The μ-invariant of isogenies

by Peter Schneider

Dedicated to S.Ramanujan

Two abelian varieties A and B over a number field k which are isogeneous have the same Hasse-Weil L-function. This fact is an immediate consequence of the definition of these L-functions. But as was pointed out by B.Perrin-Riou by way of an example the (Iwasawa theoretic) p-adic L-functions of A and B, for a fixed p, may differ if the degree of the isogeny is divisible by p. A look at their definition shows that they differ at most by a power of p. In this paper we will give a complete and explicit determination of that constant in terms of the local behaviour of the kernel of the isogeny.

§1 **Characteristic polynomials**

We fix a number field k, a prime number $p \neq 2$, and an abelian variety $A_{/k}$ which has good reduction at the primes of k above p. We furthermore fix a $\mathbb{Z}_p$-extension k_∞/k and put $\Gamma := \mathrm{Gal}(k_\infty/k)$. The idea behind p-adic L-functions is to use the action of Γ on certain cohomology groups associated with A over k_∞ in order to construct some kind of characteristic polynomial. Let o, resp. o_∞, be the ring of integers in k, resp. k_∞, and denote by $\mathcal{A}$ the Néron model of A over $\mathrm{Spec}(o)$. Those mentioned cohomology groups are the groups

$$H^i(o_\infty, \mathcal{A}(p))$$

where (as always in this paper if not indicated otherwise) cohomology is flat cohomology and $\mathcal{A}(p)$ is the p-primary torsion subsheaf of $\mathcal{A}$ with respect to the flat topology on $\mathrm{Spec}(o)$. The Pontrjagin duals $H^i(o_\infty, \mathcal{A}(p))^*$ in an obvious way are compact $\mathbb{Z}_p[[\Gamma]]$-modules where $\mathbb{Z}_p[[\Gamma]]$ is the completed group ring of Γ over $\mathbb{Z}_p$. We have the following facts ([5] §6 Prop. 5 and [7] Remark on p. 596):

- $H^i(o_\infty, \mathcal{A}(p)) = 0 \quad \text{for} \quad i \geq 3$;
- $H^0(o_\infty, \mathcal{A}(p))^*$ is a finitely generated $\mathbb{Z}_p$-module;
- $H^i(o_\infty, \mathcal{A}(p))^*$, for $i = 1$ or 2, are finitely generated $\mathbb{Z}_p[[\Gamma]]$-modules;
- $H^2(o_\infty, \mathcal{A}(p))^*$ is $\mathbb{Z}_p$-torsion free.

In order to say more we first have to recall very briefly Iwasawa's general structure theory for $\mathbb{Z}_p[[\Gamma]]$-modules (compare [11] 13.2): For any finitely generated $\mathbb{Z}_p[[\Gamma]]$-module M there exist $\mathbb{Z}_p[[\Gamma]]$-modules M_0, M_1, F and a homomorphism $\alpha : M \to M_0 \oplus M_1 \oplus F$ of $\mathbb{Z}_p[[\Gamma]]$-modules such that:

- $M_0 \cong \overset{m}{\underset{i=1}{\oplus}} \mathbb{Z}/p^{\mu_i}\mathbb{Z}[[\Gamma]]$ for some $m \geq 1$ and $\mu_i \geq 0$;
- M_1 is finitely generated over $\mathbb{Z}_p$;
- $F \cong \mathbb{Z}_p[[\Gamma]]^\rho$ with $\rho := \text{rank}_{\mathbb{Z}_p[[\Gamma]]} M$;
- kernel and cokernel of α are finite.

Furthermore, $\mu(M) := \Sigma\mu_i$ and $P(T; M) := \det(1 - \gamma^{-1}T; M_1 \underset{\mathbb{Z}_p}{\otimes} \mathbb{Q}_p)$ only depend on the module M; here γ is an once and for all fixed topological generator of Γ. We call $\mu(M)$ the <u>μ-invariant</u> of M and, in case $\rho = 0$,

$$F(T; M) := p^{\mu(M)} \cdot P(T; M)$$

the <u>characteristic polynomial</u> of M.

In order to apply these notions we would like to have that $H^i(o_\infty, \mathcal{A}(p))^*$, for $i = 1$ and 2, have $\mathbb{Z}_p[[\Gamma]]$-rank zero. It is known ([7]) that then, necessarily,

(I) A has good ordinary reduction at the set Σ of primes of k which are ramified in k_∞.

(Σ is contained in the set of primes above p.) Therefore, from now on, we always will assume that. On the other hand this condition certainly is not sufficient for the vanishing of the $\mathbb{Z}_p[[\Gamma]]$-ranks. But at least we then have ([7]):

- $\text{rank}_{\mathbb{Z}_p[[\Gamma]]} H^1(o_\infty, \mathcal{A}(p))^* = \text{rank}_{\mathbb{Z}_p[[\Gamma]]} H^2(o_\infty, \mathcal{A}(p))^*$.

And there is the following conjecture of Mazur.

Conjecture:

If k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension then $\text{rank}_{\mathbb{Z}_p[[\Gamma]]} H^1(o_\infty, \mathcal{A}(p))^* = 0$.

Since our subsequent computations will not depend on any specific property of a $\mathbb{Z}_p$-extension we make the second assumption that

(II) $\mathrm{rank}_{\mathbb{Z}_p[[\Gamma]]} H^1(o_\infty, \mathcal{A}(p))^* = 0$

but where k_∞/k remains an arbitrary $\mathbb{Z}_p$-extension. Under these assumptions the alternating product of characteristic polynomials

$$F(A, k_\infty; T) := \prod_{i\geq 0} F(T; H^i(o_\infty, \mathcal{A}(p))^*)^{(-1)^{i+1}}$$

is defined and, in fact, is equal to a power of p times an alternating product of "true" characteristic polynomials:

$$F(A, k_\infty; T) = p^{\mu(A,k_\infty)} \cdot \prod_{i\geq 0} P(T; H^i(o_\infty, \mathcal{A}(p))^*)^{(-1)^{i+1}}$$

with

$$\mu(A, k_\infty) := \mu(H^1(o_\infty, \mathcal{A}(p))^*) \quad .$$

At the moment this μ-invariant $\mu(A, k_\infty)$ still is rather mysterious. Mazur ([2] §10) has given examples with $\mu(A, k_\infty) > 0$. One tends to believe that at least for the cyclotomic $\mathbb{Z}_p$-extension it should be computable by a rather simple and explicit formula. But so far we only will be able to describe its behaviour under isogenies.

Remark:

The (Iwasawa theoretic or motivic) p-adic L-function of A is defined to be (assuming (I) and Mazur's conjecture)

$$L_p(A, s) := F(A, k_\infty; \kappa(\gamma)^{1-s})$$

where k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension and $\kappa : \Gamma \longrightarrow \mathbb{Z}_p^\times$ is the cyclotomic character.

In order to simplify notation we always will drop the k_∞ (which is fixed) in the following, e.g., writing $\mu(A)$ instead of $\mu(A, k_\infty)$.

§2 **Isogenies**

Now let $B_{/k}$ be a second abelian variety and let $f : A \to B$ be a k-isogeny. In particular, B also has good reduction at the primes above p and good ordinary reduction at the primes in Σ. By the universal property of Néron models

f extends to a homomorphism of group schemes $f_{/o} : \mathcal{A} \longrightarrow \mathcal{B}$ where $\mathcal{B}$ denotes the Néron model of B over Spec(o). Consequently f induces a $\mathbb{Z}_p[[\Gamma]]$-homomorphism

$$H^i(o_\infty, \mathcal{B}(p))^* \xrightarrow{f^*} H^i(o_\infty, \mathcal{A}(p))^* \quad .$$

If $d := \deg f$ denotes the degree of the isogeny f then we find an isogeny $g : B \longrightarrow A$ such that

$$g_{/o} \circ f_{/o} = d \cdot \mathrm{id}_{\mathcal{A}} \quad \text{and} \quad f_{/o} \circ g_{/o} = d \cdot \mathrm{id}_{\mathcal{B}} \quad .$$

From that it is clear that, with A, also B fulfills our assumption (II). We furthermore see that in case $p \not| \, d$ we have $F(B;T) = F(A;T)$ because f^* is an isomorphism. Since any isogeny can be decomposed into two others of degree prime to p and a power of p, respectively, we will assume from now on that

$$d = \deg f = p^\delta \quad .$$

We then at least have that

kernel and cokernel of f^* are annihilated by p^δ.

This implies

$$F(B;T) = p^{\mu(B)-\mu(A)} F(A;T) \quad .$$

The purpose of this paper is to give an explicit formula for the difference $\mu(B) - \mu(A)$. Our computation will be based on the observation that this difference is a kind of Euler-Poincaré characteristic: The assumption that A and B have good reduction at the primes above p implies that $f_{/o}$ is quasi-finite flat and is finite and faithfully flat over some appropriate Zariski neighbourhood U of $\{\wp \mid p\}$ in Spec(o). The kernel $\mathcal{C} := \ker(f_{/o})$ therefore is a quasi-finite flat o-group scheme such that $\mathcal{C}_{/U}$ is finite.

Definition:

$\mu(f) := \sum_{i \geq 0} (-1)^i \mu(H^i(o_\infty, \mathcal{C})^*)$ *is called the* *μ-invariant of f*.

Lemma:

$\mu(f) = \mu(B) - \mu(A)$.

Proof: Define the sheaf $\mathcal{F}$ by the exact sequence of sheaves for the flat topology on Spec(o)

$$0 \to \mathcal{C} \to \mathcal{A}(p) \xrightarrow{f} \mathcal{B}(p) \to \mathcal{F} \to 0 \quad .$$

By the usual arguments with long exact cohomology sequences our assumption (II) implies that $\mu(\ )$ is "additive" in this exact sequence, i.e., we have

$$\mu(f) - (-\mu(A)) + (-\mu(B)) - \sum_{i\geq 0}(-1)^i \mu(H^i(o_\infty, \mathcal{F})^*) = 0\ .$$

Let o', resp. o'_∞, denote the ring of p-integers in k, resp. k_∞. From $\mathcal{F}_{/U} = 0$ we immediately derive

$$H^i(o_\infty, \mathcal{F}) = H^i(o'_\infty, \mathcal{F}) \quad .$$

But over $\mathrm{Spec}(o')$ the first three terms in the above sequence are (ind-) etale group schemes. Therefore, by the theorem of Grothendieck (compare [4] III.3.9), over $\mathrm{Spec}(o'_\infty)$ the sequence remains exact after restriction to the etale topology and, for all four terms, flat and etale cohomology agree. Since the restriction of $\mathcal{F}$ to the etale topology on $\mathrm{Spec}(o')$ is a skyscraper sheaf with finite p-groups as stalks we can apply Proposition 1 below and we get

$$\sum_{i\geq 0}(-1)^i \mu(H^i(o'_\infty, \mathcal{F})^*) = 0 \quad .$$

§3 Euler-Poincaré characteristics

In this paragraph we also work with the etale topology and cohomology which will be indicated by a subscript "*et*". Furthermore we use the following notational convention: If $\mathcal{F}$ is a sheaf for the etale topology on some irreducible scheme then F denotes its stalk in a fixed geometric point above the generic point. As before o', resp. o'_∞, denotes the ring of p-integers in k, resp. k_∞. For any prime $\wp$ of k let $o_\wp$, resp. $k_\wp$, resp. $G_\wp$, be the completion of o in $\wp$, resp. the quotient field of $o_\wp$, resp. a decomposition group for $\wp$ in the absolute Galois group of k. The usual p-valuation is denoted by v (i.e., $v(p) = 1$).

Proposition 1:
For any constructible p-primary torsion sheaf $\mathcal{F}$ on $\mathrm{Spec}(o')_{et}$ we have

$$\sum_{i\geq 0}(-1)^{i+1}\mu(H^i_{et}(o'_\infty, \mathcal{F})^*) = (r_1 + r_2)\cdot v(\#F) - \sum_{\wp \text{ real}} v(\#H^0(G_\wp, F))$$

where r_1, resp. r_2, is the number of real, resp. complex, primes of k.

Proof: Since Γ has cohomological dimension 1 we conclude from the Hochschild-Serre spectral sequence

$$H^i(\Gamma, H^j_{et}(o'_\infty, \mathcal{F})) \Longrightarrow H^{i+j}_{et}(o', \mathcal{F})$$

that

$$\sum_{i\geq 0}(-1)^{i+1}\mu(H^i_{et}(o'_\infty,\mathcal{F})^*) = v(\chi(o',\mathcal{F})) \quad .$$

Here the right hand side is the usual Euler-Poincaré characteristic

$$\chi(o',\mathcal{F}) := \prod_{i\geq 0} \# H^i_{et}(o',\mathcal{F})^{(-1)^{i+1}} \quad .$$

Our assertion therefore is an immediate consequence of the following two facts:

- $\chi(o',\mathcal{F}) = \chi(V,\mathcal{F})$ for any nonempty open subset $V \subseteq \mathrm{Spec}(o')$.
- If $\mathcal{F}$ is locally constant on V_{et} then

$$\chi(V,\mathcal{F}) = (\# F)^{r_1+r_2} / \prod_{\wp \text{ real}} \# H^0(G_\wp, F) \quad .$$

The second fact is a theorem of Tate ([10] Th.2.2). In order to establish the first fact we note that an obvious argument with relative cohomology sequences shows that we have

$$\chi(o',\mathcal{F}) = \chi(V,\mathcal{F}) \cdot \prod_{\wp \,\in\, \mathrm{Spec}(o')\setminus V} \frac{\chi(o_\wp,\mathcal{F})}{\chi(k_\wp,\mathcal{F})} \quad .$$

Since $\wp$ is prime to p we know by [9] II-26 that $\chi(k_\wp,\mathcal{F}) = 1$. On the other hand the etale cohomology of a Henselian ring is equal to the etale cohomology of its residue class field. Since the absolute Galois group of the residue class field of $o_\wp$ is procyclic we see that $\chi(o_\wp,\mathcal{F}) = 1$.

Proposition 2:

Let $\wp$ be a prime of k above p, let $\mathcal{F}$ be a constructible p-primary torsion sheaf on $\mathrm{Spec}(o_\wp)_{et}$, and put $o_{\infty,\wp} := o_\infty \underset{o}{\otimes} o_\wp$; we then have

$$\sum_{i\geq 0}(-1)^i\mu(H\dot{}^i_{et}(o_{\infty,\wp},\mathcal{F})^*) = [k_\wp : \mathbb{Q}_p] \cdot v(\# F) \quad .$$

Proof: (Recall that $H\dot{}^*_{et}(o_{\infty,\wp}, .)$ is the relative cohomology with respect to the closed set of points of $\mathrm{Spec}(o_{\infty,\wp})$ above $\wp$.) If $\wp|p$ is ramified in k_∞ then $o_{\infty,\wp}$ is semi-local Henselian with finite residue class fields. This implies that the groups $H^*_{et}(o_{\infty,\wp},\mathcal{F})$ are finite and consequently, using the relative cohomology sequence, that

$$\sum_{i\geq 0}(-1)^i\mu(H\dot{}^i_{et}(o_{\infty,\wp},\mathcal{F})^*) = \sum_{i\geq 0}(-1)^{i+1}\mu(H^i_{et}(k_{\infty,\wp},\mathcal{F})^*)$$

where $k_{\wp,\infty} := o_\infty \underset{o}{\otimes} k_\wp$. The argument with the Hochschild-Serre spectral sequence applies to the right hand side and shows that

$$\sum_{i\geq 0}(-1)^{i+1}\mu(H^i_{et}(k_{\infty,\wp},\mathcal{F})^*) = v(\chi(k_\wp,\mathcal{F})) \quad .$$

If $\wp|p$ is unramified in k_∞ then the argument with the Hochschild-Serre spectral sequence applies directly and gives

$$\begin{aligned}\sum_{i\geq 0}(-1)^{i+1}\mu(H^{\cdot i}_{et}(o_{\infty,\wp},\mathcal{F})^*) &= v(\prod_{i\geq 0}\#H^{\cdot i}_{et}(o_\wp,\mathcal{F})^{(-1)^{i+1}}) \\ &= v(\chi(o_\wp,\mathcal{F})) - v(\chi(k_\wp,\mathcal{F}))\ ;\end{aligned}$$

the second identity again comes from the relative cohomology sequence. In the proof of Proposition 1 we already have seen that $\chi(o_\wp,\mathcal{F}) = 1$. To establish our assertion it remains to compute $\chi(k_\wp,\mathcal{F})$. But, according to [9] II-35, we have

$$v(\chi(k_\wp,\mathcal{F})) = [k_\wp : \mathbb{Q}_p]\cdot v(\#F) \quad .$$

Corollary 3:

For any constructible p-primary torsion sheaf $\mathcal{F}$ on $\mathrm{Spec}(o)_{et}$ *we have*

$$\sum_{i\geq 0}(-1)^i\mu(H^i_{et}(o_\infty,\mathcal{F})^*) = r_2\cdot v(\#F) + \sum_{\wp \text{ real}} v(\#H^0(G_\wp,F)) \quad .$$

Proof: Use the relative cohomology sequence.

Remark:

Let $\mathcal{F}$ be a constructible p-primary torsion sheaf on $\mathrm{Spec}(o)_{et}$ and assume that k_∞/k is the cyclotomic $\mathbb{Z}_p$-extension. Clearly,

$$H^0_{et}(o_\infty,\mathcal{F}) \quad \text{is finite.}$$

It is not difficult to prove that the etale cohomological p-dimension of $\mathrm{Spec}(o_\infty)$ is ≤ 2; we therefore have

$$H^i_{et}(o_\infty,\mathcal{F}) = 0 \quad \text{for} \quad i\geq 3 \quad .$$

There is a well-known conjecture of Iwasawa which amounts to the statement that

the etale cohomological p-dimension of $\mathrm{Spec}(o'_\infty)$ is ≤ 1

(compare [6] §3 Lemma 8). Let us assume this to hold true. It was shown in [8] §4 Lemma that then

$$H^1_{et}(o_\infty, \mathcal{F}) \quad \text{is finite} \quad .$$

Therefore we can sharpen the above Corollary to the statement that

$$\mu(H^2_{et}(o_\infty, \mathcal{F})^*) = r_2 \cdot v(\#F) + \sum_{\wp \text{ real}} v(\#H^0(G_\wp, F)) \quad .$$

From that it is straightforward to give explicit formulas for the μ-invariants $\mu(H^i_{et}(o_\infty, \mathcal{F}^\cdot)^*)$ of the individual cohomology groups of any bounded complex of sheaves on $\mathrm{Spec}(o)_{et}$ whose homology sheaves are constructible p-primary torsion sheaves. The two problems of proving Mazur's conjecture for the abelian variety A and of computing its μ-invariant come down to the problem of computing the μ-invariants

$$\mu(H^2_{et}(o_\infty, [\mathcal{A} \xrightarrow{p^m} \mathcal{A}])^*) \quad \text{for} \quad m \geq 1 \quad .$$

But, unfortunately, the homology sheaf in degree 1 of the complex $[\mathcal{A} \xrightarrow{p^m} \mathcal{A}]$ is not constructible.

Now let $\mathcal{C}$ be a quasi-finite flat (commutative) group scheme over $\mathrm{Spec}(o)$ such that $p^m\mathcal{C} = 0$ for some $m \geq 0$ and $\mathcal{C}_\wp := \mathcal{C} \underset{o}{\times} o_\wp$ is finite over $\mathrm{Spec}(o_\wp)$ for all $\wp \mid p$. For each $\mathcal{C}_\wp$ we have an exact sequence of finite flat group schemes over $\mathrm{Spec}(o_\wp)$

$$0 \longrightarrow \mathcal{C}^0_\wp \longrightarrow \mathcal{C}_\wp \longrightarrow \mathcal{C}^{et}_\wp \longrightarrow 0$$

where $\mathcal{C}^0_\wp$ is connected and $\mathcal{C}^{et}_\wp$ is etale. Let $\Omega^1_{\mathcal{C}}$ be the module of relative Kähler differentials for $\mathcal{C}$ over $\mathrm{Spec}(o)$. If s denotes the zero section of $\mathcal{C}$ then $s^*\Omega^1_{\mathcal{C}}$ is a finite o-module. Similarly, we have the finite $o_\wp$-modules $s^*\Omega^1_{\mathcal{C}_\wp} = (s^*\Omega^1_{\mathcal{C}}) \underset{o}{\otimes} o_\wp$. Since $\mathcal{C}$ is etale over $\mathrm{Spec}(o')$ we get

$$\#(s^*\Omega^1_{\mathcal{C}}) = \prod_{\wp \mid p} \#(s^*\Omega^1_{\mathcal{C}_\wp}) \quad .$$

Proposition 4:

$$\sum_{i \geq 0} (-1)^i \mu(H^i(o_\infty, \mathcal{C})^*) = \sum_{\wp \mid \infty} v(\#\mathcal{C}(k_\wp)) - \sum_{\substack{\wp \mid p \\ \wp \notin \Sigma}} v(\#s^*\Omega^1_{\mathcal{C}_\wp})$$
$$- \sum_{\wp \in \Sigma} [k_\wp : \mathbb{Q}_p] \cdot v(\text{rank } \mathcal{C}^0_\wp) \quad .$$

Proof: From the relative cohomology sequence we get

$$\begin{aligned}\sum_{i\geq 0}(-1)^i\mu(H^i(o_\infty,\mathcal{C})^*) = &\sum_{i\geq 0}(-1)^i\mu(H^i(o'_\infty,\mathcal{C})^*)\\ &+\sum_{\substack{\wp|p\\ \wp\notin\Sigma}}(\sum_{i\geq 0}(-1)^i\mu(H.^i(o_{\infty,\wp},\mathcal{C})^*))\\ &+\sum_{\wp\in\Sigma}(\sum_{i\geq 0}(-1)^i\mu(H.^i(o_{\infty,\wp},\mathcal{C})^*)) \quad .\end{aligned}$$

We will now treat the three terms on the right hand side separately.

First term: Since $\mathcal{C}$ is quasi-finite etale over $\mathrm{Spec}(o')$ it represents a constructible sheaf on $\mathrm{Spec}(o')_{et}$; furthermore, by [4] III.3.9, flat and etale cohomology of $\mathrm{Spec}(o'_\infty)$ with coefficients in $\mathcal{C}$ agree. We therefore can apply Proposition 1 which shows that our first term is equal to

$$\begin{aligned}&-(r_1+r_2)\cdot v(\mathrm{rank}\ \mathcal{C}_{/k})+\sum_{\wp\ \mathrm{real}}v(\#\mathcal{C}(k_\wp))\\ &=-[k:\mathbb{Q}]\cdot v(\mathrm{rank}\ \mathcal{C}_{/k})+\sum_{\wp|\infty}v(\#\mathcal{C}(k_\wp)) \quad .\end{aligned}$$

Second term: The argument with the Hochschild-Serre spectral sequence shows that the second term equals

$$\sum_{\substack{\wp|p\\ \wp\notin\Sigma}}\sum_{i\geq 0}(-1)^i v(\#H.^i(o_\wp,\mathcal{C})) \quad .$$

Using the results in [3] (Cor. 9.1, Prop. 8.1, and the last sentence on p.227) we compute (here $\hat{\mathcal{C}}_\wp$ denotes the Cartier dual of $\mathcal{C}_\wp$)

$$\begin{aligned}\sum_{i\geq 0}(-1)^i v(\#H.^i(o_\wp,\mathcal{C})) &= \sum_{i\geq 0}(-1)^i v(\#H^i(o_\wp,\hat{\mathcal{C}}_\wp))\\ &= v(\#s^*\Omega^1_{\hat{\mathcal{C}}_\wp})\\ &= [k_\wp:\mathbb{Q}_p]\cdot v(\mathrm{rank}\ \mathcal{C}_{/k})-v(\#s^*\Omega^1_{\mathcal{C}_\wp}) \quad .\end{aligned}$$

Third term: For $\wp\in\Sigma$ and $i\geq 0$ we have

$$H.^i(o_{\infty,\wp},\mathcal{C}) = H.^i(o_{\infty,\wp},\mathcal{C}^{et}_\wp) = H.^i_{et}(o_{\wp,\infty},\mathcal{C}^{et}_\wp) \quad ;$$

here the first equality follows from the vanishing of $H.^i(o_{\infty,\wp},\mathcal{C}^0_\wp)$ which was established in [7] (Cor. of Th. 3) whereas, for the second equality, we again use [4] III.3.9. Applying Proposition 2 to the right hand groups we see that our third term is equal to

$$\sum_{\wp\in\Sigma}[k_\wp : \mathbb{Q}_p]\cdot v(\text{rank } \mathcal{C}_\wp^{et}) =$$
$$\sum_{\wp\in\Sigma}[k_\wp : \mathbb{Q}_p]\cdot (v(\text{rank } \mathcal{C}_{/k}) - v(\text{rank } \mathcal{C}_\wp^0)) \quad .$$

We come back to the situation of §2. Let $f : A \to B$ be an isogeny of degree p^δ between abelian varieties A and B over k which have good, resp. good ordinary, reduction at the primes above p, resp. in Σ. Proposition 4 together with the Lemma in §2 applied to $\mathcal{C} := \ker(f_{/o})$ then give our main result.

Isogeny formula (First form):

Provided (II) holds true for A (or B) we have

$$\mu(B) - \mu(A) = \sum_{\wp|\infty} v(\#\mathcal{C}(k_\wp)) - v(\# s^*\Omega^1_{\mathcal{C}}) \quad .$$

Proof: Our assumption about ordinary reduction implies that $\mathcal{C}_\wp^0$, for $\wp\in\Sigma$, is of multiplicative type. In that case one has

$$\# s^*\Omega^1_{\mathcal{C}_\wp} = \# s^*\Omega^1_{\mathcal{C}_\wp^0} = (\text{rank } \mathcal{C}_\wp^0)^{[k_\wp:\mathbb{Q}_p]} \quad .$$

It is interesting to note the remarkable similarity between the above formula and Faltings' formula for the behaviour of the modular height under isogenies ([1] §4 Lemma 5). Is there a deeper connection between the μ-invariant $\mu(A)$ and differentials?

Isogeny formula (Second form):

Let k_∞/k be the cyclotomic $\mathbb{Z}_p$-extension. Provided Mazur's conjecture holds true for A (or B) we have

$$\mu(B) - \mu(A) = \sum_{\wp|\infty} v(\#\mathcal{C}(k_\wp)) - \sum_{\wp|p}[k_\wp : \mathbb{Q}_p]\cdot v(\text{rank } \mathcal{C}_\wp^0) \quad .$$

Proof: Note that $\Sigma = \{\wp \mid p\}$ for the cyclotomic $\mathbb{Z}_p$-extension.

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