\Delta'_i) \geq u(\Delta'_j)$ and $m(\Delta'_i) \leq m(\Delta'_j)$. Moreover

$$m(\Delta'_j) \leq 0 \leq u(\Delta'_j) \qquad \text{for any } j,$$

because b' is precentered. But then ${\Delta'_1}^+$ must be of maximal length among all ${\Delta'_i}^+$, and ${\Delta'_1}^-$ must be of maximal length among all ${\Delta'_i}^-$. Now we use that $r({\Delta'_i}^+) = {\Delta'}^-_{\pi(i)}$ for a certain permutation π. Hence the lengths of the ${\Delta'_i}^+$ and of the ${\Delta'_i}^-$ are the same up to a permutation. In particular, the maximal lengths are the same, which implies that Δ'_1 must be a centered segment. Therefore $\sum_{i\geq 2} \Delta'_i$ is again a precentered multisegment which satisfies the assumption of our assertion. Successively we see that b' must be centered. □

Z1-chain **Proposition 4.2.** *For a multisegment $b' = \sum_\nu \Delta'_\nu$ the following assertions are equivalent:*

i. *b' is precentered with the corresponding centered multisegment $b = (\Delta_1, \ldots, \Delta_s)$, i.e., $b' \in B(b)$;*
ii. *there exists a sequence $Z^1_1, \ldots, Z^1_r$ of operators of type Z^1 such that the multisegment $b := Z^1_1 + (Z^1_2 + \ldots (Z^1_r + b'))$ is defined and centered.*

Proof. First we assume that ii. holds true. Then $b' = -Z^1_r - (Z^1_{r-1} - \ldots (Z^1_1 - b))$. Hence b' arises from the centered multisegment b by a sequence of operators of type A^1, and the assertion i. follows from Lemma 3.10.

Now we assume i. with $b' \in B(b)$. We may assume that $b' \neq b$. Then, according to Lemma 4.1, there exists an operator of type Z^1 which is defined in b'. Writing (cf. Lemma 3.9)

$$b' = \sum_{\nu=1}^{s} \Delta'_\nu = b_\pi = \sum_{\nu=1}^{s} \Delta^+_\nu \cup \Delta^-_{\pi(\nu)} \quad \text{for some } \pi \in \mathfrak{S}_s,$$

this means that we find $\Delta'_i = \Delta^+_i \cup \Delta^-_{\pi(i)}$ and $\Delta'_j = \Delta^+_j \cup \Delta^-_{\pi(j)}$ such that $u(\Delta'_i) > u(\Delta'_j)$ and $m(\Delta'_i) > m(\Delta'_j)$. These two conditions are equivalent to the conditions

$$\ell(\Delta_i) > \ell(\Delta_j) \quad \text{and} \quad \ell(\Delta_{\pi(i)}) < \ell(\Delta_{\pi(j)}) \ .$$

In the following we will call such a pair (Δ'_i, Δ'_j) a Z^1-inversion of b'. Our strategy now is to decrease the number of Z^1-inversions by applying an appropriate operator of type Z^1 to b'. This will then lead, by iteration and Lemma 3.10, to a precentered multisegment $b'' \in B(b)$ without any Z^1-inversions. Hence there is no operator of type Z^1 which is defined in b'', and therefore $b'' = b$ by Lemma 4.1.

For the rest of the proof we fix the enumeration $b = (\Delta_1, \ldots, \Delta_s)$ in such a way that

$$i \leq j \quad \text{implies that} \quad \ell(\Delta_i) \geq \ell(\Delta_j) \ .$$

This has the consequence that any Z^1-inversion (Δ'_i, Δ'_j) of b' has the property that $i < j$. Let now $i_0 = i_0(b')$ be minimal such that $(\Delta'_{i_0}, \Delta'_j)$ is a Z^1-inversion of b'. In particular, we have $i_0 < j$.

Claim 1: $\ell(\Delta_{\pi(i_0)}) < \ell(\Delta_{\pi(j)}) \leq \ell(\Delta_{i_0})$.
The left inequality holds by assumption. For the right inequality we argue by contradiction assuming that $\ell(\Delta_{\pi(j)}) > \ell(\Delta_{i_0})$. By our assumption on the enumeration of b we then must have $\pi(j) < i_0$. We now consider the pair $(\Delta'_{\pi(j)}, \Delta'_j)$. We have $\ell(\Delta_{\pi(j)}) > \ell(\Delta_{i_0}) > \ell(\Delta_j)$. Suppose now that $\ell(\Delta_{\pi^2(j)}) < \ell(\Delta_{\pi(j)})$. Then $(\Delta'_{\pi(j)}, \Delta'_j)$ is a Z^1-inversion of b'. Since $\pi(j) < i_0$ this contradicts the minimality of i_0. Hence $\ell(\Delta_{\pi^2(j)}) \geq \ell(\Delta_{\pi(j)}) > \ell(\Delta_{i_0}) > \ell(\Delta_j)$ and $\pi^2(j) < i_0$. So we consider the pair $(\Delta'_{\pi^2(j)}, \Delta'_j)$ for which we may repeat the above argument. Proceeding inductively we deduce that $\pi^\nu(j) < i_0$ for any $\nu \geq 1$. If ν is chosen in such a way that $\pi^\nu(j) = j$ we obtain that $j < i_0$, which is a contradiction.

Claim 2: There exists a Z^1-inversion $(\Delta'_{i_0}, \Delta'_{j_0})$ of b' such that $\ell(\Delta_{\pi(j_0)}) = \ell(\Delta_{i_0})$.
First of all we note that, if $\ell(\Delta_\nu) < \ell(\Delta_{i_0})$, then we necessarily have $\ell(\Delta_{\pi(\nu)}) \leq \ell(\Delta_{i_0})$. If $(\Delta'_{i_0}, \Delta'_\nu)$ is a Z^1-inversion of b' then this is Claim 1. Otherwise we must have $\ell(\Delta_{\pi(\nu)}) \leq \ell(\Delta_{\pi(i_0)})$. In addition Claim 1 says that we always have $\ell(\Delta_{\pi(i_0)}) < \ell(\Delta_{i_0})$.

We abbreviate $\beta := \ell(\Delta_{i_0})$. We want to find j_0 such that $\ell(\Delta_{j_0}) < \beta$ and $\ell(\Delta_{\pi(j_0)}) = \beta$. Since $\beta > \ell(\Delta_{\pi(i_0)})$ by Claim 1 the pair $(\Delta'_{i_0}, \Delta'_{j_0})$ then is a Z^1-inversion as claimed. The set $J := \{\nu : \ell(\Delta_\nu) < \beta\}$ contains j and hence is nonempty. Therefore it suffices to show that the assumption that any $\nu \in J$ satisfies $\ell(\Delta_{\pi(\nu)}) < \beta$ leads to a contradiction. This assumption means that $\{\nu : \ell(\Delta_\nu) < \beta\} \subseteq \{\nu : \ell(\Delta_{\pi(\nu)}) < \beta\}$. But obviously these two sets have the same cardinality and hence are equal. The contradiction now arises because i_0 does not lie in the left hand set whereas, by Claim 1, it lies in the right hand set. This finishes the proof of Claim 2.

We now take a Z^1-inversion $(\Delta'_{i_0}, \Delta'_{j_0})$ of b' as in Claim 2. By applying the corresponding operator of type Z^1 to b' we obtain a multisegment $b'' \in B(b)$. Using Lemma 3.10 we see that

$$b'' = b_{\pi(i_0,j_0)} = \sum_\nu \left(\Delta^+_\nu \cup \Delta^-_{\pi(i_0,j_0)(\nu)}\right) = (\Delta^+_{i_0} \cup \Delta^-_{\pi(j_0)}) + (\Delta^+_{j_0} \cup \Delta^-_{\pi(i_0)}) + \sum_{\nu \neq i_0, j_0} \Delta'_\nu \ .$$

The segment $\Delta^+_{i_0} \cup \Delta^-_{\pi(j_0)}$ in b'' is centered by construction, whereas the segments Δ'_{i_0} and Δ'_{j_0} in b' are not centered, since they form a Z^1-inversion with the additional property that $\ell(\Delta_{\pi(j_0)}) = \ell(\Delta_{i_0})$. Therefore b'' contains at least one more centered segment than b'. Repeating this construction we obtain after finitely many steps a multisegment all of which segments are centered and which therefore has to be equal to b. □

As an application we derive an explicit description, for any fixed partition $\mathcal{P}'$, of all basic relations $\mathcal{P} \leq \mathcal{P}'$ where $s(\mathcal{P}) = s(\mathcal{P}')$.

Definition 4.3. *A basic relation $\mathcal{P} \leq \mathcal{P}'$ will be called of type* 1 *if $s(\mathcal{P}) = s(\mathcal{P}')$.*

basic **Proposition 4.4.** *Let $\mathcal{P}' = (\ell'_1 \geq \ldots \geq \ell'_s)$ be a partition and $b' = b(\mathcal{P}')$. Then we have the well defined surjective maps*

$$\mathfrak{S}(\ell'_1, \ldots, \ell'_s) \xrightarrow{\pi \mapsto b'_\pi} B(b') \xrightarrow{(6)} \{\mathcal{P} : \mathcal{P} \leq \mathcal{P}' \textit{ basic of type } 1\},$$

and the composite map takes π to $\mathcal{P}'_\pi := \sum_{i=1}^s [\frac{\ell'_i + \ell'_{\pi(i)}}{2}]$.

In particular, $\mathcal{P} = (\ell_1 \geq \ldots \geq \ell_s) \leq \mathcal{P}' = (\ell'_1 \geq \ldots \geq \ell'_s)$ *is a basic relation if and only if there are permutations* σ *and* π *of* $(1, \ldots, s)$ *such that* ℓ'_i *and* $\ell'_{\pi(i)}$ *have the same parity and*

$$\ell_{\sigma(i)} = \frac{\ell'_i + \ell'_{\pi(i)}}{2} \qquad \text{for all } 1 \leq i \leq s.$$

Proof. The left hand map is surjective by Lemma 3.9. For the surjectivity of the right hand map let $\mathcal{P} \leq \mathcal{P}'$ be a basic relation of type 1. This means that we have $\mathcal{P} = \mathcal{P}(b)$ for a multisegment b such that

$$b' := b(\mathcal{P}') = Z_1 + (Z_2 + \ldots + (Z_r + b))$$

for a sequence of operators $Z_1, \ldots, Z_r$ which all are of type Z^1. From Prop. 4.2 we know that $b \in B(b')$. Furthermore, it follows from Lemma 3.9 that b is of the form

$$b = \sum_{i=1}^{s} \left(\Delta(\ell'_i)^+ \cup \Delta(\ell'_{\pi(i)})^- \right)$$

for some permutation $\pi \in \mathfrak{S}(\ell'_1, \ldots, \ell'_s)$. Consider any $\ell'_i = 2m_i + 1$ in $\mathcal{P}$ which is odd. Then $\ell'_{\pi(i)} = 2m_{\pi(i)} + 1$ must be odd as well, and we obtain

$$\begin{aligned} \ell(\Delta(\ell'_i)^+ \cup \Delta(\ell'_{\pi(i)})^-) &= \ell(\Delta(\ell'_i)^+) + \ell(\Delta(\ell'_{\pi(i)})^-) - 1 \\ &= (m_i + 1) + (m_{\pi(i)} + 1) - 1 = m_i + m_{\pi(i)} + 1 \\ &= \frac{\ell'_i + \ell'_{\pi(i)}}{2} . \end{aligned}$$

A similar computation gives the same for even $\ell'_i = 2m_i$. □

cor:basic

Corollary 4.5. *For any partition* $\mathcal{P} = (\ell_1 \geq \ldots \geq \ell_s)$ *and any two permutations* $\pi_1, \pi_2 \in \mathfrak{S}(\ell_1, \ldots, \ell_s)$ *such that* $\{j : \pi_1(j) \neq j \neq \pi_2(j)\} = \emptyset$ *we have* $\mathcal{P}_{\pi_1\pi_2} \leq \mathcal{P}_{\pi_i} \leq \mathcal{P}$ *for* $i = 1, 2$.

Proof. We have $\mathcal{P}_{\pi_1\pi_2} = \sum_{i=1}^{s} [\frac{\ell_i + \ell_{\pi_1\pi_2(i)}}{2}]$ and $\mathcal{P}_{\pi_1} = \sum_{i=1}^{s} [\ell'_i]$ with $\ell'_i := \frac{\ell_i + \ell_{\pi_1(i)}}{2}$. Note that $\pi_2 \in \mathfrak{S}(\ell'_1, \ldots, \ell'_s)$ since $\ell'_i = \ell_i$ if $\pi_2(i) \neq i$ by the assumption on π_1 and π_2. Hence $\sum_i [\frac{\ell'_i + \ell'_{\pi_2(i)}}{2}] \leq \sum_i [\ell'_i]$. But it is easily checked that $\ell'_i + \ell'_{\pi_2(i)} = \ell_i + \ell_{\pi_1\pi_2(i)}$ for any i. □

c:commutation

5. Commuting Z^0-operators with Z^1- and A^1-operators

If $\mathcal{P} = \mathcal{P}(a) \leq \mathcal{P}'$ is a general basic relation, then a sequence of Zelevinsky operations which leads from a to $b(\mathcal{P}')$ will be a mixture of Zelevinsky operations of type 0 and type 1, respectively. In this section we will establish commutation relations for Zelevinsky operations which later on will enable us to show that it is always possible to find such a sequence leading from a to $b(\mathcal{P}')$ which has at the beginning only Zelevinsky operations of type 0 and after that only operations of type 1.

We now fix an operator $Z^0 := Z^0_{\{E_1, E_2\}}$ of type Z^0 as well as an operator $Z^1 := Z^1_{\{\Delta_1, \Delta_2\}}$ of type Z^1. Recall that E_i and Δ_j are segments such that

- $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 \in \mathcal{S}$, and
- $\Delta_1 \not\subseteq \Delta_2$, $\Delta_2 \not\subseteq \Delta_1$, and $\Delta_1 \cap \Delta_2 \in \mathcal{S}$.

We also recall that we write

$$\Delta_1 \cup \Delta_2 = C^- \cup (\Delta_1 \cap \Delta_2) \cup C^+$$

where the segments C^- and C^+ are characterized by the requirement that their entries are strictly smaller and strictly larger, respectively than the entries of $\Delta_1 \cap \Delta_2$.

To formulate the next lemmas we use the usual partial order $\leqslant$ on the group of divisors $\mathcal{D}$ defined by $a' \leq a$ if $a - a' \in \mathcal{D}^+$.

In the following we fix a multisegment a.

defined **Lemma 5.1.** *The composite operator $Z^0 + (Z^1 + .)$ is defined in a if and only if we are in one of the following three cases:*

(I) $E_1 + E_2 + \Delta_1 + \Delta_2 \leqslant a$;
(II$\cup$) $E_1 + E_2 + \Delta_1 + \Delta_2 \nleqslant a$, *but* $E_1 + \Delta_1 + \Delta_2 \leqslant a$ *and* $E_2 = \Delta_1 \cup \Delta_2$;
(II$\cap$) $E_1 + E_2 + \Delta_1 + \Delta_2 \nleqslant a$, *but* $E_1 + \Delta_1 + \Delta_2 \leqslant a$ *and* $E_2 = \Delta_1 \cap \Delta_2$;
(III$\cup$) $E_1 + E_2 + \Delta_1 + \Delta_2 \nleqslant a$, *but* $E_2 + \Delta_1 + \Delta_2 \leqslant a$ *and* $E_1 = \Delta_1 \cup \Delta_2$;
(III$\cap$) $E_1 + E_2 + \Delta_1 + \Delta_2 \nleqslant a$, *but* $E_2 + \Delta_1 + \Delta_2 \leqslant a$ *and* $E_1 = \Delta_1 \cap \Delta_2$.

Proof. The composite operator is defined in a if and only if

$$\Delta_1 + \Delta_2 \leqslant a \quad \text{and} \quad E_1 + E_2 \leqslant (\Delta_1 \cup \Delta_2) + (\Delta_1 \cap \Delta_2) + a - \Delta_1 - \Delta_2 \ .$$

Obviously the operator is defined in case (I). Suppose therefore that $E_1 + E_2 \nleqslant a - \Delta_1 - \Delta_2$. Since E_1 and E_2 are disjoint we cannot have $E_1 + E_2 = (\Delta_1 \cup \Delta_2) + (\Delta_1 \cap \Delta_2)$. We see that therefore the operator is defined in a if and only if either $E_2 \leqslant (\Delta_1 \cup \Delta_2) + (\Delta_1 \cap \Delta_2)$ and $E_1 \leqslant a - \Delta_1 - \Delta_2$, which are the cases (II), or the corresponding inequalities with E_1 and E_2 interchanged, which are the cases (III). □

(I) **Lemma 5.2.** *In case (I) the operator $Z^1 + (Z^0 + .)$ is defined in a as well and we have*

$$Z^0 + (Z^1 + a) = Z^1 + (Z^0 + a) \ .$$

Proof. This is obvious. □

(cup) **Lemma 5.3.** *In the cases (II$\cup$), ($III\cup$), (II$\cap$) with $E_1 \cup E_2 \subsetneqq \Delta_1$ or $\subsetneqq \Delta_2$, and (III$\cap$) with $E_1 \cup E_2 \subsetneqq \Delta_1$ or $\subsetneqq \Delta_2$ there are an operator Z'^0 of type Z^0 and an operator Z'^1 of type Z^1 such that the composite operator $Z'^1 + (Z'^0 + .)$ is defined in a and we have*

$$Z^0 + (Z^1 + a) = Z'^1 + (Z'^0 + a) \ .$$

Proof. By interchanging indices it suffices to establish the cases (II$\cup$) and (II$\cap$) with $E_1 \cup E_2 \subsetneqq \Delta_1$.

Case (II$\cup$): There are two possibilities. Either $Z'^0 := Z^0_{\{E_1,\Delta_1\}}$ or $Z'^0 := Z^0_{\{E_1,\Delta_2\}}$ is an operator of type Z^0. Correspondingly $Z'^1 := Z^1_{\{E_1 \cup \Delta_1, \Delta_2\}}$ and $Z'^1 := Z^1_{\{E_1 \cup \Delta_2, \Delta_1\}}$, respectively, are operators of type Z^1. It is a one line computation that they satisfy the assertion.

Case (II$\cap$) with $E_1 \cup E_2 \subsetneqq \Delta_1$: It is straightforward to verify that $Z'^0 := Z^0_{\{E_1,\Delta_2\}}$ and $Z'^1 := Z^1_{\{\Delta_1, E_1 \cup \Delta_2\}}$ satisfy the assertion. □

(cap) **Lemma 5.4.** *In the cases (II$\cap$) with $E_1 \cup E_2 = \Delta_1$ or $= \Delta_2$ and (III$\cap$) with $E_1 \cup E_2 = \Delta_1$ or $= \Delta_2$ there is a single operator Z'^0 of type Z^0 which is defined in a such that*

$$Z^0 + (Z^1 + a) = Z'^0 + a \ .$$

Proof. Again it suffices to consider the case (II$\cap$) with $E_1 \cup E_2 = \Delta_1$. One checks that $Z'^0 := Z^0_{\{E_1,\Delta_2\}}$ verifies the assertion. □

cap)-critical **Lemma 5.5.** *In the cases (II$\cap$) with $E_1 \cup E_2 \not\subseteq \Delta_1$ and $\not\subseteq \Delta_2$ and (III$\cap$) with $E_1 \cup E_2 \not\subseteq \Delta_1$ and $\not\subseteq \Delta_2$ there are an operator Z'^0 of type Z^0 and an operator A^1 of type A^1 such that the composite operator $A^1 + (Z'^0 + .)$ is defined in a and we have*

$$Z^0 + (Z^1 + a) = A^1 + (Z'^0 + a) \ .$$

Proof. Again it suffices to consider the case (II$\cap$) so that $E_2 = \Delta_1 \cap \Delta_2$. It is straightforward to verify that we have $i \neq j$ in $\{1, 2\}$ such that:

- $(\Delta_1 \cup \Delta_2) \cap (E_1 \cup E_2) = \Delta_i$,
- $(\Delta_1 \cup \Delta_2) \cup (E_1 \cup E_2) = E_1 \cup \Delta_j \in \mathcal{S}$,
- $\Delta_1 \cup \Delta_2 \not\subseteq E_1 \cup E_2$, $E_1 \cup E_2 \not\subseteq \Delta_1 \cup \Delta_2$, and
- $E_1 \cap \Delta_j = \emptyset$.

Hence we have the operators $Z'^0 := Z^0_{\{E_1,\Delta_j\}}$ and $A^1 := A^1_{\{E_1\cup\Delta_j,\Delta_i\}} = -Z^1_{\{\Delta_1\cup\Delta_2,E_1\cup E_2\}}$. They satisfy the assertion. ☐

Keeping the above notations we now consider the operator $A^1_{\{\Delta_1\cup\Delta_2,\Delta_1\cap\Delta_2\}}$ of type A^1.

A-defined **Lemma 5.6.** *The composite operator $Z^0 + (A^1 + .)$ is defined in a if and only if we are in one of the following three cases:*

(0) $E_1 + E_2 + \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \leqslant a$;
(1-1) $E_1 + E_2 + \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \not\leqslant a$, *but* $E_2 + \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \leqslant a$ *and* $E_1 = \Delta_1$;
(1-2) $E_1 + E_2 + \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \not\leqslant a$, *but* $E_2 + \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \leqslant a$ *and* $E_1 = \Delta_2$.
(2-1) $E_1 + E_2 + \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \not\leqslant a$, *but* $E_1 + \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \leqslant a$ *and* $E_2 = \Delta_1$;
(2-2) $E_1 + E_2 + \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \not\leqslant a$, *but* $E_1 + \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2 \leqslant a$ *and* $E_2 = \Delta_2$;

Proof. This is clear since we cannot have $E_1 + E_2 = \Delta_1 + \Delta_2$. ☐

(0) **Lemma 5.7.** *In case (0) the operator $A^1 + (Z^0 + .)$ is defined in a as well and we have*

$$Z^0 + (A^1 + a) = A^1 + (Z^0 + a) \ .$$

Proof. This is obvious. ☐

It is convenient in the following to use the notational convention that for any $i \in \{1, 2\}$ the index i' is determined by $\{i, i'\} = \{1, 2\}$.

not(0) **Lemma 5.8.** *Suppose that we are in one of the cases (1-1), (1-2), (2-1), or (2-2) of Lemma 5.6 so that $E_i = \Delta_j$ for some $i, j \in \{1, 2\}$. We then have:*

a. *If $\Delta_1 \cup \Delta_2 = E_1 \cup E_2$ then the operator $Z'^0 := Z^0_{\{E_{i'},\Delta_1\cap\Delta_2\}}$ of type Z^0 is defined in a and $Z^0 + (A^1 + a) = Z'^0 + a$.*
b. *If $\Delta_1 \cup \Delta_2 \subsetneqq E_1 \cup E_2$ then we have the operators $Z'^0 := Z^0_{\{E_{i'},\Delta_1\cap\Delta_2\}}$ of type Z^0 and $Z'^1 := Z^1_{\{\Delta_1\cup\Delta_2,(\Delta_1\cap\Delta_2)\cup E_{i'}\}}$ of type Z^1 such that the composite operator $Z'^1 + (Z'^0 + .)$ is defined in a and $Z^0 + (A^1 + a) = Z'^1 + (Z'^0 + a)$.*
c. *If $E_1 \cup E_2 \subsetneqq \Delta_1 \cup \Delta_2$ then we have the operators $Z'^0 := Z^0_{\{E_{i'},\Delta_1\cap\Delta_2\}}$ of type Z^0 and $A'^1 := A^1_{\{\Delta_1\cup\Delta_2,(\Delta_1\cap\Delta_2)\cup E_{i'}\}} = -Z^1_{\{E_1\cup E_2,\Delta_{j'}\}}$ of type A^1 such that the composite operator $A'^1 + (Z'^0 + .)$ is defined in a and $Z^0 + (A^1 + a) = A'^1 + (Z'^0 + a)$.*
d. *If $\Delta_1 \cup \Delta_2 \not\subseteq E_1 \cup E_2$ and $E_1 \cup E_2 \not\subseteq \Delta_1 \cup \Delta_2$ then we have the operators $Z'^0 := Z^0_{\{E_{i'},\Delta_1\cup\Delta_2\}}$ of type Z^0 and $A'^1 := A^1_{\{\Delta_1\cup\Delta_2\cup E_{i'},\Delta_1\cap\Delta_2\}} = -Z^1_{\{E_1\cup E_2,\Delta_{j'}\}}$ of type A^1 such that the composite operator $A'^1 + (Z'^0 + .)$ is defined in a and $Z^0 + (A^1 + a) = A'^1 + (Z'^0 + a)$.*

Proof. In all cases we have

$$Z^0 + (A^1 + a) = (E_1 \cup E_2) - E_{i'} + \Delta_{j'} - (\Delta_1 \cup \Delta_2) - (\Delta_1 \cap \Delta_2) + a \ .$$

a. The assumptions $\Delta_1 \cup \Delta_2 = E_1 \cup E_2$ and $E_i = \Delta_j$ imply that the segment $\Delta_{j'}$ is the disjoint union of the segments $E_{i'}$ and $\Delta_1 \cap \Delta_2$.

b. The assumptions $\Delta_1 \cup \Delta_2 \subsetneqq E_1 \cup E_2$ and $E_i = \Delta_j$ imply that

- the disjoint union of the segments $\Delta_1 \cap \Delta_2$ and $E_{i'}$ is a segment,
- $(\Delta_1 \cup \Delta_2) \cup ((\Delta_1 \cap \Delta_2) \cup E_{i'}) = E_1 \cup E_2$,
- $(\Delta_1 \cup \Delta_2) \cap ((\Delta_1 \cap \Delta_2) \cup E_{i'}) = \Delta_{j'}$,
- $\Delta_1 \cup \Delta_2 \not\subseteq (\Delta_1 \cap \Delta_2) \cup E_{i'}$ and $(\Delta_1 \cap \Delta_2) \cup E_{i'} \not\subseteq \Delta_1 \cup \Delta_2$.

c. The assumptions $\Delta_1 \cup \Delta_2 \supsetneqq E_1 \cup E_2$ and $E_i = \Delta_j$ imply that

- the disjoint union of the segments $\Delta_1 \cap \Delta_2$ and $E_{i'}$ is a segment,
- $(E_1 \cup E_2) \cup \Delta_{j'} = \Delta_1 \cup \Delta_2$,
- $(E_1 \cup E_2) \cap \Delta_{j'} = (\Delta_1 \cap \Delta_2) \cup E_{i'}$,
- $E_1 \cup E_2 \not\subseteq \Delta_{j'}$ and $\Delta_{j'} \not\subseteq E_1 \cup E_2$.

d. The assumptions $\Delta_1 \cup \Delta_2 \not\subseteq E_1 \cup E_2$, $E_1 \cup E_2 \not\subseteq \Delta_1 \cup \Delta_2$, and $E_i = \Delta_j$ imply that

- the disjoint union of the segments $\Delta_1 \cup \Delta_2$ and $E_{i'}$ is a segment,
- $(E_1 \cup E_2) \cup \Delta_{j'} = \Delta_1 \cup \Delta_2 \cup E_{i'}$,
- $(E_1 \cup E_2) \cap \Delta_{j'} = \Delta_1 \cap \Delta_2$,
- $E_1 \cup E_2 \not\subseteq \Delta_{j'}$ and $\Delta_{j'} \not\subseteq E_1 \cup E_2$.

Using these observations the proof is a straightforward computation. □

sec:result

6. The main result

Our main result is the following.

main

Theorem 6.1. *For two partitions $\mathcal{P}$ and $\mathcal{P}'$ the following assertions are equivalent:*

i. *$\mathcal{P} \le \mathcal{P}'$ is a basic relation;*
ii. *there exists a partition $\mathcal{P}''$ such that $\mathcal{P} \le_{ref} \mathcal{P}'' \le \mathcal{P}'$ with the right hand relation being basic of type 1.*

Proof. ii. $\Longrightarrow$ i. By assumption there is a multisegment a'' such that $\mathcal{P}'' = \mathcal{P}(a'')$ and $a'' \ge_Z b(\mathcal{P}')$. In addition, by the proof of Remark 3.6, we find a multisegment a such that $\mathcal{P} = \mathcal{P}(a)$ and $a \ge_Z a''$. Hence $a \ge_Z b(\mathcal{P}')$ which means that we have the basic relation $\mathcal{P} \le \mathcal{P}'$.

i. $\Longrightarrow$ ii. By assumption there exists a multisegment a such that $\mathcal{P} = \mathcal{P}(a)$ and $a \ge_Z b(\mathcal{P}')$. The latter means that there exists a sequence $Z_1, \ldots, Z_t$ of operators of type Z^0 or Z^1 such that the composite operator is defined in a and

$$b(\mathcal{P}') = Z_1 + (Z_2 + \ldots + (Z_t + a) \ldots) \ .$$

The lemmas in section 5 allow us to replace this sequence by another sequence of operators $Y_1, \ldots, Y_s, Z_1^0, \ldots, Z_s^0$ such that again

$$b(\mathcal{P}') = Y_1 + (Y_2 + \ldots + (Y_r + (Z_1^0 + \ldots + (Z_s^0 + a) \ldots)) \ldots)$$

but where the operators Y_i are of type Z^1 or A^1 and the Z_j^0 are of type Z^0. We define

$$a'' := Z_1^0 + (Z_2^0 + \ldots + (Z_s^0 + a) \ldots) \quad \text{and} \quad \mathcal{P}'' := \mathcal{P}(a'') \ .$$

Then, by Remark 3.6, we have $\mathcal{P} \le_{ref} \mathcal{P}''$ as well as

$$b(\mathcal{P}') = Y_1 + (Y_2 + \ldots + (Y_r + a'') \ldots) \quad \text{with} \quad s(\mathcal{P}') = s(\mathcal{P}'') \ .$$

Applying Lemma 3.10 with $b := b(\mathcal{P}')$ we deduce that the multisegment $a'' \in B(b(\mathcal{P}'))$ is precentered. Therefore Prop. 4.2 implies the existence of a sequence $Z_1^1, \ldots, Z_{r'}^1$ of operators of type Z^1 such that

$$b(\mathcal{P}') = Z_1^1 + (Z_2^1 + \ldots + (Z_{r'}^1 + a'') \ldots) \ .$$

hence we have the basic relation $\mathcal{P}'' \leq \mathcal{P}'$, which is of type 1 because $s(\mathcal{P}') = s(\mathcal{P}'')$. □

main2 **Corollary 6.2.** *The partial order $\leq$ is the transitive hull of the refinement relations and the basic relations of type* 1*.*

In view of Prop. 4.4 this Corollary gives an explicit description of the partial order $\leq$. We still want to make it explicit that our order $\leq$ is strictly contained in the dominance order, which means that the implication from $\mathcal{P} \leq \mathcal{P}'$ to $\mathcal{P} \leq_{dom} \mathcal{P}'$ can in general not be reversed. For this we consider cases where $\mathcal{P} <_{dom} \mathcal{P}'$ is a cover, which means it is impossible to refine it as $\mathcal{P} <_{dom} \mathcal{P}'' <_{dom} \mathcal{P}'$. Here we rely on Brylawski in [Bry] Prop. 2.3 who gave a complete description of possible covers for the dominance order as follows: Any cover $\mathcal{P}' = (\ell_1 \geq \cdots \geq \ell_s) >_{dom} \mathcal{P} = (m_1 \geq \cdots \geq m_t)$ is of one of the following two forms:

(i) $\mathcal{P}' = (..., \ell+2, ..., \ell, ...) >_{dom} \mathcal{P} = (..., \ell+1, ..., \ell+1, ...)$ or
(ii) $\mathcal{P}' = (..., \ell+h, \ell, ...) >_{dom} \mathcal{P} = (..., \ell+h-1, \ell+1, ...)$ for some $h \geq 3$,

where $\mathcal{P}$ and $\mathcal{P}'$ differ only at those positions which have been noted explicitly. Note here that for type (i) the constituents $\ell+2$ and ℓ need not be neighbours but might be connected by a certain number of constituents $\ell+1$. Also note that the case $\ell = 0$ may happen which brings us to the covers

$$(..., 2, 1, ..., 1, 0) >_{dom} (..., 1, 1, ...1, 1) \quad \text{and} \quad (..., h, 0) >_{dom} (..., h-1, 1) \ .$$

These are the covers which are actually refinements and therefore, according to Remark 3.6, can be realized also as $\mathcal{P}' > \mathcal{P}$. In the following we will assume that $\ell \neq 0$.

covers **Proposition 6.3.** *Let $\mathcal{P}' >_{dom} \mathcal{P}$ be a cover such that $\ell \neq 0$. Then the form (i) can be realized as $\mathcal{P}' > \mathcal{P}$ whereas the form (ii) can not.*

Proof. Assume that the cover $\mathcal{P}' >_{dom} \mathcal{P}$ can be realized as $\mathcal{P}' > \mathcal{P}$. Then of course $\mathcal{P}' > \mathcal{P}$ must be a cover for our new order and, since $\ell \neq 0$ by assumption, it must be in particular a basic relation of type 1 (see Def. 3.5). Applying Prop. 4.4 to the partition $\mathcal{P}' = (\ell_1 \geq \cdots \geq \ell_s)$ we see that $\mathcal{P} = (m_1 \geq \cdots \geq m_s) \leq \mathcal{P}'$ is equivalent to $\mathcal{P} = \mathcal{P}'_\pi$, which means $m_{\sigma(i)} = \frac{\ell_i + \ell_{\pi(i)}}{2}$, for all $i = 1, ..., s$, with a certain permutation π such that ℓ_i and $\ell_{\pi(i)}$ have always the same parity. Replacing σ by its inverse we can also write

$$m_i = \frac{\ell_{\sigma(i)} + \ell_{\pi(\sigma(i))}}{2} \qquad \text{for any } i.$$

If $\mathcal{P}' >_{dom} \mathcal{P}$ is of the form (i), we let $i < j$ be the positions such that $\ell_i = \ell+2$, $\ell_j = \ell$, and we let $\pi := (i, j)$ be the corresponding transposition. Then obviously $\mathcal{P}'_\pi = (..., \ell+1, ..., \ell+1, ...)$, and hence $\mathcal{P} < \mathcal{P}'$.

On the other hand, if $\mathcal{P}' >_{dom} \mathcal{P}$ is of the form (ii) we have to show that $\mathcal{P} \neq \mathcal{P}'_\pi$ for all π. Let i_0, i_0+1 be the positions such that $\ell_{i_0} = \ell+h$, $m_{i_0} = \ell+h-1$ and $\ell_{i_0+1} = \ell$, $m_{i_0+1} = \ell+1$, whereas $m_i = \ell_i$ for all other i. Assuming $\mathcal{P}'_\pi = \mathcal{P} = (..., \ell+h-1, \ell+1, ...)$

for some π means the existence of another permutation σ such that

$$\frac{\ell_{\sigma(i)} + \ell_{\pi\circ\sigma(i)}}{2} = \ell_i \quad \text{for all } i \neq i_0, i_0+1,$$
$$\frac{\ell_{\sigma(i_0)} + \ell_{\pi\circ\sigma(i_0)}}{2} = \ell + h - 1,$$
$$\frac{\ell_{\sigma(i_0+1)} + \ell_{\pi\circ\sigma(i_0+1)}}{2} = \ell + 1.$$

Since ℓ_1 is the maximum this implies $\ell_{\sigma(1)} = \ell_{\pi\circ\sigma(1)} = \ell_1$. Thus we may change the permutations such that $\sigma(1) = \pi(1) = 1$. Successively we may assume that $\sigma(i) = \pi(i) = i$ for all $i < i_0$. Now we consider the equation $\frac{\ell_{\sigma(i_0)}+\ell_{\pi\circ\sigma(i_0)}}{2} = \ell + h - 1$. Since we know already that both $\sigma(i_0)$ and $\pi \circ \sigma(i_0)$ must be $\geq i_0$ we are left with the following subcases:

- $\sigma(i_0) = i_0$, $\ell_{\sigma(i_0)} = \ell + h$, and
- $\pi(\sigma(i_0)) = i_0$, $\ell_{\pi\circ\sigma(i_0)} = \ell + h$.

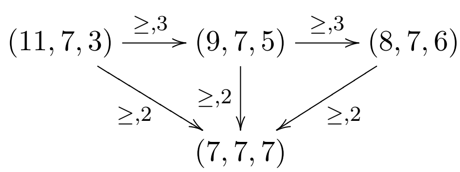
Obviously these cases exclude each other, such that in the first case we must have $\pi(\sigma(i_0)) = \pi(i_0) \geq i_0 + 1$ and hence $\ell_{\pi(\sigma(i_0))} \leq \ell$, whereas in the second case $\sigma(i_0) \geq i_0 + 1$ and hence $\ell_{\sigma(i_0)} \leq \ell$. Therefore in both cases the assumption $h \geq 3$ leads us to the contradiction

$$\frac{\ell_{\sigma(i_0)} + \ell_{\pi\circ\sigma(i_0)}}{2} \leq \frac{2\ell + h}{2} < \ell + h - 1 \ .$$

□

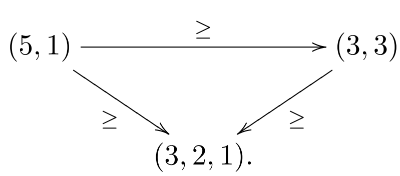
We finish with two examples related to the problem of finding the covers for our new order $\leq$. What is obvious here is that a cover must be either a refinement or a basic relation $\mathcal{P}_\pi \leq \mathcal{P}$ of type 1. Since π is uniquely the product of pairwise disjoint cycles we see from Cor. 4.5 that the relation $\mathcal{P}_\pi < \mathcal{P}$ must be realized already by one of these cycles if it is a cover.

Example 6.4. We have the following basic relations of type 1:



Here the numbers 2 and 3 indicate that the corresponding relation is induced, in the sense of Prop. 4.4, by a permutation π which is a transposition and a cycle of length 3, respectively. In particular, $(11, 7, 3) \geq (7, 7, 7)$ is not a cover for the partial order $\leq$. Moreover, the relation $(11, 7, 3) \geq (8, 7, 6)$ is not basic.

Example 6.5. We have the basic relations



Here the upper relation is basic of type 1, whereas the other two are refinement relations. This shows that $(5, 1) \geq (3, 2, 1)$, which is a cover for the refinement order $\leq_{ref}$, is not a cover for $\leq$.

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