A functor from smooth o-torsion representations to (φ, Γ)-modules

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Introduction

Let $\mathcal{G}_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ denote the absolute Galois group of the field $\mathbb{Q}_p$ of p-adic numbers. If ℓ is any prime number different from p then the local Langlands philosophy predicts a close relation between finite dimensional ℓ-adic representations of $\mathcal{G}_{\mathbb{Q}_p}$ and the (infinite dimensional) smooth representation theory in characteristic zero of reductive groups over $\mathbb{Q}_p$. In recent years it has become increasingly clear that most probably some kind of extension of this correspondence to the case $\ell = p$ exists. On the Galois side the theory of p-adic representations of $\mathcal{G}_{\mathbb{Q}_p}$ has reached maturity through the work of Colmez, Faltings, Fontaine, and others. On the reductive group side the foundations of a theory of continuous representations in p-adic locally convex vector spaces have been laid in the work of Schneider/Teitelbaum and others. Nevertheless a precise conjectural framework how these two sides correspond to each other is still missing.

It is all the more remarkable that despite a missing general picture Colmez recently has managed to establish such a correspondence for the group $GL_2(\mathbb{Q}_p)$. His starting point is Fontaine's theorem ([Fon]) that the category of p-adic Galois representations is naturally equivalent to the category of (finitely generated) etale (φ, Γ)-modules. Moreover, this equivalence arises from an analogous equivalence over any of the finite rings $\mathbb{Z}/p^m\mathbb{Z}$. The concept of a (φ, Γ)-module is a purely (semi)linear algebra notion. It is best viewed as being part of the module theory of the multiplicative monoid of p-adic integers $\mathbb{Z}_p^\bullet := \mathbb{Z}_p \setminus \{0\}$ over a certain big coefficient ring $\Lambda_F(\mathbb{Z}_p)$. This monoid can be identified with the submonoid in $GL_2(\mathbb{Q}_p)$ of dominant diagonal matrices (modulo the center). In addition, the coefficient ring $\Lambda_F(\mathbb{Z}_p)$ can be understood in terms of the unipotent radical of the standard Borel subgroup P in $GL_2(\mathbb{Q}_p)$. In fact, Colmez establishes in [Co1] and [Co2] a functorial relationship between smooth torsion P-representations and etale (φ, Γ)-modules.

This paper constitutes an attempt to understand the smooth representation theory with $\mathbb{Z}_p$-torsion coefficients of a Borel subgroup in a general reductive group in terms of new objects which we call generalized (φ, Γ)-modules. We place ourselves in the context of a $\mathbb{Q}_p$-split connected reductive group over $\mathbb{Q}_p$ (whose center we usually assume, for technical simplicity, to be connected), and we fix a Borel subgroup $P = TN$ of its group G of $\mathbb{Q}_p$-rational points with split torus T and unipotent radical N. We also fix an appropriate compact open subgroup $N_0 \subseteq N$ which in turn gives rise to the "dominant" submonoid $T_+ := \{t \in T : tN_0t^{-1} \subseteq N_0\}$ in T. On the one side we consider the abelian category $\mathcal{M}_{o-tor}(P)$ of all smooth P-representations in o-torsion modules where o is the ring of integers in a fixed finite extension $K/\mathbb{Q}_p$. On the other side we note that the monoid T_+ acts, by functoriality, on the completed group ring $\Lambda(N_0) := o[[N_0]]$. A first step towards generalizing (φ, Γ)-modules would be to consider $\Lambda(N_0)$-modules with an additional semilinear T_+-action. But technically it is preferable to introduce the monoid $P_+ := N_0 T_+$, a corresponding monoid ring $\Lambda(P_+)$ which is an overring of $\Lambda(N_0)$, and to work with the category $\mathcal{M}(\Lambda(P_+))$ of all (left unital) $\Lambda(P_+)$-modules instead. Such a module M will be called etale if every $t \in T_+$ acts, informally speaking, with slope zero on M. We show in section 1 that the etale $\Lambda(P_+)$-modules form an abelian category $\mathcal{M}_{et}(\Lambda(P_+))$.

In sections 2 to 4 we construct a universal δ-functor $V \longmapsto D^i(V)$ for $i \geq 0$ from the category $\mathcal{M}_{o-tor}(P)$ into the category $\mathcal{M}_{et}(\Lambda(P_+))$. The idea is to consider inside a representation V in $\mathcal{M}_{o-tor}(P)$ all P_+-subrepresentations which still generate V, to pass to their Pontrjagin duals as modules over $\Lambda(N_0)$, and to form the inductive limit denoted by $D(V)$. But the result is a $\Lambda(N_0)$-module which carries an additional action of the inverse monoid T_+^{-1} and not the monoid T_+ which we want. Somewhat as a miracle it turns out (section 3) that for any compactly induced V this T_+^{-1}-action has a natural right inverse T_+-action which does lead to a $\Lambda(P_+)$-module structure on $D(V)$. In section 4 we then use a functorial resolution by compactly induced representations to produce our δ-functor $D^i(V)$.

With the help of a Whittaker type functional ℓ on N we pass in section 5 from the category $\mathcal{M}_{et}(\Lambda(P_+))$ to the category $\mathcal{M}_{et}(\Lambda(S_\star))$ for the "standard" monoid $S_\star$ in $GL_2(\mathbb{Q}_p)$. We point out that our standard monoid $S_\star$ corresponds to Fontaine's original definition of (φ, Γ)-modules in which $\Gamma \cong \mathbb{Z}_p$. Colmez instead works mostly with a $\Gamma \cong \mathbb{Z}_p^\times$. Both points of view are equivalent; but the former, in our context, has less technical complications.

In section 6 we construct, in a totally elementary way, a functor from the category $\mathcal{M}_{et}(\Lambda(P_+))$ back into the category of all P-representations. Its precise relationship to the functor $D^0(V)$ remains unknown at this point.

In section 7 we show that our δ-functors are independent, up to a natural isomorphism, of the choices made for N_0 and ℓ.

An object in the category $\mathcal{M}_{et}(\Lambda(S_\star))$, even if finitely generated, is not yet a (φ, Γ)-module in the sense of Fontaine. The reason is that the base ring for the latter is not $\Lambda(\mathbb{Z}_p)$ but a certain p-adically completed localization $\Lambda_F(\mathbb{Z}_p)$ of $\Lambda(\mathbb{Z}_p)$. In section 8 we therefore take up the technically rather involved task to construct in a general framework a topological localization which when applied in section 9 to the ring $\Lambda(P_+)$ will lead to a ring $\Lambda_\ell(P_\star)$ and the corresponding abelian category $\mathcal{M}_{et}(\Lambda_\ell(P_\star))$ of etale $\Lambda_\ell(P_\star)$-modules which we view as a generalization of Fontaine's etale (φ, Γ)-modules. This construction relies in a crucial way on the interpretation in [SV2] of certain microlocalized completed group rings as skew Laurent series rings. By base extension our δ-functor $D^i(V)$ gives rise to a δ-functor $D^i_\ell(V)$ into the category $\mathcal{M}_{et}(\Lambda_\ell(P_\star))$. In section 10 we show that our specialization technique along ℓ extends to the ring $\Lambda_\ell(P_\star)$ finally leading to a δ-functor $D^i_{\Lambda_F(S_\star)}(V)$ into the category of not necessarily finitely generated (φ, Γ)-modules à la Fontaine. The question of finite generation remains the fundamental open question in this paper. We give an initial sufficient criterion, though, in Remark 11.4. We expect that for V in a suitable category of smooth o-torsion representations of G the etale (φ, Γ)-modules $D^i_{\Lambda_F(S_\star)}(V)$ indeed are finitely generated and therefore correspond to p-adic Galois representations.

As explained in section 11 Colmez' functor for the group $G = GL_2(\mathbb{Q}_p)$ (originally defined on the smooth o-torsion representations of G which are admissible, of finite length, and have a central character) coincides with our functor $D^0_{\Lambda_F(S_\star)}(V)$.

In the final section 12 we discuss the example of principal series representations.

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Contents

0 Notations and conventions

We denote by $|\ |$ the absolute value of the field $\overline{\mathbb{Q}}_p$. Let G be the group of $\mathbb{Q}_p$-rational points of a $\mathbb{Q}_p$-split connected reductive group over $\mathbb{Q}_p$. We fix a Borel subgroup $P = TN$ in G with maximal split torus T and unipotent radical N. Let Φ^+ denote, as usual, the set of roots of T positive with respect to P and let $\Delta \subseteq \Phi^+$ be the subset of simple roots. For any $\alpha \in \Phi^+$ we have the root subgroup $N_\alpha \subseteq N$. We recall that $N = \prod_{\alpha \in \Phi^+} N_\alpha$ (set-theoretically) for any total ordering of Φ^+. Let $T_0 \subseteq T$ be the maximal compact subgroup. We fix a compact open subgroup $N_0 \subseteq N$ which is totally decomposed, i.e., such that $N_0 = \prod_\alpha (N_0 \cap N_\alpha)$ for any total ordering of Φ^+. Then $P_0 := T_0 N_0$ is a group. We introduce the submonoid $T_+ \subseteq T$ of all $t \in T$ such that $t N_0 t^{-1} \subseteq N_0$, or equivalently, such that $|\alpha(t)| \leq 1$ for any $\alpha \in \Delta$. Obviously

$$P_+ := N_0 T_+ = P_0 T_+ P_0$$

then is a submonoid of P.

We also fix a finite extension $K/\mathbb{Q}_p$ with ring of integers o, prime element π, and residue class field k. By a representation we always will mean a linear action of the respective group (or monoid) in a torsion o-module V. It is called smooth if the stabilizer of each element in V is open in the group. It is called finitely generated if there are finitely many elements in V such that the smallest submodule of V which contains all translates by the group (or monoid) of these elements is V itself.

We recall that Pontrjagin duality $V \longmapsto V^* := \mathrm{Hom}_o(V, K/o)$ sets up an anti-equivalence between the category of all torsion o-modules and the category of all compact linear-topological o-modules. In particular, the functor $V \longmapsto V^*$ is exact on torsion o-modules.

For any compact open subgroup $G_0 \subseteq G$ let $\Lambda(G_0) := o[[G_0]]$, resp. $\Omega(G_0) := k[[G_0]] = \Lambda(G_0)/\pi\Lambda(G_0)$, denote the completed group ring of the profinite group G_0 over o, resp. over k. Any smooth G_0-representation V is the filtered union of its finite subrepresentations. Its Pontrjagin dual V^* therefore is a (compact) module over $\Lambda(G_0)$ (always considered as a left $\Lambda(G_0)$-module through the inversion map on G_0).

1 $\Lambda(P_+)$-modules

The monoid P_+ acts by conjugation upon itself. Let $\mathcal{N}(P_+)$ denote the set of all "open left-normal subgroups" of P_+, i. e., all open normal subgroups $Q \subseteq P_0$ which satisfy $bQb^{-1} \subseteq Q$ for any $b \in P_+$. This is a fundamental system of open neighbourhoods of the unit element in P_+. For each $Q \in \mathcal{N}(P_+)$ we may form the factor monoid $Q\backslash P_+$ as well as the corresponding monoid ring $o[Q\backslash P_+]$. The conjugation action of P_+ passes to $Q\backslash P_+$ and hence to an action on the ring $o[Q\backslash P_+]$. In the projective limit we obtain the unital o-algebra

$$\widetilde{\Lambda}(P_+) := \varprojlim_Q o[Q\backslash P_+]$$

together with an action of P_+ on it. The obvious map

$$\Lambda(P_0) \otimes_{o[P_0]} o[P_+] \longrightarrow \widetilde{\Lambda}(P_+)$$

is injective and its image is a subring $\Lambda(P_+)$ of $\widetilde{\Lambda}(P_+)$ which is invariant under the P_+-action. We will write $\phi_b : \Lambda(P_+) \longrightarrow \Lambda(P_+)$ for the ring endomorphism given by the action of the element $b \in P_+$. If $b \in P_0$ then we obviously have $\phi_b(\lambda) = b\lambda b^{-1}$ for any $\lambda \in \Lambda(P_+)$. For any ϕ_b-invariant subring $\Lambda \subseteq \Lambda(P_+)$ we denote by $\Lambda \otimes_{\Lambda,b} -$ the base change functor for left Λ-modules along the ring homomorphism ϕ_b. Let $\Theta \subseteq T_+$ be a subset of representatives for the cosets in T_+/T_0. Then, as a left $\Lambda(P_0)$-module, we have

$$(1) \qquad \Lambda(P_+) = \bigoplus_{t' \in \Theta} \Lambda(P_0) t' \ .$$

Lemma 1.1. *For any $b = n_0 t \in P_+ = N_0 T_+$ the ring endomorphism ϕ_b is injective and makes $\Lambda(P_+)$ a free right module of rank $[N_0 : bN_0 b^{-1}] = [N_0 : tN_0 t^{-1}]$ over itself; the map*

$$\begin{aligned} \Lambda(N_0) \otimes_{\Lambda(N_0),b} \Lambda(P_+) &\xrightarrow{\cong} \Lambda(P_+) \\ \nu \otimes \lambda &\longmapsto \nu \phi_b(\lambda) \end{aligned}$$

is an isomorphism.

Proof. We have $tP_0 t^{-1} = tN_0 t^{-1} T_0$ and hence the disjoint decomposition

$$P_0 = \bigcup_{n \in N_0/tN_0 t^{-1}} n(tP_0 t^{-1}) \ .$$

Conjugation by n_0 gives

$$P_0 = n_0 P_0 n_0^{-1} = \bigcup_{n \in N_0/tN_0 t^{-1}} n_0 n n_0^{-1}(bP_0 b^{-1}) = \bigcup_{n \in N_0/bN_0 b^{-1}} n(bP_0 b^{-1}) \ .$$

It follows that

$$\begin{aligned} \Lambda(P_+) = n_0 \Lambda(P_+) n_0^{-1} &= \bigoplus_{t' \in \Theta} \Lambda(P_0) n_0 t' n_0^{-1} \\ &= \bigoplus_{t' \in \Theta} \bigoplus_{n \in N_0/bN_0 b^{-1}} n\Lambda(bP_0 b^{-1}) bt' b^{-1} \\ &= \bigoplus_{n \in N_0/bN_0 b^{-1}} n \operatorname{im}(\phi_b) \ . \end{aligned}$$

□

Let $\mathcal{M}(\Lambda(P_+))$ be the abelian category of all left (unital) $\Lambda(P_+)$-modules and $D(\Lambda(P_+))$ the corresponding derived category.

Definition 1.2. *A left unital $\Lambda(P_+)$-module M is called etale if, for any $b \in P_+$, the $\Lambda(P_+)$-linear map*

$$\begin{aligned} \Lambda(P_+) \otimes_{\Lambda(P_+),b} M &\xrightarrow{\cong} M \\ \lambda \otimes x &\longmapsto \lambda b x \end{aligned}$$

is bijective.

Obviously the condition in the above definition is automatically satisfied for any $b \in P_0$ and therefore needs to be checked only for every $t' \in \Theta$. In fact, because of Lemma 1.1 a $\Lambda(P_+)$-module M is etale if and only if

$$\begin{aligned} \Lambda(N_0) \otimes_{\Lambda(N_0),t'} M &\xrightarrow{\cong} M \\ \lambda \otimes x &\longmapsto \lambda t' x \end{aligned}$$

is bijective for any $t' \in \Theta$. Let $\mathcal{M}_{et}(\Lambda(P_+))$ denote the full subcategory in $\mathcal{M}(\Lambda(P_+))$ of all etale $\Lambda(P_+)$-modules.

Proposition 1.3. *The subcategory $\mathcal{M}_{et}(\Lambda(P_+))$ of $\mathcal{M}(\Lambda(P_+))$ is closed under the formation of kernels, cokernels, extensions and arbitrary inductive and projective limits; in particular, $\mathcal{M}_{et}(\Lambda(P_+))$ is an abelian category.*

Proof. This is a straightforward consequence of Lemma 1.1. □

In general the subcategory $\mathcal{M}_{et}(\Lambda(P_+))$ of $\mathcal{M}(\Lambda(P_+))$ is not closed under the passage to submodules.

It also follows that the full subcategory $D_{et}(\Lambda(P_+))$ of all those complexes in $D(\Lambda(P_+))$ whose cohomology modules are etale is a triangulated subcategory.

In a completely symmetric way we have the ring $\widetilde{\Lambda}(P_+^{-1})$ for the monoid P_+^{-1}. The map $b \longmapsto b^{-1}$ induces an anti-isomorphism of rings $\widetilde{\Lambda}(P_+) \xrightarrow{\cong} \widetilde{\Lambda}(P_+^{-1})$.

We also will need the following straightforward variants of everything above. Let $T_\star \subseteq T_+$ be any submonoid and put $P_\star := N_0 T_\star$. We then have the subring

$$\Lambda(P_\star) := \Lambda(P_\star \cap P_0) \otimes_{o[P_\star \cap P_0]} o[P_\star] \subseteq \Lambda(P_+)$$

as well as the abelian categories $\mathcal{M}(\Lambda(P_\star))$ and $\mathcal{M}_{et}(\Lambda(P_\star))$ together with the forgetful functors

$$\mathcal{M}(\Lambda(P_+)) \longrightarrow \mathcal{M}(\Lambda(P_\star)) \qquad \text{and} \qquad \mathcal{M}_{et}(\Lambda(P_+)) \longrightarrow \mathcal{M}_{et}(\Lambda(P_\star)) \ .$$

For the latter observe that by Lemma 1.1 the map

$$\begin{aligned} \Lambda(P_\star) \otimes_{\Lambda(P_\star),b} \Lambda(P_+) &\xrightarrow{\cong} \Lambda(P_+) \\ \mu \otimes \lambda &\longmapsto \mu\phi_b(\lambda) \ , \end{aligned}$$

for any $b \in P_\star$, is an isomorphism.

Of particular interest is the case of the group $GL_2(\mathbb{Q}_p)$ and its Borel subgroup of lower triangular matrices $P_2(\mathbb{Q}_p)$. The "standard monoid" in $P_2(\mathbb{Q}_p)_+$ is

$$S_\star := \left\{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} : a \in \mathbb{Z}_p, b \in (1 + p^{\epsilon(p)}\mathbb{Z}_p)p^{\mathbb{N}_0} \right\}.$$

with $\epsilon(2) := 2$ and $\epsilon(p) := 1$ for odd p. In $S_\star$ we have the subgroups

$$S_0 := \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} : a \in \mathbb{Z}_p \right\} \qquad \text{and} \qquad \Gamma := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} : b \in 1 + p^{\epsilon(p)}\mathbb{Z}_p \right\} \cong \mathbb{Z}_p$$

and the element $\varphi := \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. The ring $\Lambda(S_\star) = \Lambda(S_0\Gamma)[\varphi; \phi_\varphi]$ is a skew polynomial ring over $\Lambda(S_0\Gamma)$. Later on we will see that these data are very closely related to Fontaine's notion of a (φ, Γ)-module.

2 P_+-subrepresentations

Fix a smooth P-representation V. An o-submodule $M \subseteq V$ will be called *generating* if $V = PM$.

Lemma 2.1. *For any P_+-subrepresentation M of V the following assertions are equivalent:*

i. M is generating;

ii. for any $v \in V$ there is a $t \in T_+$ such that $tv \in M$.

Proof. It is trivial that ii. implies i. Let M therefore be generating and let $v \in V$ be any vector. We then find elements $n_1, \ldots, n_r \in N$, $t_1, \ldots, t_r \in T$, and $v_1, \ldots, v_r \in M$ such that

$$v = \sum_{i=1}^{r} n_i t_i v_i \ .$$

We choose a $t \in T_+$ such that

$$t n_i t_i = (t n_i t^{-1})(t t_i) \in N_0 T_+$$

for any $1 \leq i \leq r$. We obtain $tv \in M$. □

Lemma 2.2. *Let $M_0, M_1 \subseteq V$ be two generating P_+-subrepresentations; then $M_0 \cap M_1$ is a generating P_+-subrepresentation as well.*

Proof. Clearly $M_0 \cap M_1$ is P_+-invariant. We check the assertion ii. in Lemma 2.1. Let $v \in V$ be any vector. Since M_i is generating we find a $t_i \in T_+$ such that $t_i v \in M_i$. Then $t_1 t_2 v = t_2 t_1 v \in M_1 \cap M_2$. □

We let $\mathcal{P}_+(V)$ denote the set of all generating P_+-subrepresentations of V. The above lemma says that the set $\mathcal{P}_+(V)$ is decreasingly filtered with respect to the partial order given by inclusion.

Lemma 2.3. *For any map $f : V_1 \longrightarrow V_2$ of smooth P-representations and any $M \in \mathcal{P}_+(V_2)$ we have $f^{-1}(M) \in \mathcal{P}_+(V_1)$.*

Proof. Obviously $f^{-1}(M)$ is P_+-invariant. Using again Lemma 2.1 let $v \in V_1$ be any vector. Since M is generating we find a $t \in T_+$ such that $tf(v) = f(tv) \in M$. Hence $tv \in f^{-1}(M)$. □

From $\mathcal{P}_+(V)$ we obtain by dualizing the filtered inductive system $\{M^*\}_{M \in \mathcal{P}_+(V)}$ of (left) $\widetilde{\Lambda}(P_+^{-1})$-modules. We define

$$D(V) := \varinjlim_{M \in \mathcal{P}_+(V)} M^*$$

as a $\widetilde{\Lambda}(P_+^{-1})$-module. We note that $D(V)$ actually is a quotient of V^* because the restriction maps in the filtered inductive system are surjective:

$$D(V) = V^*/\{x \in V^* : x|M = 0 \text{ for some } M \in \mathcal{P}_+(V)\}.$$

It follows from Lemma 2.3 that $D(V)$ is contravariantly functorial in V. Moreover, since $g^{-1}(M) \cap f^{-1}(M) \subseteq (g+f)^{-1}(M)$ for any two maps $g, f : V_1 \longrightarrow V_2$ this functor D is additive.

Remark 2.4. *Let $f : V_1 \longrightarrow V_2$ be a map of smooth P-representations; we have:*

i. If f is injective then $D(f)$ is surjective;

ii. if f is surjective then $D(f)$ is injective.

Proof. i. In fact in the commutative diagram of restriction maps

$$\begin{array}{ccc} V_2^* & \xrightarrow{f^*} & V_1^* \\ \downarrow & & \downarrow \\ D(V_2) & \xrightarrow{D(f)} & D(V_1) \end{array}$$

all four maps are surjective.

ii. This is immediate from the fact that for any $M \in \mathcal{P}_+(V_1)$ we have, by the surjectivity of the map f, that $f(M) \in \mathcal{P}_+(V_2)$. $\square$

Lemma 2.5. *Let* $0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$ *be a short exact sequence of smooth* P*-representations. Suppose that the following holds:*

(2) *For any* $M_1 \in \mathcal{P}_+(V_1)$ *we find* $M_2 \in \mathcal{P}_+(V_2)$ *such that* $M_2 \cap V_1 \subseteq M_1$.

Then the sequence $0 \longrightarrow D(V_3) \longrightarrow D(V_2) \longrightarrow D(V_1) \longrightarrow 0$ *is exact.*

Proof. Straightforward. $\square$

Example: (The Steinberg representation)
The group $P = NT$ acts on N by $(nt)(n') := ntn't^{-1}$. Consider the induced action of P on the vector space $V_{St} := C_c^\infty(N)$ of k-valued locally constant functions with compact support on N. It is straightforward to see that the subspace $C^\infty(N_0)$ of locally constant functions on N_0 is generating and P_+-invariant (cf. [Vi2] Lemme 4).

Lemma 2.6. *Any generating* P_+*-subrepresentation* $M \subseteq V_{St}$ *contains* $C^\infty(N_0)$.

Proof. By Lemma 2.1 we find a $t \in T_+$ such that $t\,\mathrm{char}_{N_0} = \mathrm{char}_{tN_0t^{-1}} \in M$ where $\mathrm{char}_?$ denotes the characteristic function of a compact open subgroup $? \subseteq N$. But $tN_0t^{-1} \subseteq N_0$ easily implies that $C^\infty(N_0) = N_0T_+ \cdot k\,\mathrm{char}_{tN_0t^{-1}}$. $\square$

It follows that $D(V_{St}) = \Lambda(N_0)/\pi\Lambda(N_0)$.

3 The case of a compactly induced representation

We now analyze in detail the following kind of P-representations. Let V_0 be a smooth P_0-representation and form the compact induction $V := \mathrm{ind}_{P_0}^P(V_0)$. This is the o-module

$$\mathrm{ind}_{P_0}^P(V_0) := \text{all compactly supported functions } \psi : P \longrightarrow V_0 \text{ such that}$$
$$\psi(bb_0) = b_0^{-1}\psi(b) \quad \text{for any } b \in P, b_0 \in P_0$$

with the group P acting by left translations. As a piece of notation we denote, for any right P_0-invariant subset $X \subseteq P$, by $\mathrm{ind}_{P_0}^X(V_0)$ the o-submodule in V of all those functions with support in X. Clearly the map

$$\begin{aligned} \mathrm{ind}_{P_0}^{P_0}(V_0) &\xrightarrow{\cong} V_0 \\ \psi &\longmapsto \psi(1) \end{aligned}$$

is a P_0-equivariant isomorphism. By abuse of notation, we let any $v \in V_0$ denote at the same time the function in the left hand side corresponding to it. For any $s \in T_+$ we have the subset $P_+s = N_0T_+sP_0$ in P so that we may introduce the P_+-subrepresentation

$$M_s := M_s(V_0) := \mathrm{ind}_{P_0}^{P_+s}(V_0)$$

of V. Containing sV_0 any M_s generates V as a P-representation. Hence $M_s \in \mathcal{P}_+(V)$. If V_0 is finite then for the same reason M_s is finitely generated as a P_+-representation. We have $M_{s'} \subseteq M_s$ whenever $s' \in T_+s$. Hence these subrepresentations are decreasingly filtered.

We also define, for each $t \in T_+$, the P_0-subrepresentation

$$M(t) := M(t)(V_0) := \mathrm{ind}_{P_0}^{N_0tP_0}(V_0) \ .$$

Since P_+ is the disjoint union

$$P_+ = N_0T_+P_0 = \bigcup_{t \in T_+/T_0} N_0tP_0 \ .$$

we have the P_0-invariant decomposition

$$M_s = \bigoplus_{t \in (T_+s)/T_0} M(t) \ .$$

Lemma 3.1. *Suppose that V_0 is finite; for any generating P_+-subrepresentation $M \subseteq V$ there is an $s \in T_+$ such that $M_s \subseteq M$.*

Proof. Since M is generating we find finitely many elements $b_1, \ldots, b_r \in P$ such that

$$V_0 \subseteq \sum_{i=1}^{r} b_iM \ .$$

Choose an $s \in T_+$ such that $sb_i \in P_+$ for any $1 \leq i \leq r$. Then

$$M_s = P_+sV_0 \subseteq \sum_{i=1}^{r} P_+sb_iM \subseteq M \ .$$

□

For general V_0 we introduce the set $\mathrm{Sub}(V_0)$, resp. $\mathrm{Fin}(V_0)$, of all, resp. all finite, P_0-subrepresentations of V_0. Both are partially ordered by inclusion. On the other hand, the monoid T_+/T_0 is preordered by $t'T_0 \leq tT_0$ if $t' \in T_+t$. Let $\sigma : T_+/T_0 \longrightarrow \mathrm{Sub}(V_0)$ be any order reversing map which satisfies

$$\bigcup_{t \in T_+/T_0} \sigma(t) = V_0 \ . \tag{3}$$

We form the subspace

$$M_\sigma := M_\sigma(V_0) := \bigoplus_{t \in T_+/T_0} M(t)(\sigma(t))$$

of $V = \mathrm{ind}_{P_0}^{P}(V_0)$.

Lemma 3.2. *i.* $M_\sigma \in \mathcal{P}_+(V)$.

ii. Given any $M \in \mathcal{P}_+(V)$ *there is a* σ *as above such that* $M_\sigma \subseteq M$.

Proof. i. Given any U_0 in $\mathrm{Sub}(V_0)$ the subspace $M(t)(U_0)$ is P_0-invariant and satisfies

$$t' M(t)(U_0) \subseteq M(t't)(U_0) \qquad \text{for any } t' \in T_+ \, .$$

It follows that

$$t'\big(M(t)(\sigma(t))\big) \subseteq M(t't)(\sigma(t)) \subseteq M(t't)(\sigma(t't))$$

since σ is order reversing. Hence M_σ is P_+-invariant. Again since σ is order reversing we have

$$M_\sigma \supseteq M_s(\sigma(s)) \qquad \text{for any } s \in T_+ \, .$$

It follows that $PM_\sigma \supseteq \mathrm{ind}_{P_0}^P(\sigma(s))$ for any $s \in T_+$ and hence, as a consequence of the condition (3), that

$$PM_\sigma \supseteq \sum_{s\in T_+} \mathrm{ind}_{P_0}^P(\sigma(s)) = \mathrm{ind}_{P_0}^P(\sum_{s\in T_+} \sigma(s)) = \mathrm{ind}_{P_0}^P(V_0) \ .$$

ii. It follows from Lemma 2.3 that, for any $U_0 \in \mathrm{Fin}(V_0)$, the P_+-subrepresentation $M \cap \mathrm{ind}_{P_0}^P(U_0)$ is generating in $\mathrm{ind}_{P_0}^P(U_0)$. Using Lemma 3.1 we find an $s(U_0) \in T_+$ such that $M_{s(U_0)}(U_0) \subseteq M$. Hence M contains

$$\begin{aligned}
\sum_{U_0\in \mathrm{Fin}(V_0)} M_{s(U_0)}(U_0) &= \sum_{U_0\in \mathrm{Fin}(V_0)} \bigoplus_{t\in T_+ s(U_0)/T_0} M(t)(U_0) \\
&= \bigoplus_{t\in T_+/T_0} \sum_{\substack{U_0\in \mathrm{Fin}(V_0)\\ t\in T_+ s(U_0)}} M(t)(U_0) \\
&= \bigoplus_{t\in T_+/T_0} M(t)\Big(\sum_{\substack{U_0\in \mathrm{Fin}(V_0)\\ t\in T_+ s(U_0)}} U_0\Big) \ .
\end{aligned}$$

We therefore set

$$\sigma(t) := \sum_{U_0\in \mathrm{Fin}(V_0), t\in T_+ s(U_0)} U_0 \ .$$

The union of all $\sigma(t)$ contains the union of all $U_0 \in \mathrm{Fin}(V_0)$ and consequently is equal to V_0. □

The dual of $\mathrm{ind}_{P_0}^P(V_0)$ can be described explicitly as follows. Let

$$\begin{gathered}
\mathrm{Ind}_{P_0}^P(V_0^*) := \text{all functions } \Phi : P \longrightarrow V_0^* \text{ such that} \\
\Phi(bb_0) = b_0^{-1}\Phi(b) \\
\text{for any } b \in P, b_0 \in P_0
\end{gathered}$$

with the group P acting by left translations. The map

$$\begin{aligned}
\mathrm{Ind}_{P_0}^P(V_0^*) &\xrightarrow{\cong} \mathrm{ind}_{P_0}^P(V_0)^* \\
\Phi &\longmapsto \big[\psi \mapsto \sum_{b\in P/P_0} \Phi(b)(\psi(b))\big]
\end{aligned}$$

is a P-equivariant o-linear isomorphism. Under this isomorphism the orthogonal complement $M_\sigma^\perp$ of M_σ in $\mathrm{ind}_{P_0}^P(V_0)^*$ corresponds to the P_+^{-1}-invariant subspace

$$J_\sigma := J_\sigma(V_0) := \{\Phi \in \mathrm{Ind}_{P_0}^P(V_0^*) : \Phi(N_0 t) \subseteq \sigma(t)^\perp \text{ for any } t \in T_+\}$$

of $\mathrm{Ind}_{P_0}^P(V_0^*)$. We therefore obtain a natural isomorphism

$$D(\mathrm{ind}_{P_0}^P(V_0)) \cong \mathrm{Ind}_{P_0}^P(V_0^*)/\bigcup_\sigma J_\sigma$$

of $\widetilde{\Lambda}(P_+^{-1})$-modules. On the other hand there is the P_0-invariant decomposition

$$\mathrm{Ind}_{P_0}^P(V_0^*) = J^+ \oplus J^-$$

with

$$J^\pm := J^\pm(V_0) := \{\Phi \in \mathrm{Ind}_{P_0}^P(V_0^*) : \Phi|\Big(\begin{cases} P \setminus P_+ \\ P_+ \end{cases}\Big) = 0\} \ .$$

We write $\Phi = \Phi^+ + \Phi^-$ for the corresponding decomposition of any function $\Phi \in \mathrm{Ind}_{P_0}^P(V_0^*)$ into functions $\Phi^\pm \in J^\pm$. The subspace J^+ in $\mathrm{Ind}_{P_0}^P(V_0^*)$ is P_+-invariant.

Since obviously $J^- \subseteq J_\sigma$ for any σ the natural map

$$J^+ \longrightarrow \mathrm{Ind}_{P_0}^P(V_0^*)/\bigcup_\sigma J_\sigma \tag{4}$$

is surjective. For any order reversing map $\sigma : T_+/T_0 \longrightarrow \mathrm{Sub}(V_0)$ satisfying (3) and any $s \in T_+$ we define the new map

$$\begin{aligned} \sigma_s : T_+/T_0 &\longrightarrow \mathrm{Sub}(V_0) \\ t &\longmapsto \begin{cases} \sigma(s^{-1}t) & \text{if } t \in T_+ s, \\ \{0\} & \text{otherwise} \end{cases} \end{aligned}$$

which again is order reversing and satisfies (3). We have $J_\sigma \subseteq J_{\sigma_s}$.

Lemma 3.3. *i. For any σ satisfying (3) and any $s \in T_+$ we have $s(J^+ \cap J_\sigma) \subseteq J^+ \cap J_{\sigma_s}$.*

ii. $J^+ \cap \bigcup_\sigma J_\sigma$ is P_+-invariant.

Proof. i. Let $\Phi \in J^+ \cap J_\sigma$. We have to show that $s\Phi$ lies in J_{σ_s}. Let $n_0 \in N_0$ and $t \in T_+$. We have

$$(s\Phi)(n_0 t) = \Phi(s^{-1} n_0 t) = \Phi((s^{-1} n_0 s)(s^{-1} t)) \ .$$

If $s^{-1} n_0 t \notin P_+$ then

$$(s\Phi)(n_0 t) = \Phi(s^{-1} n_0 t) = 0 \in \sigma_s(t)^\perp \ .$$

If $s^{-1} n_0 t \in P_+$ then $s^{-1} n_0 s \in N_0$ and $s^{-1} t \in T_+$ so that

$$(s\Phi)(n_0 t) \in \Phi(N_0(s^{-1}t)) \subseteq \sigma(s^{-1}t)^\perp = \sigma_s(t)^\perp \ .$$

ii. This is an immediate consequence of i. □

By (4) we have the $\Lambda(P_0)$-equivariant isomorphism

$$J^+/J^+ \cap \bigcup_\sigma J_\sigma \xrightarrow{\cong} \mathrm{Ind}_{P_0}^P(V_0^*)/\bigcup_\sigma J_\sigma \cong D(\mathrm{ind}_{P_0}^P(V_0)) \ .$$

The right hand side in fact carries a $\widetilde{\Lambda}(P_+^{-1})$-action whereas the left hand side, by the lemma, has a $\Lambda(P_+)$-action. By transport of structure we view this latter action also on $D(\mathrm{ind}_{P_0}^P(V_0))$. If we represent an element $x \in D(\mathrm{ind}_{P_0}^P(V_0))$ by a function $\Phi \in J^+$ then $t\Phi \in J^+$ represents tx for any $t \in T_+$. Since $t^{-1}(t\Phi) = \Phi$ we see that, for any $t \in T_+$, the operator t^{-1} on $D(\mathrm{ind}_{P_0}^P(V_0))$ is a distinguished left inverse of the operator t.

Proposition 3.4. *For any map $f : \mathrm{ind}_{P_0}^P(U_0) \longrightarrow \mathrm{ind}_{P_0}^P(V_0)$ of compactly induced smooth P-representations the map $D(f) : D(\mathrm{ind}_{P_0}^P(V_0)) \longrightarrow D(\mathrm{ind}_{P_0}^P(U_0))$ is a map of $\Lambda(P_+)$-modules.*

Proof. By Lemma 2.3 and Lemma 3.2.ii we find an order reversing map $\sigma : T_+/T_0 \longrightarrow \mathrm{Sub}(U_0)$ satisfying (3) such that

$$f(M_\sigma(U_0)) \subseteq M_1(V_0) \ . \tag{5}$$

Dually we then have

$$f^*(J^-(V_0)) \subseteq J_\sigma(U_0) \ .$$

Therefore, if $s \in T_+$ is any element, all maps in the diagram

$$\begin{array}{ccc}
J^+(V_0) & \xrightarrow{\quad s \quad} & J^+(V_0) \\
\cong \downarrow & & \downarrow \cong \\
\mathrm{Ind}_{P_0}^P(V_0^*)/J^-(V_0) & & \mathrm{Ind}_{P_0}^P(V_0^*)/J^-(V_0) \\
f^* \downarrow & & \downarrow f^* \\
\mathrm{Ind}_{P_0}^P(U_0^*)/J_\sigma(U_0) & & \mathrm{Ind}_{P_0}^P(U_0^*)/J_{\sigma_s}(U_0) \\
\cong \uparrow & & \uparrow \cong \\
J^+(U_0)/J^+(U_0) \cap J_\sigma(U_0) & \xrightarrow{\ s\ } & J^+(U_0)/J^+(U_0) \cap J_{\sigma_s}(U_0)
\end{array}$$

are well defined. It suffices to show that this diagram is commutative. But to do this we first have to observe another consequence of (5). Suppose that a function $\psi \in \mathrm{ind}_{P_0}^P(U_0)$ is supported on $NT_+ \setminus P_+$ and such that $\psi(Nt') \subseteq \sigma(t')$ for any $t' \in T_+$. Let R be a set of representatives for the cosets in N/N_0 which contains the unit element 1. Because of the disjoint decomposition

$$NT_+ = \bigcup_{n \in R} nN_0T_+$$

we may write ψ as a finite sum

$$\psi = n_1\psi_{n_1} + \ldots + n_r\psi_{n_r} \qquad \text{with } n_1, \ldots, n_r \in R \setminus \{1\} \text{ and } \psi_{n_i} \in M_\sigma(U_0).$$

It follows from (5) that

$$f(\psi) = \sum_i n_i f(\psi_{n_i}) \in \sum_i n_i f(M_\sigma(U_0)) \subseteq \sum_i n_i M_1(V_0)$$

and hence is supported on $P \setminus P_+$. Passing to orthogonal complements we obtain that

$$(6) \qquad f^*(\Phi)((N \setminus N_0)t') \subseteq \sigma(t')^\perp \qquad \text{for any } t' \in T_+ \text{ and any } \Phi \in J^+(V_0).$$

To now check the commutativity of the diagram we let $\Phi \in J^+(V_0)$. Since $J^- \subseteq J_\sigma$ its image under the composed map in the left column is $f^*(\Phi)^+$. Hence we have to prove that

$$s(f^*(\Phi)^+) - f^*(s\Phi)^+ = s(f^*(\Phi)^+) - (sf^*(\Phi))^+ \in J_{\sigma_s}(U_0)$$

or, equivalently, that

$$s(f^*(\Phi)^+)(n_0 t) - (sf^*(\Phi))^+(n_0 t) \in \sigma(s^{-1}t)^\perp \qquad \text{for any } n_0 \in N_0 \text{ and } t \in T_+ s$$

holds true. If $t = t's$ we have

$$\begin{aligned} s(f^*(\Phi)^+)(n_0 t) - (sf^*(\Phi))^+(n_0 t) &= f^*(\Phi)^+(s^{-1}n_0 st') - f^*(\Phi)(s^{-1}n_0 st') \\ &= \begin{cases} -f^*(\Phi)(s^{-1}n_0 st') & \text{if } s^{-1}n_0 s \notin N_0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This reduces us to showing that

$$f^*(\Phi)(s^{-1}n_0 st') \in \sigma(t')^\perp \qquad \text{for any } n_0 \in N_0 \setminus sN_0 s^{-1} \text{ and } t' \in T_+$$

which we already did in (6). □

Proposition 3.5. *The $\Lambda(P_+)$-module $D(\mathrm{ind}_{P_0}^{P}(V_0))$ is etale.*

Proof. Similarly as with $\mathrm{ind}_{P_0}^{X}$ we will use $\mathrm{Ind}_{P_0}^{X}(.)$, for any right P_0-invariant subset $X \subseteq P$, to denote the submodule of all functions $\Phi \in \mathrm{Ind}_{P_0}^{P}(.)$ with support in X.

We have to show that, for any $s \in T_+$, the map

$$\begin{aligned} \Lambda(P_+) \otimes_{\Lambda(P_+),s} (J^+/J^+ \cap \bigcup_\sigma J_\sigma) &\xrightarrow{\cong} J^+/J^+ \cap \bigcup_\sigma J_\sigma \\ \lambda \otimes x &\longmapsto \lambda s x \end{aligned}$$

is bijective. Let $n_1, \ldots, n_r \in N_0$ be representatives for the cosets in N_0/sN_0s^{-1}. By Lemma 1.1 the above map may be viewed as the map

$$\begin{aligned} (J^+/J^+ \cap \bigcup_\sigma J_\sigma)^r &\longrightarrow J^+/J^+ \cap \bigcup_\sigma J_\sigma \\ (x_1, \ldots, x_r) &\longmapsto \sum_{i=1}^{r} n_i s x_i \ . \end{aligned}$$

Lemma 3.3.i then reduces us to showing that, for each σ, the map

$$\begin{aligned} (J^+/J^+ \cap J_\sigma)^r &\longrightarrow J^+/J^+ \cap J_{\sigma_s} \\ (x_1, \ldots, x_r) &\longmapsto \sum_{i=1}^{r} n_i s x_i \end{aligned}$$

is bijective. But we have the N_0-equivariant decompositions

$$J^+/J^+ \cap J_\sigma = \prod_{t \in T_+/T_0} \mathrm{Ind}_{P_0}^{N_0 t P_0}((V_0/\sigma(t))^*)$$

and

$$J^+/J^+ \cap J_{\sigma_s} = \prod_{t \in T_+/T_0} \mathrm{Ind}_{P_0}^{N_0 st P_0}((V_0/\sigma(t))^*)$$

which are respected by the action of s. It finally comes down therefore to the bijectivity of the map

$$\begin{aligned} (\mathrm{Ind}_{P_0}^{N_0 t P_0}(.))^r &\longrightarrow \mathrm{Ind}_{P_0}^{N_0 st P_0}(.) \\ (\Phi_1, \ldots, \Phi_r) &\longmapsto \sum_{i=1}^{r} n_i s \Phi_i \end{aligned}$$

which is straightforward from the disjoint union

$$N_0 st P_0 = \bigcup_{i=1}^{r} n_i (s N_0 s^{-1}) st P_0 = \bigcup_{i=1}^{r} n_i s N_0 t P_0 \ .$$

$\square$

Sometimes it is technically useful to notice that $M_1(V_0)^*$ is through transport of structure along the isomorphism

$$J^+(V_0) \xrightarrow{\cong} \mathrm{Ind}_{P_0}^{P}(V_0^*)/J^-(V_0) \cong M_1(V_0)^*$$

a $\Lambda(P_+)$-module. It is not etale, though, but of course also carries a $\widetilde{\Lambda}(P_+^{-1})$-action providing distinguished left inverses. The natural surjection $M_1(V_0)^* \twoheadrightarrow D(\mathrm{ind}_{P_0}^{P}(V_0))$ respects all these structures.

4 The basic cohomological functor

Let $\mathcal{M}_{o-tor}(P)$ denote the abelian category of all smooth P-representations. Any V in $\mathcal{M}_{o-tor}(P)$ comes with the canonical epimorphism

$$\begin{aligned} \rho_V : \mathrm{ind}_{P_0}^{P}(V) &\longrightarrow V \\ \psi &\longmapsto \sum_{b \in P/P_0} b\psi(b) \end{aligned}$$

in $\mathcal{M}_{o-tor}(P)$. This leads to the canonical resolution

$$\mathcal{I}_\bullet(V) : \ldots \longrightarrow \mathcal{I}_{n+1}(V) \xrightarrow{\rho_n} \mathcal{I}_n(V) \xrightarrow{\rho_{n-1}} \ldots \xrightarrow{\rho_0} \mathcal{I}_0(V)$$

in $\mathcal{M}_{o-tor}(P)$ inductively defined by $\mathcal{I}_0(V) := \mathrm{ind}_{P_0}^{P}(V)$, $\rho_{-1} := \rho_V$,

$$\mathcal{I}_{n+1}(V) := \mathrm{ind}_{P_0}^{P}(\ker \rho_{n-1}), \text{ and } \rho_n := \rho_{\ker \rho_{n-1}} \text{ for } n \geq 0$$

such that

$$\mathcal{I}_\bullet(V) \xrightarrow[\rho_V]{\simeq} V$$

is a quasi-isomorphism.

As usual, let $C^{\geq 0}(\mathcal{M}_{et}(\Lambda(P_+)))$ denote the category of cohomological complexes in non-negative degrees in $\mathcal{M}_{et}(\Lambda(P_+))$. We have the contravariant functor

$$\begin{aligned} RD : \mathcal{M}_{o-tor}(P) &\longrightarrow C^{\geq 0}(\mathcal{M}_{et}(\Lambda(P_+))) \\ V &\longmapsto D(\mathcal{I}_\bullet(V)) \ . \end{aligned}$$

We put

$$D^i(V) := h^i(D(\mathcal{I}_\bullet(V)))$$

for $i \geq 0$. By Remark 2.4.ii there is a natural injection $D(V) \hookrightarrow D^0(V)$ but which in general is not bijective.

Lemma 4.1. *The functor $V_0 \longmapsto D(\mathrm{ind}_{P_0}^P(V_0))$ on $\mathcal{M}_{o-tor}(P_0)$ is exact.*

Proof. Clearly the functor $V_0 \longmapsto \mathrm{ind}_{P_0}^P(V_0)$ is exact. It therefore is sufficient to check the hypothesis (2) in Lemma 2.5 for any inclusion $\mathrm{ind}_{P_0}^P(V_0') \subseteq \mathrm{ind}_{P_0}^P(V_0)$ coming from a pair $V_0' \subseteq V_0$ of smooth P_0-representations. In view of Lemma 3.2 we give ourselves an order reversing map $\sigma' : T_+/T_0 \longrightarrow \mathrm{Sub}(V_0')$ satisfying $\bigcup_{t \in T_+/T_0} \sigma'(t) = V_0'$. We also pick a strictly dominant element $s_0 \in T_+$, which means that $|\alpha(s_0)| < 1$ for any $\alpha \in \Delta$. We note that the subset $\{s_0^m : m \geq 0\}$ is cofinal in the preordered set T_+. Using Zorn's lemma we find inductively, for any $m \geq 0$, a subrepresentation $\sigma(s_0^m) \in \mathrm{Sub}(V_0)$ which is maximal with respect to the properties that

$$\sigma(s_0^m) \cap V_0' = \sigma'(s_0^m) \quad \text{and} \quad \sigma(s_0^m) \supseteq \sigma(s_0^{m-1})$$

(where $\sigma(s_0^{-1}) := \{0\}$). We now define the order reversing function $\sigma : T_+/T_0 \longrightarrow \mathrm{Sub}(V_0)$ by

$$\sigma(t) := \sigma(s_0^m) \quad \text{if} \ \ t \in s_0^m T_+ \setminus s_0^{m+1} T_+ \ .$$

In order to check that

$$\bigcup_{t \in T_+/T_0} \sigma(t) = \bigcup_{m \geq 0} \sigma(s_0^m) = V_0 \tag{7}$$

holds true we consider any $U_0 \in \mathrm{Fin}(V_0)$. For any $n \geq m$ we have the obvious commutative diagram

$$\begin{array}{ccccc} [\sigma(s_0^m) + U_0] \cap V_0'/\sigma'(s_0^m) & \hookrightarrow & \sigma(s_0^m) + U_0/\sigma(s_0^m) & \xleftarrow{\cong} & U_0/U_0 \cap \sigma(s_0^m) \\ \downarrow & & \downarrow & & \downarrow \\ [\sigma(s_0^n) + U_0] \cap V_0'/\sigma'(s_0^n) & \hookrightarrow & \sigma(s_0^n) + U_0/\sigma(s_0^n) & \xleftarrow{\cong} & U_0/U_0 \cap \sigma(s_0^n) \end{array}$$

in which the left horizontal arrows are injective. In particular all members of the diagram are finite. By the finiteness of U_0 the increasing sequence of subspaces $U_0 \cap \sigma(s_0^m)$ has to stabilize. This means that the right vertical arrow in the diagram is bijective for sufficiently big m. On the other hand, since $\bigcup_{m \geq 0} \sigma'(s_0^m) = V_0'$, the left vertical arrow in the diagram is the zero map whenever the difference $n-m$ is sufficiently big. It follows that $[\sigma(s_0^m)+U_0]\cap V_0' = \sigma'(s_0^m)$ for big m which, by the maximality property of $\sigma(s_0^m)$, means that $U_0 \subseteq \sigma(s_0^m)$. This establishes (7). By construction we have

$$M_\sigma(V_0) \cap \mathrm{ind}_{P_0}^P(V_0') = M_{\sigma'}(V_0') \ .$$

$\square$

Lemma 4.2. *For any short exact sequence $0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$ in $\mathcal{M}_{o-tor}(P)$ we have the long exact sequence*

$$\begin{aligned} 0 \longrightarrow D^0(V_3) \longrightarrow D^0(V_2) \longrightarrow D^0(V_1) \longrightarrow D^1(V_3) \longrightarrow \ldots \\ \longrightarrow D^i(V_3) \longrightarrow D^i(V_2) \longrightarrow D^i(V_1) \longrightarrow \ldots \end{aligned}$$

in $\mathcal{M}_{et}(\Lambda(P_+))$.

Proof. We apply the functor D to the short exact sequence of resolutions

$$0 \longrightarrow \mathcal{I}_\bullet(V_1) \longrightarrow \mathcal{I}_\bullet(V_2) \longrightarrow \mathcal{I}_\bullet(V_3) \longrightarrow 0$$

and obtain the sequence of complexes

$$0 \longrightarrow D(\mathcal{I}_\bullet(V_3)) \longrightarrow D(\mathcal{I}_\bullet(V_2)) \longrightarrow D(\mathcal{I}_\bullet(V_1)) \longrightarrow 0$$

in $C^{\geq 0}(\mathcal{M}_{et}(\Lambda(P_+)))$. It is exact by Lemma 4.1. The associated long exact cohomology sequence is the asserted sequence. $\square$

Lemma 4.3. *For any compactly induced smooth P-representation $V = \mathrm{ind}_{P_0}^P(V_0)$ we have $D^0(V) = D(V)$ and $D^i(V) = 0$ for $i \geq 1$.*

Proof. In this case we have two more P-equivariant maps besides ρ_V. These are

$$\begin{aligned} \epsilon_V : \mathrm{ind}_{P_0}^P(V) &\longrightarrow \mathrm{ind}_{P_0}^P(V_0) = V \\ \psi &\longmapsto \epsilon_V(\psi)(b) := \psi(b)(1) \end{aligned}$$

and

$$\begin{aligned} \sigma_V : V = \mathrm{ind}_{P_0}^P(V_0) &\longrightarrow \mathrm{ind}_{P_0}^P(V) \\ \psi &\longmapsto \sigma_V(\psi)(b)(c) := \begin{cases} \psi(bc) & \text{if } c \in P_0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It is straightforward to check that

$$\rho_V \circ \sigma_V = \mathrm{id}_V = \epsilon_V \circ \sigma_V \ .$$

It follows that

$$\mathrm{ind}_{P_0}^P(V) = \mathrm{im}(\sigma_V) \oplus \ker(\epsilon_V) = \mathrm{im}(\sigma_V) \oplus \ker(\rho_V) \ .$$

In particular, we have $\ker(\rho_V) \cong \ker(\epsilon_V)$. But $\ker(\epsilon_V) = \mathrm{ind}_{P_0}^P(\mathrm{ind}_{P_0}^{P \setminus P_0}(V_0))$ is compactly induced. Proceeding inductively we obtain that each $\ker \rho_{n-1}$ is compactly induced and is a direct summand of $\mathcal{I}_n(V)$. It easily follows that $D(\mathcal{I}_\bullet(V))$ is an exact resolution of $D(V)$. $\square$

We see that the δ-functor $V \longmapsto D^i(V)$ is coeffacable and hence universal.

Corollary 4.4. *Let* $\ldots \longrightarrow \mathrm{ind}_{P_0}^{P}(V_n) \longrightarrow \ldots \longrightarrow \mathrm{ind}_{P_0}^{P}(V_0) \longrightarrow V \longrightarrow 0$ *be any exact sequence in* $\mathcal{M}_{o-tor}(P)$*; we then have* $D^i(V) \cong h^i(D(\mathrm{ind}_{P_0}^{P}(V_\bullet)))$ *for any* $i \geq 0$.

By the usual procedure the functor RD extends to an exact contravariant functor

$$\begin{aligned} RD : D^-(\mathcal{M}_{o-tor}(P)) &\longrightarrow D^+(\mathcal{M}_{et}(\Lambda(P_+))) \\ V^\bullet &\longmapsto \mathrm{Tot} D(\mathcal{I}_\bullet(V^\bullet)) \ . \end{aligned}$$

from the bounded above derived category $D^-(\mathcal{M}_{o-tor}(P))$ of the abelian category $\mathcal{M}_{o-tor}(P)$ into the bounded below derived category $D^+(\mathcal{M}_{et}(\Lambda(P_+)))$ of the abelian category $\mathcal{M}_{et}(\Lambda(P_+))$.

5 Towards (φ, Γ)-modules

At this point we fix once and for all, as part of an épinglage for the group G, isomorphisms of algebraic groups $\iota_\alpha : N_\alpha \xrightarrow{\cong} \mathbb{Q}_p$, for $\alpha \in \Delta$, such that

$$\iota_\alpha(tnt^{-1}) = \alpha(t)\iota_\alpha(n) \qquad \text{for any } n \in N_\alpha, t \in T.$$

Using that $\prod_{\alpha\in\Delta} N_\alpha$ naturally is a quotient of $N/[N,N]$ we then introduce the homomorphism of groups

$$\ell := \sum_{\alpha\in\Delta} \iota_\alpha : N \longrightarrow \mathbb{Q}_p \ .$$

It induces a (continuous) epimorphism of (compact) rings $\Lambda(N_0) \longrightarrow \Lambda(\ell(N_0)) \cong \Lambda(\mathbb{Z}_p)$, also denoted by ℓ. It is convenient to normalize the ι_α in such a way that

$$\iota_\alpha(N_0 \cap N_\alpha) = \mathbb{Z}_p \qquad \text{for any } \alpha \in \Delta$$

holds true. We then, in particular, have $\ell(N_0) = \mathbb{Z}_p$.

To avoid technicalities we assume from now on that the center of the group G is connected. Then the quotient $X^*(T)/\oplus_{\alpha\in\Delta}\mathbb{Z}\alpha$ is free. Hence we find a cocharacter $\xi \in X_*(T)$ such that $\alpha \circ \xi = \mathrm{id}_{\mathbb{G}_m}$ for any $\alpha \in \Delta$. It is injective and uniquely determined up to a central cocharacter. We once and for all fix such a ξ. It satisfies

$$\xi(\mathbb{Z}_p \setminus \{0\}) \subseteq T_+$$

and

$$\ell(\xi(a)n\xi(a^{-1})) = a\ell(n) \qquad \text{for any } a \in \mathbb{Q}_p^\times, n \in N \ . \tag{8}$$

Put

$$T_\star := \xi((1+p^{\epsilon(p)}\mathbb{Z}_p)p^{\mathbb{N}_0}) \cong (1+p^{\epsilon(p)}\mathbb{Z}_p)p^{\mathbb{N}_0} \qquad \text{and} \qquad P_\star := N_0 T_\star \ .$$

It follows from (8) that

$$\begin{aligned} P_\star = N_0 T_\star &\longrightarrow S_\star \\ n_0 t &\longmapsto \begin{pmatrix} 1 & 0 \\ \ell(n_0) & \xi^{-1}(t) \end{pmatrix} \end{aligned}$$

is an epimorphism of monoids. If we define

$$N_1 := \ker(N_0 \xrightarrow{\ell} \mathbb{Z}_p)$$

then, in fact, we have the isomorphism

$$N_1 \backslash P_\star \xrightarrow{\cong} S_\star \ .$$

It follows that

$$\Lambda(S_\star) = o \otimes_{\Lambda(N_1)} \Lambda(P_\star)$$

holds true at least as bimodules. Hence we have a natural isomorphism

$$\Lambda(S_\star) \otimes_{\Lambda(P_\star)} M \cong o \otimes_{\Lambda(N_1)} M$$

for any M in $\mathcal{M}(\Lambda(P_\star))$. But $\Lambda(P_\star)$ is free as a left $\Lambda(N_0\xi(1+p^{\epsilon(p)}\mathbb{Z}_p))$-module by (1). Furthermore, $\Lambda(N_0\xi(1+p^{\epsilon(p)}\mathbb{Z}_p))$ is flat as a left $\Lambda(N_1)$-module (cf. the proof of Lemma 5.5 in [OV]). This implies that any flat and, in particular, any projective left $\Lambda(P_\star)$-module is flat as a $\Lambda(N_1)$-module. By using $\Lambda(P_\star)$-projective resolutions the above natural isomorphism therefore extends to natural isomorphisms

$$\mathrm{Tor}_i^{\Lambda(P_\star)}(\Lambda(S_\star), M) \cong \mathrm{Tor}_i^{\Lambda(N_1)}(o, M)$$

for any $i \geq 0$ and any M in $\mathcal{M}(\Lambda(P_\star))$. Since N_1 is pro-p and torsionfree the ring $\Lambda(N_1)$ is noetherian of finite global dimension by [Neu]. It follows that there is an integer $d(N_1) \geq 0$ such that

$$\mathrm{Tor}_i^{\Lambda(P_\star)}(\Lambda(S_\star), M) = 0 \qquad \text{for any } i > d(N_1)$$

and any M in $\mathcal{M}(\Lambda(P_\star))$. From this we conclude by a standard argument (cf. [Har] I.5.3) that the total derived tensor product

$$\Lambda(S_\star) \otimes^{\mathbb{L}}_{\Lambda(P_\star)} : D(\Lambda(P_\star)) \longrightarrow D(\Lambda(S_\star))$$

is well defined on the whole derived category and respects the bounded below derived categories

$$\Lambda(S_\star) \otimes^{\mathbb{L}}_{\Lambda(P_\star)} : D^+(\Lambda(P_\star)) \longrightarrow D^+(\Lambda(S_\star)) \ .$$

Moreover, suppose that M is an etale $\Lambda(P_\star)$-module. We observe that any projective $\Lambda(P_\star)$-module necessarily is etale. Let $F_\bullet \longrightarrow M$ be a projective resolution of M. Then $F_\bullet$ is a complex of etale $\Lambda(P_\star)$-modules and, similarly, $\Lambda(S_\star) \otimes_{\Lambda(P_\star)} F_\bullet$ is a complex of etale $\Lambda(S_\star)$-modules. Hence, as a consequence of Prop. 1.3,

$$\mathrm{Tor}_i^{\Lambda(P_\star)}(\Lambda(S_\star), M) = h_i(\Lambda(S_\star) \otimes_{\Lambda(P_\star)} F_\bullet) \qquad \text{for any } i \geq 0$$

is an etale $\Lambda(S_\star)$-module. In other words, the total derived tensor product restricts to a functor

$$\Lambda(S_\star) \otimes^{\mathbb{L}}_{\Lambda(P_\star)} : D^+_{et}(\Lambda(P_\star)) \longrightarrow D^+_{et}(\Lambda(S_\star)) \ .$$

between the bounded below derived categories with etale cohomology modules.

Altogether we obtain the composed functor

$$RD_{\Lambda(S_\star)} : D^-(\mathcal{M}_{o-tor}(P)) \xrightarrow{RD} D^+_{et}(\Lambda(P_+)) \xrightarrow{\text{forget}} D^+_{et}(\Lambda(P_\star)) \xrightarrow{\Lambda(S_\star) \otimes^{\mathbb{L}}_{\Lambda(P_\star)}} D^+_{et}(\Lambda(S_\star)) \ .$$

It gives rise on $\mathcal{M}_{o-tor}(P)$ to the δ-functor

$$D^i_{\Lambda(S_\star)}(V) := h^i(RD_{\Lambda(S_\star)}(V)) \qquad \text{for } i \geq 0$$

into etale $\Lambda(S_\star)$-modules.

6 A functor in the reverse direction

As noted in section 1 the conjugation action of T_+ on P_+ induces an action of T_+ by (continuous injective) ring endomorphisms ϕ_t on $\Lambda(P_+)$.

On the other hand it is not difficult to see that, for each $t \in T_+$, there is a unique surjective continuous linear (but not multiplicative) map $\psi_t : \Lambda(P_+) \longrightarrow \Lambda(P_+)$ which on group elements $b \in P_+$ satisfies

$$\psi_t(b) = \begin{cases} t^{-1}bt & \text{if } b \in tP_+t^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

By continuity it suffices to check the following formulas on group elements where they are straightforward:

i. $\psi_t \circ \phi_t = \mathrm{id}_{\Lambda(P_+)}$ for any $t \in T_+$.

ii. $\psi_t = \phi_t^{-1}$ for $t \in T_0$.

iii. $\psi_{t_1} \circ \psi_{t_2} = \psi_{t_1 t_2}$ for any $t_1, t_2 \in T_+$.

iv. $\psi_t(\lambda\phi_t(\mu)) = \psi_t(\lambda)\mu$ and $\psi_t(\phi_t(\lambda)\mu) = \lambda\psi_t(\mu)$ for any $t \in T_+$ and $\lambda, \mu \in \Lambda(P_+)$.

Let D be any etale $\Lambda(P_+)$-module. Using the identity iv. above we may introduce, for any $t \in T_+$, the composed map

$$\begin{array}{cccccc} \psi_t : & D & \overset{\cong}{\longleftarrow} & \Lambda \otimes_{\Lambda,t} D & \longrightarrow & D \\ & \lambda t x & \longleftrightarrow & \lambda \otimes x & \longmapsto & \psi_t(\lambda)x \ . \end{array}$$

It satisfies:

v. $\psi_t \circ t = \mathrm{id}_D$ for any $t \in T_+$.

vi. $\psi_t = t^{-1}$ for $t \in T_0$.

vii. $\psi_{t_1} \circ \psi_{t_2} = \psi_{t_1 t_2}$ for any $t_1, t_2 \in T_+$.

viii. $\psi_t(\phi_t(\lambda)x) = \lambda\psi_t(x)$ for any $\lambda \in \Lambda(P_+)$ and $x \in D$.

The last is a consequence of the identity iv. above. Moreover, D carries the P_+^{-1}-action defined by

$$bx := \psi_t(n_0 x) \qquad \text{for } b = t^{-1}n_0 \in P_+^{-1} = T_+^{-1}N_0 \text{ and } x \in D.$$

Remark 6.1. *If $D = D^i(V)$ for some representation V in $\mathcal{M}_{o-tor}(P)$ then the operator ψ_t coincides with the action of t^{-1} on D given by the construction of the δ-functor D^i.*

Proof. By the construction of the δ-functor it suffices to establish the assertion for any $D = D(\mathrm{ind}_{P_0}^P(V_0))$. We have to show that

$$t^{-1}(\lambda t y) = \psi_t(\lambda)y$$

holds true for any $\lambda \in \Lambda(P_+)$ and any $y \in D$. But D is a quotient of $M_1(V_0)^*$ where $M_1(V_0) = \mathrm{ind}_{P_0}^{P_+}(V_0)$. So we are further reduced to checking the above relation for $y \in M_1(V_0)^*$. But

$M_1(V_0)^*$ is a compact $\Lambda(P_+)$-module. By continuity we therefore need to establish only the relation

$$t^{-1}(nty) = \psi_t(n)y$$

for any $n \in N_0$ and any $y \in M_1(V_0)^*$. If $n = tn't^{-1}$ for some $n' \in N_0$ then the right hand side is $n'y$. Using that t^{-1} is a left inverse of t as an operator on $M_1(V_0)^*$ we compute $t^{-1}(nty) = t^{-1}(tn't^{-1}ty) = t^{-1}(tn'y) = n'y$. If $n \notin tN_0t^{-1}$ then the right hand side is equal to zero. Evaluating the left hand side in a function $\psi \in M_1$ is the same as evaluating y in the function $\tilde{\psi}$ where

$$\tilde{\psi}(b) := \begin{cases} \psi(t^{-1}ntb) & \text{if } b \in P_+, \\ 0 & \text{if } b \in P \setminus P_+. \end{cases} \tag{9}$$

Since $t^{-1}nt \notin N_0$ the set $t^{-1}ntP_+$ is disjoint from P_+ on which ψ is supported. Hence $\tilde{\psi} = 0$. □

Generalizing notation introduced by Colmez we now define the o-module

$$\psi^{-\infty}(D) := \{(x_t)_t \in \prod_{t \in T_+} D : \psi_{t_1}(x_{t_1 t_2}) = x_{t_2} \text{ for any } t_1, t_2 \in T_+\} .$$

The monoid T_+ acts o-linearly on $\psi^{-\infty}(D)$ by

$$t' \cdot (x_t)_t := (x_{t't})_t \qquad \text{for } t' \in T_+.$$

In fact, acting by t' has the inverse

$$(x_t)_t \longmapsto (\psi_{t'}(x_t))_t .$$

Hence T_+ acts by automorphisms. Observing that any element in T is a quotient of two elements in T_+ it therefore follows that this T_+-action extends uniquely to an action of the full torus T on $\psi^{-\infty}(D)$.

On the other hand N_0 acts o-linearly on $\psi^{-\infty}(D)$ by

$$n \cdot (x_t)_t := (\phi_t(n)x_t)_t \qquad \text{for } n \in N_0.$$

To see this we compute

$$\begin{aligned} \psi_{t_1}(\phi_{t_1 t_2}(n)x_{t_1 t_2}) &= \psi_{t_1}(\phi_{t_1}(\phi_{t_2}(n))x_{t_1 t_2}) \\ &= \phi_{t_2}(n)\psi_{t_1}(x_{t_1 t_2}) \\ &= \phi_{t_2}(n)x_{t_2} \end{aligned}$$

and

$$\begin{aligned} n' \cdot (n \cdot (x_t)_t) &= n' \cdot (\phi_t(n)x_t)_t = (\phi_t(n')\phi_t(n)x_t)_t \\ &= (\phi_t(n'n)x_t)_t \\ &= (n'n) \cdot (x_t)_t . \end{aligned}$$

These two actions are compatible in the sense that, for $t' \in T_+$ and $n \in N_0$, we have

$$\begin{aligned} t' \cdot (n \cdot (x_t)_t) &= t' \cdot (\phi_t(n)x_t)_t = (\phi_{t't}(n)x_{t't})_t \\ &= (\phi_t(t'nt'^{-1})x_{t't})_t = (t'nt'^{-1}) \cdot (x_{t't})_t \\ &= (t'nt'^{-1}) \cdot (t' \cdot (x_t)_t) \ . \end{aligned}$$

In a next intermediate step we claim that for arbitrary $t' \in T$ and $n \in N_0$ but such that $t'nt'^{-1} \in N_0$ we have

$$t' \cdot n \cdot (t')^{-1} \cdot \xi = (t'nt'^{-1}) \cdot \xi \qquad \text{for any } \xi \in \psi^{-\infty}(D).$$

Write $t' = t_1 t_2^{-1}$ with $t_1, t_2 \in T_+$. Using the previous formula we compute

$$\begin{aligned} (t'nt'^{-1}) \cdot \xi &= (t_2)^{-1} \cdot t_2 \cdot (t_2^{-1} t_1 n t_1^{-1} t_2) \cdot \xi \\ &= (t_2)^{-1} \cdot (t_1 n t_1^{-1}) \cdot t_2 \cdot \xi \\ &= (t_2)^{-1} \cdot (t_1 n t_1^{-1}) \cdot t_1 \cdot (t_1)^{-1} \cdot t_2 \cdot \xi \\ &= (t_2)^{-1} \cdot t_1 \cdot n \cdot (t_1)^{-1} \cdot t_2 \cdot \xi \\ &= t' \cdot n \cdot (t')^{-1} \cdot \xi \ . \end{aligned}$$

Let now $n \in N$ be completely arbitrary. We choose a $t' \in T_+$ such that $t'nt'^{-1} \in N_0$ and define

$$n \cdot \xi := (t')^{-1} \cdot (t'nt'^{-1}) \cdot t' \cdot \xi \qquad \text{for any } \xi \in \psi^{-\infty}(D).$$

By the intermediate computation this definition is independent of the choice of t' and hence extends the N_0-action to an o-linear action of N on $\psi^{-\infty}(D)$. In fact, by construction, this N-action and the previous T-action combine into a P-action on $\psi^{-\infty}(D)$.

Everything above is natural. Hence we have constructed a covariant functor $\psi^{-\infty}$ from the category of etale $\Lambda(P_+)$-modules into the category of P-representations (in arbitrary o-modules). This functor can be viewed as a form of induction. Let

$$\begin{gathered} \mathrm{Ind}_{P_+}^{P}(D) := \text{all functions } F : P \longrightarrow D \text{ such that} \\ F(bb_+) = b_+^{-1} F(b) \\ \text{for any } b \in P, b_+ \in P_+ \end{gathered}$$

with the group P acting by left translations. Using, as above, that $P = T_+^{-1} P_+$ one checks that

$$\begin{aligned} \mathrm{Ind}_{P_+}^{P}(D) &\xrightarrow{\cong} \psi^{-\infty}(D) \\ F &\longmapsto (F(t^{-1}))_t \end{aligned}$$

is a P-equivariant isomorphism.

Lemma 6.2. *The functor $\psi^{-\infty}$ is exact.*

Proof. We pick an element $s \in T_+$ which is strictly dominant, i. e., which satisfies $|\alpha(s)| < 1$ for any $\alpha \in \Phi^+$. The subset $\{s^m\}_{m \in \mathbb{N}}$ is cofinal in the preordered set T_+. Hence $\psi^{-\infty}(D)$, for any D in $\mathcal{M}_{et}(\Lambda(P_+))$, is the projective limit of the sequence

$$\ldots \xrightarrow{\psi_s} D \xrightarrow{\psi_s} \ldots \xrightarrow{\psi_s} D \ .$$

Since ψ_s is surjective the exactness follows immediately. □

Let V be any representation in $\mathcal{M}_{o-tor}(P)$. There is the natural P_+^{-1}-equivariant map

$$\tilde{a}_V : V^* \twoheadrightarrow D(V) \hookrightarrow D^0(V) \ .$$

By Remark 6.1 we have

$$\psi^{-\infty}(D^0(V)) := \{(x_t)_t \in \prod_{t \in T_+} D^0(V) : t_1^{-1} x_{t_1 t_2} = x_{t_2} \text{ for any } t_1, t_2 \in T_+\} \ .$$

Hence the map $\tilde{a}_V$ lifts to a natural transformation

$$\begin{aligned} a_V : V^* &\longrightarrow \psi^{-\infty}(D^0(V)) \\ x &\longmapsto (\tilde{a}_V(tx))_t \ . \end{aligned}$$

of functors on $\mathcal{M}_{o-tor}(P)$.

Lemma 6.3. *The map a_V is P-equivariant.*

Proof. The T-equivariance only needs to be checked on elements $t' \in T_+^{-1}$. Using Remark 6.1 we compute

$$a_V(t'x) = (\tilde{a}_V(tt'x))_t = (t'\tilde{a}_V(tx))_t = t' \cdot (\tilde{a}_V(tx))_t = t' \cdot a_V(x) \ .$$

Knowing T-equivariance already we need to check N-equivariance only for $n \in N_0$. We compute

$$a_V(nx) = (\tilde{a}_V(tnx))_t = (\tilde{a}_V(tnt^{-1}tx))_t = (\phi_t(n)\tilde{a}_V(tx))_t = n \cdot (\tilde{a}_V(tx))_t = n \cdot a_V(x) \ .$$

□

Remark 6.4. *Suppose that, for some $M \in \mathcal{P}_+(V)$, the composed natural map $M^* \to D(V) \to D^0(V)$ is bijective; then the map a_V is an isomorphism.*

Proof. Because of Remark 6.1 we have to show that the map

$$\begin{aligned} V^* &\longrightarrow \{(x_t)_t \in \prod_{t \in T_+} M^* : t_1^{-1} x_{t_1 t_2} = x_{t_2} \text{ for any } t_1, t_2 \in T_+\} \\ x &\longmapsto ((tx)|M)_t \end{aligned}$$

is bijective. This is equivalent to the map

$$\begin{aligned} V^* &\longrightarrow \{(y_t)_t \in \prod_{t \in T_+} (t^{-1}M)^* : y_{t_1 t_2} | t_2^{-1} M = y_{t_2} \text{ for any } t_1, t_2 \in T_+\} \\ x &\longmapsto (x | t^{-1}M)_t \end{aligned}$$

being bijective which is obvious since we have $V = \bigcup_{t \in T_+} t^{-1}M$. □

7 Dependence on N_0 and ℓ

In this section we will investigate the question in which way our δ-functor D^i depends on the initial choice of the compact open subgroup N_0. We therefore make our notation more precise and write $D(N_0; V)$ and $D^i(N_0; V)$ instead of $D(V)$ and $D^i(V)$, respectively. Let $N_0' \subseteq N$ be another choice of a totally decomposed compact open subgroup. Since then $N_0 \cap N_0'$ is totally decomposed and compact open as well it suffices to treat the case

$$N_0 \subseteq N_0'$$

which we assume from now on throughout this section. Let $P_0' := T_0 N_0'$ and $P_+' := N_0' T_+$. The natural embedding of rings $\Lambda(P_+) \hookrightarrow \Lambda(P_+')$ makes $\Lambda(P_+')$ a right $\Lambda(P_+)$-module which, as a consequence of the decomposition

$$P_+' = \bigcup_{n \in N_0'/N_0} n P_+ \ ,$$

is free of rank equal to the index $[N_0' : N_0]$. We therefore have the exact base extension functor

$$\Lambda(P_+') \otimes_{\Lambda(P_+)} . : \mathcal{M}(\Lambda(P_+)) \longrightarrow \mathcal{M}(\Lambda(P_+')) \ .$$

It is straightforward to check that this functor respects etale modules. Our goal is to establish the following result.

Proposition 7.1. *There is a natural isomorphism of δ-functors*

$$\Lambda(P_+') \otimes_{\Lambda(P_+)} D^i(N_0; .) \cong D^i(N_0'; .) \ .$$

on $\mathcal{M}_{o-tor}(P)$.

Proof. The argument is formally similar to the proof of Prop. 3.5. Let V_0 be any smooth P_0-representation and $V := \mathrm{ind}_{P_0}^{P}(V_0)$ the compactly induced P-representation. Then $V_0' := \mathrm{ind}_{P_0}^{P_0'}(V_0)$ is a smooth P_0'-representation and, by the transitivity of induction, we have

$$V = \mathrm{ind}_{P_0}^{P}(V_0) = \mathrm{ind}_{P_0'}^{P}(V_0') \ .$$

On the dual side we correspondingly (and more precisely) have the isomorphisms

$$\begin{array}{ccc} & & \mathrm{Ind}_{P_0'}^{P}(\mathrm{ind}_{P_0}^{P_0'}(V_0^*)) \\ & \overset{\mathcal{I}}{\nearrow} & \downarrow \\ \mathrm{Ind}_{P_0}^{P}(V_0^*) & & \\ & \underset{\mathcal{J}}{\searrow} & \\ & & \mathrm{Ind}_{P_0'}^{P}((V_0')^*) \end{array}$$

where

$$\mathcal{I}(\Phi)(b)(b_0') := \Phi(b b_0') \qquad \text{for } b \in P,\ b_0' \in P_0'$$

and

$$\mathcal{J}(\Phi)(b)(\psi) := \sum_{b_0' \in P_0'/P_0} \Phi(b b_0')(\psi(b_0')) = \sum_{n' \in N_0'/N_0} \Phi(b n')(\psi(n')) \qquad \text{for } b \in P,\ \psi \in V_0'.$$

Clearly, for any right P'_0-invariant subset $X \subseteq P$ the map $\mathcal{J}$ restricts to a bijection

$$\operatorname{Ind}_{P_0}^{X}(V_0^*) \xrightarrow{\cong} \operatorname{Ind}_{P'_0}^{X}((V'_0)^*) \ . \tag{10}$$

In particular,

$$J^+(V_0) = \operatorname{Ind}_{P_0}^{P_+}(V_0^*) \subseteq \operatorname{Ind}_{P_0}^{P'_+}(V_0^*) \xrightarrow[\mathcal{J}]{\cong} \operatorname{Ind}_{P'_0}^{P'_+}((V'_0)^*) = J^+(V'_0) \ .$$

Notations like $J^{\pm}(V_0)$, $\operatorname{Sub}(V_0)$, and $J_\sigma(V_0)$ depend on V_0 as a representation for a specific subgroup (here P_0) of P. In order not to make the notation too heavy we agree in this proof to the abuse that when writing $J^{\pm}(V'_0)$ etc. we refer to the subgroup P'_0. We now consider any order reversing map $\sigma : T_+/T_0 \longrightarrow \operatorname{Sub}(V_0)$ satisfying (3) and define the map

$$\begin{aligned} \operatorname{ind}(\sigma) : T_+/T_0 &\longrightarrow \operatorname{Sub}(V'_0) \\ t &\longrightarrow \operatorname{ind}_{P_0}^{P'_0}(\sigma(t)) \end{aligned}$$

which again is order reversing and satisfies (3). Using that any $\Phi \in J^+(V_0) \cap J_\sigma(V_0)$ satisfies $\Phi(N'_0 t) \subseteq \sigma(t)^\perp$ for any $t \in T_+$ one easily checks that

$$\mathcal{J}\big(J^+(V_0) \cap J_\sigma(V_0)\big) \subseteq J^+(V'_0) \cap J_{\operatorname{ind}(\sigma)}(V'_0)$$

holds true. Hence the map

$$\begin{aligned} \Lambda(P'_+) \otimes_{\Lambda(P_+)} \big(J^+(V_0)/(J^+(V_0) \cap \bigcup_\sigma J_\sigma(V_0))\big) &\longrightarrow J^+(V'_0)/(J^+(V'_0) \cap \bigcup_\sigma J_{\operatorname{ind}(\sigma)}(V'_0)) \\ \lambda \otimes x &\longmapsto \lambda \mathcal{J}(x) \end{aligned} \tag{11}$$

is well defined. To check that it is bijective we let $n_1, \ldots, n_r \in N'_0$ be representatives for the cosets in N'_0/N_0. Then the above map may be viewed as the map

$$\begin{aligned} \big(J^+(V_0)/(J^+(V_0) \cap \bigcup_\sigma J_\sigma(V_0))\big)^r &\longrightarrow J^+(V'_0)/(J^+(V'_0) \cap \bigcup_\sigma J_{\operatorname{ind}(\sigma)}(V'_0)) \\ (x_1, \ldots, x_r) &\longmapsto \sum_{i=1}^r n_i \mathcal{J}(x_i) \ . \end{aligned}$$

Of course it suffices to show that, for each σ, the map

$$\begin{aligned} \big(J^+(V_0)/(J^+(V_0) \cap J_\sigma(V_0))\big)^r &\longrightarrow J^+(V'_0)/(J^+(V'_0) \cap J_{\operatorname{ind}(\sigma)}(V'_0)) \\ (x_1, \ldots, x_r) &\longmapsto \sum_{i=1}^r n_i \mathcal{J}(x_i) \end{aligned}$$

is bijective. But we have the decompositions

$$J^+(V_0)/(J^+(V_0) \cap J_\sigma(V_0)) = \prod_{t \in T_+/T_0} \operatorname{Ind}_{P_0}^{N_0 t P_0}((V_0/\sigma(t))^*)$$

and

$$J^+(V'_0)/(J^+(V'_0) \cap J_{\operatorname{ind}(\sigma)}(V'_0)) = \prod_{t \in T_+/T_0} \operatorname{Ind}_{P'_0}^{N'_0 t P'_0}((V'_0/\operatorname{ind}(\sigma)(t))^*)$$

which are respected by $\mathcal{J}$. Using (10) it therefore comes down to the bijectivity of

$$\begin{aligned}(\mathrm{Ind}_{P_0}^{N_0 t P_0}(.))^r &\longrightarrow \mathrm{Ind}_{P_0}^{N_0' t P_0'}(.)\\ (\Phi_1, \ldots, \Phi_r) &\longmapsto \sum_{i=1}^{r} n_i \Phi_i\end{aligned}$$

which is straightforward from the disjoint union

$$N_0' t P_0' = \bigcup_{i=1}^{r} n_i N_0 t P_0 \ .$$

This establishes the bijectivity of (11). In order to read (11) as an isomorphism

$$\Lambda(P_+') \otimes_{\Lambda(P_+)} D(N_0; V) \xrightarrow{\cong} D(N_0'; V) \tag{12}$$

it remains to check that the $M_{\mathrm{ind}(\sigma)}(V_0')$ with varying σ are cofinal among all generating P_+'-subrepresentations M of V. But M then a fortiori is a generating P_+-subrepresentation of V. By Lemma 3.2.ii we therefore find a σ such that $M_\sigma(V_0) \subseteq M$. It follows that $M_{\mathrm{ind}(\sigma)}(V_0') = P_+' M_\sigma(V_0) \subseteq M$.

Next we have to convince ourselves that the isomorphism (12) is natural in maps $f : \mathrm{ind}_{P_0}^{P}(U_0) \longrightarrow \mathrm{ind}_{P_0}^{P}(V_0)$ of compactly induced representations. Put $U_0' := \mathrm{ind}_{P_0}^{P_0'}(U_0)$. Viewing f as a map $\mathrm{ind}_{P_0}^{P}(U_0') \longrightarrow \mathrm{ind}_{P_0}^{P}(V_0')$ means to consider $j_{V_0} \circ f \circ j_{U_0}^{-1}$ where

$$j_{V_0} : \mathrm{ind}_{P_0}^{P}(V_0) \xrightarrow{\cong} \mathrm{ind}_{P_0}^{P}(V_0')$$

is the transitivity isomorphism such that $(j_{V_0}^*)^{-1} = \mathcal{J}$ (and correspondingly for j_{U_0}). We therefore have to check the commutativity of the diagram

$$\begin{array}{ccc}
J^+(V_0)/(J^+(V_0) \cap \bigcup_\sigma J_\sigma(V_0)) & \xrightarrow{\mathcal{J}} & J^+(V_0')/(J^+(V_0') \cap \bigcup_\sigma J_{\mathrm{ind}(\sigma)}(V_0')) \\
\cong \downarrow & & \downarrow \cong \\
\mathrm{Ind}_{P_0}^{P}(V_0^*)/\bigcup_\sigma J_\sigma(V_0) & & \mathrm{Ind}_{P_0'}^{P}((V_0')^*)/\bigcup_\sigma J_{\mathrm{ind}(\sigma)}(V_0') \\
{\scriptstyle D(N_0;f)} \downarrow & & \downarrow {\scriptstyle D(N_0';j\circ f\circ j^{-1})} \\
\mathrm{Ind}_{P_0}^{P}(U_0^*)/\bigcup_\tau J_\tau(U_0) & & \mathrm{Ind}_{P_0'}^{P}((U_0')^*)/\bigcup_\tau J_{\mathrm{ind}(\tau)}(U_0') \\
\cong \uparrow & & \uparrow \cong \\
J^+(U_0)/(J^+(U_0) \cap \bigcup_\tau J_\tau(U_0)) & \xrightarrow{\mathcal{J}} & J^+(U_0')/(J^+(U_0') \cap \bigcup_\tau J_{\mathrm{ind}(\tau)}(U_0'))
\end{array}$$

or equivalently that

$$\mathcal{J}((f^*(\Phi))^+) - ((j \circ f \circ j^{-1})^*(\mathcal{J}(\Phi)))^+ = \mathcal{J}((f^*(\Phi))^+) - (\mathcal{J}(f^*(\Phi)))^+$$

lies in $\bigcup_\tau J_{\mathrm{ind}(\tau)}(U_0')$ for any $\Phi \in J^+(V_0)$. But we know from (6) that we find in fact a single map $\tau : T_+/T_0 \longrightarrow \mathrm{Sub}(U_0)$ such that

$$f^*(\Phi)((N \setminus N_0)t) \subseteq \tau(t)^\perp \tag{13}$$

for any $t \in T_+$ and any $\Phi \in J^+(V_0)$. We claim that $\mathcal{J}((f^*(\Phi))^+) - (\mathcal{J}(f^*(\Phi)))^+ \subseteq J_{\mathrm{ind}(\tau)}(U_0')$ which amounts to

$$\mathcal{J}\big((f^*(\Phi))^+ - f^*(\Phi)\big)(n_0't) \in \mathrm{ind}_{P_0}^{P_0'}(\tau(t))^\perp$$

for any $n_0' \in N_0'$ and $t \in T_+$. Let $\psi \in \mathrm{ind}_{P_0}^{P_0'}(\tau(t))$. We compute

$$\mathcal{J}\big((f^*(\Phi))^+ - f^*(\Phi)\big)(n_0't)(\psi) = \sum_{n' \in N_0'/N_0} \big((f^*(\Phi))^+ - f^*(\Phi)\big)(n_0'tn')(\psi(n'))$$

We always have $n_0'tn' \in N_0't$. If even $n_0'tn' \in N_0t$ then in the corresponding summand the two terms coincide so that their difference is zero. Otherwise we have $(f^*(\Phi))^+(n_0'tn') = 0$ and $f^*(\Phi)(n_0'tn') \in \tau(t)^\perp$ by (13). So the value of the corresponding difference on $\psi(n')$ again is zero. This establishes the claim and hence the naturality of the isomorphism (12).

Finally we consider a general representation V in $\mathcal{M}_{o-tor}(P)$ and the corresponding resolution $\mathcal{I}_\bullet(P_0; V)$ from section 4 by representations compactly induced from P_0. There is an obvious natural homomorphism of resolutions

$$\begin{array}{ccc} \mathcal{I}_\bullet(P_0;V) & \longrightarrow & \mathcal{I}_\bullet(P_0';V) \\ \downarrow & & \downarrow \\ V & \xrightarrow{\mathrm{id}} & V. \end{array}$$

Since by the above discussion $\mathcal{I}_\bullet(P_0; V)$ can also be viewed as a resolution of V by representations compactly induced from P_0' Cor. 4.4 implies that this homomorphism induces isomorphisms

$$h^i(D(N_0'; \mathcal{I}_\bullet(P_0; V))) \xrightarrow{\cong} D^i(N_0'; V) \ .$$

On the other hand, by the above discussion the natural map

$$\Lambda(P_+') \otimes_{\Lambda(P_+)} D(N_0; \mathcal{I}_\bullet(P_0; V)) \xrightarrow{\sim} D(N_0'; \mathcal{I}_\bullet(P_0; V))$$

is an isomorphism. Together with the exactness of the functor $\Lambda(P_+')\otimes_{\Lambda(P_+)}$ this implies our assertion. □

Corollary 7.2. *For any $t \in T$ there is a natural isomorphism of δ-functors*

$$\Lambda(tP_+t^{-1}) \otimes_{\Lambda(P_+),t} D^i(N_0; .) \cong D^i(tN_0t^{-1}; .)$$

on $\mathcal{M}_{o-tor}(P)$.

Proof. First we assume that $t \in T_+$. We then have

$$\begin{aligned} \Lambda(tP_+t^{-1}) \otimes_{\Lambda(P_+),t} D^i(N_0; .) &\cong \Lambda(tP_+t^{-1}) \otimes_{\Lambda(P_+),t} \big(\Lambda(P_+) \otimes_{\Lambda(tP_+t^{-1})} D^i(tN_0t^{-1}; .)\big) \\ &= \Lambda(tP_+t^{-1}) \otimes_{\Lambda(tP_+t^{-1}),t} D^i(tN_0t^{-1}; .) \\ &\cong D^i(tN_0t^{-1}; .) \ . \end{aligned}$$

Here the first isomorphism comes from Prop. 7.1 and the last one is the fact that the $\Lambda(tP_+t^{-1})$-modules $D^i(tN_0t^{-1}; .)$ are etale. Now let $t \in T$ be arbitrary and write $t = t_1t_2^{-1}$ with $t_i \in T_+$. The above isomorphism for the group $t_2^{-1}N_0t_2$ and the elements t_1 and t_2 gives

$$\Lambda(tP_+t^{-1}) \otimes_{\Lambda(t_2^{-1}P_+t_2),t_1} D^i(t_2^{-1}N_0t_2; .) \cong D^i(tN_0t^{-1}; .)$$

and

$$\Lambda(P_+) \otimes_{\Lambda(t_2^{-1}P_+t_2),t_2} D^i(t_2^{-1}N_0t_2;.) \cong D^i(N_0;.)\ ,$$

respectively. In combination we obtain

$$\begin{aligned} D^i(tN_0t^{-1};.) &\cong \Lambda(tP_+t^{-1}) \otimes_{\Lambda(t_2^{-1}P_+t_2),t_1} D^i(t_2^{-1}N_0t_2;.) \\ &= \Lambda(tP_+t^{-1}) \otimes_{\Lambda(P_+),t} \Lambda(P_+) \otimes_{\Lambda(t_2^{-1}P_+t_2),t_2} D^i(t_2^{-1}N_0t_2;.) \\ &\cong \Lambda(tP_+t^{-1}) \otimes_{\Lambda(P_+),t} D^i(N_0;.)\ . \end{aligned}$$

□

As another consequence of the above results we can justify our specific choice of the homomorphism $\ell : N \longrightarrow \mathbb{Q}_p$ in section 5. The unipotent factor group $N/[N,N]$ is naturally a $\mathbb{Q}_p$-vector space. A homomorphism $\ell' : N \longrightarrow \mathbb{Q}_p$ is called *generic* if it induces a linear map $N/[N,N] \longrightarrow \mathbb{Q}_p$ satisfying $\ell'|N_\alpha \neq 0$ for any $\alpha \in \Delta$. We have $N/[N,N] = \prod_{\alpha\in\Delta} N_\alpha$ (cf. [BT] Prop. 4.7.(iii) and Remark 4.11). Hence the épinglage $(\iota_\alpha)_\alpha$ provides an isomorphism between $N/[N,N]$ and the standard vector space $\mathbb{Q}_p^{|\Delta|}$. A generic homomorphism then corresponds to an element in the dual vector space $(\mathbb{Q}_p^{|\Delta|})^*$ all of whose (standard) coordinates are nonzero. On the other hand, by our assumption that the center C of G is connected, the simple roots $\alpha \in \Delta$ form a basis of the character group $X^*(T/C)$. This implies that the homomorphism $\prod_{\alpha\in\Delta} \alpha : T \longrightarrow (\mathbb{Q}_p^*)^{|\Delta|}$ is surjective. It follows that the action of T on $N/[N,N]$ corresponds to the standard action of $(\mathbb{Q}_p^*)^{|\Delta|}$ on $\mathbb{Q}_p^{|\Delta|}$. The subset of vectors with nonzero coordinates is a single orbit for this action. This proves that for any generic homomorphism $\ell' : N \longrightarrow \mathbb{Q}_p$ we find a $t \in T$ such that $\ell'(.) = \ell(t.t^{-1})$.

Since $\ell'(N_0)$ will not be equal to $\mathbb{Z}_p$ in general we introduce the monoid

$$S_{\ell'} := \{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} : a \in \ell'(N_0), b \in (1 + p^{\epsilon(p)}\mathbb{Z}_p)p^{\mathbb{N}_0} \}.$$

We then have the epimorphism of monoids

$$\begin{aligned} \ell' : P_\star = N_0T_\star &\longrightarrow S_{\ell'} \\ n_0t &\longmapsto \begin{pmatrix} 1 & 0 \\ \ell'(n_0) & \xi^{-1}(t) \end{pmatrix} \end{aligned}$$

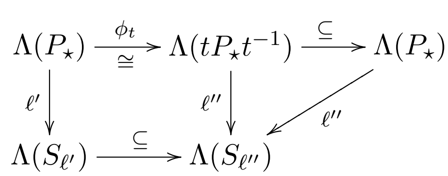
as well as the corresponding ring homomorphism $\ell' : \Lambda(P_\star) \longrightarrow \Lambda(S_{\ell'})$, which, for simplicity, we denote all by the same symbol ℓ'.

Corollary 7.3. *Let $\ell', \ell'' : N \longrightarrow \mathbb{Q}_p$ be any two generic homomorphisms such that $\ell'(N_0) \subseteq \ell''(N_0)$; then there are isomorphisms*

$$\mathrm{Tor}_i^{\Lambda(P_\star)}(\Lambda(S_{\ell''}), D^j(N_0;V)) \cong \Lambda(S_{\ell''}) \otimes_{\Lambda(S_{\ell'})} \mathrm{Tor}_i^{\Lambda(P_\star)}(\Lambda(S_{\ell'}), D^j(N_0;V))$$

for $i, j \geq 0$ which are natural in V in $\mathcal{M}_{o-tor}(P)$.

Proof. By our above discussion we find a $t \in T$ such that $\ell' = \ell''(t.t^{-1})$. Suppose first that $t \in T_+$. Having in mind the commutative diagram of rings



we then compute

$$
\begin{aligned}
&\Lambda(S_{\ell''}) \otimes_{\Lambda(S_{\ell'})} \mathrm{Tor}_i^{\Lambda(P_\star),\ell'}(\Lambda(S_{\ell'}), D^j(N_0;V)) \\
&\quad = \mathrm{Tor}_i^{\Lambda(P_\star),\ell'}(\Lambda(S_{\ell''}), D^j(N_0;V)) \\
&\quad = \mathrm{Tor}_i^{\Lambda(tP_\star t^{-1}),\ell''}(\Lambda(S_{\ell''}), \Lambda(tP_\star t^{-1}) \otimes_{\Lambda(P_\star),t} D^j(N_0;V)) \\
&\quad \cong \mathrm{Tor}_i^{\Lambda(tP_\star t^{-1}),\ell''}(\Lambda(S_{\ell''}), D^j(tN_0t^{-1};V)) \\
&\quad = \mathrm{Tor}_i^{\Lambda(P_\star),\ell''}(\Lambda(S_{\ell''}), \Lambda(P_\star) \otimes_{\Lambda(tP_\star t^{-1})} D^j(tN_0t^{-1};V)) \\
&\quad = \mathrm{Tor}_i^{\Lambda(P_\star),\ell''}(\Lambda(S_{\ell''}), \Lambda(P_+) \otimes_{\Lambda(tP_+ t^{-1})} D^j(tN_0t^{-1};V)) \\
&\quad \cong \mathrm{Tor}_i^{\Lambda(P_\star),\ell''}(\Lambda(S_{\ell''}), D^j(N_0;V)) \ .
\end{aligned}
$$

Here the two isomorphisms come from Cor. 7.2 and Prop. 7.1, respectively. Also, for greater clarity we have inserted superscripts ℓ' and ℓ'' to indicate that the respective Tor-functor is formed with respect to the corresponding ring homomorphism. A general element $t \in T$ can be written $t = t_1 t_2^{-1}$ with $t_i \in T_+$. Our claim then follows easily by using the above isomorphisms consecutively for the pairs $(\ell''(t_1.t_1^{-1}), \ell') = (\ell'(t_2.t_2^{-1}), \ell')$ and $(\ell''(t_1.t_1^{-1}), \ell'')$. $\square$

8 Some topological localizations

This section is devoted to the construction of certain topological localizations of completed group rings which are needed later on. It is entirely technical and should be skipped at first reading.

Let H_0 be a compact p-adic Lie group. In this case the ring $\Lambda(H_0)$ is well known to be noetherian. In addition we suppose given a closed normal subgroup $H_1 \subseteq H_0$ such that the factor group H_0/H_1 is isomorphic to the additive group of p-adic integers $\mathbb{Z}_p$. First of all we recall from [SV1] that $\Lambda(H_0)$ can be viewed as a skew power series ring $\Lambda(H_1)[[t_0;\sigma_0,\delta_0]]$ over $\Lambda(H_1)$. For this one picks a topological generator γ_0 of a subgroup of H_0 which maps isomorphically onto H_0/H_1. One defines $t_0 := \gamma_0 - 1$, the ring automorphism σ_0 of $\Lambda(H_1)$ by $\sigma_0(\lambda) := \gamma_0\lambda\gamma_0^{-1}$, and the σ_0-derivation $\delta_0 := \sigma_0 - \mathrm{id}$. As a consequence of [SV1] Lemma 1.6 the σ_0-derivation δ_0 is topologically nilpotent (and hence σ_0-nilpotent). Of course, $\Lambda(H_0/H_1)$ is a commutative formal power series ring in one variable over o.

From now on we also assume that H_0 is a pro-p-group without element of order p. Then $\Lambda(H_0)$ and $\Lambda(H_1)$ are integral domains ([Neu]) which are strict-local[1] with residue field k. Let $\mathfrak{m}(H_0)$ and $\mathfrak{m}(H_1)$ denote the respective maximal ideals. The ideal

$$J := J(H_0,H_1) := \ker\big(\Lambda(H_0) \longrightarrow \Lambda(H_0/H_1) \longrightarrow \Lambda(H_0/H_1)/\pi\Lambda(H_0/H_1)\big)$$

is equal to $J = \mathfrak{m}(H_1)\Lambda(H_0) = \Lambda(H_0)\mathfrak{m}(H_1)$. According to [CFKSV] Thm. 2.4 and Prop. 2.6 (and bottom of p. 203) the multiplicatively closed subset $S := S(H_0,H_1) := \Lambda(H_0) \setminus J$ of $\Lambda(H_0)$ satisfies the (left and right) Ore condition. Hence the localization $\Lambda(H_0)_S$ of $\Lambda(H_0)$ with respect to S exists. It is a strict-local integral domain with maximal ideal $J\Lambda(H_0)_S = \mathfrak{m}(H_1)\Lambda(H_0)_S$ and residue field equal to the field of fractions of $\Omega(H_0/H_1)$ (which is isomorphic to a Laurent series field in one variable over k). We now define $\Lambda_{H_1}(H_0)$ to be the $\mathfrak{m}(H_1)$-adic completion of $\Lambda(H_0)_S$. Of course, this again is a strict-local ring whose maximal ideal we

[1] A local ring is called strict-local if its quotient by the maximal ideal is a skew field.

denote by $\mathfrak{m}_{H_1}(H_0)$. We have $\mathfrak{m}_{H_1}(H_0) = J\Lambda_{H_1}(H_0) = \mathfrak{m}(H_1)\Lambda_{H_1}(H_0)$. In [SV2] Thm. 4.7 (and Lemma 4.2(ii)) it is shown that $\Lambda_{H_1}(H_0)$ is a noetherian pseudocompact ring which can be viewed as an (infinite) skew Laurent series ring in the variable t_0 over $\Lambda(H_1)$ and which is flat over $\Lambda(H_0)_S$ (and hence $\Lambda(H_0)$). Later on we need the following technical fact.

Lemma 8.1. *For any $m \geq 0$ and any $s \in S(H_0, H_1)$ there is an $l \geq 0$ such that*

$$\mathfrak{m}(H_0)^l \subseteq \big(\Lambda(H_0)\mathfrak{m}(H_1)^m + \mathfrak{m}(H_0)^m s\big) \cap \big(\mathfrak{m}(H_1)^m\Lambda(H_0) + s\mathfrak{m}(H_0)^m\big)\ .$$

Proof. Since the topology of the noetherian ring $\Lambda(H_0)$ is the $\mathfrak{m}(H_0)$-adic one it suffices to show that the left ideal $\Lambda(H_0)\mathfrak{m}(H_1)^m + \mathfrak{m}(H_0)^m s$ and the right ideal $\mathfrak{m}(H_1)^m\Lambda(H_0) + s\mathfrak{m}(H_0)^m$ both are open in $\Lambda(H_0)$. By symmetry we only discuss the former. By [CFKSV] Prop. 2.6 the $\Lambda(H_1)$-module $\Lambda(H_0)/\Lambda(H_0)s$ is finitely generated. Then also $\Lambda(H_0)/\mathfrak{m}(H_0)^m s$ is finitely generated over $\Lambda(H_1)$. It follows that $\Lambda(H_0)/\big(\Lambda(H_0)\mathfrak{m}(H_1)^m + \mathfrak{m}(H_0)^m s\big)$ is finitely generated over $\Lambda(H_1)/\mathfrak{m}(H_1)^m$ and hence is finite. The closed left ideal $\Lambda(H_0)\mathfrak{m}(H_1)^m + \mathfrak{m}(H_0)^m s$ therefore must be open. □

In this section we will consider a triple $H_1 \subseteq H_0 \subseteq H$ of pro-p p-adic Lie groups without elements of order p where H_1 and H_0 are closed normal subgroups of H and both factor groups H_0/H_1 and H/H_0 are isomorphic to $\mathbb{Z}_p$. We then have the inclusions of rings

$$\begin{array}{ccccc} & & & & \Lambda(H) \\ & & & & \uparrow \subseteq \\ \Lambda_{H_1}(H_0) & \xleftarrow{\supseteq} & \Lambda(H_0)_{S(H_0,H_1)} & \xleftarrow{\supseteq} & \Lambda(H_0). \end{array}$$

We do not know whether $S(H_0, H_1)$ also is an Ore set in the bigger ring $\Lambda(H)$. Our goal is to construct, under the additional assumption that H is a semidirect product $H \cong H_1 \rtimes (H/H_1)$, a topological ring which contains the bimodule $\Lambda_{H_1}(H_0) \otimes_{\Lambda(H_0)} \Lambda(H)$ as a dense subgroup.

We pick a topological generator γ of a subgroup of H which maps isomorphically onto H/H_0, and we define $t := \gamma - 1$, $\sigma(\lambda) := \gamma\lambda\gamma^{-1}$ for $\lambda \in \Lambda(H_0)$, and $\delta := \sigma - \mathrm{id}_{\Lambda(H_0)}$. Then $\Lambda(H) = \Lambda(H_0)[[t; \sigma, \delta]]$. Since H_1 is normal in H the ring automorphism σ of $\Lambda(H_0)$ respects the ideals $\mathfrak{m}(H_1)$ and $J(H_0, H_1)$ and the Ore set $S(H_0, H_1)$. Hence σ extends first to a ring automorphism of $\Lambda(H_0)_{S(H_0,H_1)}$ and then further to a ring automorphism of the completion $\Lambda_{H_1}(H_0)$ which we also denote by σ. Correspondingly δ extends to the σ-derivation $\delta := \sigma - \mathrm{id}_{\Lambda_{H_1}(H_0)}$. Recall from the above that $\Lambda_{H_1}(H_0)$ is pseudocompact and noetherian. In particular, the pseudocompact topology is the $\mathfrak{m}_{H_1}(H_0)$-adic one. Hence σ necessarily is a topological automorphism of $\Lambda_{H_1}(H_0)$ and δ is continuous. But unfortunately δ is not topologically nilpotent on $\Lambda_{H_1}(H_0)$ for the pseudocompact topology. So the formalism of [SV1] §1 (in particular Remark 1.1.i) to construct the skew power series ring $\Lambda_{H_1}(H_0)[[t; \sigma, \delta]]$ does not apply formally.

But there is the following coarser topology on $\Lambda_{H_1}(H_0)$, introduced and called the weak topology in [SV2] §1.2. It is given by the fundamental system of open zero neighbourhoods

$$B_m := \mathfrak{m}_{H_1}(H_0)^m + \mathfrak{m}(H_0)^m$$

for $m \geq 0$. Since each B_m is a $\big(\Lambda(H_0), \Lambda(H_0)\big)$-sub-bimodule of $\Lambda_{H_1}(H_0)$ this certainly makes $\Lambda_{H_1}(H_0)$ an additive topological group. We obviously have

$$B_k B_l \subseteq B_{\min(k,l)}\ .$$

Lemma 8.2. *i. The weak topology is a ring topology.*

ii. The weak topology on $\Lambda_{H_1}(H_0)$ induces the compact topology on the subring $\Lambda(H_0)$.

iii. $\bigcap_m B_m = \{0\}$.

iv. The weak topology is complete.

v. Each $\mathfrak{m}_{H_1}(H_0)^m$, for $m \geq 0$, is closed for the weak topology.

vi. σ is a topological automorphism for the weak topology.

Proof. i. It remains to show that for any $m \geq 0$ and any $\mu \in \Lambda_{H_1}(H_0)$ there is an $l \geq 0$ such that $\mu B_l \cup B_l \mu \subseteq B_m$. The other one being analogous we only show the inclusion $\mu B_l \subseteq B_m$. We may write

$$\mu = \mu' + s^{-1}\lambda \qquad \text{with } \mu' \in \mathfrak{m}_{H_1}(H_0)^m, s \in S(H_0, H_1), \text{ and } \lambda \in \Lambda(H_0).$$

According to Lemma 8.1 we find an $l \geq m$ such that $\mathfrak{m}(H_0)^l \subseteq \mathfrak{m}(H_1)^m \Lambda(H_0) + s\mathfrak{m}(H_0)^m$. It follows that

$$s^{-1}\mathfrak{m}(H_0)^l \subseteq \mathfrak{m}_{H_1}(H_0)^m + \mathfrak{m}(H_0)^m \ .$$

Altogether we obtain

$$\mu B_l \subseteq \mathfrak{m}_{H_1}(H_0)^l + \mu\mathfrak{m}(H_0)^l \subseteq \mathfrak{m}_{H_1}(H_0)^m + s^{-1}\mathfrak{m}(H_0)^l \subseteq B_m \ .$$

ii., iii., and iv. According to [SV2] Lemma 4.3.i we may identify $\Lambda_{H_1}(H_0)$ as a (left) $\Lambda(H_1)$-module with

$$\{(\lambda_j)_j \in \prod_{j\in\mathbb{Z}} \Lambda(H_1) : \lim_{j\to-\infty} \lambda_j = 0\}$$

in such a way that

$$\mathfrak{m}_{H_1}(H_0)^m = \{(\lambda_j)_j \in \prod_{j\in\mathbb{Z}} \mathfrak{m}(H_1)^m : \lim_{j\to-\infty} \lambda_j = 0\},$$
$$\Lambda(H_0) = \prod_{j\geq 0} \Lambda(H_1), \text{ and}$$
$$\mathfrak{m}(H_1)^m \Lambda(H_0) = \prod_{j\geq 0} \mathfrak{m}(H_1)^m$$

(loc. cit. Lemma 1.12 and Prop. 2.26.i). We easily read off from this that

$$\mathfrak{m}_{H_1}(H_0)^m \cap \Lambda(H_0) = \mathfrak{m}(H_1)^m \Lambda(H_0) \subseteq \mathfrak{m}(H_0)^m \tag{14}$$

and that

$$\bigcap_{m\geq 0} \big(\mathfrak{m}_{H_1}(H_0)^m + \Lambda(H_0)\big) = \Lambda(H_0) \ .$$

We deduce from the former equation that

$$B_m \cap \Lambda(H_0) = \big(\mathfrak{m}_{H_1}(H_0)^m \cap \Lambda(H_0)\big) + \mathfrak{m}(H_0)^m = \mathfrak{m}(H_0)^m$$

which is ii. Together with the latter equation we get

$$\bigcap_{m\geq 0} B_m = \bigcap_{m\geq 0} B_m \cap \Lambda(H_0) = \bigcap_{m\geq 0} \mathfrak{m}(H_0)^m = \{0\}$$

hence iii. As a $\Lambda(H_1)$-module we have

$$\Lambda_{H_1}(H_0) = \Lambda^-_{H_1}(H_0) \oplus \Lambda(H_0)$$

where, with the above identification,

$$\Lambda^-_{H_1}(H_0) := \{(\lambda_j)_j \in \prod_{j<0} \Lambda(H_1) : \lim_{j\to-\infty} \lambda_j = 0\}$$

and

$$B_m = \left(\mathfrak{m}_{H_1}(H_0)^m \cap \Lambda^-_{H_1}(H_0)\right) \oplus \mathfrak{m}(H_0)^m \ .$$

This means that the weak topology is the direct sum topology of the subspace topology on $\Lambda^-_{H_1}(H_0)$ induced by the pseudocompact topology on $\Lambda_{H_1}(H_0)$ on the one hand and the compact topology on $\Lambda(H_0)$ on the other hand. The latter clearly is complete and the former as well once we show that $\Lambda^-_{H_1}(H_0)$ is closed in $\Lambda_{H_1}(H_0)$ with respect to the pseudocompact topology. But one easily checks that the maps

$$\begin{aligned} \Lambda_{H_1}(H_0) &\longrightarrow \Lambda(H_1) \\ (\lambda_j)_j &\longmapsto \lambda_{j_0} \ , \end{aligned}$$

for any $j_0 \in \mathbb{Z}$, are continuous for the pseudocompact topologies, and $\Lambda^-_{H_1}(H_0)$ is the simultaneous kernel of these maps for $j_0 \geq 0$.

v. Using the above descriptions we have

$$\begin{aligned} \bigcap_{l\geq 0}(\mathfrak{m}_{H_1}(H_0)^m + B_l) &= \bigcap_{l\geq m}(\mathfrak{m}_{H_1}(H_0)^m + \mathfrak{m}(H_0)^l) \\ &= \left(\mathfrak{m}_{H_1}(H_0)^m \cap \Lambda^-_{H_1}(H_0)\right) \oplus \bigcap_{l\geq m} \left((\mathfrak{m}_{H_1}(H_0)^m \cap \Lambda(H_0)) + \mathfrak{m}(H_0)^l\right) \\ &= \left(\mathfrak{m}_{H_1}(H_0)^m \cap \Lambda^-_{H_1}(H_0)\right) \oplus \bigcap_{l\geq m} \left(\mathfrak{m}(H_1)^m \Lambda(H_0) + \mathfrak{m}(H_0)^l\right) \ . \end{aligned}$$

But as a finitely generated ideal $\mathfrak{m}(H_1)^m\Lambda(H_0)$ is closed in the compact ring $\Lambda(H_0)$. This implies that

$$\bigcap_{l\geq m} \left(\mathfrak{m}(H_1)^m \Lambda(H_0) + \mathfrak{m}(H_0)^l\right) = \mathfrak{m}(H_1)^m \Lambda(H_0)$$

and hence that

$$\bigcap_{l\geq 0}(\mathfrak{m}_{H_1}(H_0)^m + B_l) = \mathfrak{m}_{H_1}(H_0)^m \ .$$

vi. This is obvious. □

With σ, of course, also δ is continuous for the weak topology. We are able to say more when the group H is a semidirect product $H \cong H_1 \rtimes (H/H_1)$. In this case we may pick the above elements $\gamma \in H$ and $\gamma_0 \in H_0$ in such a way that we have

$$\gamma\gamma_0\gamma^{-1} = \gamma_0^x \qquad \text{for some } x \in 1 + p\mathbb{Z}_p.$$

Lemma 8.3. *Suppose that $H \cong H_1 \rtimes (H/H_1)$ is a semidirect product; for any $i \geq 1$ we have:*

i. $\delta^{m+i}(t_0^{-i}) \in B_m$ *for any* $m \geq 1$;

ii. $t^i t_0 \in t_0 \mathfrak{m}(H)^i$.

Proof. Inside $\Lambda(H_0)$ we have the subring $o[[t_0]]$. Let $\mathfrak{n}$ denote the maximal ideal of this latter ring. It is generated by π and t_0. Since $o[[t_0]]/\mathfrak{n} = k$ we must have $\delta(o[[t_0]]) \subseteq \mathfrak{n}$. We compute

$$\sigma(t_0) = \gamma(\gamma_0 - 1)\gamma^{-1} = \gamma_0^x - 1 = (t_0 + 1)^x - 1 = \sum_{j \geq 1} \binom{x}{j} t_0^j$$

and see that

$$\sigma(t_0) \in x t_0 + o[[t_0]] t_0^2 = x t_0 (1 + o[[t_0]] t_0) \ .$$

This implies first of all that

$$\delta(t_0) \in (x-1) t_0 + o[[t_0]] t_0^2 \subseteq \big(\pi o + o[[t_0]] t_0\big) t_0 = \mathfrak{n} t_0 \subseteq \mathfrak{n}^2 \ ,$$

hence $\delta(\mathfrak{n}) \subseteq \mathfrak{n}^2$, and then inductively

$$\delta(\mathfrak{n}^j) \subseteq \mathfrak{n}^{j+1} \qquad \text{for any } j \geq 0. \tag{15}$$

Because of $1 + o[[t_0]] t_0 \subseteq o[[t_0]]^\times$ it also implies that

$$\sigma(t_0^{-1}) \in x^{-1} t_0^{-1} (1 + o[[t_0]] t_0) = x^{-1} t_0^{-1} + o[[t_0]] \ .$$

It follows inductively that

$$\sigma(t_0^{-i}) \in x^{-i} t_0^{-i} + o[[t_0]] t_0^{-(i-1)}$$

and hence that

$$\delta(t_0^{-i}) \in \pi o t_0^{-i} + o[[t_0]] t_0^{-(i-1)} = \mathfrak{n} t_0^{-i} \ .$$

Using (15) we deduce by another induction that

$$\delta^j(t_0^{-i}) \in \mathfrak{n}^j t_0^{-i} = \sum_{l=0}^{j} \pi^l t_0^{j-l-i} o[[t_0]] \qquad \text{for any } j \geq 1.$$

In this last sum the summands for $l \geq m$ lie in $\mathfrak{m}_{H_1}(H_0)^m$. On the other hand, for $l < m$ and $j \geq m + i$ we have $j - l - i \geq 0$ and $l + (j - l - i) \geq m$. Hence the corresponding summand in this case lies in $\mathfrak{m}(H_0)^m$. This proves the first assertion.

For the second assertion it suffices, by induction, to consider the case $i = 1$. We compute

$$t t_0 = \sigma(t_0) t + \delta(t_0) \in t_0 o[[t_0]]^\times t + t_0 \mathfrak{n} \subseteq t_0 \mathfrak{m}(H) \ .$$

□

Proposition 8.4. *Suppose that $H \cong H_1 \rtimes (H/H_1)$ is a semidirect product; then δ is locally topologically nilpotent, i. e., for any $\mu \in \Lambda_{H_1}(H_0)$ and any $m \geq 1$ there is a $k \geq 1$ such that $\delta^l(\mu) \in B_m$ for any $l \geq k$.*

Proof. We fix m. We know from [SV2] Prop. 1.2 and Remark 1.11 that $\Theta := \{t_0^j\}_{j\geq 0}$ is an Ore set in $\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0)$ and that

$$\Lambda_{H_1}(H_0)/\mathfrak{m}_{H_1}(H_0)^m = \big(\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0)\big)_\Theta$$

is the corresponding localization. Hence any element $\mu \in \Lambda_{H_1}(H_0)$ can be written, modulo $\mathfrak{m}_{H_1}(H_0)^m$, in the form

$$\mu = \mu_i t_0^{-i} + \mu_{i-1} t_0^{-(i-1)} + \ldots + \mu_1 t_0^{-1} + \mu_0$$

for some $i \geq 0$, $\mu_1, \ldots, \mu_i \in \Lambda(H_1)$, and $\mu_0 \in \Lambda(H_0)$. Since $\mathfrak{m}_{H_1}(H_0)^m$ is δ-invariant and since δ is topologically nilpotent on $\Lambda(H_0)$ ([SV1] Lemma 1.6) it therefore suffices to consider elements μ of the form

$$\mu = \mu_1 t_0^{-i} \qquad \text{for some } i \geq 1 \text{ and } \mu_1 \in \Lambda(H_1).$$

As δ is a σ-derivation and $\Lambda(H_1)$ is σ-invariant one easily verifies by induction that

$$\delta^l(\Lambda(H_1)t_0^{-i}) \subseteq \sum_{j=0}^{l} \delta^{l-j}(\Lambda(H_1))\delta^j(t_0^{-i}) \qquad \text{for any } l \geq 1.$$

Again since δ is topologically nilpotent on $\Lambda(H_0)$ and hence on $\Lambda(H_1)$ we have $\delta^{l-j}(\Lambda(H_1)) \subseteq \mathfrak{m}(H_1)^m$ and hence $\delta^{l-j}(\Lambda(H_1))\delta^j(t_0^{-i}) \subseteq B_m$ whenever $l-j$ is sufficiently big. This reduces us finally to the case $\mu = t_0^{-i}$ for any $i \geq 1$ which we have dealt with in Lemma 8.3.i. $\square$

It will be convenient to write elements in the countable direct product $\prod_{i\geq 0} \Lambda_{H_1}(H_0)$, viewed as a left $\Lambda_{H_1}(H_0)$-module, as formal power series

$$\sum_{i\geq 0} \mu_i t^i \qquad \text{with } \mu_i \in \Lambda_{H_1}(H_0).$$

The ring we want to construct will be a certain submodule of this direct product. For its definition we first have to recall the notion of boundedness in topological rings as well as some of its elementary properties.

Definition 8.5. *Let R be a Hausdorff topological ring; a subset $A \subseteq R$ is called bounded if for any neighbourhood of zero $U' \subseteq R$ there is another neighbourhood of zero $U \subseteq R$ such that $U \cdot A \cup A \cdot U \subseteq U'$ (where $X \cdot Y := \{xy : x \in X, y \in Y\}$).*

Remark 8.6. *Let A, A_1, and A_2 be subsets of a Hausdorff topological ring R; we then have:*

i. If A is bounded then any subset $A_1 \subseteq A$ also is bounded;

ii. with A_1 and A_2 also $A_1 \cup A_2$, $A_1 + A_2$, and $A_1 \cdot A_2$ are bounded;

iii. any compact A (in particular, any finite A) is bounded;

iv. if A is bounded then also its closure $\overline{A}$ is bounded;

v. suppose that R has a fundamental system of neighbourhoods of zero consisting of additive subgroups; then with A also the additive subgroup generated by A is bounded;

vi. any convergent sequence in R forms a bounded subset.

Proof. i. and v. are obvious. vi. is easy. See [War] Thm. 12.3 for iii. and Cor. 12.5 for ii. and iv. $\square$

Lemma 8.7. *Let $A \subseteq \Lambda_{H_1}(H_0)$ be a bounded subset (for the weak topology); then the smallest σ-invariant additive subgroup containing A is bounded as well (and is δ-invariant).*

Proof. Using the fact that $\sigma(B_m) = B_m$ one easily checks that $\bigcup_{j \geq 0} \sigma^j(A)$ is bounded. Now apply Remark 8.6.v. $\square$

Lemma 8.8. *For any subset $A \subseteq \Lambda_{H_1}(H_0)$ the following conditions are equivalent:*

i. A is bounded;

ii. for any $m \geq 1$ there is an $l \geq 1$ such that $t_0^l A \subseteq \mathfrak{m}_{H_1}(H_0)^m + \Lambda(H_0)$;

iii. for any $m \geq 1$ there is an $l \geq 1$ such that $A t_0^l \subseteq \mathfrak{m}_{H_1}(H_0)^m + \Lambda(H_0)$.

Proof. By definition the set A is bounded if and only if for any $m \geq 1$ there is an $l \geq m$ such that

$$B_l \cdot A \cup A \cdot B_l \subseteq B_m \ .$$

Since $\mathfrak{m}_{H_1}(H_0)^l$ is a (two-sided) ideal in $\Lambda_{H_1}(H_0)$ this inclusion is equivalent to the inclusion

$$\mathfrak{m}(H_0)^l \cdot A \cup A \cdot \mathfrak{m}(H_0)^l \subseteq B_m$$

and then also to the inclusion

$$\mathfrak{m}(H_0)^l \cdot A \cup A \cdot \mathfrak{m}(H_0)^l \subseteq \mathfrak{m}_{H_1}(H_0)^m + \Lambda(H_0)$$

(for a possibly different l). The latter trivially implies that

$$t_0^l A \cup A t_0^l \subseteq \mathfrak{m}_{H_1}(H_0)^m + \Lambda(H_0) \ . \tag{16}$$

Suppose, vice versa, that (16) holds true for some l (depending on m) which we then may assume to satisfy $l \geq m$. By Lemma 8.1 applied to $s := t_0^l$ we find an $l' \geq 0$ such that

$$\mathfrak{m}(H_0)^{l'} \subseteq \big(J(H_0, H_1)^l + \Lambda(H_0) t_0^l\big) \cap \big(J(H_0, H_1)^l + t_0^l \Lambda(H_0)\big) \ .$$

It easily follows that

$$\mathfrak{m}(H_0)^{l'} \cdot A \cup A \cdot \mathfrak{m}(H_0)^{l'} \subseteq \mathfrak{m}_{H_1}(H_0)^m + \Lambda(H_0) \ .$$

It remains to show that the conditions ii. and iii. are equivalent. In fact, we fix m. As recalled earlier we know from [SV2] Remark 1.11 that

$$\Lambda_{H_1}(H_0)/\mathfrak{m}_{H_1}(H_0)^m = \big(\Lambda(H_1)/\mathfrak{m}(H_1)^m\big)((t_0; \overline{\sigma}_0, \overline{\delta}_0))$$

is a skew Laurent series (with finite negative parts) ring where $\overline{\sigma}_0$ and $\overline{\delta}_0$ denote the maps on the coefficients induced by σ_0 and δ_0, respectively. The condition $t_0^l A \subseteq \mathfrak{m}_{H_1}(H_0)^m + \Lambda(H_0)$

therefore means that the image in $\Lambda_{H_1}(H_0)/\mathfrak{m}_{H_1}(H_0)^m$ of each element in A can be written in the form

$$\sum_{i\geq -l} t_0^i r_i \qquad \text{with } r_i \in \Lambda(H_1)/\mathfrak{m}(H_1)^m \ .$$

But $\overline{\delta}_0$ is nilpotent on $\Lambda(H_1)/\mathfrak{m}(H_1)^m$, say $\overline{\delta}_0^{N+1} = 0$. The formula (1.4) in [SV2] then shows an identity of the form

$$\sum_{i\geq -l} t_0^i r_i = \sum_{j\geq -l-N} r'_j t_0^j$$

whatever the coefficients r_i are. It therefore follows that $At_0^{l+N} \subseteq \mathfrak{m}_{H_1}(H_0)^m + \Lambda(H_0)$. In the other direction the argument works in the same way (cf. (1.3) in [SV2]). $\square$

Remark 8.9. *The proof of Lemma 8.8 in particular shows that a subset $A \subseteq \Lambda_{H_1}(H_0)$ is bounded if and only if it is left bounded if and only if it is right bounded.*

Corollary 8.10. *Suppose that $H \cong H_1 \rtimes (H/H_1)$ is a semidirect product; for any bounded subset $A \subseteq \Lambda_{H_1}(H_0)$ and any $m \geq 1$ there is a $k \geq 1$ such that $\delta^k(A) \subseteq B_m$.*

Proof. By Lemma 8.8 we find, for any $m \geq 1$, an $i \geq 1$ such that

$$A \subseteq \Lambda(H_1)t_0^{-i} + \ldots + \Lambda(H_1)t_0^{-1} + \Lambda(H_0) + \mathfrak{m}_{H_1}(H_0)^m \ .$$

From here on the argument proceeds as in the proof of Prop. 8.4. $\square$

We now define

$$\Lambda_{H_0,H_1}(H) := \{\sum_{i\geq 0} \mu_i t^i : \{\mu_i\}_i \text{ is bounded in } \Lambda_{H_1}(H_0)\} \ .$$

By Remark 8.6.ii/iii this is a $\Lambda_{H_1}(H_0)$-submodule of the direct product. We view $\Lambda_{H_1}(H_0)$ as being contained in $\Lambda_{H_0,H_1}(H)$ through $\mu \mapsto \mu + 0t + 0t^2 + \ldots$ On the other hand viewing $\Lambda(H) = \Lambda(H_0)[[t;\sigma,\delta]]$ as a skew power series ring in t and noting that $\Lambda(H_0)$ is compact and therefore, by Lemma 8.2.ii and Remark 8.6.iii, bounded in $\Lambda_{H_1}(H_0)$ we see that the ring $\Lambda(H)$ in an obvious way is contained in $\Lambda_{H_0,H_1}(H)$ as well. More generally, by Remark 8.6.ii the map

$$\begin{aligned} \Lambda_{H_1}(H_0) \otimes_{\Lambda(H_0)} \Lambda(H) &\longrightarrow \Lambda_{H_0,H_1}(H) \\ \lambda \otimes (\sum_{i\geq 0} \mu_i t^i) &\longmapsto \sum_{i\geq 0} \lambda\mu_i t^i \end{aligned} \tag{17}$$

is well defined.

Remark 8.11. *The above map* (17) *is injective.*

Proof. Suppose that the element $\sum_{k=0}^r \lambda_k \otimes (\sum_{i\geq 0} \mu_{k,i} t^i)$ lies in the kernel of (17). This means that

$$\sum_{k=0}^r \lambda_k \mu_{k,i} = 0 \qquad \text{for any } i \geq 0.$$

Let X denote the right $\Lambda(H_0)$-submodule of $\Lambda(H_0)^r$ generated by the vectors $(\mu_{k,i})_k$ for $i \geq 0$. Since $\Lambda(H_0)$ is noetherian this module X is finitely generated. Hence we find vectors $(\nu_{1,k})_k, \ldots, (\nu_{s,k})_k \in X$ and elements $c_{1,i}, \ldots, c_{s,i} \in \Lambda(H_0)$ for $i \geq 0$ such that

$$(\mu_{k,i})_k = (\nu_{1,k})_k c_{1,i} + \ldots + (\nu_{s,k})_k c_{s,i} \qquad \text{for any } i \geq 0.$$

Of course we have

$$\sum_{k=0}^{r} \lambda_k \nu_{1,k} = \ldots = \sum_{k=0}^{r} \lambda_k \nu_{s,k} = 0\ .$$

We now compute

$$\begin{aligned}
\sum_{k=0}^{r} \lambda_k \otimes (\sum_{i\geq 0} \mu_{k,i} t^i) &= \sum_{k=0}^{r} \lambda_k \otimes \sum_{i \geq 0} (\nu_{1,k} c_{1,i} + \ldots + \nu_{s,k} c_{s,i}) t^i \\
&= \sum_{k=0}^{r} \sum_{l=0}^{s} \lambda_k \otimes \nu_{l,k} (\sum_{i\geq 0} c_{l,i} t^i) \\
&= \sum_{l=0}^{s} (\sum_{k=0}^{r} \lambda_k \nu_{l,k}) \otimes (\sum_{i \geq 0} c_{l,i} t^i) \\
&= 0\ .
\end{aligned}$$

□

On $\Lambda_{H_0,H_1}(H)$ we have the descending filtration

$$F^m \Lambda_{H_0,H_1}(H) := \{\sum_{i\geq 0} \mu_i t^i \in \Lambda_{H_0,H_1}(H) : \{\mu_i\}_i \subseteq \mathfrak{m}_{H_1}(H_0)^m\} \qquad \text{for } m \geq 0$$

by $\Lambda_{H_1}(H_0)$-submodules. By taking this filtration as a fundamental system of open zero neighbourhoods we obtain the *strong* topology on $\Lambda_{H_0,H_1}(H)$. The fact that $\mathfrak{m}_{H_1}(H_0)^m \Lambda_{H_0,H_1}(H) \subseteq F^m \Lambda_{H_0,H_1}(H)$ implies that $\Lambda_{H_0,H_1}(H)$ with the strong topology is a (left) topological module over $\Lambda_{H_1}(H_0)$ with its pseudocompact topology.

Lemma 8.12. *i. $\Lambda_{H_0,H_1}(H)$ is Hausdorff and complete in the strong topology.*

ii. $\mathfrak{m}_{H_1}(H_0)^m \Lambda_{H_0,H_1}(H)$, for any $m \geq 0$, is dense in $F^m \Lambda_{H_0,H_1}(H)$ for the strong topology.

iii. $\Lambda_{H_1}(H_0) \otimes_{\Lambda(H_0)} \Lambda(H)$ is dense in $\Lambda_{H_0,H_1}(H)$ for the strong topology.

iv. For any $m \geq 0$ the natural map

$$\big(\Lambda_{H_1}(H_0)/\mathfrak{m}_{H_1}(H_0)^m\big) \otimes_{\Lambda(H_0)} \Lambda(H) \xrightarrow{\cong} \Lambda_{H_0,H_1}(H)/F^m \Lambda_{H_0,H_1}(H)$$

is an isomorphism.

Proof. i. We have

$$\begin{aligned}
\Lambda_{H_0,H_1}(H) \subseteq \varprojlim_m \big(\Lambda_{H_0,H_1}(H)/F^m \Lambda_{H_0,H_1}(H)\big) &\subseteq \varprojlim_m \big(\prod_{i\geq 0} \Lambda_{H_1}(H_0)/\prod_{i\geq 0} \mathfrak{m}_{H_1}(H_0)^m\big) \\
&= \prod_{i \geq 0} \varprojlim_m \big(\Lambda_{H_1}(H_0)/\mathfrak{m}_{H_1}(H_0)^m\big) \\
&= \prod_{i\geq 0} \Lambda_{H_1}(H_0)
\end{aligned}$$

Therefore it suffices to show that

$$\bigcap_{m\geq 0}\big(\Lambda_{H_0,H_1}(H)+\prod_{i\geq 0}\mathfrak{m}_{H_1}(H_0)^m\big)=\Lambda_{H_0,H_1}(H)\ .$$

Let $\sum_{i\geq 0}\mu_i t^i$ be a power series contained in the left hand side, and let $k\geq 0$. By assumption we find a power series $\sum_{i\geq 0}\nu_i t^i\in\Lambda_{H_0,H_1}(H)$ such that $\mu_i-\nu_i\in\mathfrak{m}_{H_1}(H_0)^k\subseteq B_k$ for any $i\geq 0$. There is an $l\geq k$ such that $B_l\cdot\{\nu_i\}_i\subseteq B_k$. It follows that

$$B_l\cdot\{\mu_i\}_i\subseteq B_l\cdot\{\mu_i-\nu_i\}_i+B_l\cdot\{\nu_i\}_i\subseteq B_k\ .$$

Hence $\{\mu_i\}_i$ is left bounded and therefore bounded by Remark 8.9.

ii. and iii. Let $\lambda=\sum_{i\geq 0}\lambda_i t^i\in F^m\Lambda_{H_0,H_1}(H)$, and let $m'\geq m$. By Lemma 8.8.ii we find an $l\geq 1$ and elements $\mu_i\in\Lambda(H_0)$ such that

$$\lambda_i-t_0^{-l}\mu_i\in\mathfrak{m}_{H_1}(H_0)^{m'}\qquad\text{for any } i\geq 0.$$

We put $\mu:=\sum_{i\geq 0}\mu_i t^i\in\Lambda(H)$ and obtain $\lambda-t_0^{-l}\mu\in F^{m'}\Lambda_{H_0,H_1}(H)$. In particular, we have $t_0^{-l}\mu\in F^m\Lambda_{H_0,H_1}(H)$ and hence

$$\mu\in F^m\Lambda_{H_0,H_1}(H)\cap\Lambda(H)=\mathfrak{m}(H_1)^m\Lambda(H)$$

where the right hand identity comes from (14) in the proof of Lemma 8.2. We see that λ, modulo $F^{m'}\Lambda_{H_0,H_1}(H)$, is congruent to $t_0^{-l}\mu\in\mathfrak{m}_{H_1}(H_0)^m\Lambda(H)$. This proves both assertions.

iv. Surjectivity is immediate from iii., and the injectivity is a computation totally analogous to the one in the proof of Remark 8.11. $\square$

Obviously, the skew polynomial ring $\Lambda_{H_1}(H_0)[t;\sigma,\delta]$ is contained as a $\Lambda_{H_1}(H_0)$-submodule in $\Lambda_{H_0,H_1}(H)$. The multiplication in this skew polynomial ring is given by the formula

$$(18)\qquad (\sum_{i\geq 0}\lambda_i t^i)(\sum_{j\geq 0}\mu_j t^j)=\sum_{l\geq 0}\big(\sum_{k=0}^{l}\sum_{i\geq k}\binom{i}{k}\lambda_i\delta^{i-k}(\sigma^k(\mu_{l-k}))\big)t^l\ .$$

Proposition 8.13. *Suppose that $H\cong H_1\rtimes(H/H_1)$ is a semidirect product; then the formula* (18) *defines (via convergence of the sums on the right hand side for the weak topology on $\Lambda_{H_1}(H_0)$) a multiplication map on $\Lambda_{H_0,H_1}(H)$ making the latter into a ring in which the $F^m\Lambda_{H_0,H_1}(H)$ are two-sided ideals.*

Proof. We begin by checking that the coefficients on the right hand side of (18) are well defined in $\Lambda_{H_1}(H_0)$ whenever the two factors on the left hand side lie in $\Lambda_{H_0,H_1}(H)$. Because of Lemma 8.2.iv it suffices to show that, for any bounded sequence $\{\nu_1,\nu_2,\ldots\}\subseteq\Lambda_{H_1}(H_0)$ and any singleton $\nu\in\Lambda_{H_1}(H_0)$, the sequence $\nu_1\delta(\nu),\nu_2\delta^2(\nu),\nu_3\delta^3(\nu),\ \ldots$ tends to zero with respect to the weak topology. Given any neighbourhood of zero B_m we find, by boundedness, some $m'\geq 0$ such that $\bigcup_{k\geq 0}\nu_k B_{m'}\subseteq B_m$. But according to Prop. 8.4 we have $\delta^k(\nu)\in B_{m'}$ for any sufficiently big k. It follows that $\nu_k\delta^k(\nu)\in B_m$ for any sufficiently big k.

That with $\{\lambda_i\}_i$ and $\{\mu_j\}_j$ also the sequence of coefficients on the right hand side of (18) is bounded is a straightforward consequence of Lemma 8.7 and Remark 8.6.ii/iv/v.

The resulting multiplication map

$$\cdot:\Lambda_{H_0,H_1}(H)\times\Lambda_{H_0,H_1}(H)\longrightarrow\Lambda_{H_0,H_1}(H)$$

clearly is bi-additive. Since the $\mathfrak{m}_{H_1}(H_0)^m$ are σ- and hence δ-invariant (two-sided) ideals in $\Lambda_{H_1}(H_0)$ which are closed with respect to the weak topology by Lemma 8.2.v we easily see that this multiplication satisfies

$$F^m\Lambda_{H_0,H_1}(H)\cdot\Lambda_{H_0,H_1}(H)\cup\Lambda_{H_0,H_1}(H)\cdot F^m\Lambda_{H_0,H_1}(H)\subseteq F^m\Lambda_{H_0,H_1}(H)\quad\text{for any } m\geq 0\,.$$

It remains to establish the associativity of this multiplication. Let $\lambda=\sum_i\lambda_i t^i$, $\mu=\sum_j\mu_j t^j$, and $\nu=\sum_k\nu_k t^k$ be three elements in $\Lambda_{H_0,H_1}(H)$. We have

$$\alpha:=\lambda\cdot\mu=\sum_m\alpha_m t^m\qquad\text{with}\quad\alpha_m:=\sum_{a=0}^{m}\sum_b\binom{b}{a}\lambda_b\delta^{b-a}(\sigma^a(\mu_{m-a}))$$

$$\beta:=\mu\cdot\nu=\sum_n\beta_n t^n\qquad\text{with}\quad\beta_n:=\sum_{c=0}^{n}\sum_f\binom{f}{c}\mu_f\delta^{f-c}(\sigma^c(\nu_{n-c}))$$

and

$$(\lambda\cdot\mu)\cdot\nu=\alpha\cdot\nu=\sum_l\big(\sum_{e=0}^{l}\sum_f\binom{f}{e}\alpha_f\delta^{f-e}(\sigma^e(\nu_{l-e}))\big)t^l$$

$$\lambda\cdot(\mu\cdot\nu)=\lambda\cdot\beta=\sum_l\big(\sum_{g=0}^{l}\sum_b\binom{b}{g}\lambda_b\delta^{b-g}(\sigma^g(\beta_{l-g}))\big)t^l.$$

Hence we have to show the identity

$$\sum_{e=0}^{l}\sum_f\binom{f}{e}\alpha_f\delta^{f-e}(\sigma^e(\nu_{l-e}))=\sum_{g=0}^{l}\sum_b\binom{b}{g}\lambda_b\delta^{b-g}(\sigma^g(\beta_{l-g}))\tag{19}$$

for any $l\geq 0$. We first compute the right hand side. By inserting the definition of β_n and using that σ and hence also δ are continuous for the weak topology (Lemma 8.2.vi) we obtain

$$\begin{aligned}&\sum_{g=0}^{l}\sum_b\binom{b}{g}\lambda_b\sum_{c=0}^{l-g}\sum_f\binom{f}{c}\delta^{b-g}(\sigma^g(\mu_f\delta^{f-c}(\sigma^c(\nu_{l-g-c}))))\\&=\sum_b\lambda_b\sum_f\sum_{g=0}^{l}\sum_{c=0}^{l-g}\binom{b}{g}\binom{f}{c}\delta^{b-g}(\sigma^g(\mu_f)\delta^{f-c}(\sigma^{g+c}(\nu_{l-(g+c)})))\\&=\sum_b\lambda_b\sum_f\sum_{g=0}^{l}\sum_{e=g}^{l}\binom{b}{g}\binom{f}{e-g}\delta^{b-g}(\sigma^g(\mu_f)\delta^{f-e+g}(\sigma^e(\nu_{l-e}))).\end{aligned}$$

where in the last identity we have substituted e for $g+c$. By applying the general Leibniz type rule for the σ-derivation δ which, since σ and δ commute, reads

$$\delta^r(ab)=\sum_{s=0}^{r}\binom{r}{s}\delta^{r-s}(\sigma^s(a))\delta^s(b)$$

we continue computing

$$= \sum_b \lambda_b \sum_f \sum_{g=0}^{l} \sum_{e=g}^{l} \binom{b}{g} \binom{f}{e-g} \sum_{s=0}^{b-g} \binom{b-g}{s} \delta^{b-g-s}(\sigma^{g+s}(\mu_f))\delta^{f-e+g+s}(\sigma^e(\nu_{l-e}))$$

$$= \sum_b \lambda_b \sum_f \sum_{g=0}^{l} \sum_{a=g}^{b} \sum_{e=g}^{l} \binom{b}{g} \binom{f}{e-g} \binom{b-g}{a-g} \delta^{b-a}(\sigma^{a}(\mu_f))\delta^{f-e+a}(\sigma^e(\nu_{l-e}))$$

$$= \sum_b \lambda_b \sum_f \sum_{a=0}^{b} \sum_{e=0}^{l} [\sum_{g=0}^{\min(a,e)} \binom{b}{g} \binom{f}{e-g} \binom{b-g}{a-g}] \delta^{b-a}(\sigma^{a}(\mu_f))\delta^{f-e+a}(\sigma^e(\nu_{l-e}))$$

where in the middle identity we have substituted a for $g+s$. One easily checks that

$$\sum_{g=0}^{\min(a,e)} \binom{b}{g} \binom{f}{e-g} \binom{b-g}{a-g} = \binom{b}{a} \sum_{g=0}^{e} \binom{a}{g} \binom{f}{e-g} = \binom{b}{a} \binom{f+a}{e} .$$

Hence the right side of (19) is equal to

$$\sum_b \lambda_b \sum_f \sum_{a=0}^{b} \sum_{e=0}^{l} \binom{b}{a} \binom{f+a}{e} \delta^{b-a}(\sigma^{a}(\mu_f))\delta^{f+a-e}(\sigma^e(\nu_{l-e})) .$$

Next we argue that in this latter multi-sum the summations over b and over f can be interchanged. For this we first check that, for any given $m \geq 0$, all but finitely many of the elements

$$x_{b,f} := \sum_{a=0}^{b} \sum_{e=0}^{l} \binom{b}{a} \binom{f+a}{e} \delta^{b-a}(\sigma^{a}(\mu_f))\delta^{f+a-e}(\sigma^e(\nu_{l-e}))$$

(recall that l is arbitrary but fixed) lie in B_m. By Lemma 8.7 the set

$$A := \{\delta^{b-a}(\sigma^a(\mu_f))\}_{a,b,f} \cup \{\delta^{f+a-e}(\sigma^e(\nu_{l-e}))\}_{a,e,f} .$$

is bounded. Hence we find an $m' \geq 0$ such that

$$A \cdot B_{m'} \cup B_{m'} \cdot A \subseteq B_m .$$

By Prop. 8.4 there is an $N_1 \geq 0$ such that $\delta^{f+a-e}(\sigma^e(\nu_{l-e})) \in B_{m'}$ whenever $f+a \geq N_1$. By Cor. 8.10 there is an $N_2 \geq 0$ such that $\delta^{b-a}(\sigma^a(\mu_f)) \in B_{m'}$ whenever $b-a \geq N_2$. We conclude that

$$\delta^{b-a}(\sigma^a(\mu_f))\delta^{f+a-e}(\sigma^e(\nu_{l-e})) \in B_m$$

provided $f+a \geq N_1$ or $b-a \geq N_2$. Since $f+a < N_1$ and $b-a < N_2$ together imply $f+b < N_1+N_2$ we finally see that $x_{b,f} \in B_m$ for $b+f \geq N_1+N_2$.

Since $\{\lambda_b\}_b$ is bounded the family $\{\lambda_b x_{b,f}\}_{b,f}$ also has the property that, for any given $m \geq 0$, all but finitely many of its elements lie in B_m. It follows that the right hand side of

(19) is equal to

$$\sum_f \sum_b \sum_{a=0}^{b} \sum_{e=0}^{l} \binom{b}{a} \binom{f+a}{e} \lambda_b \delta^{b-a}(\sigma^a(\mu_f)) \delta^{f+a-e}(\sigma^e(\nu_{l-e}))$$
$$= \sum_f \sum_b \sum_{a=0}^{f} \sum_{e=0}^{l} \binom{b}{a} \binom{f}{e} \lambda_b \delta^{b-a}(\sigma^a(\mu_{f-a})) \delta^{f-e}(\sigma^e(\nu_{l-e}))$$
$$= \sum_{e=0}^{l} \sum_f \binom{f}{e} \big(\sum_{a=0}^{f} \sum_b \binom{b}{a} \lambda_b \delta^{b-a}(\sigma^a(\mu_{f-a})) \big) \delta^{f-e}(\sigma^e(\nu_{l-e}))$$
$$= \sum_{e=0}^{l} \sum_f \binom{f}{e} \alpha_f \delta^{f-e}(\sigma^e(\nu_{l-e}))$$

which is the left hand side of (19). □

For the remainder of this section we *assume that $H \cong H_1 \rtimes (H/H_1)$ is a semidirect product.* In the proof of Prop. 8.4 we had recalled already from [SV2] that $\Theta = \{t_0^j\}_{j \geq 0}$ is an Ore set in $\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0)$, for any $m \geq 1$ with

$$\Lambda_{H_1}(H_0)/\mathfrak{m}_{H_1}(H_0)^m = \big(\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0)\big)_\Theta \ .$$

Lemma 8.14. *The set Θ consists of regular elements and satisfies the (left and right) Ore condition in $\Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H)$ for any $m \geq 1$.*

Proof. According to Lemma 8.3.ii we have $t^i t_0 = t_0 \nu_i$ with $\nu_i \in \mathfrak{m}(H)^i$. We fix an $m \geq 1$, and we let $\lambda = \sum_{i \geq 0} t^i \lambda_i \in \Lambda(H)$ with $\lambda_i \in \Lambda(H_0)$ be an arbitrary element. By the "right" version of [SV2] Lemma 1.1.ii there exist an $M \geq 0$ and $\mu_i \in \Lambda(H_0)$ such that

$$\lambda_i t_0^M \equiv t_0 \mu_i \mod \mathfrak{m}(H_1)^m \Lambda(H_0)$$

for any $i \geq 0$. We obtain

$$\lambda t_0^M = \sum_{i \geq 0} t^i \lambda_i t_0^M \equiv \sum_{i \geq 0} t^i t_0 \mu_i = t_0 (\sum_{i \geq 0} \nu_i \mu_i) \mod \mathfrak{m}(H_1)^m \Lambda(H)$$

where $\mu := \sum_{i \geq 0} \nu_i \mu_i \in \Lambda(H)$ is well defined because of $\nu_i \in \mathfrak{m}(H)^i$; note that $\mathfrak{m}(H_1)^m\Lambda(H) = \Lambda(H)\mathfrak{m}(H_1)^m$. By a straightforward induction we deduce from this that for any $j \geq 1$ and any $\lambda \in \Lambda(H)$ we find a $\mu \in \Lambda(H)$ such that

$$\lambda t_0^{Mj} \equiv t_0^j \mu \mod \mathfrak{m}(H_1)^m \Lambda(H) \ .$$

This in particular gives the asserted right Ore condition.

For right regularity we write an arbitrary element $\lambda \in \Lambda(H)$ as $\lambda = \sum_{i \geq 0} \lambda_i t^i$ with $\lambda_i = \sum_{k \geq 0} t_0^k \mu_{i,k} \in \Lambda(H_0)$ and $\mu_{i,k} \in \Lambda(H_1)$. If $t_0^j \lambda \in \mathfrak{m}(H_1)^m \Lambda(H)$ then $t_0^j \lambda_i = \sum_{k \geq 0} t_0^{j+k} \mu_{i,k} \in \mathfrak{m}(H_1)^m \Lambda(H_0) = \Lambda(H_0)\mathfrak{m}(H_1)^m$ for any $i \geq 0$, hence $\mu_{i,k} \in \mathfrak{m}(H_1)^m$ for any $i, k \geq 0$, and therefore $\lambda \in \mathfrak{m}(H_1)^m \Lambda(H)$.

The left Ore condition and left regularity follow by analogous arguments. □

We have the obvious injective ring homomorphisms

$$\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0) \hookrightarrow \Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H) \hookrightarrow \Lambda_{H_0,H_1}(H)/F^m\Lambda_{H_0,H_1}(H)$$

where the injectivity of the right hand map is a consequence of (14). By the universal property of localization they extend to injective ring homomorphisms

$$\big(\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0)\big)_\Theta \hookrightarrow \big(\Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H)\big)_\Theta \hookrightarrow \Lambda_{H_0,H_1}(H)/F^m\Lambda_{H_0,H_1}(H)\ .$$

Proposition 8.15. *We have as rings:*

i. $\big(\Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H)\big)_\Theta = \Lambda_{H_0,H_1}(H)/F^m\Lambda_{H_0,H_1}(H)$*;*

ii. $\Lambda_{H_0,H_1}(H) = \varprojlim_m \big(\Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H)\big)_\Theta$*;*

iii. The ring $\Lambda_{H_0,H_1}(H)$ *is, up to isomorphism, independent of the choice of the variables* t_0 *and* t*.*

Proof. i. Because of

$$\begin{aligned}
&\big(\Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H)\big)_\Theta \\
&\quad = \big(\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0) \otimes_{\Lambda(H_0)} \Lambda(H)\big)_\Theta \\
&\quad = \big(\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0)\big)_\Theta \otimes_{\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0)} \big(\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0) \otimes_{\Lambda(H_0)} \Lambda(H)\big) \\
&\quad = \big(\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0)\big)_\Theta \otimes_{\Lambda(H_0)} \Lambda(H) \\
&\quad = \Lambda_{H_1}(H_0)/\mathfrak{m}_{H_1}(H_0)^m \otimes_{\Lambda(H_0)} \Lambda(H)
\end{aligned}$$

this follows from Lemma 8.12.iv.

ii. As $\Lambda_{H_0,H_1}(H) = \varprojlim_m \Lambda_{H_0,H_1}(H)/F^m\Lambda_{H_0,H_1}(H)$ by Lemma 8.12.i this is a consequence of i.

iii. Because of ii. it suffices to show that $\big(\Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H)\big)_\Theta$ is independent of the choice of t_0. Let $\widetilde{\Theta} := \{\widetilde{t_0}^j\}_{j\geq 0}$ for some other choice. We have

$$\big(\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0)\big)_\Theta = \Lambda_{H_1}(H_0)/\mathfrak{m}_{H_1}(H_0)^m = \big(\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0)\big)_{\widetilde{\Theta}}\ .$$

Hence $\widetilde{t_0}$, resp. t_0, is a unit in $\big(\Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H)\big)_\Theta$, resp. $\big(\Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H)\big)_{\widetilde{\Theta}}$. This implies

$$\big(\Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H)\big)_\Theta = \big(\Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H)\big)_{\widetilde{\Theta}}\ .$$

□

It follows from Prop. 8.15 that any element $\mu \in \Lambda_{H_0,H_1}(H)$ can be written in the form $\mu = \sum_{i\geq 0} t^i\mu_i$ with $\{\mu_i\}_i \subseteq \Lambda_{H_1}(H_0)$ a bounded subset.

We want to define and investigate a "weak" topology on the ring $\Lambda_{H_0,H_1}(H)$ which actually will be more important than the strong topology. To motivate the definition we point out that as a consequence of Lemma 8.8 we have

$$\Lambda_{H_0,H_1}(H) = \bigcup_{j\geq 0} F^m\Lambda_{H_0,H_1}(H) + t_0^{-j}\Lambda(H) \qquad \text{for any } m \geq 0. \tag{20}$$

An additive subgroup $C \subseteq \Lambda_{H_0,H_1}(H)$ will be called open for the weak topology if

– $F^m\Lambda_{H_0,H_1}(H) \subseteq C$ for some $m \geq 0$ and

– for any $j \geq 0$ there is an $\ell(j) \geq 0$ such that $C \supseteq t_0^{-j}\mathfrak{m}(H)^{\ell(j)}$.

Correspondingly the *weak* topology on $\Lambda_{H_0,H_1}(H)$ is defined to be the topology for which the additive subgroups

$$C_{m,\ell} := F^m\Lambda_{H_0,H_1}(H) + \sum_{j\geq 0} t_0^{-j}\mathfrak{m}(H)^{\ell(j)} \ ,$$

with $m \geq 0$ and $\ell : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ any function, form a fundamental system of open zero neighbourhoods. This certainly makes $\Lambda_{H_0,H_1}(H)$ into an additive topological group. But one easily checks that multiplication by t_0^k, for any $k \in \mathbb{Z}$, is a topological automorphism.

Remark 8.16. *The weak topology on $\Lambda_{H_0,H_1}(H)$ is independent of the choice of the variable t_0.*

Proof. For the purposes of this proof we write $C_{m,\ell}(t_0) := C_{m,\ell}$. Let $\widetilde{t_0}$ be another choice of variable. For any $j \geq 0$ there is an $i(j,m) \geq 0$ such that

$$\widetilde{t_0}^{-j} \in \mathfrak{m}_{H_1}(H_0)^m + t_0^{-i(j,m)}\Lambda(H_0) \ .$$

We define a new function $\widetilde{\ell} : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ by $\widetilde{\ell}(j) := \ell(i(j,m))$ and obtain $C_{m,\widetilde{\ell}}(\widetilde{t_0}) \subseteq C_{m,\ell}(t_0)$. Hence, by symmetry, the two neighbourhood bases $\{C_{m,\ell}(\widetilde{t_0})\}_{m,\ell}$ and $\{C_{m,\ell}(t_0)\}_{m,\ell}$ define the same topology. □

Lemma 8.17. *Given any $m' \geq m \geq 0$ there is a function ℓ such that for any $\lambda = \sum_{i\geq 0}\lambda_i t^i \in C_{m,\ell}$ we have*

$$\lambda_i \in \mathfrak{m}_{H_1}(H_0)^m + \mathfrak{m}(H_0)^{m'} \subseteq B_m \qquad \text{for any } 0 \leq i \leq m'.$$

Proof. Applying Lemma 8.1 to m' and the element $s = t_0^j$, for any $j \geq 0$, we find an $\ell'(j) \geq 0$ such that

$$\mathfrak{m}(H_0)^{\ell'(j)} \subseteq \mathfrak{m}(H_1)^{m'}\Lambda(H_0) + t_0^j\mathfrak{m}(H_0)^{m'} \subseteq \mathfrak{m}_{H_1}(H_0)^{m'} + t_0^j\mathfrak{m}(H_0)^{m'}$$

and hence

$$t_0^{-j}\mathfrak{m}(H_0)^{\ell'(j)} \subseteq \mathfrak{m}_{H_1}(H_0)^{m'} + \mathfrak{m}(H_0)^{m'} \subseteq \mathfrak{m}_{H_1}(H_0)^m + \mathfrak{m}(H_0)^{m'} \ .$$

Since the $\mathfrak{m}(H)$-adic topology on $\Lambda(H) = \Lambda(H_0)[[t;\sigma,\delta]] \cong \prod_{i\geq 0}\Lambda(H_0)$ coincides with the direct product topology (cf. [SV1] §1) we then may pick $\ell(j) \geq 0$ in such a way that

$$\mathfrak{m}(H)^{\ell(j)} \subseteq (\sum_{i=0}^{m'}\mathfrak{m}(H_0)^{\ell'(j)}t^i) + \Lambda(H)t^{m'+1} \ .$$

Suppose now that $\lambda = \sum_{i\geq 0}\lambda_i t^i \in C_{m,\ell}$. Then λ_i, for any $0 \leq i \leq m'$, lies in

$$\mathfrak{m}_{H_1}(H_0)^m + \sum_{j\geq 0} t_0^{-j}\mathfrak{m}(H_0)^{\ell'(j)} \subseteq \mathfrak{m}_{H_1}(H_0)^m + \mathfrak{m}(H_0)^{m'} \ .$$

□

Proposition 8.18. *i. The weak topology of $\Lambda_{H_0,H_1}(H)$ induces on $\Lambda_{H_1}(H_0)$, resp. on $\Lambda(H)$, the weak, resp. compact, topology.*

ii. $F^m\Lambda_{H_0,H_1}(H)$, for any $m \geq 0$, is closed in $\Lambda_{H_0,H_1}(H)$ for the weak topology.

iii. $\Lambda_{H_0,H_1}(H)$ is Hausdorff and complete in the weak topology.

Proof. i. We obviously have $B_{\max(m,\ell(0))} \subseteq C_{m,\ell}$. On the other hand, if for a given $m \geq 0$ we choose the function ℓ as in the above Lemma 8.17 then we obtain

$$C_{m,\ell} \cap \Lambda_{H_1}(H_0) \subseteq B_m \ .$$

To determine the topology induced on $\Lambda(H)$ we note that obviously $\mathfrak{m}(H)^{\ell(0)} \subseteq C_{m,\ell}$. On the other hand, given any $m \geq 0$, we find, by the direct product topology argument, an $m' \geq 0$ such that

$$(\sum_{i=0}^{m'} \mathfrak{m}(H_0)^{m'} t^i) + \Lambda(H) t^{m'+1} \subseteq \mathfrak{m}(H)^m \ .$$

We consider now any $\lambda = \sum_{i \geq 0} \lambda_i t^i \in C_{m',\ell} \cap \Lambda(H)$ where the function ℓ is chosen as in Lemma 8.17 for the pair (m', m'). Then λ_i, for any $0 \leq i \leq m'$, lies in

$$\big(\mathfrak{m}_{H_1}(H_0)^{m'} + \mathfrak{m}(H_0)^{m'}\big) \cap \Lambda(H_0) = \big(\mathfrak{m}_{H_1}(H_0)^{m'} \cap \Lambda(H_0)\big) + \mathfrak{m}(H_0)^{m'} = \mathfrak{m}(H_0)^{m'}$$

where the last identity uses the fact (cf. formula (14) in the proof of Lemma 8.2) that

$$\mathfrak{m}_{H_1}(H_0)^{m'} \cap \Lambda(H_0) = \mathfrak{m}(H_1)^{m'} \Lambda(H_0) \subseteq \mathfrak{m}(H_0)^{m'} \ .$$

By the choice of m' this means that $\lambda \in \mathfrak{m}(H)^m$.

ii. We choose for any $m' \geq m$ a function $\ell_{m',m}$ as in Lemma 8.17. Any $\lambda = \sum_{i \geq 0} \lambda_i t^i \in \bigcap_{m' \geq m} C_{m,\ell_{m',m}}$ satisfies

$$\lambda_i \in \bigcap_{m' \geq 0} \big(\mathfrak{m}_{H_1}(H_0)^m + \mathfrak{m}(H_0)^{m'}\big) \qquad \text{for any } i \geq 0.$$

But we know from Lemma 8.2.v that this intersection is equal to $\mathfrak{m}_{H_1}(H_0)^m$. It follows that

$$\bigcap_{\ell} C_{m,\ell} = F^m \Lambda_{H_0,H_1}(H) \ .$$

iii. That the weak topology is Hausdorff follows from ii. and Lemma 8.12.i. To establish the completeness let $(\lambda^{(\alpha)})_{\alpha \in \Xi}$ with $\lambda^{(\alpha)} = \sum_{i \geq 0} \lambda_i^{(\alpha)} t^i$ be a Cauchy net in $\Lambda_{H_0,H_1}(H)$ for the weak topology. Lemma 8.17 immediately implies that each $(\lambda_i^{(\alpha)})_{\alpha \in \Xi}$, for $i \geq 0$, is a Cauchy net for the weak topology in $\Lambda_{H_1}(H_0)$. The latter is complete by Lemma 8.2.iv. Hence each of these Cauchy nets has a limit $\lambda_i \in \Lambda_{H_1}(H_0)$. We will show that $\lambda := \sum_{i \geq 0} \lambda_i t^i$ lies in $\Lambda_{H_0,H_1}(H)$ and is the limit, for the weak topology, of the original Cauchy net $(\lambda^{(\alpha)})_{\alpha \in \Xi}$.

As a piece of notation we let $\mathcal{L}$ denote the set of all functions $\ell : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$. It is partially ordered by $\ell_1 \leq \ell_2$ if $\ell_1(j) \leq \ell_2(j)$ for any $j \geq 0$.

In a first step we will establish, for each $m \geq 0$, the existence of a $j(m) \geq 0$ such that for any function ℓ the subset

$$\Xi(m,\ell) := \{\alpha \in \Xi : \lambda^{(\alpha)} \in C_{m,\ell} + t_0^{-j(m)} \Lambda(H)\}$$

is cofinal in Ξ. Fixing m there otherwise is, for any $k \geq 0$, a function ℓ_k and an index $\alpha_k \in \Xi$ such that

$$\lambda^{(\alpha)} \notin C_{m,\ell_k} + t_0^{-k}\Lambda(H) \qquad \text{for all } \alpha \geq \alpha_k.$$

We certainly may assume that $\ell_0 \leq \ell_1 \leq \ell_2 \leq \ldots$ We now define a new function ℓ by $\ell(j) := \ell_j(j)$. For $k \leq j$ we have $\ell(j) = \ell_j(j) \geq \ell_k(j)$ and hence

$$t_0^{-j}\mathfrak{m}(H)^{\ell(j)} \subseteq t_0^{-j}\mathfrak{m}(H)^{\ell_k(j)} \subseteq C_{m,\ell_k} .$$

For $k \geq j$ we have

$$t_0^{-j}\mathfrak{m}(H)^{\ell(j)} \subseteq t_0^{-j}\Lambda(H) \subseteq t_0^{-k}\Lambda(H) .$$

It follows that

$$C_{m,\ell} + t_0^{-k}\Lambda(H) \subseteq C_{m,\ell_k} + t_0^{-k}\Lambda(H) \qquad \text{for any } k \geq 0.$$

In particular, for any $k \geq 0$, we obtain

$$\lambda^{(\alpha)} \notin C_{m,\ell} + t_0^{-k}\Lambda(H) \qquad \text{for all } \alpha \geq \alpha_k.$$

Now we choose an index $\beta \in \Xi$ such that

$$\lambda^{(\beta_1)} - \lambda^{(\beta_2)} \in C_{m,\ell} \qquad \text{for any } \beta_1, \beta_2 \geq \beta.$$

We also choose, by (20), the integer k large enough so that $\lambda^{(\beta)} \in C_{m,\ell} + t_0^{-k}\Lambda(H)$. Then

$$\lambda^{(\alpha)} \in C_{m,\ell} + t_0^{-k}\Lambda(H) \qquad \text{for all } \alpha \geq \beta.$$

Since Ξ is directed this is a contradiction.

In the next step we show that $\lambda \in \Lambda_{H_0,H_1}(H)$. We have

$$\lambda_0^{(\alpha)}, \ldots, \lambda_{m'}^{(\alpha)} \in \mathfrak{m}_{H_1}(H_0)^m + t_0^{-j(m)}\Lambda(H_0) \qquad \text{for any } \alpha \in \Xi(m, \ell_{m',m}).$$

Since $\mathfrak{m}_{H_1}(H_0)^m + t_0^{-j(m)}\Lambda(H_0)$ is closed in $\Lambda_{H_1}(H_0)$ for the weak topology by Lemma 8.2 the cofinality of $\Xi(m, \ell_{m',m})$ implies that

$$\lambda_0, \ldots, \lambda_{m'} \in \mathfrak{m}_{H_1}(H_0)^m + t_0^{-j(m)}\Lambda(H_0) .$$

But m' was arbitrary. We therefore obtain

$$\{\lambda_i\}_{i\geq 0} \subseteq \mathfrak{m}_{H_1}(H_0)^m + t_0^{-j(m)}\Lambda(H_0) \qquad \text{for any } m \geq 0.$$

This means, by Lemma 8.8, that $\{\lambda_i\}_{i\geq 0}$ is bounded and hence that $\lambda \in \Lambda_{H_0,H_1}(H)$.

For the time being we fix an $m \geq 0$. The product set $\Xi \times \mathcal{L}$ is a directed partially ordered set by $(\alpha, \ell) \geq (\beta, \ell')$ if $\alpha \geq \beta$ and $\ell \geq \ell'$. We construct a net $\{\nu^{(\alpha,\ell)}\}_{(\alpha,\ell)\in\Xi\times\mathcal{L}}$ in $t_0^{-j(m)}\Lambda(H)$ in the following way. By cofinality we may pick an index $\alpha' \geq \alpha$ in $\Xi(m, \ell)$. We then find a $\nu^{(\alpha,\ell)} = \sum_{i\geq 0} \nu_i^{(\alpha,\ell)} t^i \in t_0^{-j(m)}\Lambda(H)$ such that $\lambda^{(\alpha')} - \nu^{(\alpha,\ell)} \in C_{m,\ell}$. Let us check that the net $\{\nu_i^{(\alpha,\ell)}\}_{(\alpha,\ell)\in\Xi\times\mathcal{L}}$, for any $i \geq 0$, converges to λ_i in the quotient $\Lambda_{H_1}(H_0)/\mathfrak{m}_{H_1}(H_0)^m$. Given any $m' \geq \max(m, i)$ we choose an $\alpha \in \Xi$ such that

$$\lambda_i^{(\beta)} - \lambda_i \in B_{m'} \subseteq \mathfrak{m}_{H_1}(H_0)^m + \mathfrak{m}(H_0)^{m'} \qquad \text{for any } \beta \geq \alpha.$$

By construction we find, for any $(\beta, \ell) \geq (\alpha, \ell_{m',m})$, an index $\beta' \geq \beta$ such that

$$\lambda^{(\beta')} - \nu^{(\beta,\ell)} \in C_{m,\ell} \subseteq C_{m,\ell_{m',m}} \ .$$

In particular, by Lemma 8.17, we have

$$\lambda_i^{(\beta')} - \nu_i^{(\beta,\ell)} \in \mathfrak{m}_{H_1}(H_0)^m + \mathfrak{m}(H_0)^{m'} \ .$$

It follows that

$$\nu_i^{(\beta,\ell)} - \lambda_i = (\lambda_i^{(\beta')} - \lambda_i) - (\lambda_i^{(\beta')} - \nu_i^{(\beta,\ell)}) \in \mathfrak{m}_{H_1}(H_0)^m + \mathfrak{m}(H_0)^{m'}$$

for any $(\beta, \ell) \geq (\alpha, \ell_{m',m})$.

Now we observe that the weak topology induces, by i., on the quotient

$$\big(t_0^{-j(m)}\Lambda(H) + F^m\Lambda_{H_0,H_1}(H)\big)/F^m\Lambda_{H_0,H_1}(H) \cong \prod_{i\geq 0} t_0^{-j(m)}\Lambda(H_0)/\mathfrak{m}(H_1)^m\Lambda(H_0)$$

the (compact) direct product topology. It follows that the net $\{\nu^{(\alpha,\ell)}\}_{(\alpha,\ell)\in\Xi\times\mathfrak{L}}$ converges to λ for the weak topology in the quotient $\Lambda_{H_0,H_1}(H)/F^m\Lambda_{H_0,H_1}(H)$.

Next we claim that with $\{\nu^{(\alpha,\ell)}\}_{(\alpha,\ell)\in\Xi\times\mathfrak{L}}$ also $(\lambda^{(\alpha)})_{\alpha\in\Xi}$ converges to λ in the quotient $\Lambda_{H_0,H_1}(H)/F^m\Lambda_{H_0,H_1}(H)$. Given any function ℓ_0 we choose an $(\alpha, \ell_1) \in \Xi \times \mathcal{L}$ such that

$$\nu^{(\beta,\ell)} - \lambda \in C_{m,\ell_0} \qquad \text{for any } (\beta,\ell) \geq (\alpha,\ell_1)$$

and

$$\lambda^{(\beta_1)} - \lambda^{(\beta_2)} \in C_{m,\ell_0} \qquad \text{for any } \beta_1, \beta_2 \geq \alpha.$$

We put $\ell_2 := \max(\ell_0, \ell_1)$ and pick the $\alpha' \geq \alpha$ such that

$$\lambda^{(\alpha')} - \nu^{(\alpha,\ell_2)} \in C_{m,\ell_2} \ .$$

Then

$$\lambda^{(\beta)} - \lambda = (\lambda^{(\beta)} - \lambda^{(\alpha')}) + (\lambda^{(\alpha')} - \nu^{(\alpha,\ell_2)}) + (\nu^{(\alpha,\ell_2)} - \lambda) \in C_{m,\ell_0} + C_{m,\ell_2} + C_{m,\ell_0} \subseteq C_{m,\ell_0}$$

for any $\beta \geq \alpha$.

We now have shown that our original Cauchy net $(\lambda^{(\alpha)})_{\alpha\in\Xi}$ converges to λ for the weak topology in the quotient $\Lambda_{H_0,H_1}(H)/F^m\Lambda_{H_0,H_1}(H)$ for any $m \geq 0$. It is clear from the explicit definition of the weak topology that this means that $(\lambda^{(\alpha)})_{\alpha\in\Xi}$ converges to λ in $\Lambda_{H_0,H_1}(H)$. □

Behind the above proof is the general principle that a (countable) strict inductive limit of complete topological abelian groups again is complete. But the notion of an inductive limit for topological algebraic structures is not entirely straightforward in the sense that it has the tendency to depend on the precise category one is working in. Since we did not want to get into a discussion of these questions we preferred to explicitly work out the argument in our case.

Lemma 8.19. *Suppose that $H \cong H_1 \rtimes (H/H_1)$ is a semidirect product; for any $m, k \geq 0$ there are $N(m) \geq 0$ and $l(k), l'(k) \geq 0$ such that we have:*

i. $\mathfrak{m}(H_0)^{l(k)} t_0^{-1} \subseteq \mathfrak{m}_{H_1}(H_0)^m + t_0^{-1-N(m)} \mathfrak{m}(H_0)^k$;

ii. $\mathfrak{m}(H)^{l'(k)} \cdot t_0^{-1} \subseteq F^m \Lambda_{H_0,H_1}(H) + t_0^{-1-N(m)} \mathfrak{m}(H)^k$.

Proof. Since δ_0 is topologically nilpotent we find an $N(m) \geq 0$ such that $\delta_0^{N(m)+1}(\Lambda(H_1)) \subseteq \mathfrak{m}(H_1)^m$. i. Let $k \geq 1$. The $\mathfrak{m}(H_0)$-adic topology on $\Lambda(H_0) = \Lambda(H_1)[[t_0; \sigma_0, \delta_0]] \cong \prod_{i \geq 0} \Lambda(H_1)$ coincides with the direct product topology. Hence $\mathfrak{m}(H_1)^k + \mathfrak{m}(H_0)^{k-1} t_0$ is open which means that it contains some $\mathfrak{m}(H_0)^{l(k)}$. We in fact consider any $\lambda = \nu + \mu t_0$ with $\nu \in \mathfrak{m}(H_1)^k$ and $\mu \in \mathfrak{m}(H_0)^{k-1}$. Using formula (1.5) in [SV2] we obtain

$$\lambda t_0^{-1} = \nu t_0^{-1} + \mu = \big(\sum_{i \leq -1} t_0^i \sigma_0 \delta_0^{-i-1}(\nu) \big) + \mu \ .$$

Due to the choice of $N(m)$ the right hand side is contained in

$$\begin{aligned}
&\big(\sum_{i=-1-N(m)}^{-1} t_0^i \sigma_0 \delta_0^{-i-1}(\nu) \big) + \mu + \mathfrak{m}_{H_1}(H_0)^m \\
&\quad \subseteq \big(\sum_{i=-1-N(m)}^{-1} t_0^i \mathfrak{m}(H_1)^k \big) + \mathfrak{m}(H_0)^{k-1} + \mathfrak{m}_{H_1}(H_0)^m \\
&\quad = t_0^{-1-N(m)} \big(\big(\sum_{i=0}^{N(m)} t_0^i \mathfrak{m}(H_1)^k \big) + t_0^{1+N(m)} \mathfrak{m}(H_0)^{k-1} \big) + \mathfrak{m}_{H_1}(H_0)^m \\
&\quad \subseteq t_0^{-1-N(m)} \mathfrak{m}(H_0)^k + \mathfrak{m}_{H_1}(H_0)^m \ .
\end{aligned}$$

In the case $k = 0$ the same computation actually gives

$$\Lambda(H_0) t_0^{-1} \subseteq t_0^{-1-N(m)} \Lambda(H_0) + \mathfrak{m}_{H_1}(H_0)^m \ .$$

ii. We now have to use, for $\Lambda(H)$, the direct product topology argument twice. First we observe that there is a $k' \geq 0$ such that

$$\sum_{l=0}^{k'} \mathfrak{m}(H_0)^{k'} t^l + \Lambda(H) t^{k'+1} \subseteq \mathfrak{m}(H)^k \ .$$

We then have available the integer $l(k')$ form the first assertion. Secondly we note that

$$\sum_{i=0}^{l(k')+k'} \mathfrak{m}(H_0)^{l(k')+k'-i} t^i + \Lambda(H) t^{l(k')+k'+1}$$

is open in $\Lambda(H)$ and hence contains some $\mathfrak{m}(H)^{l'(k)}$. We actually will show that

$$\begin{aligned}
&\Big(\sum_{i=0}^{l(k')+k'} \mathfrak{m}(H_0)^{l(k')+k'-i} t^i + \Lambda(H) t^{l(k')+k'+1} \Big) \cdot t_0^{-1} \\
&\qquad \subseteq F^m \Lambda_{H_0,H_1}(H) + t_0^{-1-N(m)} \Big(\sum_{l=0}^{k'} \mathfrak{m}(H_0)^{k'} t^l + \Lambda(H) t^{k'+1} \Big)
\end{aligned}$$

holds true. We therefore consider any

$$\lambda = \sum_{i\geq 0} \lambda_i t^i \qquad \text{with } \lambda_i \in \mathfrak{m}(H_0)^{l(k')+k'-i} \text{ for } 0 \leq i \leq l(k') + k'.$$

By construction we have

$$\lambda \cdot t_0^{-1} = \sum_{l\geq 0} \Big(\sum_{i \geq l} \binom{i}{l} \lambda_i \delta^{i-l}(\sigma^l(t_0^{-1})) \Big) t^l \ ,$$

and we claim that the coefficients on the right hand side lie in $\mathfrak{m}(H_0)^{l(k')} t_0^{-1}$, resp. $\Lambda(H_0) t_0^{-1}$, for $0 \leq l \leq l(k')$, resp. $l > l(k')$. We know from the proof of Lemma 8.3 that

$$\delta^{i-l}(t_0^{-1}) \in \mathfrak{n}^{i-l} t_0^{-1} \ ,$$

where $\mathfrak{n}$ denotes the maximal ideal in $o[[t_0]]$, and that $\sigma(t_0^{-1}) \subseteq o[[t_0]] t_0^{-1}$. We deduce that

$$\delta^{i-l}(\sigma^l(t_0^{-1})) = \sigma^l(\delta^{i-l}(t_0^{-1})) \subseteq \sigma^l(\mathfrak{n}^{i-l} t_0^{-1}) \subseteq \mathfrak{n}^{i-l} t_0^{-1} \ .$$

For any $l \leq i$ we therefore always have $\lambda_i \delta^{i-l}(\sigma^l(t_0^{-1})) \in \Lambda(H_0) t_0^{-1}$. If in addition $l \leq k'$ then

$$\lambda_i \delta^{i-l}(\sigma^l(t_0^{-1})) \in \mathfrak{m}(H_0)^{l(k')+k'-i} \mathfrak{n}^{i-l} t_0^{-1} \subseteq \mathfrak{m}(H_0)^{l(k')+k'-l} t_0^{-1} \subseteq \mathfrak{m}(H_0)^{l(k')} t_0^{-1}$$

for $l \leq i \leq l(k') + k'$ and

$$\lambda_i \delta^{i-l}(\sigma^l(t_0^{-1})) \in \mathfrak{m}(H_0)^{i-l} t_0^{-1} \subseteq \mathfrak{m}(H_0)^{l(k')+k'-l} t_0^{-1} \subseteq \mathfrak{m}(H_0)^{l(k')} t_0^{-1} \ .$$

for $i > l(k') + k'$. Since any $\mathfrak{m}(H_0)^j t_0^{-1}$ is compact and hence closed for the weak topology this establishes our claim, i. e., we have

$$\lambda \cdot t_0^{-1} = \sum_{l \geq 0} \mu_l t_0^{-1} t^l \quad \text{with } \mu_l \in \mathfrak{m}(H_0)^{l(k')} \text{ for } 0 \leq l \leq k' \text{ and} \in \Lambda(H_0) \text{ for } l > k'.$$

Applying now the first assertion we obtain that

$$\mu_l t_0^{-1} \in t_0^{-1-N(m)} \begin{Bmatrix} \mathfrak{m}(H_0)^{k'} \\ \Lambda(H_0) \end{Bmatrix} + \mathfrak{m}_{H_1}(H_0)^m \qquad \begin{cases} \text{if } 0 \leq l \leq k', \\ \text{if } l > k'. \end{cases}$$

This means that

$$\lambda \cdot t_0^{-1} \in F^m \Lambda_{H_0,H_1}(H) + t_0^{-1-N(m)} \Big(\sum_{l=0}^{k'} \mathfrak{m}(H_0)^{k'} t^l + \Lambda(H) t^{k'+1} \Big) \ .$$

□

Proposition 8.20. *Suppose that $H \cong H_1 \rtimes (H/H_1)$ is a semidirect product; the multiplication map in the ring $\Lambda_{H_0,H_1}(H)$ is separately continuous for the weak topology.*

Proof. We first consider the left multiplication by some $\lambda \in \Lambda_{H_0,H_1}(H)$. For any $m \geq 0$ and $j \geq 0$ we find, by applying Lemma 8.8 to the set A of coefficients of $\lambda \cdot t_0^{-j}$, a $k(\lambda, m, j) \geq 0$ such that

$$\lambda \cdot t_0^{-j} \subseteq F^m\Lambda_{H_0,H_1}(H) + t_0^{-k(\lambda,m,j)}\Lambda(H) \ .$$

Hence

$$\begin{aligned}\lambda \cdot (F^m\Lambda_{H_0,H_1}(H) + t_0^{-j}\mathfrak{m}(H)^k) &\subseteq F^m\Lambda_{H_0,H_1}(H) + \lambda \cdot t_0^{-j}\mathfrak{m}(H)^k \\ &\subseteq F^m\Lambda_{H_0,H_1}(H) + t_0^{-k(\lambda,m,j)}\mathfrak{m}(H)^k\end{aligned}$$

for any $k \geq 0$. Suppose given now any open $C_{m,\ell}$. If we define a new function ℓ' by $\ell'(j) := \ell(k(\lambda, m, j))$ then

$$\lambda \cdot C_{m,\ell'} \subseteq C_{m,\ell} \ .$$

The argument for the right multiplication by λ is similar but in addition is crucially based on Lemma 8.19.ii. Let $l'' := l' \circ \ldots \circ l'$ denote the $k(\lambda, m, 0)$-fold iteration of the function l' in that lemma. By a correspondingly iterated application of that lemma we obtain

$$\begin{aligned}&(F^m\Lambda_{H_0,H_1}(H) + t_0^{-j}\mathfrak{m}(H)^{l''(k)}) \cdot \lambda \\ &\quad\subseteq \big(F^m\Lambda_{H_0,H_1}(H) + t_0^{-j}\mathfrak{m}(H)^{l''(k)}\big) \cdot \big(F^m\Lambda_{H_0,H_1}(H) + t_0^{-k(\lambda,m,0)}\Lambda(H)\big) \\ &\quad\subseteq F^m\Lambda_{H_0,H_1}(H) + t_0^{-j}\mathfrak{m}(H)^{l''(k)} \cdot t_0^{-k(\lambda,m,0)}\Lambda(H) \\ &\quad\subseteq F^m\Lambda_{H_0,H_1}(H) + t_0^{-j-k(\lambda,m,0)(1+N(m))}\mathfrak{m}(H)^k\end{aligned}$$

for any $k \geq 0$. If we this time, given any function ℓ, define a new function ℓ' by

$$\ell'(j) := l''(\ell(j + k(\lambda, m, 0)(1 + N(m))))$$

then we have

$$C_{m,\ell'} \cdot \lambda \subseteq C_{m,\ell} \ .$$

□

Under our standing assumption that $H \cong H_1 \rtimes (H/H_1)$ we now have the commutative diagram of rings

(21)
$$\begin{array}{ccc}\Lambda_{H_0,H_1}(H) & \xleftarrow{\supseteq} & \Lambda(H) \\ \uparrow{\scriptstyle\subseteq} & & \uparrow{\scriptstyle\subseteq} \\ \Lambda_{H_1}(H_0) & \xleftarrow{\supseteq} & \Lambda(H_0)\end{array}$$

where, in addition, all maps are topological inclusions for the weak, resp. compact, topologies on the rings in the left, resp. right, column.

Proposition 8.21. *Suppose that $H \cong H_1 \rtimes (H/H_1)$ is a semidirect product; we then have:*

i. $\Lambda_{H_0,H_1}(H)$ *is (left and right) noetherian;*

ii. $\Lambda_{H_0,H_1}(H)$ *is flat as a left as well as right* $\Lambda_{H_1}(H_0)$*-module;*

iii. $\Lambda_{H_0,H_1}(H)$ *is flat as a left as well as right* $\Lambda(H)$*-module.*

Proof. *Step 1:* Let $H_1' \subseteq H_1$ be an open subgroup which is normal in H. We put

$$H_0' := \overline{< H_1', \gamma_0 >} \quad \text{and} \quad H' := \overline{< H_1', \gamma_0, \gamma >} .$$

Then H' is an open subgroup of H such that $H' \cong H_1' \rtimes (H/H_1)$. Obviously $\Lambda(H_1)$ is free of rank $[H_1 : H_1']$ as a left or right $\Lambda(H_1')$-module. Each of the rings $A := \Lambda(H_0), \Lambda(H), \Lambda_{H_1}(H_0)$, or $\Lambda_{H_0,H_1}(H)$ contains the corresponding ring $A' := \Lambda(H_0'), \Lambda(H'), \Lambda_{H_1'}(H_0')$, or $\Lambda_{H_0',H_1'}(H')$. We claim that in each case

$$A = \Lambda(H_1) \otimes_{\Lambda(H_1')} A' = A' \otimes_{\Lambda(H_1')} \Lambda(H_1)$$

holds true. For $A = \Lambda(H_0)$ and $A = \Lambda(H)$ this follows immediately from their descriptions

$$\Lambda(H_0) = \{\sum_{i\geq 0} \mu_i t_0^i : \mu_i \in \Lambda(H_1)\} = \{\sum_{i\geq 0} t_0^i \mu_i : \mu_i \in \Lambda(H_1)\}$$

and

$$\Lambda(H) = \{\sum_{i\geq 0} \lambda_i t^i : \lambda_i \in \Lambda(H_0)\} = \{\sum_{i\geq 0} t^i \lambda_i : \lambda_i \in \Lambda(H_0)\}$$

in terms of skew power series. Similarly, using that

$$\begin{aligned}\Lambda_{H_1}(H_0) &= \{\sum_{i\in\mathbb{Z}} \mu_i t_0^i : \mu_i \in \Lambda(H_1), \lim_{i\to-\infty} \mu_i = 0\} \\ &= \{\sum_{i\in\mathbb{Z}} t_0^i \mu_i : \mu_i \in \Lambda(H_1), \lim_{i\to-\infty} \mu_i = 0\}\end{aligned}$$

together with the fact that on $\Lambda(H_1)$ the $\mathfrak{m}(H_1)$-adic and the $\mathfrak{m}(H_1')\Lambda(H_1)$-adic topology coincide the claim is clear for the ring $A = \Lambda_{H_1}(H_0)$ as well. Finally, for the ring $A = \Lambda_{H_0,H_1}(H)$ we use Prop. 8.15 and obtain

$$\begin{aligned}\Lambda_{H_0',H_1'}(H') \otimes_{\Lambda(H_1')} \Lambda(H_1) &= \big(\varprojlim \big(\Lambda(H')/\mathfrak{m}(H_1')^m\Lambda(H')\big)_\Theta\big) \otimes_{\Lambda(H_1')} \Lambda(H_1) \\ &= \varprojlim \big(\big(\Lambda(H')/\mathfrak{m}(H_1')^m\Lambda(H')\big)_\Theta \otimes_{\Lambda(H_1')} \Lambda(H_1)\big) \\ &= \varprojlim \big(\Lambda(H')/\mathfrak{m}(H_1')^m\Lambda(H') \otimes_{\Lambda(H_1')} \Lambda(H_1)\big)_\Theta \\ &= \varprojlim \big(\Lambda(H)/\mathfrak{m}(H_1')^m\Lambda(H)\big)_\Theta \\ &= \varprojlim \big(\Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H)\big)_\Theta \\ &= \Lambda_{H_0,H_1}(H)\end{aligned}$$

where the second, resp. the second last, identity is due to the fact that $\Lambda(H_1)$ is free over $\Lambda(H_1')$ of rank $[H_1 : H_1']$, resp. to the cofinality of $\{\mathfrak{m}(H_1)^m\}_{m\geq 1}$ and $\{\mathfrak{m}(H_1')^m\Lambda(H_1)\}_{m\geq 1}$. The symmetric identity follows in the same way. This establishes our claim and shows that in order to prove our assertion we may replace, whenever convenient, the triple $H_1 \subseteq H_0 \subseteq H$ by the triple $H_1' \subseteq H_0' \subseteq H'$.

Step 2: The above commutative diagram of rings (21) in fact is a diagram of filtered rings with complete and separated filtrations defined by the two-sided ideals

$$F^m\Lambda(H_0) := \mathfrak{m}(H_1)^m\Lambda(H_0), \ F^m\Lambda(H) := \mathfrak{m}(H_1)^m\Lambda(H),$$
$$F^m\Lambda_{H_1}(H_0) := \mathfrak{m}_{H_1}(H_0)^m = \mathfrak{m}(H_1)^m\Lambda_{H_1}(H_0),$$

and $F^m\Lambda_{H_0,H_1}(H)$ as before. We obtain a corresponding commutative diagram of graded rings

$$\begin{array}{ccc} \mathrm{gr}^\bullet \Lambda_{H_0,H_1}(H) & \longleftarrow & \mathrm{gr}^\bullet \Lambda(H) \\ \uparrow & & \uparrow \\ \mathrm{gr}^\bullet \Lambda_{H_1}(H_0) & \longleftarrow & \mathrm{gr}^\bullet \Lambda(H_0). \end{array} \tag{22}$$

By the way, all four maps in this diagram again are injective (cf. (14) and [SV2] Lemma 1.12.i). By [LvO] Prop.s II.1.2.1 and II.1.2.3 our assertions follow from:

iv. All four graded rings in the diagram (22) are left and right noetherian;

v. $\mathrm{gr}^\bullet \Lambda_{H_0,H_1}(H)$ is flat as a left and as a right $\mathrm{gr}^\bullet \Lambda_{H_1}(H_0)$-module;

vi. $\mathrm{gr}^\bullet \Lambda_{H_0,H_1}(H)$ is flat as a left and as a right $\mathrm{gr}^\bullet \Lambda(H)$-module.

Ad v.: As a consequence of Lemma 8.12.iv and of the flatness of $\Lambda(H)$ over $\Lambda(H_0)$ we have

$$\mathrm{gr}^\bullet \Lambda_{H_1}(H_0) \otimes_{\Lambda(H_0)/\mathfrak{m}(H_1)\Lambda(H_0)} \Lambda(H)/\mathfrak{m}(H_1)\Lambda(H) \xrightarrow{\cong} \mathrm{gr}^\bullet \Lambda_{H_0,H_1}(H) \ .$$

The same flatness then implies, by base extension, that $\mathrm{gr}^\bullet \Lambda_{H_0,H_1}(H)$ is flat as a left $\mathrm{gr}^\bullet \Lambda_{H_1}(H_0)$-module. Using Prop. 8.15.i one sees that one also has

$$\Lambda(H)/\mathfrak{m}(H_1)\Lambda(H) \otimes_{\Lambda(H_0)/\mathfrak{m}(H_1)\Lambda(H_0)} \mathrm{gr}^\bullet \Lambda_{H_1}(H_0) \xrightarrow{\cong} \mathrm{gr}^\bullet \Lambda_{H_0,H_1}(H)$$

which implies the asserted flatness as a right module.
Ad vi.: Because of [SV2] lemma 1.12.ii we have

$$F^m\Lambda(H) = \{\sum_{i\geq 0} t^i\nu_i \in \Lambda(H) : \nu_i \in \mathfrak{m}(H_1)^m\Lambda(H_0)\}$$

and

$$\mathfrak{m}(H_1)^m\Lambda(H_0) = \{\sum_{i\geq 0} \alpha_i t_0^i \in \Lambda(H_0) : \alpha_i \in \mathfrak{m}(H_1)^m\}.$$

It follows first of all that the set $\Theta = \{t_0^j\}_{j\geq 0}$ is mapped, by the symbol map, injectively into $\mathrm{gr}^0 \Lambda(H)$. We therefore will not distinguish, in the notation, between the elements in Θ and their symbols. Secondly, let $\nu = \sum_{i\geq 0} t^i\nu_i \in \Lambda(H)$ and assume that $\nu t_0^j \in F^m\Lambda(H)$ for some $j \geq 0$. Then $\nu_i t_0^j \in \mathfrak{m}(H_1)^m\Lambda(H_0)$ and hence $\nu_i \in \mathfrak{m}(H_1)^m\Lambda(H_0)$ for any $i \geq 0$. We conclude that $\nu \in F^m\Lambda(H)$. It follows that the elements in Θ are left regular in $\mathrm{gr}^0 \Lambda(H)$. In particular, Θ as a subset of $\mathrm{gr}^\bullet \Lambda(H)$ is multiplicatively closed. Right regularity follows by a symmetric argument. We consider now any element $a = \overline{\lambda_0} + \ldots + \overline{\lambda_r} \in \mathrm{gr}^\bullet \Lambda(H)$ with $\overline{\lambda_j} = \lambda_j + F^{j+1}\Lambda(H) \in \mathrm{gr}^j \Lambda(H)$. By Lemma 8.14 we find, after choosing some $m > r$, an integer $M > 0$ and elements $\mu_0, \ldots, \mu_r \in \Lambda(H)$ such that

$$t_0^M\lambda_j \equiv \mu_j t_0 \mod F^m\Lambda(H) \qquad \text{for any } 0 \leq j \leq r.$$

The regularity of t_0 then implies that $\mu_j \in F^j\Lambda(H)$ and that

$$t_0^M\overline{\lambda_j} = \overline{\mu_j}t_0 \qquad \text{with } \overline{\mu_j} := \mu_j + F^{j+1}\Lambda(H) \in \mathrm{gr}^j \Lambda(H).$$

By setting $b := \overline{\mu_0} + \ldots + \overline{\mu_r}$ we obtain

$$t_0^M a = bt_0 \qquad \text{in } \mathrm{gr}^\bullet \Lambda(H).$$

This means that $\Theta \subseteq \mathrm{gr}^\bullet \Lambda(H)$ satisfies the left Ore condition. Again the right Ore condition holds as well by a symmetric argument. Since t_0 is invertible in $\mathrm{gr}^\bullet \Lambda_{H_0,H_1}(H)$ the injective homomorphism $\mathrm{gr}^\bullet \Lambda(H) \longrightarrow \mathrm{gr}^\bullet \Lambda_{H_0,H_1}(H)$ extends to an injective ring homomorphism

$$\big(\mathrm{gr}^\bullet \Lambda(H)\big)_\Theta \longrightarrow \mathrm{gr}^\bullet \Lambda_{H_0,H_1}(H)\ .$$

As a straightforward consequence of Prop. 8.15.i it also is surjective and hence is an isomorphism. As a localization in Θ the ring $\mathrm{gr}^\bullet \Lambda_{H_0,H_1}(H)$ of course is (left and right) flat over $\mathrm{gr}^\bullet \Lambda(H)$.

Step 3: For this proof we do not need to establish the assertion iv. in full generality. Because of Step 1 it in fact suffices to do this after replacing the given triple $H_1 \subseteq H_0 \subseteq H$ by an appropriate "smaller" one. By [Wil] Prop. 8.5.2 there is an open normal subgroup $\widetilde{H} \subseteq H$ which is extra-powerful. Then $H_1' := H_1 \cap \widetilde{H}$ is extra-powerful, too, since $\widetilde{H}/H_1'$ is torsionfree. Hence it suffices to prove iv. under the additional assumption that H_1 is extra-powerful.

We have seen that $\mathrm{gr}^\bullet \Lambda_{H_0,H_1}(H)$ is a localization of $\mathrm{gr}^\bullet \Lambda(H)$. By exactly analogous arguments $\mathrm{gr}^\bullet \Lambda_{H_1}(H_0)$ is a localization of $\mathrm{gr}^\bullet \Lambda(H_0)$. Since being noetherian is preserved by localization (cf. [MCR] Prop. 2.1.16.iii) we therefore need only to consider the rings $\mathrm{gr}^\bullet \Lambda(H)$ and $\mathrm{gr}^\bullet \Lambda(H_0)$. They contain the graded ring $\mathrm{gr}^\bullet \Lambda(H_1)$ for the filtration $F^m\Lambda(H_1) := \mathfrak{m}(H_1)^m$. Using [SV2] Lemma 1.12.i one checks that

$$\mathrm{gr}^\bullet \Lambda(H) = \mathrm{gr}^\bullet \Lambda(H_1) \otimes_k \Omega(H/H_1) = \Omega(H/H_1) \otimes_k \mathrm{gr}^\bullet \Lambda(H_1)$$

and

$$\mathrm{gr}^\bullet \Lambda(H_0) = \mathrm{gr}^\bullet \Lambda(H_1) \otimes_k \Omega(H_0/H_1) = \Omega(H_0/H_1) \otimes_k \mathrm{gr}^\bullet \Lambda(H_1)$$

hold true (at least as bimodules). We see, first of all, that if $H' \subseteq H$ is an open subgroup containing H_1 then $\mathrm{gr}^\bullet \Lambda(H)$ and $\mathrm{gr}^\bullet \Lambda(H_0)$ are finitely generated free modules over $\mathrm{gr}^\bullet \Lambda(H')$ and $\mathrm{gr}^\bullet \Lambda(H_0 \cap H')$, respectively. Due to [DDMS] Lemma 3.4 and Cor. 8.34 there is an open normal subgroup $N \subseteq H_1$ such that each element of N is a p-th power in H_1. Since, by the above observation, we may replace H by any appropriate open $H' \supseteq H_1$ it suffices to consider the special case where conjugation by γ_0 and by γ both induce the identity on H_1/N. This implies that the commutators $[\gamma_0, h]$ and $[\gamma, h]$, for any $h \in H_1$, are p-th powers in H_1. The computation in the proof of [SV2] Lemma 4.3.ii then shows that the two factors $\mathrm{gr}^\bullet \Lambda(H_1)$ and $\Omega(H/H_1)$ in the above tensor product representation of $\mathrm{gr}^\bullet \Lambda(H)$ centralize each other.

At this point we make use of our additional assumption that H_1 is extra-powerful. Then $\mathrm{gr}^\bullet \Lambda(H_1)$ is a finitely generated commutative k-algebra by [SV2] Lemma 4.3.iii. It follows that $\mathrm{gr}^\bullet \Lambda(H)$, resp. $\mathrm{gr}^\bullet \Lambda(H_0)$, is an almost normalizing extension of the noetherian ring $\Omega(H/H_1)$, resp. $\Omega(H_0/H_1)$, and hence is noetherian by [MCR] Thm. 1.6.14. □

Consider any finitely generated $\Lambda_{H_1}(H_0)$-module M. We choose a presentation of M as a quotient $\Lambda_{H_1}(H_0)^n \twoheadrightarrow M$ of a finitely generated free $\Lambda_{H_1}(H_0)$-module. On $\Lambda_{H_1}(H_0)^n$ we have the product topology of the weak topology on each factor $\Lambda_{H_1}(H_0)$, and then on M we consider the corresponding quotient topology. The latter is easily shown to be independent of the particular presentation of M used and to make M into a topological $\Lambda_{H_1}(H_0)$-module. It will be called the *weak* topology on M.

Lemma 8.22. *For any finitely generated $\Lambda_{H_1}(H_0)$-module M we have:*

i. Every submodule $L \subseteq M$ is closed for the weak topology;

ii. M is is complete and Hausdorff in its weak topology.

Proof. i. By the definition of the weak topology we only need to consider the case $M = \Lambda_{H_1}(H_0)^n$. Note that in this case M is a $\Lambda_{H_1}(H_0)$-bimodule so that we may multiply by any ring element from the right. We recall that the ring $\Lambda_{H_1}(H_0)$ is noetherian and pseudocompact. Hence M is a finitely generated pseudocompact $\Lambda_{H_1}(H_0)$-module. The general theory of pseudocompact rings then tells us that L is (finitely generated and hence) closed for the pseudocompact topology. As a consequence we have by [Gab] IV.3 Prop. 11 that

$$L = \bigcap_{m \geq 0} (L + \mathfrak{m}_{H_1}(H_0)^m M) \ .$$

Note that $\mathfrak{m}_{H_1}(H_0)^m M$ is closed in M by Lemma 8.2.v. Let now $\{x_i\}_{i \in \mathbb{N}}$ be a sequence in L which converges to some x in M. It suffices to show that $x \in L + \mathfrak{m}_{H_1}(H_0)^m M$ for any $m \geq 0$. We fix some $m \geq 0$. By Remark 8.6.vi and Lemma 8.8 we find an $l \geq 0$ such that all $x_i t_0^l$ as well as $x t_0^l$ lie in $\Lambda(H_0)^n + \mathfrak{m}_{H_1}(H_0)^m M$. Hence modulo $\mathfrak{m}_{H_1}(H_0)^m M$ the sequence $\{x_i t_0^l\}_i$ lies in

$$\big((L t_0^l + \mathfrak{m}_{H_1}(H_0)^m M)/\mathfrak{m}_{H_1}(H_0)^m M\big) \cap \big((\Lambda(H_0)^n + \mathfrak{m}_{H_1}(H_0)^m M)/\mathfrak{m}_{H_1}(H_0)^m M\big)$$

and converges to $x t_0^l$ in $(\Lambda(H_0)^n + \mathfrak{m}_{H_1}(H_0)^m M)/\mathfrak{m}_{H_1}(H_0)^m M$ with respect to the topology induced by the weak topology. By Lemma 8.2.ii this topology coincides with the natural compact topology on the latter as a finitely generated module over the noetherian compact ring $\Lambda(H_0)$. The former is a (necessarily finitely generated) $\Lambda(H_0)$-submodule of the latter and as such has to be closed. This shows that $x t_0^l$ lies in $L t_0^l + \mathfrak{m}_{H_1}(H_0)^m M$ and hence that $x \in L + \mathfrak{m}_{H_1}(H_0)^m M$.

ii. The assertion i. immediately implies that M is Hausdorff for the weak topology. By construction the weak topology on $\Lambda_{H_1}(H_0)^n$ has a countable fundamental system of open neighbourhoods of zero. Hence it is metrizable (cf. [War] Thm. 6.12). It also is complete by Lemma 8.2.iv. In this situation any factor group of $\Lambda_{H_1}(H_0)^n$ by a closed subgroup, so in particular M by i., is complete as well (cf. [War] Thm. 6.12). □

Remark 8.23. *Let M be a complete Hausdorff linear-topological o-module; any continuous (left) H-action on M extends uniquely to a continuous (left) $\Lambda(H)$-module structure on M.*

Proof. [Laz] Thm. II.2.2.6. □

Let $\Gamma \subseteq H$ denote the closed subgroup topologically generated by our choice of γ; in particular, $\Gamma \xrightarrow{\cong} H/H_0 \cong \mathbb{Z}_p$. Since H_1 is normal in H the conjugation action of Γ on H_0 induces an action of Γ on the ring $\Lambda_{H_1}(H_0)$. We let $\sigma_{\gamma'}$ denote the ring automorphism corresponding to $\gamma' \in \Gamma$.

Remark 8.24. *i. The $\sigma_.$-action of Γ on $\Lambda_{H_1}(H_0)$ is continuous for the weak topology.*

ii. We have $\sigma_{\gamma'}(\lambda) = \gamma' \cdot \lambda \cdot \gamma'^{-1}$ for any $\gamma' \in \Gamma$ and $\lambda \in \Lambda_{H_1}(H_0)$ (where $\cdot$ denotes the multiplication in the ring $\Lambda_{H_0,H_1}(H)$.

Proof. i. The Γ-action respects the rings $\Lambda(H_0) \supseteq \Lambda(H_1)$ and hence their unique maximal ideals. It follows immediately that

$$\sigma_{\gamma'}(B_m) = B_m \qquad \text{for any } \gamma' \in \Gamma \text{ and any } m \geq 0.$$

For the asserted continuity it therefore remains to show that for any $\mu \in \Lambda_{H_1}(H_0)$ and any $m \geq 0$ there is an open subgroup $\Gamma' \subseteq \Gamma$ such that

$$\sigma_{\gamma'}(\mu) \in \mu + B_m \qquad \text{for any } \gamma' \in \Gamma'.$$

Since this relation only depends on μ modulo B_m we may assume that μ is of the form $\mu = t_0^{-l}\nu$ for some $l \geq 0$ and some $\nu \in \Lambda(H_0)$. We fix an $m' \geq m$ such that $t_0^{-l}B_{m'} \subseteq B_m$. First of all, contemplating the diagram

$$\begin{array}{ccc} \Gamma \times \Lambda(H_0) & \xrightarrow{\sigma.} & \Lambda(H_0) \\ \downarrow{\scriptstyle (\gamma,\mu)\mapsto(\gamma,\mu,\gamma^{-1})} & & \downarrow{\scriptstyle \subseteq} \\ \Lambda(H) \times \Lambda(H_0) \times \Lambda(H) & \xrightarrow{\text{product}} & \Lambda(H) \end{array}$$

we see that the $\sigma.$-action on $\Lambda(H_0)$ is continuous. Hence there is an open subgroup $\Gamma_1 \subseteq \Gamma$ such that

$$\sigma_{\gamma'}(\nu) \in \nu + \mathfrak{m}(H_0)^{m'} \subseteq \nu + B_{m'} \qquad \text{for any } \gamma' \in \Gamma_1.$$

Secondly we have to revisit the computation in Lemma 8.3. We recall that $\mathfrak{n}$ denotes the maximal ideal of the subring $o[[t_0]]$ in $\Lambda(H_0)$. Define the continuous homomorphism $e : \Gamma \longrightarrow 1 + p\mathbb{Z}_p$ by

$$\gamma'\gamma_0\gamma'^{-1} = \gamma_0^{e(\gamma')}$$

and put $\Gamma_2 := e^{-1}(1 + p^{m+l+1}\mathbb{Z}_p)$. We have

$$\sigma_{\gamma'}(t_0) = e(\gamma')t_0 + \sum_{j\geq 2} \binom{e(\gamma')}{j} t_0^j \ .$$

If v_p denotes the p-adic valuation then, for $j \geq 2$ and $y \in \mathbb{Z}_p$, one has (cf. [Sch] Prop. 47.4)

$$v_p(\binom{y}{j}) \geq v_p(y-1) - j \ .$$

It follows that for $\gamma' \in \Gamma_2$ we have

$$\begin{aligned} \sigma_{\gamma'}(t_0) &= e(\gamma')t_0\big(1 + \sum_{j\geq 2} e(\gamma')^{-1}\binom{e(\gamma')}{j} t_0^{j-1}\big) \\ &\in e(\gamma')t_0\big(1 + \sum_{j\geq 2} (\pi o)^{m+l+1-j} t_0^{j-1}\big) \\ &\subseteq e(\gamma')t_0\big(1 + \sum_{j=2}^{l+1} (\pi o)^m t_0^{j-1} + \mathfrak{n}^m t_0^l\big) \\ &\subseteq e(\gamma')t_0\big(1 + (\pi o)^m o[[t_0]] + \mathfrak{n}^m t_0^l\big) \ , \end{aligned}$$

hence

$$\sigma_{\gamma'}(t_0^l) \in e(\gamma')^l t_0^l \big(1 + (\pi o)^m o[[t_0]] + \mathfrak{n}^m t_0^l\big) \ ,$$

and therefore

$$\begin{aligned}
\sigma_{\gamma'}(t_0^{-l}) &\in e(\gamma')^{-l} t_0^{-l} \big(1 + (\pi o)^m o[[t_0]] + \mathfrak{n}^m t_0^l\big) \\
&\subseteq e(\gamma')^{-l} t_0^{-l} + \mathfrak{m}_{H_1}(H_0)^m + \mathfrak{m}(H_0)^m \\
&\subseteq e(\gamma')^{-l} t_0^{-l} + B_m = t_0^{-l} + (e(\gamma')^{-l} - 1) t_0^{-l} + B_m \\
&= t_0^{-l} + B_m \ .
\end{aligned}$$

Together we obtain

$$\sigma_{\gamma'}(t_0^{-l}\nu) \in (t_0^{-l} + B_m)(\nu + B_{m'}) \subseteq t_0^{-l}\nu + t_0^{-l} B_{m'} + B_m + B_m B_{m'} \subseteq t_0^{-l}\nu + B_m$$

for any $\gamma' \in \Gamma' := \Gamma_1 \cap \Gamma_2$.

ii. Since

$$\Lambda_{H_0,H_1}(H) = \varprojlim_m \big(\Lambda_{H_0,H_1}(H)/F^m \Lambda_{H_0,H_1}(H)\big)$$

we may do the comparison modulo $F^m \Lambda_{H_0,H_1}(H)$. By (20) we therefore may assume that $\lambda = t_0^{-j}\mu$ for some $j \geq 0$ and some $\mu \in \Lambda(H)$. At this point we emphasize that, for any fixed γ', we assert the equality of two ring automorphisms. Hence we are reduced to showing that the assertion holds true in the two cases $\lambda = t_0$ and $\lambda = \mu$. The first case, of course, is subsumed by the second one. But for $\lambda \in \Lambda(H)$ our assertion is clear since the multiplication $\cdot$ restricts to the usual multiplication in $\Lambda(H)$. □

Definition 8.25. *A* $(\Lambda_{H_1}(H_0), \Gamma)$ *-module is a finitely generated (left)* $\Lambda_{H_1}(H_0)$*-module with a* $\sigma_.$*-linear (left)* Γ*-action which is continuous for the weak topology.*

Let us consider a (left) $\Lambda_{H_0,H_1}(H)$-module M such that

– M is finitely generated over $\Lambda_{H_1}(H_0)$ and

– the module multiplication $\Lambda_{H_0,H_1}(H) \times M \longrightarrow M$ is separately continuous for the weak topologies.

By applying Prop. 8.18.i to a commutative diagram

$$\begin{array}{ccc}
\Lambda_{H_0,H_1}(H)^n & & \\
\uparrow \subseteq & \searrow\!\!\!\searrow & M \\
\Lambda_{H_1}(H_0)^n & \nearrow\!\!\!\nearrow &
\end{array}$$

of presentations of M the latter requirement implies that the weak topology on M coincides with the quotient topology derived from some presentation $\Lambda_{H_0,H_1}(H)^n \twoheadrightarrow M$. As another consequence of Prop. 8.18.i the compact ring $\Lambda(H)$ acts separately continuously on M. Since both, $\Lambda(H)$ and M, are complete metrizable abelian groups by Lemma 8.2 this action, in fact, has to be continuous ([CF] Thm. 2). In particular, by further restriction we obtain a continuous Γ-action on M. It is σ_*-linear by Remark 8.24.ii. We see that M is a $(\Lambda_{H_1}(H_0), \Gamma)$-module. By Lemma 8.12.ii the (left) ideal $\mathfrak{m}_{H_1}(H_0)^m \Lambda_{H_0,H_1}(H)$ is dense in $F^m\Lambda_{H_0,H_1}(H)$ for the strong and hence the weak topology. On the other hand $\mathfrak{m}_{H_1}(H_0)^m M$ is closed in M by Lemma 8.22.i. We point out that therefore M has the additional property that

$$F^m\Lambda_{H_0,H_1}(H) \cdot M = \mathfrak{m}_{H_1}(H_0)^m M \qquad \text{for any } m \geq 0.$$

Vice versa, let us start now with a $(\Lambda_{H_1}(H_0), \Gamma)$-module M. By Lemma 8.2.ii the compact ring $\Lambda(H_0)$ and hence the group H_0 act continuously on M. Therefore $H = H_0 \rtimes \Gamma$ acts continuously on M. Because of Lemma 8.22.ii we may apply Remark 8.23 and see that the $\Lambda(H_0)$-action extends to a continuous action of $\Lambda(H)$ on M. We want to see that the actions of $\Lambda_{H_1}(H_0)$ and $\Lambda(H)$ on M, which we have so far, combine and further extend to a separately continuous action of the ring $\Lambda_{H_0,H_1}(H)$. For this it is useful to first make the following observation. Being finitely generated over the noetherian pseudocompact ring $\Lambda_{H_1}(H_0)$ the module M is pseudocompact for the $\mathfrak{m}_{H_1}(H_0)$-adic topology. It therefore follows from [Gab] IV.3 Prop. 10 that the natural map

$$M \xrightarrow{\cong} \varprojlim_m M/\mathfrak{m}_{H_1}(H_0)^m M \tag{23}$$

is an isomorphism of $\Lambda_{H_1}(H_0)$-modules. The σ_*-action, of course, respects the maximal ideal $\mathfrak{m}_{H_1}(H_0)$. Hence σ_*-linearity implies that Γ respects the submodules $\mathfrak{m}_{H_1}(H_0)^m M$. In particular, (23) is an isomorphism of $(\Lambda_{H_1}(H_0), \Gamma)$-modules.

In order to construct an action by $\Lambda_{H_0,H_1}(H)$ on M we therefore may assume, provided we do this in a functorial way, that

$$\mathfrak{m}_{H_1}(H_0)^m M = 0 \qquad \text{for some } m \geq 0. \tag{24}$$

Let now $\lambda \in \Lambda_{H_0,H_1}(H)$. By (20) we may write $\lambda = \mu + t_0^{-j}\nu$ for appropriate $j \geq 0$, $\mu \in F^m\Lambda_{H_0,H_1}(H)$, and $\nu \in \Lambda(H)$. We define

$$\lambda \cdot x := t_0^{-j}(\nu x) \qquad \text{for any } x \in M.$$

In order to see that this is well defined let $\lambda = \mu' + t_0^{-j'}\nu'$ be another such decomposition. We may assume that $j \geq j'$. Then $t_0^{-j}(\nu - t_0^{j-j'}\nu') = \mu' - \mu \in F^m\Lambda_{H_0,H_1}(H)$ and hence $\nu - t_0^{j-j'}\nu' \in F^m\Lambda_{H_0,H_1}(H) \cap \Lambda(H) = \mathfrak{m}(H_1)^m\Lambda(H)$ where the last identity comes from (14) in the proof of Lemma 8.2. Because of (24) it follows that $(\nu - t_0^{j-j'}\nu')x = 0$ and consequently that

$$t_0^{-j}(\nu x) = t_0^{-j}((t_0^{j-j'}\nu')x) = t_0^{-j}(t_0^{j-j'}(\nu' x)) = t_0^{-j'}(\nu' x) \ .$$

One easily deduces from this computation that our definition also is independent of the choice of a specific m in (24). It is straightforward to check that the resulting map

$$\cdot : \Lambda_{H_0,H_1}(H) \times M \longrightarrow M$$

is o-bilinear and functorial in M (at least as long as M satisfies (24)).

Lemma 8.26. *Assuming* (24) *the map* $\cdot$ *is associative.*

Proof. *Step 1:* Let $\lambda, \lambda' \in \Lambda_{H_0,H_1}(H)$ and $x \in M$ be any elements. We have to show that

$$\lambda \cdot (\lambda' \cdot x) = (\lambda \cdot \lambda') \cdot x$$

holds true. Choose $j \geq 0$ and $\nu, \nu' \in \Lambda(H)$ such that

$$\lambda - t_0^{-j}\nu, \lambda' - t_0^{-j}\nu' \in F^m\Lambda_{H_0,H_1}(H) \ .$$

Then

$$\lambda \cdot \lambda' - (t_0^{-j}\nu) \cdot (t_0^{-j}\nu') \in F^m\Lambda_{H_0,H_1}(H) \ .$$

Hence the above identity amounts to

$$t_0^{-j}(\nu(t_0^{-j}(\nu' x))) = ((t_0^{-j}\nu) \cdot (t_0^{-j}\nu')) \cdot x \ . \tag{25}$$

Step 2: We claim that both sides of (25) depend continuously on $\nu, \nu' \in \Lambda(H)$. For the left hand side this is an immediate consequence of the continuity of the $\Lambda_{H_1}(H_0)$- and $\Lambda(H)$-actions on M. To see this on the right hand side we must rewrite it. By Lemma 8.19.ii there are integers $N = N(m,j) \geq 0$ and $l = l(j) \geq 0$ such that

$$t_0^l\Lambda(H) \cdot t_0^{-j} \subseteq \mathfrak{m}(H)^l \cdot t_0^{-j} \subseteq F^m\Lambda_{H_0,H_1}(H) + t_0^{-N}\Lambda(H) \ .$$

Moreover, the same lemma says that the resulting map

$$\begin{aligned} \mathfrak{m}(H)^l &\longrightarrow \Lambda(H)/\mathfrak{m}(H_1)^m\Lambda(H) \\ \nu &\longmapsto \tilde{\nu} + \mathfrak{m}(H_1)^m\Lambda(H) \text{ where } \nu \cdot t_0^{-j} - t_0^{-N}\tilde{\nu} \in F^m\Lambda_{H_0,H_1}(H) \end{aligned}$$

is continuous. It follows that

$$((t_0^{-j}\nu) \cdot (t_0^{-j}\nu')) \cdot x = t_0^{-j-l-N}(((\widetilde{t_0^l\nu})\nu')x) \ ,$$

and that the right hand side is continuous in ν and ν'.

Step 3: The elements in H span a dense o-submodule of $\Lambda(H)$. By the continuity property established in Step 2 it therefore suffices to prove the identity (25) for group elements $\nu = h$ and $\nu' = h'$. Write $h = h_0\gamma_1$ and $h' = h_0'\gamma_2$ with $h_0, h_0' \in H_0$ and $\gamma_1, \gamma_2 \in \Gamma$. Then (25) becomes a special case of the identity

$$\alpha(\gamma_1(\beta(\gamma_2 x))) = (\alpha \cdot \gamma_1 \cdot \beta \cdot \gamma_2) \cdot x \qquad \text{for any } \alpha, \beta \in \Lambda_{H_1}(H_0).$$

Using the σ.-linearity of the Γ-action the left hand side is equal to

$$\alpha(\sigma_{\gamma_1}(\beta)(\gamma_1(\gamma_2 x))) = \alpha\sigma_{\gamma_1}(\beta)((\gamma_1\gamma_2)x) = (\alpha\sigma_{\gamma_1}(\beta)) \cdot ((\gamma_1\gamma_2) \cdot x) \ .$$

Using the Remark 8.24.ii the right hand side is equal to

$$(\alpha \cdot \gamma_1 \cdot \beta \cdot \gamma_1^{-1} \cdot \gamma_1 \cdot \gamma_2) \cdot x = (\alpha\sigma_{\gamma_1}(\beta) \cdot \gamma_1\gamma_2) \cdot x \ .$$

This reduces us to the special case of associativity dealt with in the subsequent last step.

Step 4: For $\mu \in \Lambda_{H_1}(H_0)$ and $\nu \in \Lambda(H)$ we have

$$(\mu \cdot \nu) \cdot x = \mu \cdot (\nu \cdot x) \ .$$

Write $\mu = \mu' + t_0^{-l}\nu'$ for appropriate $l \geq 0$, $\mu' \in \mathfrak{m}_{H_1}(H_0)^m$, and $\nu' \in \Lambda(H_0)$. Then $\mu \cdot \nu = \mu' \cdot \nu + t_0^{-l}\nu'\nu$ with $\mu' \cdot \nu \in F^m\Lambda_{H_0,H_1}(H)$ and $\nu'\nu \in \Lambda(H)$. Hence

$$(\mu \cdot \nu) \cdot x = t_0^{-l}((\nu'\nu)x) = t_0^{-l}(\nu'(\nu x)) = (t_0^{-l}\nu')(\nu x) = \mu \cdot (\nu \cdot x) \ .$$

□

By using (23) our construction extends in an obvious way to arbitrary $(\Lambda_{H_1}(H_0), \Gamma)$-modules M. We leave it to the reader to check that this construction is functorial. We have achieved in this way a fully faithful embedding of the category of $(\Lambda_{H_1}(H_0), \Gamma)$-modules into the category of $\Lambda_{H_0,H_1}(H)$-modules. (Of course, we always keep supposing that $H \cong H_1 \rtimes (H/H_1)$ is a semidirect product.) The image of this embedding is characterized by the next proposition.

Remark 8.27. *The map* (23) *is a topological isomorphism (with the right hand side given the projective limit topology of the weak topologies).*

Proof. Since $\mathfrak{m}_{H_1}(H_0)$ is an ideal in $\Lambda_{H_1}(H_0)$ any open neighbourhood of zero in M contains some $\mathfrak{m}_{H_1}(H_0)^m M$. □

Proposition 8.28. *For any $(\Lambda_{H_1}(H_0), \Gamma)$-module M the corresponding $\Lambda_{H_0,H_1}(H)$-action on M is separately continuous for the weak topologies.*

Proof. By the Remark 8.27 we again may assume that M satisfies (24). Then the multiplication by any $\lambda \in \Lambda_{H_0,H_1}(H)$ is the composite of the multiplication by some $\nu \in \Lambda(H)$ and the multiplication by some $t_0^{-j} \in \Lambda_{H_1}(H_0)$ both of which are already known to be continuous. On the other hand let $x \in M$ be a fixed element. The o-linear map

$$\begin{aligned} \rho_x : \Lambda_{H_0,H_1}(H) &\longrightarrow M \\ \lambda &\longmapsto \lambda \cdot x \end{aligned}$$

whose continuity remains to be seen, by construction, vanishes on $F^m\Lambda_{H_0,H_1}(H)$. Let $U \subseteq M$ be any open neighbourhood of zero which we may assume to be an additive subgroup. For any $j \geq 0$ there is an open neighbourhood of zero $U_j \subseteq M$ such that $t_0^{-j} U_j \subseteq U$. Moreover, since $\rho_x|\Lambda(H)$ is continuous we find an $\ell(j) \geq 0$ such that $\rho_x(\mathfrak{m}(H)^{\ell(j)}) \subseteq U_j$. It follows that

$$\rho_x(t_0^{-j}\mathfrak{m}(H)^{\ell(j)}) = t_0^{-j}\rho_x(\mathfrak{m}(H)^{\ell(j)}) \subseteq t_0^{-j}U_j \subseteq U$$

and hence that $\rho_x(C_{m,\ell}) \subseteq U$.

□

9 Generalized (φ, Γ)-modules

It cannot be expected that the modules $D^i(V)$ have good properties in general. To improve the situation we propose to pass to a specific topological localization. To do so we will apply the construction of the previous section to the situation introduced at the beginning of section 5. Specifically we put $H_1 := N_1 \subseteq H_0 := N_0$. As γ_0 we choose any element in $N_0 \cap N_\alpha$ for some $\alpha \in \Delta$ such that $\ell(\gamma_0) = 1$. We also put $\Gamma := \xi(1 + p^{\epsilon(p)}\mathbb{Z}_p)$, let γ be any topological generator of Γ, and define $H := N_0\Gamma$. The semidirect product condition needed for most of the previous section is satisfied and we have available the diagram of rings

$$\begin{array}{ccc} \Lambda_\ell(N_0\Gamma) := \Lambda_{N_0,N_1}(N_0\Gamma) & \xleftarrow{\supseteq} & \Lambda(N_0\Gamma) \\ \uparrow{\scriptstyle\subseteq} & & \uparrow{\scriptstyle\subseteq} \\ \Lambda_\ell(N_0) := \Lambda_{N_1}(N_0) & \xleftarrow{\supseteq} & \Lambda(N_0). \end{array}$$

The $\sigma_.$-action of Γ on $\Lambda_\ell(N_0)$ extends the Γ-action on $\Lambda(N_0)$ denoted by $\phi_.$ in section 1. Remark 8.24.ii says that these Γ-actions are induced by the conjugation by Γ on $\Lambda_\ell(N_0\Gamma)$.

We want to go one step further. The group $N_0\Gamma$ is the group part of the monoid $P_\star = N_0\Gamma\varphi^{\mathbb{N}_0}$ where $\varphi := \xi(p)$. Correspondingly we have the inclusion of rings $\Lambda(N_0\Gamma) \subseteq \Lambda(P_\star)$. More precisely, if we let σ_φ denote the (injective but not surjective) continuous ring endomorphism of $\Lambda(N_0\Gamma)$ induced by the conjugation by φ on $N_0\Gamma$ then $\Lambda(P_\star) = \Lambda(N_0\Gamma)[\varphi;\sigma_\varphi]$ is the skew polynomial ring over $\Lambda(N_0\Gamma)$ with respect to σ_φ. We note that σ_φ fixes the subring $\Lambda(\Gamma)$ which means that in $\Lambda(P_\star)$ the two variables t and φ commute. The endomorphism σ_φ respects the subrings $\Lambda(N_1) \subseteq \Lambda(N_0)$ and their maximal ideals and, since it still is injective on $\Omega(N_0/N_1)$, also the Ore set $S := S(N_0, N_1)$. It therefore extends to a ring endomorphism first of the localization $\Lambda(N_0)_S$ and then of its $\mathfrak{m}(N_1)$-adic completion $\Lambda_\ell(N_0)$, still denoted by σ_φ. Since φ and γ commute in $T_\star$ the endomorphism σ_φ commutes with $\sigma = \sigma_\gamma$ and $\delta = \sigma - \mathrm{id}$. We visibly have $\sigma_\varphi(B_m) \subseteq B_m$ for any $m \geq 0$ which implies that σ_φ is continuous for the weak topology on $\Lambda_\ell(N_0)$.

Lemma 9.1. *i.* $\sigma_\varphi(t_0) = (t_0 + 1)^p - 1$.

ii. $t_0 = u\sigma_\varphi(t_0)$ *for some unit* u *in* $\Lambda_{\{1\}}(N_0 \cap N_\alpha) \subseteq \Lambda_\ell(N_0)$.

iii. σ_φ *respects bounded subsets for the weak topology on* $\Lambda_\ell(N_0)$.

Proof. i. This follows immediately from our choice of γ_0 and the fact that $\alpha \circ \xi = \mathrm{id}_{\mathbb{G}_m}$.

ii. The ring $\Lambda_{\{1\}}(N_0 \cap N_\alpha)$ is a commutative local ring with maximal ideal generated by π. By i. we have $\sigma_\varphi(t_0) = t_0 v$ where $v := \sum_{i=1}^p \binom{p}{i} t_0^{i-1} = p + \ldots + t_0^{p-1} \in o[t_0]$ does not lie in this maximal ideal. Hence its inverse $u := v^{-1}$ exists.

iii. Let $A \subseteq \Lambda_\ell(N_0)$ be any bounded subset. For a given $m \geq 0$ let $l \geq 0$ be such that $t_0^l A \subseteq \mathfrak{m}_{N_1}(N_0)^m + \Lambda(N_0)$ (cf. Lemma 8.8). Applying σ_φ and using ii. we obtain

$$t_0^l \sigma_\varphi(A) = u^l \sigma_\varphi(t_0^l A) \subseteq \mathfrak{m}_{N_1}(N_0)^m + u^l \Lambda(N_0) \ .$$

If we choose $k \geq 0$ such that $t_0^k u \in \pi^m \Lambda_{\{1\}}(N_0 \cap N_\alpha) + o[[t_0]]$ then $t_0^{(k+1)l} \sigma_\varphi(A) \subseteq \mathfrak{m}_{N_1}(N_0)^m + \Lambda(N_0)$. It now follows from Lemma 8.8 that $\sigma_\varphi(A)$ is bounded.

□

We therefore may define the map

$$\begin{aligned} \sigma_\varphi : \Lambda_\ell(N_0\Gamma) &\longrightarrow \Lambda_\ell(N_0\Gamma) \\ \sum_{i\geq 0} \mu_i t^i &\longmapsto \sum_{i \geq 0} \sigma_\varphi(\mu_i) t^i \ . \end{aligned}$$

It is immediate from (18), the continuity of σ_φ on $\Lambda_\ell(N_0)$, and its commutation with σ and δ that this extended σ_φ in fact is an endomorphism of the ring $\Lambda_\ell(N_0\Gamma)$.

Remark 9.2. *The endomorphism* σ_φ *is continuous for the strong as well as the weak topology on* $\Lambda_\ell(N_0\Gamma)$.

Proof. The case of the strong topology is obvious since σ_φ respects the filtration $F^m\Lambda_\ell(N_0\Gamma)$. The case of the weak topology is a straightforward consequence of the following computation based on Lemma 9.1.ii. Again let $k \geq 0$ be such that $t_0^k u \in \pi^m \Lambda_{\{1\}}(N_0 \cap N_\alpha) + o[[t_0]]$. For any $j, l \geq 0$ we then have

$$\begin{aligned}\sigma_\varphi(t_0^{-j}\mathfrak{m}(N_0\Gamma)^l) &\subseteq \sigma_\varphi(t_0)^{-j}\mathfrak{m}(N_0\Gamma)^l \\ &= t_0^{-j}u^j\mathfrak{m}(N_0\Gamma)^l \\ &= t_0^{-j(1+k)}t_0^{kj}u^j\mathfrak{m}(N_0\Gamma)^l \\ &\subseteq t_0^{-j(1+k)}(\pi^m\Lambda_{\{1\}}(N_0 \cap N_\alpha) + o[[t_0]])\mathfrak{m}(N_0\Gamma)^l \\ &\subseteq F^m\Lambda_\ell(N_0\Gamma) + t_0^{-j(1+k)}\mathfrak{m}(N_0\Gamma)^l\end{aligned}$$

□

This allows us to form the skew polynomial ring

$$\Lambda_\ell(P_\star) := \Lambda_\ell(N_0\Gamma)[\varphi; \sigma_\varphi] \ .$$

As a bimodule it satisfies

$$\Lambda_\ell(P_\star) = \Lambda_\ell(N_0\Gamma) \otimes_{\Lambda(N_0\Gamma)} \Lambda(P_\star) \ . \qquad (26)$$

Our basic diagram for the following now is

$$\begin{array}{ccc} \Lambda_\ell(P_\star) & \xleftarrow{\supseteq} & \Lambda(P_\star) \\ \uparrow \subseteq & & \uparrow \subseteq \\ \Lambda_\ell(N_0) & \xleftarrow{\supseteq} & \Lambda(N_0). \end{array}$$

Definition 9.3. *A $(\Lambda_\ell(N_0), \Gamma, \varphi)$-module M is a $(\Lambda_\ell(N_0), \Gamma)$-module with an additional σ_φ-linear endomorphism φ_M which commutes with the Γ-action.*

We point out that, as σ_φ is continuous on $\Lambda_\ell(N_0)$, the σ_φ-linear endomorphism φ_M of a $(\Lambda_\ell(N_0), \Gamma, \varphi)$-module M, which by definition is finitely generated over $\Lambda_\ell(N_0)$, automatically is continuous for the weak topology on the module M.

In Lemma 8.26 we have seen that the $(\Lambda_\ell(N_0), \Gamma)$-modules form a full subcategory of all $\Lambda_\ell(N_0\Gamma)$-modules.

Lemma 9.4. *Let M be a $(\Lambda_\ell(N_0), \Gamma, \varphi)$-module viewed as a $\Lambda_\ell(N_0\Gamma)$-module; the endomorphism φ_M is σ_φ-linear with respect to the $\Lambda_\ell(N_0\Gamma)$-action.*

Proof. We may assume that $\mathfrak{m}_{N_1}(N_0)^m M = 0$ for some $m \geq 0$. By the definition of the $\Lambda_\ell(N_0\Gamma)$-module structure we then have to show the identity

$$\varphi_M((t_0^{-j}\nu) \cdot x) = \sigma_\varphi(t_0^{-j}\nu) \cdot \varphi_M(x)$$

for any $j \geq 0$, $\nu \in \Lambda(N_0\Gamma)$, and $x \in M$. As a consequence of Prop. 8.28, Remark 9.2, and the continuity of φ_M both sides of this identity depend continuously on ν. Hence it suffices to consider any ν of the form $\nu = \nu_0 + \nu_1 t + \ldots + \nu_k t^k$ with $\nu_i \in \Lambda(N_0)$. By assumption φ_M is σ_φ-linear with respect to scalars in $\Lambda_\ell(N_0)$. This, in fact, reduces us to the identity

$$\varphi_M(t \cdot x) = \sigma_\varphi(t) \cdot \varphi_M(x) \qquad \text{for any } x \in M.$$

On the left hand side φ_M commutes with the Γ-action by assumption. On the right hand side we have $\sigma_\varphi(t) = t$. Hence both sides are equal to $t \cdot \varphi_M(x)$. □

This lemma implies that by letting $\varphi \in \Lambda(P_\star)$ act as φ_M on a $(\Lambda_\ell(N_0), \Gamma, \varphi)$-module M we obtain a $\Lambda_\ell(P_\star)$-module. In this way the $(\Lambda_\ell(N_0), \Gamma, \varphi)$-modules form a full subcategory of all $\Lambda_\ell(P_\star)$-modules.

We recall from section 1 that a $\Lambda(P_\star)$-module M is etale if the $\Lambda(P_\star)$-linear map

$$\begin{aligned} \Lambda(P_\star) \otimes_{\Lambda(P_\star),\sigma_\varphi} M &\xrightarrow{\cong} M \\ \mu \otimes x &\longmapsto \mu\varphi x \end{aligned}$$

is an isomorphism. We observe that the endomorphisms σ_φ of $\Lambda_\ell(N_0\Gamma)$ and of $\Lambda(P_\star)$ both come by restriction from the ring endomorphism

$$\begin{aligned} \sigma_\varphi : \Lambda_\ell(P_\star) = \Lambda_\ell(N_0\Gamma)[\varphi;\sigma_\varphi] &\longrightarrow \Lambda_\ell(P_\star) = \Lambda_\ell(N_0\Gamma)[\varphi;\sigma_\varphi] \\ \sum_{k\geq 0} \lambda_k \varphi^k &\longrightarrow \sum_{k\geq 0} \sigma_\varphi(\lambda_k)\varphi^k \ . \end{aligned}$$

This suggests the following definition.

Definition 9.5. *A $\Lambda_\ell(P_\star)$-module M is called etale if the $\Lambda_\ell(P_\star)$-linear map*

$$\begin{aligned} \Lambda_\ell(P_\star) \otimes_{\Lambda_\ell(P_\star),\sigma_\varphi} M &\xrightarrow{\cong} M \\ \mu \otimes x &\longmapsto \mu\varphi x \end{aligned}$$

is an isomorphism.

Proposition 9.6. *The endomorphism σ_φ of $\Lambda_\ell(P_\star)$ is injective and makes $\Lambda_\ell(P_\star)$ a free right module of rank $[N_0 : \varphi N_0 \varphi^{-1}]$ over itself; the map*

$$\begin{aligned} \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\sigma_\varphi} \Lambda_\ell(P_\star) &\xrightarrow{\cong} \Lambda_\ell(P_\star) \\ \nu \otimes \mu &\longmapsto \nu\sigma_\varphi(\mu) \end{aligned}$$

is an isomorphism.

Proof. Preliminary observation: In the general situation of section 8 let $H_0' \subseteq H_0$ be an open subgroup and put $H_1' := H_1 \cap H_0'$. We then have of course $\Lambda(H_0') \subseteq \Lambda(H_0)$ and $J(H_0', H_1') \subseteq J(H_0, H_1)$. But since $\Omega(H_0'/H_1') \subseteq \Omega(H_0/H_1)$ we also have $S(H_0', H_1') \subseteq S(H_0, H_1)$. By localization and completion we therefore obtain a natural ring homomorphism $\Lambda_{H_1'}(H_0') \longrightarrow \Lambda_{H_1}(H_0)$ which gives rise to a natural homomorphism of bimodules

$$\Lambda(H_0) \otimes_{\Lambda(H_0')} \Lambda_{H_1'}(H_0') \longrightarrow \Lambda_{H_1}(H_0) \ .$$

Step 1: We claim that the natural map

$$\Lambda(N_0) \otimes_{\Lambda(\varphi N_0 \varphi^{-1})} \Lambda_{\varphi N_1 \varphi^{-1}}(\varphi N_0 \varphi^{-1}) \xrightarrow{\cong} \Lambda_{N_1}(N_0) = \Lambda_\ell(N_0)$$

is bijective. (Note that $\varphi N_1 \varphi^{-1} = N_1 \cap \varphi N_0 \varphi^{-1}$.) We choose an open subgroup $N_0' \subseteq \varphi N_0 \varphi^{-1} \subseteq N_0$ which is normal in N_0. We now apply (a symmetric version of) [SV2] Prop. 4.5 to the pairs $N_0' \trianglelefteq \varphi N_0 \varphi^{-1}$ and $N_0' \trianglelefteq N_0$ obtaining the isomorphisms

$$\Lambda(\varphi N_0 \varphi^{-1}) \otimes_{\Lambda(N_0')} \Lambda_{N_1 \cap N_0'}(N_0') \xrightarrow{\cong} \Lambda_{N_1 \cap \varphi N_0 \varphi^{-1}}(\varphi N_0 \varphi^{-1})$$

and

$$\Lambda(N_0) \otimes_{\Lambda(N_0')} \Lambda_{N_1 \cap N_0'}(N_0') \xrightarrow{\cong} \Lambda_{N_1}(N_0) \ .$$

The combination of the two gives our claim. In addition we know that $\Lambda_{\varphi N_1 \varphi^{-1}}(\varphi N_0 \varphi^{-1})$ is flat as a (left) $\Lambda(\varphi N_0 \varphi^{-1})$-module. It follows that the natural ring homomorphism

$$\begin{aligned}\Lambda_{\varphi N_1 \varphi^{-1}}(\varphi N_0 \varphi^{-1}) = \Lambda(\varphi N_0 \varphi^{-1}) \otimes_{\Lambda(\varphi N_0 \varphi^{-1})} \Lambda_{\varphi N_1 \varphi^{-1}}(\varphi N_0 \varphi^{-1}) \\ \longrightarrow \Lambda(N_0) \otimes_{\Lambda(\varphi N_0 \varphi^{-1})} \Lambda_{\varphi N_1 \varphi^{-1}}(\varphi N_0 \varphi^{-1}) = \Lambda_\ell(N_0)\end{aligned}$$

is injective. In an obvious reformulation we have shown so far that the ring endomorphism σ_φ of $\Lambda_\ell(N_0)$ is injective and that the bimodule map

$$\begin{aligned}\Lambda(N_0) \otimes_{\Lambda(N_0),\sigma_\varphi} \Lambda_\ell(N_0) &\xrightarrow{\cong} \Lambda_\ell(N_0) \\ \nu \otimes \mu &\longmapsto \nu \sigma_\varphi(\mu)\end{aligned}$$

is bijective.

Step 2: We let $r := [N_0 : \varphi N_0 \varphi^{-1}]$ and we fix representatives $n_1, \ldots, n_r \in N_0$ for the cosets in $N_0/\varphi N_0 \varphi^{-1}$. In the previous step we have seen that the map

$$\begin{aligned}I: \qquad \Lambda_\ell(N_0)^r &\xrightarrow{\cong} \Lambda_\ell(N_0) \\ (\mu^{(1)}, \ldots \mu^{(r)}) &\longmapsto n_1 \sigma_\varphi(\mu^{(1)}) + \ldots + n_r \sigma_\varphi(\mu^{(r)})\end{aligned}$$

is bijective. We claim that I also is a homeomorphism for the weak topology, resp. the direct product of the weak topologies, on the right, resp. left, hand side. To see this we pick an open subgroup $N' \subseteq \varphi N_1 \varphi^{-1}$ which is normal in N_0. In particular, N' is normal in $\varphi N_0 \varphi^{-1}$. It then follows from [SV2] Lemma 4.4 that in $\Lambda_\ell(N_0)$, resp. in $\Lambda_{\varphi N_1 \varphi^{-1}}(\varphi N_0 \varphi^{-1})$ the $\mathfrak{m}_{N_1}(N_0)$-adic and the $\mathfrak{m}(N')\Lambda_\ell(N_0)$-adic filtrations, resp. the $\mathfrak{m}_{\varphi N_1 \varphi^{-1}}(\varphi N_0 \varphi^{-1})$-adic and the $\mathfrak{m}(N')\Lambda_\ell(\varphi N_0 \varphi^{-1})$-adic filtrations, are equivalent. In fact, this means that in $\Lambda_\ell(N_0)$ all three filtrations, the $\mathfrak{m}_{N_1}(N_0)$-adic one, the $\mathfrak{m}(N')\Lambda_\ell(N_0)$-adic one, and the $\mathfrak{m}(\varphi^{-1} N' \varphi)\Lambda_\ell(N_0)$-adic one are equivalent. Visibly under the map I the product filtration $\big(\mathfrak{m}(\varphi^{-1} N' \varphi)^m \Lambda_\ell(N_0)\big)^r$ on the left hand side corresponds to the filtration

$$\begin{aligned}n_1 \mathfrak{m}(N')^m \sigma_\varphi(\Lambda_\ell(N_0)) + \ldots + n_r \mathfrak{m}(N')^m \sigma_\varphi(\Lambda_\ell(N_0)) = \\ \mathfrak{m}(N')^m \big(n_1 \sigma_\varphi(\Lambda_\ell(N_0)) + \ldots + n_r \sigma_\varphi(\Lambda_\ell(N_0))\big) = \mathfrak{m}(N')^m \Lambda_\ell(N_0) \ .\end{aligned}$$

on the right hand side. It remains to observe that the restriction of the map I to $\Lambda(N_0)^r \xrightarrow{\cong} \Lambda(N_0)$ is a homeomorphism by compactness.

Step 3: It follows immediately from the first step that both, the ring endomorphism σ_φ of $\Lambda_\ell(N_0\Gamma)$ as well as the bimodule map

$$\begin{aligned}\Lambda(N_0) \otimes_{\Lambda(N_0),\sigma_\varphi} \Lambda_\ell(N_0\Gamma) &\longrightarrow \Lambda_\ell(N_0\Gamma) \\ \nu \otimes \mu = \nu \otimes \sum_{i \geq 0} \mu_i t^i &\longmapsto \nu \sigma_\varphi(\mu) = \sum_{i \geq 0} \nu \sigma_\varphi(\mu_i) t^i \ ,\end{aligned} \tag{27}$$

are injective. To establish the surjectivity of the latter map let $\lambda = \sum_{i \geq 0} \lambda_i t^i \in \Lambda_\ell(N_0\Gamma)$ be any element. According to the first step there are, for any $i \geq 0$, uniquely determined elements $\mu_i^{(1)}, \ldots, \mu_i^{(r)} \in \Lambda_\ell(N_0)$ such that

$$\lambda_i = n_1 \sigma_\varphi(\mu_i^{(1)}) + \ldots + n_r \sigma_\varphi(\mu_i^{(r)}) \ .$$

We claim that, for each $1 \leq k \leq r$, the subset $\{\mu_i^{(k)}\}_{i\geq 0} \subseteq \Lambda_\ell(N_0)$ is bounded for the weak topology. Let $m' \geq 0$. By the second step we find an $m'' \geq 0$ such that $I(B_{m'}^r) \supseteq B_{m''}$. The boundedness of the set $\{\lambda_i\}_{i\geq 0}$ implies the existence of an $m \geq 0$ such that $\lambda_i B_m \subseteq B_{m''}$ for any $i \geq 0$. For any $\nu \in B_m$ we then obtain $I\big((\mu_i^{(1)}\nu, \ldots, \mu_i^{(r)}\nu)\big) = \lambda_i \sigma_\varphi(\nu) \in \lambda_i \sigma_\varphi(B_m) \subseteq \lambda_i B_m \subseteq B_{m''}$, hence $\mu_i^{(1)}\nu, \ldots, \mu_i^{(r)}\nu \in B_{m'}$, and therefore

$$\mu_i^{(k)} B_m \subseteq B_{m'} \qquad \text{for any } i \geq 0 \text{ and } 1 \leq k \leq r.$$

It follows that the elements $\mu^{(k)} := \sum_{i\geq 0} \mu_i^{(k)} t^k$ are well defined in $\Lambda_\ell(N_0\Gamma)$ and that we have

$$\lambda = n_1 \sigma_\varphi(\mu^{(1)}) + \ldots + n_r \sigma_\varphi(\mu^{(r)}) \ .$$

This proves the surjectivity and hence bijectivity of (27).

Step 4: The injectivity of σ_φ on $\Lambda_\ell(P_\star)$ is a trivial consequence of the injectivity of σ_φ on $\Lambda_\ell(N_0\Gamma)$ established in the third step. The isomorphisms (26) and (27) combine into the isomorphisms

$$\Lambda(N_0) \otimes_{\Lambda(N_0),\sigma_\varphi} \Lambda_\ell(P_\star) \xrightarrow{\cong} \Lambda_\ell(P_\star) \quad \text{and} \quad \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\sigma_\varphi} \Lambda_\ell(P_\star) \xrightarrow{\cong} \Lambda_\ell(P_\star) \ .$$

□

Corollary 9.7. *Let M be a $(\Lambda_\ell(N_0), \Gamma, \varphi)$-module; viewed as a $\Lambda_\ell(P_\star)$-module M is etale if and only if the map*

$$\begin{aligned} \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\sigma_\varphi} M &\xrightarrow{\cong} M \\ \nu \otimes x &\longmapsto \nu\varphi_M(x) \end{aligned}$$

is bijective.

Proof. By Prop. 9.6 we have

$$\Lambda_\ell(P_\star) \otimes_{\Lambda_\ell(P_\star),\sigma_\varphi} M = \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\sigma_\varphi} \Lambda_\ell(P_\star) \otimes_{\Lambda_\ell(P_\star)} M = \Lambda_\ell(N_0) \otimes_{\Lambda_\ell(N_0),\sigma_\varphi} M \ .$$

□

Let $\mathcal{M}(\Lambda_\ell(P_\star))$ be the abelian category of (left unital) $\Lambda_\ell(P_\star)$-modules and $D^+(\Lambda_\ell(P_\star))$ the corresponding bounded below derived category. Let $\mathcal{M}_{et}(\Lambda_\ell(P_\star))$ denote the full subcategory in $\mathcal{M}(\Lambda_\ell(P_\star))$ of etale modules and $D^+_{et}(\Lambda_\ell(P_\star))$ the full subcategory in $D^+(\Lambda_\ell(P_\star))$ of all complexes whose cohomology modules are etale.

Corollary 9.8. *i. The subcategory $\mathcal{M}_{et}(\Lambda_\ell(P_\star))$ of $\mathcal{M}(\Lambda_\ell(P_\star))$ is closed under the formation of kernels, cokernels, extensions and arbitrary inductive and projective limits; in particular, $\mathcal{M}_{et}(\Lambda_\ell(P_\star))$ is an abelian category.*

ii. $D^+_{et}(\Lambda_\ell(P_\star))$ is a triangulated subcategory of $D^+(\Lambda_\ell(P_\star))$.

The base change functor for modules, which by Prop. 8.21.iii and (26) is exact, obviously restricts to an exact functor

$$\Lambda_\ell(P_\star) \otimes_{\Lambda(P_\star)} : \mathcal{M}_{et}(\Lambda(P_\star)) \longrightarrow \mathcal{M}_{et}(\Lambda_\ell(P_\star)) \ .$$

and then extends to the functor

$$\Lambda_\ell(P_\star)\otimes_{\Lambda(P_\star)} : D^+_{et}(\Lambda(P_\star)) \longrightarrow D^+_{et}(\Lambda_\ell(P_\star)) \ .$$

between derived categories. We introduce the composed functor

$$RD_\ell : D^-(\mathcal{M}_{o-tor}(P)) \xrightarrow{RD} D^+_{et}(\Lambda(P_+)) \xrightarrow{\text{forget}} D^+_{et}(\Lambda(P_\star)) \xrightarrow{\Lambda_\ell(P_\star)\otimes_{\Lambda(P_\star)}} D^+_{et}(\Lambda_\ell(P_\star)) \ .$$

as well as the δ-functor

$$D^i_\ell(V) := h^i(RD_\ell(V)) = \Lambda_\ell(P_\star) \otimes_{\Lambda(P_\star)} D^i(V) \qquad \text{for } i \geq 0$$

from $\mathcal{M}_{o-tor}(P)$ into etale $\Lambda_\ell(P_\star)$-modules. The **fundamental open question** in this context is for which representations V in $\mathcal{M}_{o-tor}(P)$ the etale module $D^0_\ell(V)$ (or even any $D^i_\ell(V)$) is a $(\Lambda_\ell(N_0), \Gamma, \varphi)$-module.

A first result in this direction can be obtained by generalizing to our noncommutative setting the arguments in [Eme].

Proposition 9.9. *Let V be an admissible representation in $\mathcal{M}_{o-tor}(G)$; suppose that for some $M \in \mathcal{P}_+(V)$ there is an exact sequence of P_+-representations of the form*

$$\ldots \longrightarrow \mathrm{ind}^{P_+}_{P_0}(V_n) \longrightarrow \ldots \longrightarrow \mathrm{ind}^{P_+}_{P_0}(V_0) \longrightarrow M \longrightarrow 0$$

with V_n finite for any $n \geq 0$; then M^* and hence $D(V)$ are finitely generated $\Lambda(N_0)$-modules, and $\mathcal{P}_+(V)$ contains a unique minimal element.

Proof. Step 1: At first we let $M \in \mathcal{P}_+(V)$ be arbitrary. Since $D(V)$ is a quotient of M^* and since $\Lambda(N_0)$ is noetherian it suffices to show that the compact $\Lambda(N_0)$-module M^* is finitely generated. By the topological Nakayama lemma ([BH]) this reduces to the finiteness of $M^*/\mathfrak{m}(N_0)M^*$. The latter is the Pontrjagin dual of the group $H^0(N_0, M^{\pi=0})$ of N_0-invariants in the k-vector space $M^{\pi=0}$ consisting of all elements in M which are annihilated by π. We therefore will show that $H^0(N_0, M^{\pi=0})$ is finite.

The N_0-invariants $H^0(N_0, V)$ in V do not form a P_+-subrepresentation. But the monoid T_+ acts on $H^0(N_0, V)$ via the so called Hecke action which is defined by

$$t \cdot v := \sum_{n \in N_0/tN_0t^{-1}} ntv \qquad \text{for } t \in T_+ \text{ and } v \in V.$$

Since for T_0 the Hecke action coincides with the group action we see that the Hecke action extends to a $\Lambda(P_+)$-module structure. By [Em] Thm. 3.2.3(1) the admissibility of V implies that $H^0(N_0, V)$ is a union of Hecke invariant finitely generated o-submodules. It follows that $H^0(N_0, M^{\pi=0})$ is a union of Hecke invariant finite k-vector spaces. Hence $H^0(N_0, M^{\pi=0})$ is finite if and only if it is finitely generated as a $\Lambda(P_+)$-module (for the Hecke action).

Step 2: Next we apply duality to $H^0(N_0, M^{\pi=0})$. First of all we observe that

$$H^0(N_0, M^{\pi=0}) = \mathrm{Hom}_{\Lambda(N_0)}(k, M) \ .$$

Let d denote the dimension of the p-adic Lie group N_0. The ring $\Lambda(N_0)$ is a regular local noetherian integral domain of global dimension $d+1$ ([Neu]). We therefore have, for any finitely generated $\Lambda(N_0)$-module X, the natural duality isomorphism

$$\mathrm{Ext}^*_{\Lambda(N_0)}(X, .) = \mathrm{Tor}^{\Lambda(N_0)}_{d+1-*}(\mathcal{D}_{\Lambda(N_0)}(X), .)$$

between functors on the category of all (left) $\Lambda(N_0)$-modules where the dualizing complex

$$\mathcal{D}_{\Lambda(N_0)}(X) := \mathrm{RHom}_{\Lambda(N_0)}(X, \Lambda(N_0))$$

is placed in degrees $-(d+1)$ up to 0. On the other hand N_0 is a Poincaré group ([Laz] Thm. V.2.5.8). Therefore ([NSW] Cor. 5.4.15(ii)) the dualizing complex $\mathcal{D}_{\Lambda(N_0)}(k)$ in fact is quasi-isomorphic to the trivial module k placed in degree zero. (We note that the character $\chi : N_0 \longrightarrow \mathbb{Z}_p^\times$ which describes the action of N_0 on the dualizing module $I \cong \mathbb{Q}_p/\mathbb{Z}_p$ has values in $1+p\mathbb{Z}_p$ so that $I^{p=0}$ is a trivial N_0-module.) It follows that the duality isomorphism specializes to

$$\mathrm{Ext}^*_{\Lambda(N_0)}(k, .) = \mathrm{Tor}^{\Lambda(N_0)}_{d+1-*}(k, .) \ ,$$

and we obtain in particular that

$$H^0(N_0, M^{\pi=0}) = \mathrm{Tor}^{\Lambda(N_0)}_{d+1}(k, M) \tag{28}$$

for any $M \in \mathcal{P}_+(V)$.

Step 3: In this step we identify the Hecke action on the right hand side of (28). We begin with a completely general observation. Let H_0 be any profinite group and $H_1 \subseteq H_0$ any open subgroup. Then

$$\begin{aligned} \mathrm{Hom}_{\Lambda(H_1)}(\Lambda(H_0), \Lambda(H_1)) &\xrightarrow{\cong} \Lambda(H_0) \\ f &\longmapsto \sum_{g \in H_0/H_1} g f(g^{-1}) \end{aligned}$$

is an isomorphism of left $\Lambda(H_0)$-modules. Hence

$$\mathrm{Hom}_{\Lambda(H_1)}(\Lambda(H_0), Y) = \mathrm{Hom}_{\Lambda(H_1)}(\Lambda(H_0), \Lambda(H_1)) \otimes_{\Lambda(H_1)} Y \cong \Lambda(H_0) \otimes_{\Lambda(H_1)} Y$$

for any left $\Lambda(H_1)$-module Y. It follows that the exact scalar extension functor $\Lambda(H_0) \otimes_{\Lambda(H_1)} .$ preserves injective modules and is right adjoint to the scalar restriction functor. We consequently have the adjunction isomorphism

$$\begin{aligned} \mathrm{Hom}_{\Lambda(H_1)}(X, Y) &\xrightarrow{\cong} \mathrm{Hom}_{\Lambda(H_0)}(X, \Lambda(H_0) \otimes_{\Lambda(H_1)} Y) \\ f &\longmapsto [x \longmapsto \sum_{g \in H_0/H_1} g \otimes f(g^{-1}x)] \end{aligned} \tag{29}$$

for any $\Lambda(H_0)$-module X and any $\Lambda(H_1)$-module Y. More generally, by using an injective resolution of Y, we obtain

$$\mathrm{RHom}_{\Lambda(H_1)}(X, Y) \cong \mathrm{RHom}_{\Lambda(H_0)}(X, \Lambda(H_0) \otimes_{\Lambda(H_1)} Y) \ .$$

For $X = \Lambda(H_0)$ one checks by a straightforward computation that the isomorphism (29) is compatible with the above duality isomorphism. For purely formal reasons the same then holds true for any X which is finitely generated and projective over $\Lambda(H_0)$ and hence over $\Lambda(H_1)$.

Let $t \in T_+$. We apply the above discussion to the groups $H_0 := N_0$ and $H_1 := tN_0t^{-1}$ and obtain the commutative diagram of isomorphisms

$$\begin{CD}
\mathrm{Ext}^*_{\Lambda(tN_0t^{-1})}(X,Y) @>\cong>> \mathrm{Tor}^{\Lambda(tN_0t^{-1})}_{d+1-*}(\mathcal{D}_{\Lambda(tN_0t^{-1})}(X),Y) \\
@V\cong VV @VV\cong V \\
\mathrm{Ext}^*_{\Lambda(N_0)}(X,\Lambda(N_0)\otimes_{\Lambda(tN_0t^{-1})} Y) @>\cong>> \mathrm{Tor}^{\Lambda(N_0)}_{d+1-*}(\mathcal{D}_{\Lambda(N_0)}(X),\Lambda(N_0)\otimes_{\Lambda(tN_0t^{-1})} Y)
\end{CD}$$

for any finitely generated $\Lambda(N_0)$-module X and any $\Lambda(tN_0t^{-1})$-module Y where the horizontal, resp. perpendicular, arrows come from duality, resp. adjunction. In the special case $X = k$ we may rewrite this as a commutative diagram of isomorphisms

$$\begin{CD}
\mathrm{Ext}^*_{\Lambda(N_0)}(k,Y) @>\cong>> \mathrm{Tor}^{\Lambda(N_0)}_{d+1-*}(k,Y) \\
@V\cong VV @VV\cong V \\
\mathrm{Ext}^*_{\Lambda(N_0)}(k,\Lambda(N_0)\otimes_{\Lambda(N_0),t} Y) @>\cong>> \mathrm{Tor}^{\Lambda(N_0)}_{d+1-*}(k,\Lambda(N_0)\otimes_{\Lambda(N_0),t} Y)
\end{CD}$$

for any $\Lambda(N_0)$-module Y.

We now suppose that Y is a $\Lambda(P_+)$-module. We then have the $\Lambda(N_0)$-equivariant map

$$\begin{aligned} \Lambda(N_0)\otimes_{\Lambda(N_0),t} Y &\longrightarrow Y \\ \lambda\otimes y &\longmapsto \lambda t y\ . \end{aligned} \tag{30}$$

Hence the naturality of the duality isomorphism together with the above commutative diagram gives rise to the commutative diagram

$$\begin{CD}
\mathrm{Ext}^*_{\Lambda(N_0)}(k,Y) @>\cong>> \mathrm{Tor}^{\Lambda(N_0)}_{d+1-*}(k,Y) \\
@VVV @VVV \\
\mathrm{Ext}^*_{\Lambda(N_0)}(k,Y) @>\cong>> \mathrm{Tor}^{\Lambda(N_0)}_{d+1-*}(k,Y)
\end{CD}$$

where the vertical arrows are induced by (30). We see that the duality isomorphism respects the natural $\Lambda(P_+)$-actions on both sides. Moreover, using (29) one easily checks that under the identification $H^0(N_0, M^{\pi=0}) = \mathrm{Hom}_{\Lambda(N_0)}(k,M)$ the Hecke action on the left hand side corresponds to the natural $\Lambda(P_+)$-action on the right hand side.

Step 4: It remains to show that under the assumption imposed on M in our assertion $\mathrm{Tor}^{\Lambda(N_0)}_{d+1}(k,M)$ is finitely generated over $\Lambda(P_+)$ (for the natural action). In fact, since N_0 of course acts trivially on $\mathrm{Tor}^{\Lambda(N_0)}_{*}(k,M)$, it is the factor ring $\Lambda(P_+)/\mathfrak{m}(N_0)\Lambda(P_+)$ by the two-sided ideal $\mathfrak{m}(N_0)\Lambda(P_+)$ which really acts. This factor ring is isomorphic to the ring $\Omega(T_+)$ which is formally constructed in exactly the same way as $\Lambda(P_+)$ but starting from the monoid rings over k of the factor monoids of T_+. In fact, in this situation we simply have

$$\Omega(T_+) = \Omega(T_0)[T_+/T_0]$$

which obviously is a commutative ring. Moreover, since the factor monoid T_+/T_0 is finitely generated the ring $\Omega(T_+)$ is a finitely generated algebra over the noetherian ring $\Omega(T_0)$ and therefore is noetherian.

We now compute $\mathrm{Tor}_*^{\Lambda(N_0)}(k, M)$ as a $\Lambda(T_+)$-module in the following way. Let

$$\ldots \longrightarrow \bigoplus_{I_n} \Lambda(P_+) \longrightarrow \ldots \longrightarrow \bigoplus_{I_0} \Lambda(P_+) \longrightarrow M \longrightarrow 0$$

be any resolution of M by free $\Lambda(P_+)$-modules. Since $\Lambda(P_+)$ is free as a left $\Lambda(P_0)$-module and $\Lambda(P_0)$ is flat over $\Lambda(N_0)$ this in particular is a resolution of M by flat $\Lambda(N_0)$-modules. Hence

$$\begin{aligned}\mathrm{Tor}_*^{\Lambda(N_0)}(k, M) &= h_*\big(k \otimes_{\Lambda(N_0)} \big(\bigoplus_{I_\bullet} \Lambda(P_+)\big)\big) = h_*\big(\bigoplus_{I_\bullet} \Lambda(P_+)/\mathfrak{m}(N_0)\Lambda(P_+)\big) \\ &= h_*\big(\bigoplus_{I_\bullet} \Omega(T_+)\big).\end{aligned}$$

It follows that if the index set I_{d+1} is finite then, since $\Omega(T_+)$ is noetherian, $\mathrm{Tor}_{d+1}^{\Lambda(N_0)}(k, M)$ is a finitely generated $\Omega(T_+)$-module. Therefore it suffices to show that our assumption on M guarantees the existence of a resolution of M by finitely generated free $\Lambda(P_+)$-modules. A double complex argument further reduces us to showing that each representation $\mathrm{ind}_{P_0}^{P_+}(V)$ with finite V has a resolution by finitely generated free $\Lambda(P_+)$-modules. But, $\Lambda(P_0)$ being noetherian, V certainly has a resolution by finitely generated free $\Lambda(P_0)$-modules. We only have to apply the exact functor $\mathrm{ind}_{P_0}^{P_+}(.) = \Lambda(P_+) \otimes_{\Lambda(P_0)} -$ to this latter resolution. □

Having Lemma 8.12.iv in mind it also seems interesting to investigate the "completed" base change functor which sends M to

$$\begin{aligned}&\varprojlim_m \Big(\big(\Lambda_\ell(N_0\Gamma)/F^m\Lambda_\ell(N_0\Gamma)\big) \otimes_{\Lambda(N_0\Gamma)} M \Big) \\ &\quad = \varprojlim_m \Big(\big(\Lambda_\ell(N_0\Gamma)/F^m\Lambda_\ell(N_0\Gamma)\big) \otimes_{\Lambda(N_0\Gamma)} \big(M/\mathfrak{m}(N_1)^m M\big) \Big) \\ &\quad = \varprojlim_m \Big(\big(\Lambda_\ell(N_0)/\mathfrak{m}_{N_1}(N_0)^m\big) \otimes_{\Lambda(N_0)} M \Big) .\end{aligned}$$

Unfortunately it has no apparent exactness properties.

We finish this section by showing that by an appropriate choice of the subgroup N_0 one can improve the properties of the ring $\Lambda_\ell(N_0)$.

Proposition 9.10. *i. If the pro-p group N_0 satisfies $[N_0, N_0] \subseteq N_0^{p^2}$ then $\Lambda_\ell(N_0)$ is an integral domain.*

ii. Let

$$\iota_\alpha(N_0 \cap N_\alpha) = p^{n(\alpha)}\mathbb{Z}_p \qquad \text{for any } \alpha \in \Phi^+$$

and suppose that the function $n : \Phi^+ \longrightarrow \mathbb{Z}$ satisfies $n(i\alpha + j\beta) < in(\alpha) + jn(\beta) - 1$ for any $\alpha, \beta \in \Phi^+$ and $i, j > 0$ such that $i\alpha + j\beta \in \Phi^+$; we then have $[N_0, N_0] \subseteq N_0^{p^2}$.

Proof. i. Pro-p groups satisfying the commutator condition in our assertion are called extra-powerful in the literature. Since N_0/N_1 is torsionfree the subgroup N_1 is extra-powerful as well. Since even N_0 is torsionfree and since N_0 and N_1 are topologically finitely generated we know from [DDMS] Thm. 4.5 that N_0 and N_1 are uniform pro-p groups. It suffices to show that the graded ring $\mathrm{gr}^\bullet \Lambda_\ell(N_0)$ for the $\mathfrak{m}(N_1)$-adic filtration is an integral domain. It is known (cf. [Ven] Thm. 3.22) that for a uniform and extra-powerful N_1 the graded ring $\mathrm{gr}^\bullet \Lambda(N_1)$ for the $\mathfrak{m}(N_1)$-adic filtration is a polynomial ring in finitely many variables over k and hence is an integral domain. On the other hand in our situation [SV2] Lemma 4.3(ii) and (iii) hold and say that the assumptions in [SV2] Prop. 1.15 are satisfied the proof of which then tells us that $\mathrm{gr}^\bullet \Lambda_\ell(N_0)$ is a subring of the commutative Laurent series ring $(\mathrm{gr}^\bullet \Lambda(N_1))((t_0))$. With $\mathrm{gr}^\bullet \Lambda(N_1)$ also $(\mathrm{gr}^\bullet \Lambda(N_1))((t_0))$ is an integral domain.

ii. Since N_0 is topologically finitely generated it suffices, by [DDMS] §3.1, to show that $[N_0, N_0]$ is contained in the subgroup of N_0 generated by the p^2-powers (every element in this subgroup then in fact is a p^2-power). It further suffices to consider the commutators $[N_0 \cap N_\alpha, N_0 \cap N_\beta]$ for any $\alpha, \beta \in \Phi^+$. By the standard commutation rules in N together with our assumption on n we obtain

$$\begin{aligned} [N_0 \cap N_\alpha, N_0 \cap N_\beta] &= [\iota_\alpha^{-1}(p^{n(\alpha)}\mathbb{Z}_p), \iota_\beta^{-1}(p^{n(\beta)}\mathbb{Z}_p)] \\ &\subseteq \prod_{\substack{i,j>0 \\ i\alpha+j\beta\in\Phi}} \iota_{i\alpha+j\beta}^{-1}(p^{in(\alpha)+jn(\beta)}\mathbb{Z}_p) \\ &\subseteq \prod_{\substack{i,j>0 \\ i\alpha+j\beta\in\Phi}} \iota_{i\alpha+j\beta}^{-1}(p^{2+n(i\alpha+j\beta)}\mathbb{Z}_p) \\ &= \prod_{\substack{i,j>0 \\ i\alpha+j\beta\in\Phi}} (N_0 \cap N_{i\alpha+j\beta})^{p^2} \end{aligned}$$

(where some order in the product is fixed). □

10 (φ, Γ)-modules

Everything in the preceding two sections applies in particular to the standard monoid $S_\star$ from section 1. (To be precise apply it to $P = P_2(\mathbb{Q}_p)$ and

$$\ell(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}) := a \quad \text{and} \quad \xi(b) := \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}).$$

It is more convenient to use an independent notation in this case. We put

$$\Lambda_F(S_0) := \Lambda_{\{1\}}(S_0),\ \Lambda_F(S_0\Gamma) := \Lambda_{S_0,\{1\}}(S_0\Gamma),\ \text{and}\ \Lambda_F(S_\star) := \Lambda_F(S_0\Gamma)[[\varphi; \sigma_\varphi]]\ .$$

We have $\Lambda(S_0) = o[[t_0]]$, and

$$\Lambda_F(S_0) = \{\sum_{j\in\mathbb{Z}} a_j t_0^j : a_j \in o \text{ and } \lim_{j\to-\infty} a_j = 0\}$$

is the π-adic completion of the localization of $o[[t_0]]$ in the multiplicative subset $o[[t_0]]\setminus \pi o[[t_0]]$; it is a complete discrete valuation ring with residue field $k((t_0))$. The element $\left(\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right) \in \Gamma$ acts on $\Lambda_F(S_0)$ by sending t_0 to $(t_0+1)^b - 1$. The endomorphism σ_φ sends t_0 to $(t_0+1)^p - 1$. We see that an etale $(\Lambda_F(S_0), \Gamma, \varphi)$-module in our sense is exactly the same as an etale (φ, Γ)-module in the sense of Fontaine ([Fon]).

We have the exact base change functor

$$\Lambda_F(S_\star)\otimes_{\Lambda(S_\star)} : \mathcal{M}_{et}(\Lambda(S_\star)) \longrightarrow \mathcal{M}_{et}(\Lambda_F(S_\star)) \ .$$

as well as the functor

$$\Lambda_F(S_\star)\otimes_{\Lambda(S_\star)} : D^+_{et}(\Lambda(S_\star)) \longrightarrow D^+_{et}(\Lambda_F(S_\star)) \ .$$

between derived categories.

Remark 10.1. *For any $\Lambda(S_\star)$-module M such that $\pi^m M = 0$ for some $m \geq 0$ we have*

$$\Lambda_F(S_\star) \otimes_{\Lambda(S_\star)} M = \Lambda_F(S_0) \otimes_{\Lambda(S_0)} M \ .$$

Proof. This follows from (26) and Lemma 8.12.iv using the fact that in the present situation we have $F^m\Lambda_F(S_0\Gamma) = \pi^m\Lambda_F(S_0\Gamma)$. □

By composition we obtain the functor

$$RD_{\Lambda_F(S_\star)} := \Lambda_F(S_\star)\otimes^{\mathbb{L}}_{\Lambda(P_\star)} RD = \Lambda_F(S_\star)\otimes_{\Lambda(S_\star)} RD_{\Lambda(S_\star)} : D^-(\mathcal{M}_{o-tor}(P)) \longrightarrow D^+_{et}(\Lambda_F(S_\star)).$$

The corresponding δ-functor on $\mathcal{M}_{o-tor}(P)$ is

$$D^i_{\Lambda_F(S_\star)}(V) := h^i(RD_{\Lambda_F(S_\star)}(V)) = \Lambda_F(S_\star) \otimes_{\Lambda(S_\star)} D^i_{\Lambda(S_\star)}(V) \qquad \text{for } i \geq 0.$$

By the universal properties of localization and adic completion the homomorphism $\ell : \Lambda(N_0) \longrightarrow \Lambda(S_0)$ from section 5 extends naturally to a surjective homomorphism of pseudocompact rings $\ell : \Lambda_\ell(N_0) \longrightarrow \Lambda_\ell(S_0)$ (in terms of Laurent series it is given by applying the augmentation map $\Lambda(N_1) \longrightarrow o$ to the coefficients). One easily checks that the weak topology on $\Lambda_\ell(S_0)$ is the quotient topology with respect to the map ℓ of the weak topology on $\Lambda_\ell(N_0)$. Using our boundedness criterion Lemma 8.8 one sees that ℓ further extends to the surjective map

$$\begin{aligned} \ell : \Lambda_\ell(N_0\Gamma) &\longrightarrow \Lambda_F(S_0\Gamma) \\ \sum_{i\geq 0} \mu_i t^i &\longmapsto \sum_{i \geq 0} \ell(\mu_i) t^i \ . \end{aligned}$$

Since, as a consequence of (8), the original map ℓ respects σ and δ this extended map ℓ in fact is a ring homomorphism. Again it is easy to check that this extension still is strict for the weak topologies. But ℓ also respects σ_φ (again by (8)). So we finally obtain the surjective ring homomorphism

$$\begin{aligned} \ell : \quad \Lambda_\ell(P_\star) &\longrightarrow \Lambda_F(S_\star) \\ \sum_{k\geq 0} \lambda_k \varphi^k &\longmapsto \sum_{k \geq 0} \ell(\lambda_k)\varphi^k \ . \end{aligned}$$

This allows us to introduce the right exact base change functor

$$\Lambda_F(S_\star)\otimes_{\Lambda_\ell(P_\star)} : \mathcal{M}(\Lambda_\ell(P_\star)) \longrightarrow \mathcal{M}(\Lambda_F(S_\star))$$

as well as its left derived functor

$$\Lambda_F(S_\star)\otimes^{\mathbb{L}}_{\Lambda_\ell(P_\star)} : D(\Lambda_\ell(P_\star)) \longrightarrow D(\Lambda_F(S_\star))$$

between the corresponding unbounded derived categories ([KS] Thm. 14.4.3). We recall that to compute the latter for a complex $M^\bullet$ in $D(\Lambda_\ell(P_\star))$ one chooses a homotopically projective resolution $P^\bullet \xrightarrow{\sim} M^\bullet$ and one has

$$\Lambda_F(S_\star) \otimes^{\mathbb{L}}_{\Lambda_\ell(P_\star)} M^\bullet \simeq \Lambda_F(S_\star) \otimes_{\Lambda_\ell(P_\star)} P^\bullet \ .$$

Obviously the former functor restricts to

$$\Lambda_F(S_\star)\otimes_{\Lambda_\ell(P_\star)} : \mathcal{M}_{et}(\Lambda_\ell(P_\star)) \longrightarrow \mathcal{M}_{et}(\Lambda_F(S_\star)) \ .$$

As far as the derived functor is concerned let $D_{et}(\Lambda_\ell(P_\star))$ and $D_{et}(\Lambda_F(S_\star))$ denote the respective full triangulated subcategories of complexes with etale cohomology modules. Let $M^\bullet$ be a complex in $D_{et}(\Lambda_\ell(P_\star))$. As a consequence of Prop. 9.6 the natural map

$$\Lambda_\ell(P_\star) \otimes_{\Lambda_\ell(P_\star),\sigma_\varphi} M^\bullet \xrightarrow{\simeq} M^\bullet$$

then is a quasi-isomorphism. By functoriality we obtain the isomorphisms

$$h^\bullet\big(\Lambda_F(S_\star) \otimes^{\mathbb{L}}_{\Lambda_\ell(P_\star)} \big(\Lambda_\ell(P_\star) \otimes_{\Lambda_\ell(P_\star),\sigma_\varphi} M^\bullet\big)\big) \cong h^\bullet\big(\Lambda_F(S_\star) \otimes^{\mathbb{L}}_{\Lambda_\ell(P_\star)} M^\bullet\big) \ .$$

To further compute the left hand term we fix a homotopically projective resolution $P^\bullet \xrightarrow{\sim} M^\bullet$. Since any base change has the restriction functor as an exact right adjoint

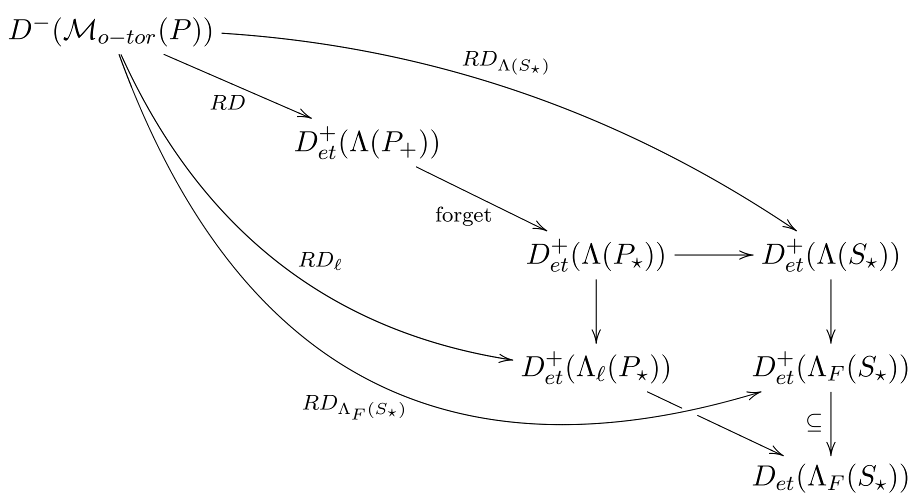
$$\Lambda_\ell(P_\star) \otimes_{\Lambda_\ell(P_\star),\sigma_\varphi} P^\bullet \xrightarrow{\sim} \Lambda_\ell(P_\star) \otimes_{\Lambda_\ell(P_\star),\sigma_\varphi} M^\bullet$$

is a homotopically projective resolution as well. We compute

$$\begin{aligned} h^\bullet\big(\Lambda_F(S_\star) \otimes^{\mathbb{L}}_{\Lambda_\ell(P_\star)} \big(\Lambda_\ell(P_\star) \otimes_{\Lambda_\ell(P_\star),\sigma_\varphi} M^\bullet\big)\big) &= h^\bullet\big(\Lambda_F(S_\star) \otimes_{\Lambda_\ell(P_\star)} \big(\Lambda_\ell(P_\star) \otimes_{\Lambda_\ell(P_\star),\sigma_\varphi} P^\bullet\big)\big) \\ &= h^\bullet\big(\Lambda_F(S_\star) \otimes_{\Lambda_\ell(P_\star),\sigma_\varphi} P^\bullet\big) \\ &= h^\bullet\big(\Lambda_F(S_\star) \otimes_{\Lambda_F(S_\star),\sigma_\varphi} \big(\Lambda_F(S_\star) \otimes_{\Lambda_\ell(P_\star)} P^\bullet\big)\big) \\ &= \Lambda_F(S_\star) \otimes_{\Lambda_F(S_\star),\sigma_\varphi} h^\bullet\big(\Lambda_F(S_\star) \otimes_{\Lambda_\ell(P_\star)} P^\bullet\big) \\ &= \Lambda_F(S_\star) \otimes_{\Lambda_F(S_\star),\sigma_\varphi} h^\bullet\big(\Lambda_F(S_\star) \otimes^{\mathbb{L}}_{\Lambda_\ell(P_\star)} M^\bullet\big) \end{aligned}$$

where the second to last identity uses Prop. 9.6. It follows that $\Lambda_F(S_\star) \otimes^{\mathbb{L}}_{\Lambda_\ell(P_\star)} M^\bullet$ lies in $D_{et}(\Lambda_F(S_\star))$.

The following commutative diagram displays all the functors we have constructed:



The unnamed arrows are the respective (derived) base change functors. For the commutativity of the corresponding box of base changes see [KS] Prop. 14.4.7 (and Ex. 13.3.9).

Lemma 10.2. *For any $\Lambda(P_\star)$-module M such that $\pi^m M = 0$ for some $m \geq 0$ we have*

$$\mathrm{Tor}_i^{\Lambda(P_\star)}(\Lambda_F(S_\star), M) = \Lambda_F(S_0) \otimes_{\Lambda(S_0)} \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), M) \qquad \textit{for any } i \geq 0.$$

Proof. Since $\Lambda_F(S_0\Gamma)$ is flat over $\Lambda(S_0\Gamma)$, $\Lambda(P_\star)$ is flat over $\Lambda(N_0\Gamma)$, and

$$\Lambda_F(S_\star) = \Lambda_F(S_0\Gamma) \otimes_{\Lambda(S_0\Gamma)} \Lambda(S_\star) \quad \text{as well as} \quad \Lambda(S_\star) = \Lambda(S_0\Gamma) \otimes_{\Lambda(N_0\Gamma)} \Lambda(P_\star)$$

we have

$$\mathrm{Tor}_i^{\Lambda(P_\star)}(\Lambda_F(S_\star), M) = \Lambda_F(S_0\Gamma) \otimes_{\Lambda(S_0\Gamma)} \mathrm{Tor}_i^{\Lambda(N_0\Gamma)}(\Lambda(S_0\Gamma), M) \ .$$

With M each $\mathrm{Tor}_i^{\Lambda(N_0\Gamma)}(\Lambda(S_0\Gamma), M)$ is annihilated by π^m. Hence we may apply Lemma 8.12.iv (cf. also Remark 10.1) to the right hand side and obtain

$$\Lambda_F(S_0\Gamma) \otimes_{\Lambda(S_0\Gamma)} \mathrm{Tor}_i^{\Lambda(N_0\Gamma)}(\Lambda(S_0\Gamma), M) = \Lambda_F(S_0) \otimes_{\Lambda(S_0)} \mathrm{Tor}_i^{\Lambda(N_0\Gamma)}(\Lambda(S_0\Gamma), M) \ .$$

Finally we note that $\Lambda(N_0\Gamma)$ is flat over $\Lambda(N_0)$ and that

$$\Lambda(S_0\Gamma) = \Lambda(S_0) \otimes_{\Lambda(N_0)} \Lambda(N_0\Gamma)$$

which imply

$$\mathrm{Tor}_i^{\Lambda(N_0\Gamma)}(\Lambda(S_0\Gamma), M) = \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), M) \ .$$

□

For finitely generated compactly induced representations the functor $RD_{\Lambda_F(S_\star)}$ has an interesting stability property. Let $V = \mathrm{ind}_{P_0}^{P}(V_0)$ with a finite V_0. Using the notations from section 3 we have

$$D(V) = \varinjlim_{s \in T_+} M_s^* \ ,$$

as $\widetilde{\Lambda}(P_+^{-1})$-modules, by Lemma 3.1. As noted before, via the identification

$$J^+(V_0) \xrightarrow{\cong} \mathrm{Ind}_{P_0}^{P}(V_0^*)/J^-(V_0) = M_1(V_0^*)$$

the Pontrjagin dual of $M_1(V_0) = \mathrm{ind}_{P_0}^{P_+}(V_0)$ is a $\Lambda(P_+)$-module, and the natural map

$$M_1(V_0)^* \longrightarrow D(V)$$

is a map of $\Lambda(P_+)$-modules both of which are annihilated by some power of π.

Proposition 10.3. *For any representation compactly induced from a finite V_0 the map*

$$\Lambda_F(S_\star) \otimes^{\mathbb{L}}_{\Lambda(P_\star)} M_1(V_0)^* \xrightarrow{\simeq} \Lambda_F(S_\star) \otimes^{\mathbb{L}}_{\Lambda(P_\star)} D(\mathrm{ind}_{P_0}^{P}(V_0))$$

is a quasi-isomorphism.

Proof. The assertion is that the maps

$$\mathrm{Tor}_i^{\Lambda(P_\star)}(\Lambda_F(S_\star), M_1(V_0)^*) \xrightarrow{\cong} \mathrm{Tor}_i^{\Lambda(P_\star)}(\Lambda_F(S_\star), D(V))$$

are isomorphisms. By Lemma 10.2 we are reduced to showing that

$$\Lambda_F(S_0) \otimes_{\Lambda(S_0)} \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), M_1(V_0)^*) \xrightarrow{\cong} \Lambda_F(S_0) \otimes_{\Lambda(S_0)} \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), D(V))$$

are isomorphisms. But Tor-functors commute with inductive limits. Hence it suffices to show that the maps

$$\Lambda_F(S_0) \otimes_{\Lambda(S_0)} \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), M_1(V_0)^*) \xrightarrow{\cong} \Lambda_F(S_0) \otimes_{\Lambda(S_0)} \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), M_s(V_0)^*)$$

are isomorphisms, or equivalently, that

$$\Lambda_F(S_0) \otimes_{\Lambda(S_0)} \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), (M_1(V_0)/M_s(V_0))^*) = 0$$

for any $s \in T_+$ and any $i \geq 0$. But as an N_0-representation we have

$$M_1(V_0)/M_s(V_0) = \bigoplus_{t \in (T_+ - T_+ s)/T_0} M(t)(V_0) \ . \tag{31}$$

For each direct summand there is the N_0-equivariant isomorphism

$$\begin{aligned} M(t)(V_0) = \mathrm{ind}_{P_0}^{N_0 t P_0}(V_0) &\xrightarrow{\cong} \mathrm{ind}_{tN_0t^{-1}}^{N_0}(t_* V_0) \\ \psi &\longmapsto \phi(n_0) := \psi(n_0 t) \ , \end{aligned}$$

and so

$$M(t)(V_0)^* \cong \mathrm{ind}_{tN_0t^{-1}}^{N_0}(t_*V_0)^* = \Lambda(N_0) \otimes_{\Lambda(tN_0t^{-1})} (t_*V_0)^* = \Lambda(N_0) \otimes_{\Lambda(tN_0t^{-1})} t_*V_0^* \ .$$

Since the tensor product with a finitely generated module over a noetherian ring commutes with arbitrary direct products we obtain

$$\mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), (M_1(V_0)/M_s(V_0))^*) \cong \prod_{t \in (T_+ - T_+ s)/T_0} \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), \Lambda(N_0) \otimes_{\Lambda(tN_0t^{-1})} t_*V_0^*) \ .$$

Since in the commutative diagram of rings

$$\begin{array}{ccc} \Lambda(tN_0t^{-1}) & \longrightarrow & \Lambda(tN_0t^{-1}N_1/N_1) \\ \subseteq\big\downarrow & & \big\downarrow\subseteq \\ \Lambda(N_0) & \longrightarrow & \Lambda(S_0) \end{array}$$

the vertical maps make the lower ring a free module of finite rank over the upper ring each term in the right hand direct product, as a $\Lambda(S_0)$-module, can be rewritten as

$$\begin{aligned} &\mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), \Lambda(N_0) \otimes_{\Lambda(tN_0t^{-1})} t_*V_0^*) \\ &\quad = \Lambda(S_0) \otimes_{\Lambda(tN_0t^{-1}N_1/N_1)} \mathrm{Tor}_i^{\Lambda(tN_0t^{-1})}(\Lambda(tN_0t^{-1}N_1/N_1), t_*V_0^*) \\ &\quad = \Lambda(S_0) \otimes_{\Lambda(tN_0t^{-1}N_1/N_1)} t_* \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(N_0t^{-1}N_1t/t^{-1}N_1t), V_0^*) \\ &\quad = \Lambda(S_0) \otimes_{\Lambda(tN_0t^{-1}N_1/N_1)} t_* \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(N_0/N_0 \cap t^{-1}N_1t), V_0^*) \\ &\quad = t_*\big(\Lambda(t^{-1}N_0t/t^{-1}N_1t) \otimes_{\Lambda(N_0/N_0 \cap t^{-1}N_1t)} \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(N_0/N_0 \cap t^{-1}N_1t), V_0^*)\big) \ . \end{aligned}$$

To go further we divide up the index set $T_+ - T_+s$ into finitely many subsets $T_{s,\alpha}$ indexed by the simple roots $\alpha \in \Delta$ by defining

$$T_{s,\alpha} := \{t \in T_+ : |\alpha(t)| > |\alpha(s)|\}.$$

Since $t \notin T_+$ if and only if $|\alpha(t)| > 1$ for some $\alpha \in \Delta$ it is clear that

$$T_+ - T_+s = \bigcup_{\alpha\in\Delta} T_{s,\alpha} \ .$$

We claim that, for any given $\alpha \in \Delta$, we find an open subgroup $N''_\alpha \subseteq N_0 \cap N_\alpha$ which, through the injective projection map $N''_\alpha \hookrightarrow N_0/N_0 \cap t^{-1}N_1t$, acts trivially on $\mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(N_0/N_0 \cap t^{-1}N_1t), V_0^*)$ for any $t \in T_+$.

To establish this claim we let $N_0^c := \prod_{\beta\in\Phi^+\setminus\Delta} N_0 \cap N_\beta$. It follows from the standard commutator relations in N (cf. [BT] Prop. 4.7.(iii) and Remark 4.11) that N_0/N_0^c is commutative. Moreover, $N_0^c \subseteq N_0 \cap t^{-1}N_1t$ for any $t \in T_+$, and we have the spectral sequence

$$\mathrm{Tor}_i^{\Lambda(N_0/N_0^c)}\big(\Lambda(N_0/N_0 \cap t^{-1}N_1t), \mathrm{Tor}_j^{\Lambda(N_0)}(\Lambda(N_0/N_0^c), V_0^*)\big) \\ \Longrightarrow \mathrm{Tor}_{i+j}^{\Lambda(N_0)}(\Lambda(N_0/N_0 \cap t^{-1}N_1t), V_0^*) \ .$$

We first investigate the o-modules $\mathrm{Tor}_j^{\Lambda(N_0)}(\Lambda(N_0/N_0^c), V_0^*)$ under the action of $N_{\alpha,0} := N_0 \cap N_\alpha$. First of all, the ring $\Lambda(N_0)$ having finite global dimension ([Neu]) at most finitely many of them can be nonzero. Secondly, since V_0^* is finite they are killed by some power of π. Since $\Lambda(N_0/N_0^c) = o \otimes_{\Lambda(N_0^c)} \Lambda(N_0)$ and $\Lambda(N_0)$ is flat over $\Lambda(N_0^c)$ (cf. the proof of Lemma 5.5 in [OV]) we have

$$\mathrm{Tor}_j^{\Lambda(N_0)}(\Lambda(N_0/N_0^c), V_0^*) \cong \mathrm{Tor}_j^{\Lambda(N_0^c)}(o, V_0^*)$$

as o-modules. Using a resolution

$$\ldots \longrightarrow \Lambda(N_0^c)^{m_j} \longrightarrow \ldots \longrightarrow \Lambda(N_0^c)^{m_0} \longrightarrow V_0^* \longrightarrow 0$$

by finitely generated free modules over the noetherian ring $\Lambda(N_0^c)$ we compute

$$\begin{aligned}\mathrm{Tor}_j^{\Lambda(N_0^c)}(o, V_0^*) &= h_j(o \otimes_{\Lambda(N_0^c)} \Lambda(N_0^c)^{m_\bullet}) \\ &= h_j(o^{m_\bullet})\end{aligned}$$

which shows that these groups are finitely generated over o. Altogether we see that the o-modules $\mathrm{Tor}_j^{\Lambda(N_0)}(\Lambda(N_0/N_0^c), V_0^*)$ in fact are finite and therefore are fixed by some open subgroup $N''_\alpha \subseteq N_{\alpha,0}$. The ring $\Lambda(N_0/N_0 \cap t^{-1}N_1t)$ being commutative we conclude that all terms in the above spectral sequence are fixed by N''_α which, in particular, establishes our claim.

We deduce from this claim that

$$\mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), \Lambda(N_0) \otimes_{\Lambda(tN_0t^{-1})} t_*V_0^*) \\ = t_*\big(\Lambda(t^{-1}N_0t/t^{-1}N_1t) \otimes_{\Lambda(N_0/N_0\cap t^{-1}N_1t)} \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(N_0/N_0 \cap t^{-1}N_1t), V_0^*)\big) \ ,$$

for any $t \in T_+$, is fixed by $tN''_\alpha t^{-1}$. But

$$N'_\alpha := \bigcap_{t\in T_{s,\alpha}} tN''_\alpha t^{-1}$$

still is open in $N_{\alpha,0}$. We pick an element $\gamma_\alpha \in S_0$ which is the image of a topological generator of N'_α, and we finally obtain that the $\Lambda(S_0)$-module

$$\prod_{t \in T_{s,\alpha}/T_0} \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), \Lambda(N_0) \otimes_{\Lambda(tN_0t^{-1})} t_* V_0^*)$$

is killed by $\gamma_\alpha - 1$, which in particular implies that its base change to $\Lambda_F(S_0)$ vanishes. Forming the finite direct sum over the $\alpha \in \Delta$ then gives the vanishing of $\Lambda_F(S_0) \otimes_{\Lambda(S_0)} \mathrm{Tor}_i^{\Lambda(N_0)}(\Lambda(S_0), (M_1(V_0)/M_s(V_0))^*)$. □

11 The case $GL_2(\mathbb{Q}_p)$

Throughout this section we let G be the group $GL_2(\mathbb{Q}_p)$, and we make our choices of $P = P_2(\mathbb{Q}_p), \ldots$ as at the end of section 1. This case is particularly simple since we obviously have

$$D^i_{\Lambda_F(S_\star)}(V) = \Lambda_F(S_\star) \otimes_{\Lambda(S_\star)} D^i(V) \qquad \text{for any } i \geq 0 \text{ and any } V \text{ in } \mathcal{M}_{o-tor}(P).$$

Proposition 11.1. *If V in $\mathcal{M}_{o-tor}(P)$ is finitely generated then the map*

$$\Lambda_F(S_0) \otimes_{\Lambda(S_0)} M^* \xrightarrow{\cong} \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V) = \Lambda_F(S_\star) \otimes_{\Lambda(S_\star)} D(V)$$

is an isomorphism for any sufficiently small M in $\mathcal{P}_+(V)$.

Proof. We write V as a quotient

$$\mathrm{ind}_{P_0}^P(V_0) \xrightarrow{f} V \longrightarrow 0$$

of a representation compactly induced from a finite V_0. Let $M := f(M_1(V_0)) \in \mathcal{P}(V)$, and consider any $M' \in \mathcal{P}(V)$ which is contained in M. Then f induces an isomorphism

$$M_1(V_0)/f^{-1}(M') \cap M_1(V_0) \xrightarrow{\cong} M/M' \ .$$

We pick an $s \in T_+$ such that $M_s(V_0) \subseteq f^{-1}(M')$. By the proof of Prop. 10.3 we have $\Lambda_F(S_0) \otimes_{\Lambda(S_0)} \big(M_1(V_0)/M_s(V_0)\big)^* = 0$ and a fortiori $\Lambda_F(S_0) \otimes_{\Lambda(S_0)} (M/M')^* = \Lambda_F(S_0) \otimes_{\Lambda(S_0)} \big(M_1(V_0)/f^{-1}(M') \cap M_1(V_0)\big)^* = 0$ which implies that

$$\Lambda_F(S_0) \otimes_{\Lambda(S_0)} M^* \xrightarrow{\cong} \Lambda_F(S_0) \otimes_{\Lambda(S_0)} M'^*$$

is an isomorphism. Since the tensor product commutes with inductive limits the assertion follows. For the additional identity in the assertion we recall Remark 10.1. □

A representation V in $\mathcal{M}_{o-tor}(P)$ is called *finitely presented* if there is an exact sequence in $\mathcal{M}_{o-tor}(P)$ of the form

$$\mathrm{ind}_{P_0}^P(U_1) \xrightarrow{\rho} \mathrm{ind}_{P_0}^P(U_0) \longrightarrow V \longrightarrow 0$$

with finite U_0 and U_1. According to Cor. 4.4 we then have the exact sequence of (etale) $\Lambda(S_\star)$-modules

$$0 \longrightarrow D^0(V) \longrightarrow D(\mathrm{ind}_{P_0}^P(U_0)) \xrightarrow{D(\rho)} D(\mathrm{ind}_{P_0}^P(U_1)) \ .$$

Using Propositions 10.3 and 11.1 (rather their proofs) we see that $D^0_{\Lambda_F(S_\star)}(V)$ can be computed as the kernel

$$D^0_{\Lambda_F(S_\star)}(V) = \Lambda_F(S_0) \otimes_{\Lambda(S_0)} \ker\big(M_s(U_0)^* \xrightarrow{\rho^*} M_{s'}(U_1)^*\big) \tag{32}$$

for any $s, s' \in T_+$ such that $\rho(M_{s'}(U_1)) \subseteq M_s(U_0))$.

Lemma 11.2. *For any finite subset $X \subseteq P$ and any sufficiently big $n \geq 0$ we have*

$$P_+\varphi^{2n}X \subseteq P_+\varphi^n \quad \textit{and} \quad (P \setminus P_+)X \subseteq P \setminus P_+\varphi^n \ .$$

Proof. We choose n big enough so that $\varphi^n X \cap \varphi^n X^{-1} \subseteq P_+$ holds true. Then, of course, $P_+\varphi^{2n}X \subseteq P_+\varphi^n P_+ = P_+\varphi^n$. Moreover, if $b \in P \setminus P_+$ satisfies $bx = b_+\varphi^n$ for some $x \in X$ and $b_+ \in P_+$ then $b = b_+\varphi^n x^{-1} \in P_+$ which is a contradiction. □

Proposition 11.3. *If V in $\mathcal{M}_{o-tor}(P)$ is finitely presented then the map*

$$\Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V) \xrightarrow{\cong} \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D^0(V) = D^0_{\Lambda_F(S_\star)}(V)$$

is an isomorphism.

Proof. Consider any finite presentation

$$\mathrm{ind}_{P_0}^P(U_1) \xrightarrow{\rho} \mathrm{ind}_{P_0}^P(U_0) \xrightarrow{f} V \longrightarrow 0$$

of V. By (32) we have

$$\Lambda_F(S_0) \otimes_{\Lambda(S_0)} D^0(V) = \Lambda_F(S_0) \otimes_{\Lambda(S_0)} \ker\big(M_{\varphi^n}(U_0)^* \xrightarrow{\rho^*} M_{\varphi^m}(U_1)^*\big)$$

for any $m, n \geq 0$ such that $\rho(M_{\varphi^m}(U_1)) \subseteq M_{\varphi^n}(U_0))$. We now choose n big enough so that $M := f(M_{\varphi^n}(U_0)) \subseteq V$ satisfies Prop. 11.1 so that we obtain (also using the flatness of $\Lambda_F(S_0)$ over $\Lambda(S_0)$) the exact sequence

$$0 \to \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V) \longrightarrow \Lambda_F(S_0) \otimes_{\Lambda(S_0)} M_{\varphi^n}(U_0)^* \longrightarrow \Lambda_F(S_0) \otimes_{\Lambda(S_0)} \rho^{-1}(M_{\varphi^n}(U_0))^* \ .$$

We see that to prove our assertion it is sufficient to establish that

$$\Lambda_F(S_0) \otimes_{\Lambda(S_0)} \big(\rho^{-1}(M_{\varphi^n}(U_0))/M_{\varphi^m}(U_1)\big)^* = 0$$

vanishes for appropriate choices of m, n. There is a finite subset $X \subseteq P$ such that $\rho(U_1) \subseteq XU_0$. Hence by possibly enlarging m, n we may, according to Lemma 11.2, assume that

$$P_+\varphi^m X \subseteq P_+\varphi^n \quad \text{and} \quad (P \setminus P_+)X \subseteq P \setminus P_+\varphi^n$$

holds true. It follows that in the decompositions

$$\mathrm{ind}_{P_0}^P(U_1) = \mathrm{ind}_{P_0}^{P\setminus P_+}(U_1) \oplus \mathrm{ind}_{P_0}^{P_+\setminus P_+\varphi^m}(U_1) \oplus M_{\varphi^m}(U_1)$$

and

$$\mathrm{ind}_{P_0}^P(U_0) = \mathrm{ind}_{P_0}^{P\setminus P_+\varphi^n}(U_0) \oplus M_{\varphi^n}(U_0)$$

the homomorphism ρ respects the first and the last summands. We deduce from this that $\rho^{-1}(M_{\varphi^n}(U_0))/M_{\varphi^m}(U_1)$ is isomorphic to a P_+-subrepresentation of $\mathrm{ind}_{P_0}^{P_+\setminus P_+\varphi^m}(U_1)$. Hence we are further reduced to proving that

$$\Lambda_F(S_0) \otimes_{\Lambda(S_0)} \mathrm{ind}_{P_0}^{P_+\setminus P_+\varphi^m}(U_1)^* = 0 \ .$$

But this we have done already in the proof of Prop. 10.3 (cf. (31)). □

Remark 11.4. *For a general split group G arguments as above show that for any finitely presented V in $\mathcal{M}_{o-tor}(P)$ the map*

$$\Lambda_F(S_0) \otimes_{\Lambda(N_0)} D(V) \longrightarrow \Lambda_F(S_0) \otimes_{\Lambda(N_0)} D^0(V)$$

at least is surjective. It therefore follows from Prop. 9.9 that for general G and any admissible V in $\mathcal{M}_{o-tor}(G)$ the following holds: If for some $M \in \mathcal{P}_+(V)$ there is an exact sequence of P_+-representations of the form

$$\ldots \longrightarrow \mathrm{ind}_{P_0}^{P_+}(V_n) \longrightarrow \ldots \longrightarrow \mathrm{ind}_{P_0}^{P_+}(V_0) \longrightarrow M \longrightarrow 0$$

with V_n finite for any $n \geq 0$ then V is finitely presented for P and $\Lambda_F(S_0) \otimes_{\Lambda(N_0)} D^0(V)$ is finitely generated over $\Lambda_F(S_0)$.

Corollary 11.5. *Let $(0 \longrightarrow)V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$ be an exact sequence in $\mathcal{M}_{o-tor}(P)$; if V_3 is finitely presented then the sequence*

$$0 \longrightarrow \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V_3) \longrightarrow \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V_2) \longrightarrow \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V_1)(\longrightarrow 0)$$

is exact.

Proof. We contemplate the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V_3) & \longrightarrow & \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V_2) & \longrightarrow & \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V_1) \\
 & & \downarrow \cong & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D^0(V_3) & \longrightarrow & \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D^0(V_2) & \longrightarrow & \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D^0(V_1)
\end{array}$$

in which the lower row is exact by construction, the columns are exact by Remark 2.4.ii, and the upper row is a complex which is exact at the left spot again by Remark 2.4.ii (always using in addition the flatness of $\Lambda_F(S_0)$ over $\Lambda(S_0)$). But by our assumption and Prop. 11.3 the left vertical map is an isomorphism. In this situation an easy diagram chase shows that the upper row has to be exact as well. For the additional zero we use Remark 2.4.i. $\square$

Corollary 11.6. *Let $V_1 \longrightarrow V_2 \longrightarrow V_3$ be an exact sequence in $\mathcal{M}_{o-tor}(P)$; we have:*

i. If V_1 is finitely generated and V_2 is finitely presented then the sequence

$$\Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V_3) \longrightarrow \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V_2) \longrightarrow \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V_1)$$

is exact;

ii. if all three representations V_1, V_2, and V_3 are finitely presented then the sequence

$$D^0_{\Lambda_F(S_\star)}(V_3) \longrightarrow D^0_{\Lambda_F(S_\star)}(V_2) \longrightarrow D^0_{\Lambda_F(S_\star)}(V_1)$$

is exact.

Proof. i. Let $V_3' := \mathrm{im}(V_2 \longrightarrow V_3)$. We leave it to the reader to check that the quotient of a finitely presented representation by a finitely generated subrepresentation is finitely presented. Hence we may apply the previous corollary to the exact sequence $V_1 \longrightarrow V_2 \longrightarrow V_3' \longrightarrow 0$. In addition we observe the Remark 2.4.i.

ii. This is an immediate consequence of i. and Prop. 11.3. □

Colmez in [Co1] investigates particularly nice finite presentations of a modified form. To review his result let, at first, V be an arbitrary finitely generated smooth G-representation which has a central character. We note right away that, as a consequence of the Iwasawa decomposition, V then also is finitely generated as a P-representation. Let Z denote the center of G, put $G_0 := GL_2(\mathbb{Z}_p)$, and recall the notation $\varphi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. Let G_1 denote the normalizer of $G_0 \cap \varphi^{-1} G_0 \varphi$ in G. It contains $(G_0 \cap \varphi^{-1} G_0 \varphi) Z$ with index two and with $\left(\begin{smallmatrix} 0 & p \\ 1 & 0 \end{smallmatrix}\right)$ representing the nontrivial coset. We pick a finite $G_0 Z$-subrepresentation $U_0 \subseteq V$ which generates V so that we obtain a surjection

$$\mathrm{ind}_{G_0 Z}^{G}(U_0) \xrightarrow{f} V \longrightarrow 0 \ .$$

One checks that $U_0 \cap \varphi U_0$ is $\varphi G_1 \varphi^{-1}$-invariant ([Co1] Lemma 2.6). To avoid confusion we denote, in this section, the natural inclusion $U_0 \hookrightarrow \mathrm{ind}_{G_0 Z}^{G}(U_0)$ by $v \mapsto \tilde{v}$. The finite o-submodule

$$U_1 := \{\widetilde{\varphi^{-1} v} - \varphi^{-1} \tilde{v} : v \in U_0 \cap \varphi U_0\}$$

of $\mathrm{ind}_{G_0 Z}^{G}(U_0)$ lies in the kernel of f and is G_1-invariant. Hence we obtain a complex

$$\mathrm{ind}_{G_1}^{G}(U_1) \xrightarrow{\rho} \mathrm{ind}_{G_0 Z}^{G}(U_0) \xrightarrow{f} V \longrightarrow 0 \qquad (33)$$

in $\mathcal{M}_{o-tor}(G)$. Colmez calls (33) a standard presentation of V provided that it is exact. We emphasize that the center Z acts on all three terms of this sequence through the same central character.

Theorem 11.7. *(Colmez) For any smooth G-representation V which is admissible and of finite length and which has a central character U_0 can be chosen in such a way that*

$$0 \longrightarrow \mathrm{ind}_{G_1}^{G}(U_1) \xrightarrow{\rho} \mathrm{ind}_{G_0 Z}^{G}(U_0) \xrightarrow{f} V \longrightarrow 0 \qquad (34)$$

is a short exact sequence.

Proof. [Co1] Thm. 3.1 and Lemma 2.8 (compare also [Oll] and [Vi3] for modulo p representations). □

By using the Iwasawa decompositions $G = PG_0 = PG_1$ as well as $G_0 Z \cap P = G_1 Z \cap P = P_0 Z$ we may rewrite (34) as

$$0 \longrightarrow \mathrm{ind}_{P_0 Z}^{P}(U_1) \xrightarrow{\rho} \mathrm{ind}_{P_0 Z}^{P}(U_0) \xrightarrow{f} V \longrightarrow 0 \ . \qquad (35)$$

We have

$$P_0 Z = P_0 \times \zeta^{\mathbb{Z}} \qquad \text{with } \zeta := \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right).$$

Let $\mathcal{M}_{G-good}(P)$ denote the full subcategory of $\mathcal{M}_{o-tor}(P)$ whose objects are all V which arise by restriction from an admissible finite length smooth G-representation which has a central character.

Lemma 11.8. *For any smooth P_0Z-representation U_0 we have*

i. $D^i(\mathrm{ind}_{P_0Z}^{P}(U_0)) = 0$ *for any* $i \geq 1$*;*

ii. if ζ *acts as a scalar on* U_0 *then* $D^0(\mathrm{ind}_{P_0Z}^{P}(U_0)) = D(\mathrm{ind}_{P_0Z}^{P}(U_0))$*.*

Proof. We view

$$0 \longrightarrow o[\zeta^{\mathbb{Z}}] \xrightarrow{(\zeta-1)\cdot} o[\zeta^{\mathbb{Z}}] \xrightarrow{\zeta \mapsto 1} o \longrightarrow 0$$

as a short exact sequence of smooth (but not o-torsion) P_0Z-representations with P_0Z acting through its projection onto $\zeta^{\mathbb{Z}}$. It splits as a sequence of o-modules. With respect to the diagonal action

$$0 \longrightarrow U_0 \otimes_o o[\zeta^{\mathbb{Z}}] \xrightarrow{\rho_0} U_0 \otimes_o o[\zeta^{\mathbb{Z}}] \longrightarrow U_0 \longrightarrow 0 \ ,$$

where $\rho_0 := \mathrm{id} \otimes (\zeta - 1)$, therefore is a short exact sequence of smooth (and o-torsion) P_0Z-representations. It gives rise to the short exact sequence of smooth P-representations

$$0 \longrightarrow \mathrm{ind}_{P_0Z}^{P}(U_0 \otimes_o o[\zeta^{\mathbb{Z}}]) \xrightarrow{\mathrm{ind}(\rho_0)} \mathrm{ind}_{P_0Z}^{P}(U_0 \otimes_o o[\zeta^{\mathbb{Z}}]) \longrightarrow \mathrm{ind}_{P_0Z}^{P}(U_0) \longrightarrow 0 \ . \tag{36}$$

But the map

$$\begin{aligned} \mathrm{ind}_{P_0}^{P_0Z}(U_0) &\xrightarrow{\cong} U_0 \otimes_o o[\zeta^{\mathbb{Z}}] \\ \psi &\longmapsto \sum_{n \in \mathbb{Z}} \zeta^n(\psi(\zeta^n)) \otimes \zeta^n \end{aligned}$$

is a P_0Z-equivariant isomorphism. Inserting this into (36) and using the transitivity of compact induction we arrive at an exact sequence of smooth P-representations of the form

$$0 \longrightarrow \mathrm{ind}_{P_0}^{P}(U_0) \longrightarrow \mathrm{ind}_{P_0}^{P}(U_0) \longrightarrow \mathrm{ind}_{P_0Z}^{P}(U_0) \longrightarrow 0 \ .$$

Applying Cor. 4.4 and Lemma 2.4.i to it gives the first assertion.

To establish the second assertion it suffices to check the condition (2) in Lemma 2.5 for the short exact sequence (36). We first make some general observations. Let V_0 be any smooth P_0Z-representation on which ζ acts as a scalar. Since $P_+ \supseteq P_0Z$ the P_+-subrepresentations of $\mathrm{ind}_{P_0Z}^{P}(V_0)$ defined by

$$M'_s(V_0) := \mathrm{ind}_{P_0Z}^{P_+s}(V_0) \qquad \text{for any } s \in T_+$$

and

$$M'_\sigma(V_0) := \oplus_{t \in T_+/T_0Z}\, \mathrm{ind}_{P_0Z}^{N_0tP_0Z}(\sigma(t))$$

for any order reversing map $\sigma : T_+/T_0Z \longrightarrow \mathrm{Sub}(V_0)$ (note that by our assumption any element in $\mathrm{Sub}(V_0)$ automatically is a P_0Z-subrepresentation) satisfying (3) make perfect sense. It is easily checked that analogs of Lemma 3.1 and Lemma 3.2 hold true. Given this background the verification of (2) for (36) proceeds in exactly the same way as the proof of Lemma 4.1. ∎

Remark 11.9. *Any* V *in an exact sequence* $\mathrm{ind}_{P_0Z}^{P}(U_1) \longrightarrow \mathrm{ind}_{P_0Z}^{P}(U_0) \longrightarrow V \longrightarrow 0$ *with finite* U_0 *and* U_1 *is of finite presentation.*

Proof. In the proof of Lemma 11.8 we have seen that $\mathrm{ind}_{P_0Z}^{P}(U_0)$ is of finite presentation. Hence V being the quotient of a finitely presented representation by a finitely generated subrepresentation is finitely presented as well. $\square$

Proposition 11.10. *i. For any V in $\mathcal{M}_{G-good}(P)$ we have $D^i(V) = 0$ for any $i \geq 1$.*

ii. The functor D^0 restricted to $\mathcal{M}_{G-good}(P)$ is exact.

Proof. ii. is an immediate consequence of i. Lemma 11.8 says that we can use (35) to compute the δ-functor D^i on V. Hence

$$D^i(V) = h^i(D(\mathrm{ind}_{P_0Z}^{P}(U_0)) \xrightarrow{D(\rho)} D(\mathrm{ind}_{P_0Z}^{P}(U_1)) \longrightarrow 0 \longrightarrow \ldots) .$$

By Remark 2.4.i the map $D(\rho)$ is surjective. It follows that $D^i(V) = 0$ for $i \geq 1$. $\square$

In view of Remark 10.1 and our discussion of etale $(\Lambda_F(S_0), \Gamma, \varphi)$-modules in section 9 our functor $D^0_{\Lambda_F(S_\star)} = \Lambda_F(S_\star) \otimes_{\Lambda(S_\star)} D^0$ restricted to $\mathcal{M}_{G-good}(P)$ coincides with the functor constructed by Colmez in [Co1].

Proposition 11.11. *For every representation V in $\mathcal{M}_{G-good}(P)$ the set $\mathcal{P}_+(V)$ has a (unique) minimal element M_0; in particular, we have $D(V) = M_0^*$.*

Proof. According to [Co1] Lemma 4.25 there is an $M \in \mathcal{P}_+(V)$ such that the map $M^* \longrightarrow \Lambda_F(S_0) \otimes_{\Lambda(S_0)} D(V)$ and a fortiori the map $M^* \longrightarrow D(V)$ have a finite kernel. Let now $M' \subseteq M$ be some other element in $\mathcal{P}_+(V)$. Then

$$\ker(M^* \longrightarrow M'^*) \subseteq \ker(M^* \longrightarrow D(V)) .$$

Hence the finite groups $\ker(M^* \longrightarrow M'^*)$ for decreasing M' must stabilize. It follows that M'^* and M' stabilize. $\square$

12 Subquotients of principal series

Let $\chi : P \longrightarrow k^\times$ be a locally constant character and

$$\mathrm{Ind}_P^G(\chi) := \{F \in C^\infty(G) : F(gb) = \chi(b)^{-1}F(g) \text{ for any } g \in G, b \in P\}$$

the corresponding principal series representation of G (by left translation). We recall that, for any topological space X we let $C^\infty(X)$, resp. $C_c^\infty(X)$, denote the k-vector space of all k-valued locally constant, resp. locally constant with compact support, functions on X. As a matter of further notation we write

$$\mathrm{Ind}_P^X(\chi) := \{F \in \mathrm{Ind}_P^G(\chi) : F|(G \setminus X) = 0\}$$

for any open right P-invariant subset $X \subseteq G$. Furthermore, we fix a representative $\dot{w}$ in the normalizer $N(T)$ of T in G of every element w in the Weyl group $W := N(T)/T$.

As a P-representation $\mathrm{Ind}_P^G(\chi)$ is best understood by using the Bruhat decomposition $G = \bigcup_{w \in W} PwP$. Choosing once and for all a total order on W refining the Bruhat order we obtain the P-invariant filtration $\{\mathrm{Ind}_P^{PwP}(\chi)\}_{w \in W}$ of $\mathrm{Ind}_P^G(\chi)$. Its bottom term is $\mathrm{Ind}_P^{Pw_0P}(\chi)$, corresponding to the Steinberg representation of G, where w_0 denotes the longest element in

W. Each filtration step is, via $F \longmapsto F(\cdot\dot{w})$, isomorphic to $V(w,\chi) := C_c^\infty(N/N_w)$, where $N_w := N \cap \dot{w}N\dot{w}^{-1}$, with N acting by left translation and T acting by

$$(t\phi)(n) := \chi(w^{-1}tw)\phi(t^{-1}nt) \ .$$

In particular, $V(w,\chi)$ is a character twist of $V(w,1)$. In $V(w,\chi)$ we have the generating P_+-subrepresentation $M(w,\chi) := C^\infty(N_0/N_{w,0})$ where $N_{w,0} := N_0 \cap \dot{w}N\dot{w}^{-1}$.

Proposition 12.1. *$M(w,\chi)$ is the (unique) minimal element in $\mathcal{P}_+(V(w,\chi))$; in particular, we have $D(V(w,\chi)) = M(w,\chi)^* = \Lambda(N_0) \otimes_{\Lambda(N_{w,0})} k$.*

Proof. This is the same argument as for Lemma 2.6 (cf. [Vi2] Lemma 4). □

Since $N_{w_0w,0} \xrightarrow{\cong} N_0/N_{w,0}$ one can write $D(V(w,\chi)) = \Omega(N_{w_0w,0})$; but this does not reflect the $\Lambda(N_0)$-action very well.

Proposition 12.2. *$\Lambda_F(S_0) \otimes_{\Lambda(N_0)} D(V(w,\chi)) = 0$ when $w \neq w_0$ and is equal to $\Lambda_F(S_0) \otimes_o k$ when $w = w_0$.*

Proof. The case $w = w_0$ being clear let $w \neq w_0$. Then (cf. [MS] Lemma 4.1) there is a simple root α such that $N_\alpha \subseteq N \cap \dot{w}N\dot{w}^{-1}$. It follows that S_0, being the image of $N_{\alpha,0}$, acts trivially on

$$\Lambda(S_0) \otimes_{\Lambda(N_0)} D(V(w,\chi)) = \Lambda(S_0) \otimes_{\Lambda(N_{w,0})} k$$

(compare the end of the proof of Prop. 10.3). □

Both propositions remain true (with the same proofs) when the coefficient ring is $o/\pi^m o$, for some $m \geq 1$, instead of k.

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