p-adic Fourier theory

P. Schneider, J. Teitelbaum

In the early sixties, Amice ([Am1], [Am2]) studied the space of K-valued, locally analytic functions on $\mathbb{Z}_p$ and formulated a complete description of its dual, the ring of K-valued, locally $\mathbb{Q}_p$-analytic distributions on $\mathbb{Z}_p$, when K is a complete subfield of $\mathbb{C}_p$. She found an isomorphism between the ring of distributions and the space of global functions on a rigid variety over K parameterizing K-valued, locally analytic characters of $\mathbb{Z}_p$. This rigid variety is in fact the open unit disk, a point z of $\mathbb{C}_p$ with $|z| < 1$ corresponding to the locally $\mathbb{Q}_p$-analytic character $\kappa_z(a) = (1+z)^a$ for $a \in \mathbb{Z}_p$. The rigid function F_λ corresponding to a distribution λ is determined by the formula $F_\lambda(z) = \lambda(\kappa_z)$. Amice's description of the ring of $\mathbb{Q}_p$-analytic distributions was complemented by results of Lazard ([Laz]). He described a divisor theory for the ring of functions on the open disk and proved that, when K is spherically complete, the classes of closed, finitely generated, and principal ideals in this ring coincide.

In this paper we generalize the work of Amice and Lazard by studying the space $C^{an}(o, K)$ of K-valued, locally L-analytic functions on o, and the corresponding ring of distributions $D(o, K)$, when $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ with L finite over $\mathbb{Q}_p$ and K complete and $o = o_L$ the additive group of the ring of integers in L. To clarify this, observe that, as a $\mathbb{Q}_p$-analytic manifold, the ring o is a product of $[L : \mathbb{Q}_p]$ copies of $\mathbb{Z}_p$. The K-valued, $\mathbb{Q}_p$-analytic functions on o are thus given locally by power series in $[L : \mathbb{Q}_p]$ variables, with coefficients in K. The L-analytic functions in $C^{an}(o, K)$ are given locally by power series in *one* variable; they form a subspace of the $\mathbb{Q}_p$-analytic functions cut out by a set of "Cauchy-Riemann" differential equations. These facts are treated in Section 1.

Like Amice, we develop a Fourier theory for the locally L-analytic functions on o. We construct (Section 2) a rigid group variety $\widehat{o}$, defined over L, whose closed points z in a field K parameterize K-valued locally L-analytic characters κ_z of o. We then show that, for K a complete subfield of $\mathbb{C}_p$, the ring of rigid functions on $\widehat{o}/K$ is isomorphic to the ring $D(o, K)$, where the isomorphism $\lambda \mapsto F_\lambda$ is defined by $\lambda(\kappa_z) = F_\lambda(z)$, just as in Amice's situation.

The most novel aspect of this situation is the variety $\widehat{o}$. We prove (Section 3) that $\widehat{o}$ is a rigid variety defined over L that becomes isomorphic over $\mathbb{C}_p$ to the open unit disk, but is not isomorphic to a disk over any discretely valued extension field of L. The ring of rigid functions on $\widehat{o}$ has the property that the classes of closed, finitely generated, and invertible ideals coincide; but we show that unless $L = \mathbb{Q}_p$ (Lazard's situation) there are non-principal, finitely generated ideals, even over spherically complete coefficient fields.

The "uniformization" of $\widehat{o}$ by the open unit disk follows from a result of Tate's in his famous paper on p-divisible groups ([Tat]). We show that over $\mathbb{C}_p$ the group

$\widehat{o}$ becomes isomorphic to the group of $\mathbb{C}_p$-valued points of a Lubin-Tate formal group associated to L. The Galois cocycle that gives the descent data on the open unit disk yielding the twisted form $\widehat{o}$ comes directly out of the Lubin-Tate group. The period of the Lubin-Tate group plays a crucial role in the explicit form of our results; in an Appendix we use results of Fontaine [Fon] to obtain information on the valuation of this period, generalizing work of Boxall ([Box]).

We give two applications of our Fourier theory. The first is a generalized Mahler expansion for locally L-analytic functions on o (Section 4). The second is a construction of a p-adic L-function for a CM elliptic curve at a supersingular prime (Section 5). Although the method by which we obtain it is more natural, and we obtain stronger analyticity results, the L-function we construct is essentially that studied by Katz ([Ka1]) and by Boxall ([Box]). The paper by Katz in particular was a major source of inspiration in our work.

Our original motivation for studying this problem came from our work on locally analytic representation theory. In the paper [ST] we classified the locally analytic principal series representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. The results of Amice and Lazard played a key role in the proof, and in seeking to generalize those results to the principal series of $\mathrm{GL}_2(L)$ we were led to consider the problems discussed in this paper. The results of this paper are sufficient to extend the methods of [ST] to the groups $\mathrm{GL}_2(L)$, though to keep the paper self-contained we do not give the proof here.

The relationship between formal groups and p-adic integration has been known and exploited in some form by many authors. We have already mentioned the work of Katz [Ka1] and Boxall [Box]. Height one formal groups and their connection to p-adic integration is systematically used in [dS] and we have adapted this approach to the height two case in Section 5 of our paper. Some other results of a similar flavor were obtained in [SI]. Finally, we point out that the appearance of p-adic Hodge theory in our work raises the interesting question of relating our results to the work of Colmez ([Col]).

We would like to thank John Coates, Robert Coleman, Pierre Colmez, and Ehud deShalit for interesting discussions on this work. The first, resp. second, author thanks the Landau Center, resp. the Forcheimer Fellowship Fund, and Hebrew University for its support and hospitality during the writing of this paper. The second author was also supported by a grant from the NSA Mathematical Sciences Program.

1. Preliminaries on restriction of scalars

We fix fields $\mathbb{Q}_p \subseteq L \subseteq K$ such that $L/\mathbb{Q}_p$ is finite and K is complete with respect to a nonarchimedean absolute value $|\ |$ extending the one on L. We also fix a commutative d-dimensional locally L-analytic group G. Then the locally convex K-vector space $C^{an}(G,K)$ of all K-valued locally analytic functions on G is defined ([Fe2] 2.1.10).

We consider now an intermediate field $\mathbb{Q}_p \subseteq L_0 \subseteq L$ and let G_0 denote the locally L_0-analytic group obtained from G by restriction of scalars ([B-VAR] §5.14). The dimension of G_0 is $d[L : L_0]$. There is the obvious injective continuous K-linear map

$$(*) \qquad\qquad C^{an}(G,K) \longrightarrow C^{an}(G_0,K)\ .$$

We want to describe the image of this map. The Lie algebra $\mathfrak{g}$ of G can naturally be identified with the Lie algebra of G_0 ([B-VAR] 5.14.5). We fix an exponential map $\exp : \mathfrak{g} - - - > G$ for G; it, in particular, is a local isomorphism, and can be viewed as an exponential map for G_0 as well. The Lie algebra $\mathfrak{g}$ acts in a compatible way on both sides of the above map by continuous endomorphisms defined by

$$(\mathfrak{x}f)(g) := \frac{d}{dt} f(\exp(-t\mathfrak{x})g)_{|t=0}$$

([Fe2] 3.1.2 and 3.3.4). By construction the map $\mathfrak{x} \to \mathfrak{x}f$ on $\mathfrak{g}$, for a fixed $f \in C^{an}(G_0,K)$, resp. $f \in C^{an}(G,K)$, is L_0-linear, resp. L-linear.

Lemma 1.1:

The image of $(*)$ is the closed subspace of all $f \in C^{an}(G_0,K)$ such that $(t\mathfrak{x})f = t \cdot (\mathfrak{x}f)$ for any $\mathfrak{x} \in \mathfrak{g}$ and any $t \in L$.

Proof: We fix an L-basis $\mathfrak{x}_1, \ldots, \mathfrak{x}_d$ of $\mathfrak{g}$ as well an orthogonal basis $v_1 = 1, v_2, \ldots, v_e$ of L as a normed L_0-vector space. Then $v_1\mathfrak{x}_1, \ldots, v_e\mathfrak{x}_d$ is an L_0-basis of $\mathfrak{g}$. Using the corresponding canonical coordinates of the second kind ([B-GAL] Chap.III,§4.3) we have, for a given $f \in C^{an}(G_0,K)$ and a given $g \in G$, the convergent expansion

$$f(\exp(t_{11}v_1\mathfrak{x}_1 + \ldots + t_{ed}v_e\mathfrak{x}_d)g) = \sum_{n_{11},\ldots,n_{ed}\geq 0} c_{\underline{n}} t_{11}^{n_{11}} \ldots t_{ed}^{n_{ed}}\ ,$$

with $c_{\underline{n}} \in K$, in a neighborhood of g (i.e., for $t_{ij} \in L_0$ small enough). We now assume that

$$(v_i\mathfrak{x}_j)f = v_i \cdot (\mathfrak{x}_j f) = v_i \cdot ((v_1\mathfrak{x}_j)f)$$

holds true for all i and j. Computing both sides in terms of the above expansion and comparing coefficients results in the equations

$$(n_{ij}+1)c_{(n_{11},\ldots,n_{ij}+1,\ldots,n_{ed})} = v_i(n_{1j}+1)c_{(n_{11},\ldots,n_{1j}+1,\ldots,n_{ed})}\ .$$

Introducing the tuple $\underline{m}(\underline{n}) = (m_1, \ldots, m_d)$ defined by $m_j := n_{1j} + \ldots + n_{ej}$ and the new coefficients $b_{\underline{m}(\underline{n})} := c_{(m_1,0,\ldots,m_2,0,\ldots,m_d,0,\ldots)}$ we deduce from this by induction that

$$c_{\underline{n}} = b_{\underline{m}(\underline{n})} \frac{m_1!}{n_{11}! \ldots n_{e1}!} \cdots \frac{m_d!}{n_{1d}! \ldots n_{ed}!} v_1^{n_{11}+\ldots n_{1d}} \ldots v_e^{n_{e1}+\ldots+n_{ed}} .$$

Inserting this back into the above expansion and setting $t_j := t_{1j}v_1 + \ldots + t_{ej}v_e$ we obtain the new expansion

$$f(\exp(t_1\mathfrak{x}_1 + \ldots + t_d\mathfrak{x}_d)g) = \sum_{m_1,\ldots,m_d \geq 0} b_{\underline{m}} t_1^{m_1} \ldots t_d^{m_d}$$

which shows that f is locally analytic on G.

Lemma 1.2:

The map $(*)$ *is a homeomorphism onto its (closed) image.*

Proof: Let $H \subseteq G$ be a compact open subgroup. According to [Fe2] 2.2.4) we then have

$$C^{an}(G, K) = \prod_{g \in G/H} C^{an}(H, K) .$$

A corresponding decomposition holds for G_0. This shows that it suffices to consider the case where G is compact. In this case $(*)$ is a compact inductive limit of isometries between Banach spaces ([Fe2] 2.3.2), and the assertion follows from [GKPS] 3.1.16.

The continuous dual $D(G, K) := C^{an}(G, K)'$ is the algebra of K-valued distributions on G. The multiplication is the convolution product $*$ ([Fe1] 4.4.2 and 4.4.4).

We assume from now on that G is compact. To describe the correct topology on $D(G, K)$ we need to briefly recall the construction of $C^{an}(G, K)$. Let $G \supseteq H_0 \supseteq H_1 \supseteq \ldots \supseteq H_n \supseteq \ldots$ be a fundamental system of open subgroups such that each H_n corresponds under the exponential map to an L-affinoid disk. We then have, for each $g \in G$ and $n \in \mathbb{N}$, the K-Banach space $\mathcal{F}_{gH_n}(K)$ of K-valued L-analytic functions on the coset gH_n viewed as an L-affinoid disk. The space $C^{an}(G, K)$ is the locally convex inductive limit

$$C^{an}(G, K) = \varinjlim_n \mathcal{F}_n(G, K)$$

of the Banach spaces

$$\mathcal{F}_n(G, K) := \prod_{g \in G/H_n} \mathcal{F}_{gH_n}(K) .$$

The dual $D(G,K)$ therefore coincides as a vector space with the projective limit

$$D(G,K) = \varprojlim_n \mathcal{F}_n(G,K)'$$

of the dual Banach spaces. We always equip $D(G,K)$ with the corresponding projective limit topology. (Using [GKPS] 3.1.7(vii) and the open mapping theorem one can show that this topology in fact coincides with the strong dual topology.) In particular, $D(G,K)$ is a commutative K-Fréchet algebra. The dual of the map $(*)$ is a continuous homomorphism of Fréchet algebras

$$(*)' \qquad D(G_0,K) \longrightarrow\!\!\!\!\rightarrow D(G,K) \ .$$

It is surjective since $C^{an}(G_0,K)$ as a compact inductive limit is of countable type ([GKPS] 3.1.7(viii)) and hence satisfies the Hahn-Banach theorem ([Sh] 4.2 and 4.4). By the open mapping theorem $(*)'$ then is a quotient map.

The action of $\mathfrak{g}$ on $C^{an}(G_0,K)$ induces an action of $\mathfrak{g}$ on $D(G_0,K)$ by $(\mathfrak{x}\lambda)(f) := \lambda(-\mathfrak{x}f)$. This action is related to the algebra structure through the L_0-linear inclusion

$$\begin{array}{rccl} \iota : & \mathfrak{g} & \longrightarrow & D(G_0,K) \\ & \mathfrak{x} & \longmapsto & [f \mapsto (-\mathfrak{x}(f))(1)] \end{array}$$

which satisfies

$$\iota(\mathfrak{x}) * \lambda = \mathfrak{x}\lambda \quad \text{for } \mathfrak{x} \in \mathfrak{g} \text{ and } \lambda \in D(G_0,K)$$

(see the end of section 2 in [ST]). Followed by $(*)'$ this inclusion becomes L-linear.

Let $\widehat{G_0}(K) \subseteq C^{an}(G_0,K)$ denote the subset of all K-valued locally analytic characters on G_0. Any $\chi \in \widehat{G_0}(K)$ induces the L_0-linear map

$$\begin{array}{rccl} d\chi : & \mathfrak{g} & \longrightarrow & K \\ & \mathfrak{x} & \longmapsto & \frac{d}{dt}\chi(\exp(t\mathfrak{x}))_{|t=0} \ . \end{array}$$

Lemma 1.3:

$\widehat{G}(K) = \{\chi \in \widehat{G_0}(K) : d\chi \text{ is } L\text{-linear}\}$.

Proof: Because of

$$(-\mathfrak{x}\chi)(g) = \chi(g) \cdot d\chi(\mathfrak{x})$$

this is a consequence of Lemma 1.1.

The lemma says that the diagram

$$\begin{array}{ccc} \widehat{G}(K) & \xrightarrow{\subseteq} & \widehat{G_0}(K) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ \mathrm{Hom}_L(\mathfrak{g}, K) & \xrightarrow{\subseteq} & \mathrm{Hom}_{L_0}(\mathfrak{g}, K) \end{array}$$

is cartesian.

We suppose from now on that K is a subfield of $\mathbb{C}_p$ (= the completion of an algebraic closure of $\mathbb{Q}_p$). There is the natural strict inclusion

$$\begin{array}{rcl} D(G_0, K) & = & \varprojlim_n \mathrm{Hom}_K^{cont}(\mathcal{F}_n(G_0, K), K) \\ & & \downarrow \\ D(G_0, \mathbb{C}_p) & = & \varprojlim_n \mathrm{Hom}_{\mathbb{C}_p}^{cont}(\mathcal{F}_n(G_0, \mathbb{C}_p), \mathbb{C}_p) \ . \end{array}$$

The <u>*Fourier transform*</u> F_λ of a $\lambda \in D(G_0, K)$, by definition, is the function

$$\begin{array}{rccc} F_\lambda : & \widehat{G_0}(\mathbb{C}_p) & \longrightarrow & \mathbb{C}_p \\ & \chi & \longmapsto & \lambda(\chi) \ . \end{array}$$

Proposition 1.4:

i. For any $\lambda \in D(G_0, K)$ we have $\lambda = 0$ if and only if $F_\lambda = 0$;

ii. $F_{\mu * \lambda} = F_\mu F_\lambda$ for any two $\mu, \lambda \in D(G_0, K)$.

Proof: [Fe1] Thm. 5.4.8 (recall that G is assumed to be compact). For the convenience of the reader we sketch the proof of the first assertion in the case of the additive group $G_0 = G = o_L$. (This is the only case in which we actually will use this result in the next section. Moreover, the general proof is just an elaboration of this special case.) Let $\mathfrak{b} \subseteq o_L$ be an arbitrary nonzero ideal viewed as an additive subgroup. We use the convention to denote by $f|a + \mathfrak{b}$, for any function f on o_L and any coset $a + \mathfrak{b} \subseteq o_L$, the function on o_L which is equal to f on the coset $a + \mathfrak{b}$ and which vanishes elsewhere. Suppose now that $F_\lambda = 0$, i.e., that $\lambda(\chi) = 0$ for any $\chi \in \widehat{G}(\mathbb{C}_p)$. Using the character theory of finite abelian groups one easily concludes that

$$\lambda(\chi | a + \mathfrak{b}) = 0 \quad \text{ for any } \chi \in \widehat{G}(\mathbb{C}_p) \text{ and any coset } a + \mathfrak{b} \subseteq o_L \ .$$

We apply this to the character $\chi_y(x) := \exp(yx)$ where $y \in o_{\mathbb{C}_p}$ is small enough and obtain by continuity that

$$0 = \lambda(\chi_y | a + \mathfrak{b}) = \sum_{n \geq 0} \frac{y^n}{n!} \lambda(x^n | a + \mathfrak{b}) \ .$$

Viewing the right hand side as a power series in y in a small neighbourhood of zero it follows that

$$\lambda(x^n | a + \mathfrak{b}) = 0 \quad \text{for any } n \geq 0 \text{ and any coset } a + \mathfrak{b} \subseteq o_L \ .$$

Again from continuity we see that $\lambda = 0$.

Corollary 1.5:

i. $\widehat{G}(\mathbb{C}_p) = \{\chi \in \widehat{G_0}(\mathbb{C}_p) : F_{\iota(t\mathfrak{x}) - t\iota(\mathfrak{x})}(\chi) = 0$ *for any* $\mathfrak{x} \in \mathfrak{g}$ *and* $t \in L\}$*;*

ii. the kernel of $(*)'$ *is the ideal* $I(G) := \{\lambda \in D(G_0, K) : F_\lambda | \widehat{G}(\mathbb{C}_p) = 0\}$.

Proof: The assertion i. is a consequence of Lemma 1.3 and the identity

$$F_{\iota(t\mathfrak{x}) - t\iota(\mathfrak{x})}(\chi) = ((-t\mathfrak{x})\chi)(1) - t(-\mathfrak{x}\chi)(1) = d\chi(t\mathfrak{x}) - t \cdot d\chi(\mathfrak{x}) \ .$$

The assertion ii. follows from Prop. 1.4.i (applied to G).

2. The Fourier transform for $G = o_L$

Let $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ again be a chain of complete fields and let $o := o_L$ denote the ring of integers in L. The aim of this section is to determine the image of the Fourier transform for the compact additive group $G := o$. The restriction of scalars G_0 will always be understood with respect to the extension $L/\mathbb{Q}_p$.

First we have to discuss briefly a certain way to write rigid analytic polydisks in a coordinate free manner. Let $\mathbf{B}_1$ denote the rigid L-analytic open disk of radius one around the point $1 \in L$; its K-points are $\mathbf{B}_1(K) = \{z \in K : |z - 1| < 1\}$. We note that the group $\mathbb{Z}_p$ acts on $\mathbf{B}_1$ via the rigid analytic automorphisms

$$\begin{array}{ccc} \mathbb{Z}_p \times \mathbf{B}_1 & \longrightarrow & \mathbf{B}_1 \\ (a, z) & \longmapsto & z^a := \sum\limits_{n \geq 0} \binom{a}{n} (z-1)^n \end{array}$$

(compare [Sch] §§32 and 47). Hence, given any free $\mathbb{Z}_p$-module M of finite rank r, we can in an obvious sense form the rigid L-analytic variety $\mathbf{B}_1 \otimes_{\mathbb{Z}_p} M$ whose K-points are $\mathbf{B}_1(K) \otimes_{\mathbb{Z}_p} M$. Any choice of an $\mathbb{Z}_p$-basis of M defines an isomorphism between $\mathbf{B}_1 \otimes_{\mathbb{Z}_p} M$ and an r-dimensional open polydisk over L. In particular, the family of all affinoid subdomains in $\mathbf{B}_1 \otimes_{\mathbb{Z}_p} M$ has a countable cofinal subfamily. Writing the ring $\mathcal{O}(\mathbf{B}_1 \otimes_{\mathbb{Z}_p} M)$ of global holomorphic functions on $\mathbf{B}_1 \otimes_{\mathbb{Z}_p} M$ as the projective limit of the corresponding affinoid algebras we see that $\mathcal{O}(\mathbf{B}_1 \otimes_{\mathbb{Z}_p} M)$ in a natural way is an L-Fréchet algebra.

After this preliminary discussion we recall that the maps

$$\begin{array}{ccc} \widehat{\mathbb{Z}_p}(K) & \longleftrightarrow & \mathbf{B}_1(K) \\ \\ \chi & \longmapsto & \chi(1) \\ \chi_z(a) := z^a & \longleftarrow\!\shortmid & z \end{array}$$

are bijections inverse to each other (compare [Am2] 1.1 and [B-GAL] III.8.1). They straightforwardly generalize to the bijection

$$\begin{array}{ccc} \mathbf{B}_1(K) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) & \xrightarrow{\sim} & \widehat{G_0}(K) \\ \\ z \otimes \beta & \longmapsto & \chi_{z\otimes\beta}(g) := z^{\beta(g)} \ . \end{array}$$

By transport of structure the right hand side therefore can and will be considered as the K-points of a rigid analytic group variety $\widehat{G_0}$ over L (which is non-canonically isomorphic to an open polydisk of dimension $[L : \mathbb{Q}_p]$). By construction the Lie algebra of $\widehat{G_0}$ is equal to $\mathrm{Hom}_{\mathbb{Q}_p}(\mathfrak{g}, L)$. One easily checks that

$$d\chi_{z\otimes\beta} = \log(z) \cdot \beta \ .$$

If we combine this identity with the commutative diagram after Lemma 1.3 we arrive at the following fact which is recorded here for use in the next section.

Lemma 2.1

The diagram

$$\begin{array}{ccc} \widehat{G}(K) & \xrightarrow{\ \subseteq\ } & \mathbf{B}_1(K) \otimes \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) \\ \big\downarrow d & & \big\downarrow \log\otimes\mathrm{id} \\ \mathrm{Hom}_L(\mathfrak{g}, K) & \xrightarrow{\ \subseteq\ } \mathrm{Hom}_{\mathbb{Q}_p}(\mathfrak{g}, K) = & K \otimes \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) \end{array}$$

is cartesian.

We denote by $\mathcal{O}(\widehat{G_0}/K)$ the K-Fréchet algebra of global holomorphic functions on the base extension of the variety $\widehat{G_0}$ to K. The main result of Fourier analysis over the field $\mathbb{Q}_p$ is the following.

Theorem 2.2: (Amice)

The Fourier transform is an isomorphism of K-Fréchet algebras

$$\begin{array}{rcl} \mathcal{F} : D(G_0, K) & \xrightarrow{\cong} & \mathcal{O}(\widehat{G_0}/K) \\ \lambda & \longmapsto & F_\lambda \ . \end{array}$$

Proof: This is a several variable version of [Am2] 1.3 (compare also [Sc]) based on [Am1].

Next we want to compute the ideal $J(o) := \mathcal{F}(I(o))$ in $\mathcal{O}(\widehat{G_0}/K)$. Let $\mathfrak{x}_1 := 1 \in \mathfrak{g} = L$ and $F_t := F_{\iota(t\mathfrak{x}_1) - t\iota(\mathfrak{x}_1)} \in \mathcal{O}(\widehat{G_0}/K)$ for $t \in L$. A straightforward computation shows that

$$F_t(\chi_{z\otimes\beta}) = (\beta(t) - t \cdot \beta(1)) \cdot \log(z) \ .$$

By Cor. 1.5 we have the following facts:

1) $\widehat{G}(\mathbb{C}_p)$ is the analytic subset of the variety $\widehat{G_0}/K$ defined by $F_t = 0$ for $t \in L$.

2) $J(o)$ is the ideal of all global holomorphic functions which vanish on $\widehat{G}(\mathbb{C}_p)$.

In 1) one can replace the family of all F_t by finitely many $F_{t_1}, \ldots, F_{t_e}$ if $t_1, \ldots, t_e$ runs through a $\mathbb{Q}_p$-basis of L.

According to [BGR] 9.5.2 Cor.6 the sheaf of ideals $\mathcal{J}$ in the structure sheaf $\mathcal{O}_{\widehat{G_0}}$ of the variety $\widehat{G_0}$ consisting of all germs of functions vanishing on the analytic subset $\widehat{G}(\mathbb{C}_p)$ is coherent. Moreover, [BGR] 9.5.3 Prop.4 says that the analytic subset $\widehat{G}(\mathbb{C}_p)$ carries the structure of a reduced closed L-analytic subvariety $\widehat{G} \subseteq \widehat{G_0}$ such that for the structure sheaves we have $\mathcal{O}_{\widehat{G}} = \mathcal{O}_{\widehat{G_0}}/\mathcal{J}$. Since $\widehat{G_0}$ is a Stein space the global section functor is exact on coherent sheaves. All this remains true of course after base extension to K. Hence, if $\mathcal{O}(\widehat{G}/K)$ denotes the ring of global holomorphic functions on the base extension of the variety $\widehat{G}$ to K then, by 2), we have

$$(+) \qquad \mathcal{O}(\widehat{G}/K) = \mathcal{O}(\widehat{G_0}/K)/J(o) \ .$$

The ideal $J(o)$ being closed $\mathcal{O}(\widehat{G}/K)$ in particular is in a natural way a K-Fréchet algebra as well. It is clear from the open mapping theorem that this quotient topology on $\mathcal{O}(\widehat{G}/K)$ coincides with the topology as a projective limit of affinoid algebras.

Theorem 2.3:

The Fourier transform is an isomorphism of K-Fréchet algebras

$$\begin{array}{rcl} \mathcal{F} : D(G,K) & \xrightarrow{\cong} & \mathcal{O}(\widehat{G}/K) \\ \lambda & \longmapsto & F_\lambda \ . \end{array}$$

Proof: This follows from Thm. 2.2, (+), and the surjection $(*)'$ in section 1.

We remark that by construction we (noncanonically) have a cartesian diagram of rigid L-analytic varieties of the form

$$\begin{array}{ccc} \widehat{G} & \longrightarrow & (\mathbf{B}_1)^d \\ \Big\downarrow{\scriptstyle d} & & \Big\downarrow{\scriptstyle \log} \\ \mathbb{A}^1 & \longrightarrow & \mathbb{A}^d \end{array}$$

with $\mathbb{A}^m$ denoting affine m-space where the horizontal arrows are closed immersions and the vertical arrows are etale. Hence the variety $\widehat{G}$ is smooth and quasi-Stein (in the sense of [Kie]).

Lemma 2.4:

$\widehat{G}$ is a smooth rigid analytic group variety over L.

Proof: We have constructed $\widehat{G}$ as a reduced closed subvariety of the rigid analytic group variety $\widehat{G_0}$ over L. With $\widehat{G}$ also $\widehat{G} \times \widehat{G}$ is smooth. In particular, $\widehat{G} \times \widehat{G}$ is a reduced closed subvariety of $\widehat{G_0} \times \widehat{G_0}$. Since the multiplication and the inverse on $\widehat{G_0}$ preserve $\widehat{G}(\mathbb{C}_p)$ they restrict to morphisms between these reduced subvarieties.

3. Lubin-Tate formal groups and twisted unit disks

Keeping the notations of the previous section we will give in this section a different description of the rigid variety $\widehat{G}$. We will show that the character variety $\widehat{G}$ becomes isomorphic to the open unit disk after base change to $\mathbb{C}_p$. As a corollary, the ring of functions $\mathcal{O}(\widehat{G}/\mathbb{C}_p)$ is the same for any group $G = o$. This result originates in the observation that the character group $\widehat{G}(\mathbb{C}_p)$ can be parametrized with the help of Lubin-Tate theory.

Fix a prime element π of o and let $\mathcal{G} = \mathcal{G}_\pi$ denote the corresponding Lubin-Tate formal group over o. It is commutative and has dimension one and height $[L : \mathbb{Q}_p]$. Most importantly, $\mathcal{G}$ is a formal o-module which means that the ring

o acts on $\mathcal{G}$ in such a way that the induced action of o on the tangent space $t_{\mathcal{G}}$ is the one coming from the natural o-module structure on the latter ([LT]). We always identify $\mathcal{G}$ with the rigid L-analytic open unit disk $\mathbf{B}$ around zero in L. In this way $\mathbf{B}$ becomes an o-module object, and we will denote the action $o \times \mathbf{B} \longrightarrow \mathbf{B}$ by $(g, z) \longmapsto [g](z)$. This identification, of course, also trivializes the tangent space $t_{\mathcal{G}}$.

Let $\mathcal{G}'$ denote the p-divisible group dual to $\mathcal{G}$ and let $T' = T(\mathcal{G}')$ be the Tate module of $\mathcal{G}'$. Lubin-Tate theory tells us that T' is a free o-module of rank one and that the Galois action on T' is given by a continuous character $\tau : \mathrm{Gal}(\mathbb{C}_p/L) \longrightarrow o^{\times}$. From [Tat] p.177 we know that, by Cartier duality, T' is naturally identified with the group of homomorphisms of formal groups over $o_{\mathbb{C}_p}$ between $\mathcal{G}$ and the formal multiplicative group. This gives rise to natural Galois equivariant and o-invariant pairings

$$< \, , \, > : \, T' \underset{o}{\otimes} \mathbf{B}(\mathbb{C}_p) \longrightarrow \mathbf{B}_1(\mathbb{C}_p)$$

and on tangent spaces

$$(\, , \,) \; : \; T' \underset{o}{\otimes} \mathbb{C}_p \longrightarrow \mathbb{C}_p \; .$$

To describe them explicitly we will denote by $F_{t'}(Z) = \Omega_{t'} Z + \ldots \in Z o_{\mathbb{C}_p}[[Z]]$, for any $t' \in T'$, the power series giving the corresponding homomorphism of formal groups. Then

$$< t', z > = 1 + F_{t'}(z) \quad \text{and} \quad (t', x) = \Omega_{t'} x \; .$$

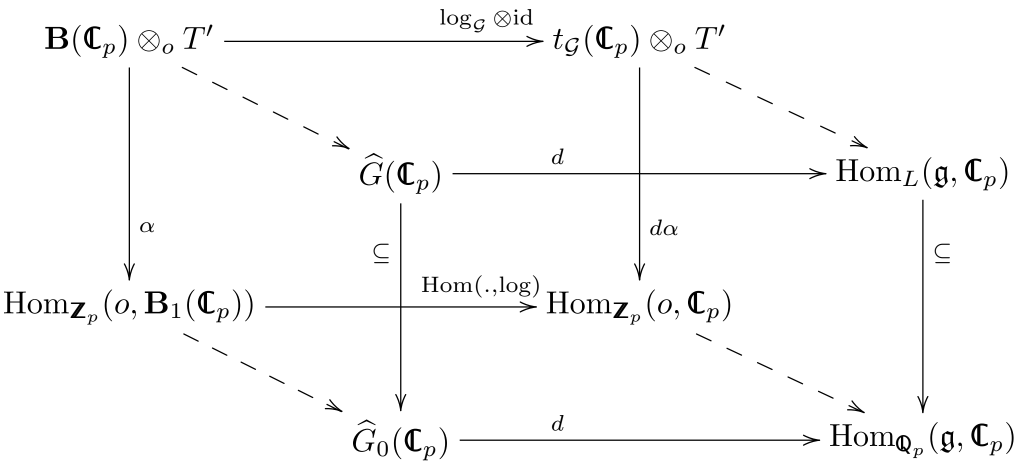
Proposition 3.1:

The map

$$(\diamond) \qquad \begin{array}{rcl} \mathbf{B}(\mathbb{C}_p) \otimes_o T' & \longrightarrow & \widehat{G}(\mathbb{C}_p) \\ z \otimes t' & \longmapsto & \kappa_{z \otimes t'}(g) := < t', [g](z) > \end{array}$$

is a well defined isomorphism of groups.

Proof: We will study the following diagram:



Here:

a. The rear face of the cube in this diagram is the tensorization by T' of a portion of the map of exact sequences labelled $(*)$ in [Tat] §4. We use for this the identification

$$\mathrm{Hom}_{\mathbf{Z}_p}(T', .) \otimes_o T' = \mathrm{Hom}_{\mathbf{Z}_p}(o, .)\ .$$

By $\log_{\mathcal{G}}$ we denote the logarithm map of the formal group $\mathcal{G}$. The map α, resp. $d\alpha$, associates to an element $z \otimes t'$, resp. $\mathfrak{x} \otimes t'$, the map $g \mapsto < gt', z >$, resp. $g \mapsto (gt', \mathfrak{x})$, for $g \in o$.

b. The front face of the cube is the diagram after Lemma 1.3.

c. The dashed arrows on the bottom face of the cube come from the discussion before Lemma 2.1.

d. The dashed arrow in the upper left of the diagram is the one we want to establish.

e. The formal o-module property of $\mathcal{G}$ says that the induced o-action on $t_{\mathcal{G}}(\mathbb{C}_p)$ is the same as the action by linearity and the inclusion $o \subseteq \mathbb{C}_p$. It means that we have $(gt', \mathfrak{x}) = (t', g\mathfrak{x}) = g \cdot (t', \mathfrak{x})$ for $g \in o$ and hence that any map in the image of $d\alpha$ is o-linear. This defines the dashed arrow in the upper right of the diagram.

The back face of the cube is commutative by [Tat] §4. The front and bottom faces are commutative by Lemma 1.3 and Lemma 2.1. Furthermore, the dashed arrows on the bottom of the diagram are isomorphisms, the left one by the discussion before Lemma 2.1 and the right one for trivial reasons.

Consider now the right side of the cube. It is commutative by construction. Since, by [Tat] Prop. 11, $d\alpha$ is injective and since the lower dashed arrow is bijective the upper dashed arrow must at least be injective. But by a comparison of dimensions we see that the dashed arrow in the upper right of the cube is an isomorphism as well.

In this situation we now may use the fact that, by Lemma 1.3, the front of the cube is cartesian to obtain that the upper left dashed arrow is well defined (making the whole cube commutative) and is given by ($\diamond$). But according to [Tat] Prop. 11 the back of the cube also is cartesian. Therefore the map ($\diamond$) in fact is an isomorphism. $\square$

Fixing a generator t'_o of the o-module T' the isomorphism ($\diamond$) becomes

$$(\diamond\diamond) \qquad \begin{array}{ccc} \mathbf{B}(\mathbb{C}_p) & \xrightarrow{\cong} & \widehat{G}(\mathbb{C}_p) \\ z & \longmapsto & \kappa_z := \kappa_{z\otimes t'_o} \ . \end{array}$$

The main purpose of this section is to see that this latter isomorphism derives from an isomorphism $\mathbf{B}/\mathbb{C}_p \xrightarrow{\cong} \widehat{G}/\mathbb{C}_p$ between rigid $\mathbb{C}_p$-analytic varieties. In fact, we will exhibit compatible admissible coverings by affinoid open subsets on both sides.

Let us begin with the left hand side. For any $r \in p^{\mathbb{Q}}$ we have the affinoid disk

$$\mathbf{B}(r) := \{z : |z| \le r\}$$

over L. Clearly the disks $\mathbf{B}(r)$ for $r < 1$ form an admissible covering of $\mathbf{B}$. It actually will be convenient to normalize the absolute value $|\ |$ and we do this by the requirement that $|p| = p^{-1}$. The numerical invariants of the finite extension $L/\mathbb{Q}_p$ which will play a role are the ramification index e and the cardinality q of the residue class field of L. Recall that o acts on $\mathbf{B}$ since we identify $\mathbf{B}$ with the formal group $\mathcal{G}_\pi$. We need some information how this covering behaves with respect to the action of π.

Lemma 3.2:

For any $r \in p^{\mathbb{Q}}$ such that $p^{-q/e(q-1)} \le r < 1$ we have

$$[\pi]^{-1}(\mathbf{B}(r)) = \mathbf{B}(r^{1/q}) \quad \textit{and} \quad [p]^{-1}(\mathbf{B}(r)) = \mathbf{B}(r^{1/q^e}) \ ;$$

further, in this situation the map $[\pi^n] : \mathbf{B}(r^{1/q^n}) \to \mathbf{B}(r)$, for any $n \in \mathbb{N}$, is a finite etale affinoid map.

Proof: The second identity is a consequence of the first since π^e and p differ by a unit in o, and for any unit $g \in o^\times$ one has $|[g](z)| = |z|$. Moreover, up to isomorphism, we may assume ([Lan] §8.1) that the action of the prime element π on $\mathbf{B}$ is given by

$$[\pi](z) = \pi z + z^q \ .$$

In this case the first identity follows by a straightforward calculation of absolute values. The finiteness and etaleness of the map $[\pi^n]$ also follows from this explicit formula together with the fact that a composition of finite etale affinoid maps is finite etale.

Now we consider the right side of $(\diamond\diamond)$. The disk $\mathbf{B}_1$ has the admissible covering by the L-affinoid disks $\mathbf{B}_1(r) := \{z : |z-1| \leq r\}$ for $r \in p^{\mathbb{Q}}$ such that $r < 1$. They are $\mathbb{Z}_p$-submodules so that the L-affinoids $\mathbf{B}_1(r) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p)$ form an admissible covering of $\mathbf{B}_1 \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) \cong \widehat{G_0}$. We therefore have the admissible covering of $\widehat{G}$ by the L-affinoids $\widehat{G}(r) := \widehat{G} \cap (\mathbf{B}_1(r) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p))$. We emphasize that on both sides the covering is defined over L.

Lemma 3.3:

For any $r \in p^{\mathbb{Q}}$ such that $p^{-p/(p-1)} \leq r < 1$ we have

$$\{\chi \in \widehat{G} : \chi^p \in \widehat{G}(r)\} = \widehat{G}(r^{1/p}) ;$$

further, in this situation the map $[p^n] : \widehat{G}(r^{1/p^n}) \to \widehat{G}(r)$, for any $n \in \mathbb{N}$, is a finite etale affinoid map.

Proof: This follows from a corresponding identity between the affinoids $\mathbf{B}_1(r)$. It is, in fact, a special case of the previous lemma.

In order to see in which way the isomorphism $(\diamond\diamond)$ respects these coverings, we need more detailed information on the power series $F_{t'}$ representing $t' \in T'$. We summarize the facts that we require in the following lemma.

Lemma 3.4:

Suppose $t' \in T'$ is non-zero; the power series $F_{t'}(Z) = \Omega_{t'} Z + \ldots \in o_{\mathbb{C}_p}[[Z]]$ has the following properties:

a. $\Omega_{gt'} = \Omega_{t'} g$ for $g \in o$;

b. if t' generates T' as an o-module, then

$$|\Omega_{t'}| = p^{-s} \quad \text{with} \quad s = \frac{1}{p-1} - \frac{1}{e(q-1)} ;$$

c. for any $r < p^{-1/e(q-1)}$, the power series $F_{t'}(Z)$ gives an analytic isomorphism between $\mathbf{B}(r)$ and $\mathbf{B}(r|\Omega_{t'}|)$.

Proof: Part (a) is a restatement of the o-linearity of the pairing (,) introduced at the start of this section. Part (b) follows from work of Fontaine ([Fon]) on p-adic Hodge theory. We give a proof in the appendix. For part (c), recall that

$F_{t'}(Z)$ is a formal group homomorphism. Therefore if $F_{t'}(z) = 0$, $F_{t'}$ vanishes on the entire subgroup of $\mathbf{B}(\mathbb{C}_p)$ generated by z. The point z belongs to some $\mathbf{B}(r)(\mathbb{C}_p)$, and therefore so does the entire subgroup generated by z. If this subgroup were infinite, $F_{t'}$ would have infinitely many zeroes in the affinoid $\mathbf{B}(r)(\mathbb{C}_p)$ and would therefore be zero. Consequently z must be a torsion point of the group $\mathcal{G}$. But other than zero, there are no torsion points inside the disk $\mathbf{B}(r)(\mathbb{C}_p)$ if $r < p^{-1/e(q-1)}$ ([Lan] §8.6 Lemma 4 and 5). It follows that the power series $F_{t'}(Z)/\Omega_{t'}Z = 1 + c_1 Z + c_2 Z^2 + \ldots$ has no zeroes inside $\mathbf{B}(r)(\mathbb{C}_p)$. Suppose that some coefficient c_n in this expansion satisfies $|c_n| > p^{n/e(q-1)}$. Then by considering the Newton polygon of the power series $F_{t'}(Z)/\Omega_{t'}Z$ one sees that the power series in question must have a zero of absolute value less than $p^{-1/e(q-1)}$, which we have seen is impossible. Therefore $|c_n| \leq p^{n/e(q-1)}$, from which part (c) follows immediately.

To simplify the notation, we write $\Omega = \Omega_{t'_o}$ for the "period" of the Lubin-Tate group associated to our fixed generator of T'.

By trivializing the tangent space as well as identifying $\mathrm{Hom}_L(\mathfrak{g}, \mathbb{C}_p)$ with $\mathbb{C}_p$ by evaluation at 1 we may simplify the upper face of the cubical diagram in the proof of Prop. 3.1 to the following:

(*)
$$\begin{array}{ccc} \mathbf{B}(\mathbb{C}_p) & \xrightarrow{\ \log_{\mathcal{G}}\ } & \mathbb{C}_p \\ {\scriptstyle z\mapsto\kappa_z}\downarrow & & \downarrow{\scriptstyle x\mapsto \Omega x} \\ \widehat{G}(\mathbb{C}_p) & \xrightarrow[\kappa_z\mapsto \log\kappa_z(1)]{} & \mathbb{C}_p \end{array}$$

Let us examine the map $\kappa(z) := \kappa_z$ in coordinates. Choose for the moment a $\mathbb{Z}_p$-basis $e_1, \ldots, e_d$ for o and let $e_1^*, \ldots, e_d^*$ be the dual basis. In coordinates, the map $\kappa : \mathbf{B} \to \widehat{G}_0$ is given by

(**)
$$\kappa_z = \sum_{i=1}^{d} (1 + F_{e_i t'_o}(z)) \otimes e_i^* \ .$$

Note first that this map is explicitly rigid $\mathbb{C}_p$-analytic and we know by Prop. 3.1 that this map factorizes through the subvariety $\widehat{G}/\mathbb{C}_p \subset \widehat{G}_0/\mathbb{C}_p$. We now also see that, if $r = p^{-q/e(q-1)} < p^{-1/e(q-1)}$, the three parts of Lemma 3.4 together imply that this map carries $\mathbf{B}(r)/\mathbb{C}_p$ into $\widehat{G}(r|\Omega|)/\mathbb{C}_p$.

Lemma 3.5:

Let $r = p^{-q/e(q-1)}$; the map

$$\begin{array}{ccc} \mathbf{B}(r)/\mathbb{C}_p & \xrightarrow{\ \cong\ } & \widehat{G}(r|\Omega|)/\mathbb{C}_p \\ z & \longmapsto & \kappa_z \end{array}$$

is a rigid isomorphism.

Proof: In the discussion preceeding the statement of the lemma we saw that this is a well-defined rigid map. Consider now the other maps in the diagram $(*)$.

– For $r = p^{-q/e(q-1)} < p^{-1/e(q-1)}$, the logarithm $\log_{\mathcal{G}}$ of the formal group $\mathcal{G}$ restricts to a rigid isomorphism

$$\log_{\mathcal{G}} : \mathbf{B}(r) \xrightarrow{\cong} \mathbf{B}(r) \ .$$

([Lan] §8.6 Lemma 4).

– Because $|\Omega| r = p^{-1/(p-1)-1/e} < p^{-1/(p-1)}$, the usual logarithm restricts to a rigid isomorphism

$$\log : \mathbf{B}_1(r|\Omega|) \xrightarrow{\cong} \mathbf{B}(r|\Omega|) \ .$$

All of this information, together with the diagram $(*)$, tells us that the following diagram of rigid morphisms commutes:

$$\begin{array}{ccc}
\mathbf{B}(r)/\mathbb{C}_p & \xrightarrow{\cong} & \mathbf{B}(r)/\mathbb{C}_p \\
{\scriptstyle z\mapsto\kappa_z}\downarrow & & \downarrow{\scriptstyle\cong} \\
\widehat{G}(r|\Omega|)/\mathbb{C}_p & \xrightarrow[\kappa_z\mapsto\log\kappa_z(1)]{} & \mathbf{B}(r|\Omega|)/\mathbb{C}_p
\end{array}$$

We claim that the lower arrow in this diagram is injective on $\mathbb{C}_p$-points. Assume that $\log \kappa_z(1) = 0$; we then have $\kappa_z(1) = 1$ which, by the local L-analyticity of κ_z, means that κ_z is locally constant and hence of finite order. But for our value of r we know that $\mathbf{B}_1(r)(\mathbb{C}_p)$ is torsionfree so it follows that κ_z must be the trivial character.

Because the upper horizontal and the right vertical map are rigid isomorphisms and the other two maps at least are injective on $\mathbb{C}_p$-points, all the maps in this diagram must be isomorphisms on $\mathbb{C}_p$-points. Because $\widehat{G}$ is reduced, it follows that the other arrows are rigid isomorphisms as well.

This lemma provides a starting point for the proof of the main theorem of this section.

Theorem 3.6:

The map

$$\kappa : \mathbf{B}/\mathbb{C}_p \xrightarrow{\cong} \widehat{G}/\mathbb{C}_p$$

is an isomorphism of rigid varieties over $\mathbb{C}_p$; more precisely, if $r = p^{-q/e(q-1)}$ and $n \in \mathbb{N}_0$, then κ is a rigid isomorphism between the affinoids

$$\kappa : \mathbf{B}(r^{1/q^{en}})/\mathbb{C}_p \xrightarrow{\cong} \widehat{G}((r|\Omega|)^{1/p^n})/\mathbb{C}_p \ .$$

Proof: We remark first that the second statement is in fact stronger than the first, because as n runs through $\mathbb{N}_0$ the given affinoids form admissible coverings of $\mathbf{B}/\mathbb{C}_p$ and $\widehat{G}/\mathbb{C}_p$ respectively.

Lemma 3.5 is the case $n = 0$. To obtain the result for all n, fix $n > 0$ and consider the diagram:

$$\begin{array}{ccc} \mathbf{B}(r^{1/q^{en}})/\mathbb{C}_p & \xrightarrow{z \mapsto \kappa_z} & \widehat{G}((r|\Omega|)^{1/p^n})/\mathbb{C}_p \\ \Big\downarrow{\scriptstyle [p^n]} & & \Big\downarrow{\scriptstyle \chi \mapsto \chi^{p^n}} \\ \mathbf{B}(r)/\mathbb{C}_p & \xrightarrow{z \mapsto \kappa_z} & \widehat{G}(r|\Omega|)/\mathbb{C}_p \end{array}$$

By Lemma 3.2, the left-hand vertical arrow is a well-defined finite etale affinoid map of degree $q^{ne} = p^{nd}$. By Lemma 3.4, part (b), $1 > r|\Omega| = p^{-1/(p-1)-1/e} \geq p^{-p/(p-1)}$ so that Lemma 3.3 applies to the right-hand vertical arrow and it enjoys the same properties. Lemma 3.5 shows that the lower arrow is a rigid analytic isomorphism. The upper horizontal arrow then is a well-defined bijective map on points because of Proposition 3.1 and the first assertions in Lemma 3.2 and Lemma 3.3. It is a rigid morphism because it is given in coordinates by the same formula as in the $n = 0$ case (see $(**)$).

To complete the argument, let A and B be the affinoid algebras of $\mathbf{B}(r^{1/q^{en}})/\mathbb{C}_p$ and $\widehat{G}((r|\Omega|)^{1/p^n})/\mathbb{C}_p$ respectively. Let D be the affinoid algebra of $\widehat{G}(r|\Omega|)/\mathbb{C}_p$. The rings A and B are finite etale D-algebras of the same rank. The map $f : B \to A$ induced by the upper arrow in the diagram is a map of D-algebras. Because f is bijective on maximal ideals and B is reduced (because $\widehat{G}$ is reduced), this map is injective. To see that f also is surjective it suffices, by [B-CA] II§3.3 Prop. 11, to check that the induced map $B/\mathfrak{m}B \longrightarrow A/\mathfrak{m}A$ is surjective for any maximal ideal $\mathfrak{m} \subseteq D$. But the latter is a map of finite etale algebras over $D/\mathfrak{m} = \mathbb{C}_p$ of the same dimension which is bijective on points. Hence it clearly must be bijective.

Corollary 3.7:

The ring of functions $\mathcal{O}(\widehat{G}/\mathbb{C}_p)$ is isomorphic to the ring $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ of $\mathbb{C}_p$-analytic functions on the open unit disk in $\mathbb{C}_p$; in particular, the distribution algebra $D(G, K)$ is an integral domain.

Let us remark that a careful examination of the proofs shows that these results in fact hold true over any complete intermediate field between L and $\mathbb{C}_p$ which contains the period Ω.

The ring $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ is the ring of power series $F(z) = \sum_{n\geq 0} a_n z^n$ over $\mathbb{C}_p$ which converge on $\{z : |z| < 1\}$. Let $G_L := \mathrm{Gal}(\overline{L}/L)$ be the absolute Galois group of the field L and let G_L act on $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ by

$$F^\sigma(z) := \sum_{n\geq 0} \sigma(a_n) z^n \quad \text{for } \sigma \in G_L \ .$$

By Tate's theorem ([Tat]) we have

$${\mathbb{C}_p}^{G_L} = L \ .$$

Hence the ring $\mathcal{O}(\mathbf{B})$ coincides with the ring of Galois fixed elements

$$\mathcal{O}(\mathbf{B}) = \mathcal{O}(\mathbf{B}/\mathbb{C}_p)^{G_L}$$

with respect to this action. This principle which here can be seen directly on power series in fact holds true for any quasi-separated rigid L-analytic variety $\mathcal{X}$; i.e., one has

$$\mathcal{O}(\mathcal{X}) = \mathcal{O}(\mathcal{X}/\mathbb{C}_p)^{G_L} \ .$$

By the way the base extension $\mathcal{X}/\mathbb{C}_p$ is constructed by pasting the base extension of affinoids ([BGR] 9.3.6) this identity immediately is reduced to the case of an affinoid variety. But for any L-affinoid algebra A we may consider an orthonormal base of A and apply Tate's theorem to the coefficients to obtain that

$$A = (A \mathbin{\widehat{\otimes}_L} \mathbb{C}_p)^{G_L} \ .$$

Since, according to our above theorem, $\mathbf{B}/\mathbb{C}_p$ also is the base extension of $\widehat{G}$ the ring $\mathcal{O}(\widehat{G})$ must be isomorphic to the subring of Galois fixed elements in the power series ring $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ with respect to a certain twisted Galois action. To work this out we first note that the natural Galois action on $\widehat{G}(\mathbb{C}_p)$ is given by composition $\kappa \mapsto \sigma \circ \kappa$ for $\sigma \in G_L$ and $\kappa \in \widehat{G}(\mathbb{C}_p)$ viewed as a character $\kappa : G \longrightarrow {\mathbb{C}_p}^\times$. Suppose that $\kappa = \kappa_z$ is the image of $z \in \mathbf{B}(\mathbb{C}_p)$ under the map ($\diamond\diamond$). The twisted Galois action $z \mapsto \sigma * z$ on $\mathbf{B}(\mathbb{C}_p)$ which we want to consider then is defined by

$$\kappa_{\sigma * z} = \sigma \circ \kappa_z$$

and we have

$$\mathcal{O}(\widehat{G}) \cong \{F \in \mathcal{O}(\mathbf{B}/\mathbb{C}_p) : F = F^\sigma(\sigma(\sigma^{-1} * z)) \text{ for any } \sigma \in G_L\} \ .$$

Recalling that $\tau : G_L \longrightarrow o^\times$ denotes the character which describes the Galois action on T' we compute

$$\begin{aligned} \sigma^{-1} \circ \kappa_z(g) &= \ < \sigma^{-1}(t'_\mathrm{o}), \sigma^{-1}([g](z)) > \ = \ < \tau(\sigma^{-1}) \cdot t'_\mathrm{o}, [g](\sigma^{-1}(z)) > \\ &= \ < t'_\mathrm{o}, [g]([\tau(\sigma^{-1})](\sigma^{-1}(z))) > \ = \ \kappa_{[\tau(\sigma^{-1})](\sigma^{-1}(z))}(g) \end{aligned}$$

for any $g \in G = o$. Hence

$$\sigma^{-1} * z = [\tau(\sigma^{-1})](\sigma^{-1}(z)) \quad \text{and} \quad \sigma(\sigma^{-1} * z) = [\tau(\sigma^{-1})](z) \ .$$

Corollary 3.8:

$\mathcal{O}(\widehat{G}) \cong \{F \in \mathcal{O}(\mathbf{B}/\mathbb{C}_p) : F = F^\sigma \circ [\tau(\sigma^{-1})] \text{ for any } \sigma \in G_L\}$.

The following two negative facts show that our above results cannot be improved much.

Lemma 3.9:

Suppose that K is discretely valued; If $L \neq \mathbb{Q}_p$ then $\widehat{G}/K$ and $\mathbf{B}/K$ are not isomorphic as rigid K-analytic varieties.

Proof: We consider the difference $\delta_1 - \delta_0 \in D(G, K)$ of the Dirac distributions in the elements 1 and 0 in $G = o$, respectively. Since the image of any character in $\widehat{G}(\mathbb{C}_p)$ lies in the 1-units of $o_{\mathbb{C}_p}$ we see that the Fourier transform of $\delta_1 - \delta_0$ as a function on $\widehat{G}(\mathbb{C}_p)$ is bounded. On the other hand every torsion point in $\widehat{G}(\mathbb{C}_p)$ corresponding to a locally constant character on the quotient $o/\mathbb{Z}_p$ is a zero of this function. Since $o \neq \mathbb{Z}_p$ we therefore have a nonzero function in $\mathcal{O}(\widehat{G}/K)$ which is bounded and has infinitely many zeroes. If $\widehat{G}/K$ and $\mathbf{B}/K$ were isomorphic this would imply the existence of a nonzero power series in $\mathcal{O}(\mathbf{B}/K)$ which as a function on the open unit disk in $\mathbb{C}_p$ is bounded and has infinitely many zeroes. By the maximum principle the former means that the coefficients of this power series are bounded in K. But according to the Weierstrass preparation theorem ([B-CA] VII§3.8 Prop. 6) a nonzero bounded power series over a discretely valued field can have at most finitely many zeroes. So we have arrived at a contradiction.

According to Lazard ([Laz]) the ring $\mathcal{O}(\mathbf{B}/K)$, for K spherically complete, is a so called Bezout domain which is the non-noetherian version of a principal ideal domain and which by definition means that any finitely generated ideal is principal. We show that this also fails in our setting as soon as $L \neq \mathbb{Q}_p$.

Lemma 3.10:

Suppose that K is discretely valued; if $L \neq \mathbb{Q}_p$ then the ideal of functions in $\mathcal{O}(\widehat{G}/K)$ vanishing in the trivial character $\kappa_0 \in \widehat{G}(L)$ is finitely generated but not principal.

Proof: The ideal in question is a quotient of the corresponding ideal for the polydisk $\widehat{G_0}/K$ which visibly is finitely generated. Reasoning by contradiction let f be a generator of the ideal in the assertion. As a consequence of Theorems A and B ([Kie] Satz 2.4) for the quasi-Stein variety $\widehat{G}/K$ we then have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\widehat{G}/K} \xrightarrow{f\cdot} \mathcal{O}_{\widehat{G}/K} \longrightarrow K \longrightarrow 0$$

on $\widehat{G}/K$ where the third term is a skyscraper sheaf in the point κ_0. The corresponding sequence of sections in any affinoid subdomain is split-exact and hence remains exact after base extension to $\mathbb{C}_p$. It follows that we have a corresponding exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\widehat{G}/\mathbb{C}_p} \xrightarrow{f\cdot} \mathcal{O}_{\widehat{G}/\mathbb{C}_p} \longrightarrow \mathbb{C}_p \longrightarrow 0$$

on $\widehat{G}/\mathbb{C}_p$. Using Theorem B we deduce from it that f also generates the ideal of functions vanishing in κ_0 in $\mathcal{O}(\widehat{G}/\mathbb{C}_p)$. Consider f now as a rigid map $f : \widehat{G}/K \longrightarrow \mathbb{A}^1/K$ into the affine line. The composite $f \circ \kappa : \mathbf{B}/\mathbb{C}_p \longrightarrow \widehat{G}/\mathbb{C}_p \longrightarrow \mathbb{A}^1/\mathbb{C}_p$ then is given by a power series $F \in \mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ which generates the maximal ideal of functions vanishing in the point 0. Hence F is of the form

$$F(z) = az(1 + b_1 z + b_2 z^2 + \ldots) \quad \text{with } a \in \mathbb{C}_p \text{ and } b_i \in o_{\mathbb{C}_p}$$

and gives an isomorphism

$$\mathbf{B}/\mathbb{C}_p \xrightarrow{\simeq} \mathbf{B}^-(|a|)/\mathbb{C}_p$$

between the open unit disk and the open disk $\mathbf{B}^-(|a|)$ of radius $|a|$ over $\mathbb{C}_p$. We see that f in fact is a rigid map

$$\widehat{G}/K \longrightarrow \mathbf{B}^-(|a|)/K$$

which becomes an isomorphism after base extension to $\mathbb{C}_p$. It follows from the general descent principle we have noted earlier that f induces an isomorphism of rings

$$\mathcal{O}(\mathbf{B}^-(|a|)/K) \xrightarrow{\cong} \mathcal{O}(\widehat{G}/K)$$

which obviously respects bounded functions. This leads to a contradiction by repeating the argument in the proof of the previous lemma. By a more refined descent argument one can in fact show that f already is an isomorphism of rigid K-analytic varieties which is in direct contradiction to Lemma 3.9.

We close this section by remarking that, because $\widehat{G}/K$ is a smooth 1-dimensional quasi-Stein rigid variety, one has, for any K, the following positive results about the integral domain $\mathcal{O}(\widehat{G}/K)$:

1. For ideals I of $\mathcal{O}(\widehat{G}/K)$, the three properties "$I$ is closed", "I is finitely generated", and "I is invertible" are equivalent.

2. The closed ideals in $\mathcal{O}(\widehat{G}/K)$ are in bijection with the divisors of $\widehat{G}/K$. (A divisor is an infinite sum of closed points, having only finite support in any affinoid subdomain). In addition a Baire category theory argument shows that given a divisor D there is a function $F \in \mathcal{O}(\widehat{G}/K)$ whose divisor is of the form $D + D'$ where D and D' have disjoint support.

3. Any finitely generated submodule in a finitely generated free $\mathcal{O}(\widehat{G}/K)$-module is closed.

In particular, $\mathcal{O}(\widehat{G}/K)$ is a Prüfer domain and consequently a coherent ring. We omit the proofs which consist of rather standard applications of Theorems A and B ([Kie]).

4. Generalized Mahler expansions

In this section we apply the Fourier theory to obtain a generalization of the Mahler expansion ([Am1] Cor. 10.2) for locally L-analytic functions. Crucial to our computations is the observation that the power series $F_{t'_o}(Z)$ introduced before Prop. 3.1, which gives the formal group homomorphism $\mathcal{G} \to \mathbb{G}_m$ associated to t'_o, is given as a formal power series by the formula

$$F_{t'_o}(Z) = \exp(\Omega \log_{\mathcal{G}}(Z)) - 1.$$

Throughout the following, we let ∂ denote the invariant differential on the formal group $\mathcal{G}$.

Definition 4.1: *For $m \in \mathbb{N}_0$, let $P_m(Y) \in L[Y]$ be the polynomial defined by the formal power series expansion*

$$\sum_{m=0}^{\infty} P_m(Y) Z^m = \exp(Y \log_{\mathcal{G}}(Z)) \ .$$

Observe that in the case $\mathcal{G} = \mathbb{G}_m$, we have

$$\exp(Y \log(1+Z)) = \sum_{m=0}^{\infty} \binom{Y}{m} Z^m$$

so in that case $P_m(Y) = \binom{Y}{m}$.

Lemma 4.2:

The polynomials $P_m(Y)$ satisfy the following properties:

1. $P_0(Y) = 1$ *and* $P_1(Y) = Y$*;*
2. $P_m(0) = 0$ *for all* $m \geq 1$*;*
3. *the degree of* P_m *is exactly* m*, and the leading coefficient of* P_m *is* $1/m!$*;*
4. $P_m(Y + Y') = \sum_{i+j=m} P_i(Y)P_j(Y')$*;*
5. $P_m(a\Omega) \in o_{\mathbb{C}_p}$ *for all* $a \in o_L$*;*
6. *for* $f(x) \in \mathbb{C}_p[[x]]$*, we have the identity*

$$(P_m(\partial)f(x))|_{x=0} = \frac{1}{m!}\frac{d^m f}{dx^m}|_{x=0} \ .$$

Proof: The first four properties are clear from the definition. The fifth property follows from the fact that, for $a \in o_L$, the power series

$$F_{at'_o}(Z) = F_{t'_o}([a](Z)) = \exp(a\Omega \log_{\mathcal{G}}(Z)) = \sum_{m=0}^{\infty} P_m(a\Omega)Z^m$$

has coefficients in $o_{\mathbb{C}_p}$. For the last property, let $\delta = \frac{d}{dx}$ be the invariant differential on the additive formal group. Then Taylor's formula says that

$$\exp(\delta b)h(a) = \sum_m (\frac{\delta^m}{m!}h(a))b^m = h(a+b) \ .$$

Using the fact that $\log_{\mathcal{G}}$ and $\exp_{\mathcal{G}}$ are inverse isomorphisms between $\mathcal{G}$ and the additive group over L, Taylor's formula for $\mathcal{G}$ can be obtained by making the substitutions

$$\begin{aligned} a &= \log_{\mathcal{G}}(x) \\ b &= \log_{\mathcal{G}}(y) \\ h &= f \circ \exp_{\mathcal{G}} \ . \end{aligned}$$

It follows easily that $\delta h(a) = \partial f(x)$, and Taylor's formula becomes

$$\exp(\partial \log_{\mathcal{G}}(y))f(x) = f(x +_{\mathcal{G}} y) \ .$$

Comparing coefficients after expanding both sides in y and setting $x = 0$ gives the result.

Remark: The identity (6) is part of the theory of Cartier duality, as sketched for example in Section 1 of [Ka2]. Corollary 1.8 of [Ka2] shows that $P_m(\partial)$ is the invariant differential operator called there $D(m)$. Comparing this fact with Formula 1.1 of [Ka2] yields the claim in part (6) for the functions $f(x) = x^n$, and the general fact then follows by linearity.

Our goal now is to study the functions $P_m(Y\Omega)$ as elements of the locally convex vector space $C^{an}(G, \mathbb{C}_p)$, where as always $G = o_L$ as a locally L-analytic group. The Banach space $\mathcal{F}_{a+\pi^n o_L}(K)$, for any complete intermediate field $L \subseteq K \subseteq \mathbb{C}_p$ and any coset $a + \pi^n o_L$ in o_L, is equipped with the norm

$$\|\sum_{i=0}^{\infty} c_i(x-a)^i\|_{a,n} := \max_i\{|c_i \pi^{ni}|\} \ .$$

This Banach space is the same as the Tate algebra of K-valued rigid analytic functions on the disk $a + \pi^n o_{\mathbb{C}_p}$, and the norm, by the maximum principle, has the alternative definition

$$\|f\|_{a,n} = \max_{x \in a+\pi^n o_{\mathbb{C}_p}} |f(x)|$$

which is sometimes more convenient for computation. We recall that

$$\mathcal{F}_n(o_L, K) = \prod_{a \bmod \pi^n o_L} \mathcal{F}_{a+\pi^n o_L}(K) \quad \text{and} \quad C^{an}(G, K) = \varinjlim_n \mathcal{F}_n(o_L, K) \ .$$

Lemma 4.3:

For all $a \in o_L$ and all $m \geq 0$, we have

$$\|P_m(Y\Omega)\|_{a,n} \leq \max_{0 \leq i \leq m} \|P_i(Y\Omega)\|_{0,n} \ .$$

Proof: By Property (4) of Lemma 4.2, we have

$$P_m((a+\pi^n x)\Omega) = \sum_{i+j=m} P_i(a\Omega) P_j(\pi^n \Omega x) \ .$$

Therefore, using Property (5) of Lemma 4.2, we have

$$\begin{aligned}\|P_m(Y\Omega)\|_{a,n} &= \max\nolimits_{z \in a+\pi^n o_{\mathbb{C}_p}} |P_m(z\Omega)| \\ &\leq \max\nolimits_{0 \leq i \leq m} \max\nolimits_{x \in o_{\mathbb{C}_p}} |P_i(\pi^n \Omega x)| \\ &= \max\nolimits_{0 \leq i \leq m} \|P_i(Y\Omega)\|_{0,n} \ .\end{aligned}$$

Lemma 4.4:

The following estimate holds for $P_m(Y\Omega)$ and $m \geq 1$:

$$||P_m(Y\Omega)||_{0,n} < p^{-1/(p-1)} p^{\frac{m}{eq^{n-1}(q-1)}}.$$

Proof: Let $r = p^{-q/e(q-1)}$. As shown in [Lan] §8.6 Lemma 4, and as we have used earlier, the functions $\log_{\mathcal{G}}$ and $\exp_{\mathcal{G}}$ are inverse isomorphisms from $\mathbf{B}(r)$ to itself, and $\log_{\mathcal{G}}$ is rigid analytic on all of $\mathbf{B}$. Furthermore by Lemma 3.2, $[\pi^n]^{-1}\mathbf{B}(r) = \mathbf{B}(r^{1/q^n})$ for any $n \in \mathbb{N}$. It follows that

$$\| \log_{\mathcal{G}}(x)\|_{\mathbf{B}(r^{1/q^n})} = rp^{n/e}$$

where $\|f\|_{\mathcal{X}}$ denotes the spectral semi-norm on an affinoid $\mathcal{X}$. The function $H(x,y) = y\Omega \log_{\mathcal{G}}(x)$ is a rigid function of two variables on the affinoid domain $\mathbf{B}(r^{1/q^n}) \times \mathbf{B}(p^{-n/e})$ satisfying

$$\begin{aligned} \|H(x,y)\|_{\mathbf{B}(r^{1/q^n})\times\mathbf{B}(p^{-n/e})} &\leq p^{-n/e} p^{-1/(p-1)+1/e(q-1)} rp^{n/e} \\ &= p^{-1/(p-1)-1/e} < p^{-1/(p-1)} . \end{aligned}$$

We conclude from this that $\exp(H(x,y))$ is rigid analytic on $\mathbf{B}(r^{1/q^n})\times\mathbf{B}(p^{-n/e})$ and that

$$\| \exp(H(x,y)) - 1\|_{\mathbf{B}(r^{1/q^n})\times\mathbf{B}(p^{-n/e})} < p^{-1/(p-1)}.$$

The power series expansion of $\exp(H(x,y))$ is

$$\exp(H(x,y)) = \sum_{m=0}^{\infty} P_m(y\Omega)x^m,$$

and so we conclude that, for all $m \geq 1$ and for all $y \in \mathbf{B}(p^{-n/e})$, we have

$$|P_m(y\Omega)| r^{m/q^n} < p^{-1/(p-1)}.$$

Therefore we obtain

$$\|P_m(Y\Omega)\|_{0,n} < p^{-1/(p-1)} p^{\frac{m}{eq^{n-1}(q-1)}}$$

as desired.

From these lemmas we may deduce the following proposition.

Proposition 4.5:

Given a sequence $\{c_m\}_{m\geq 0}$ *of elements of* $\mathbb{C}_p$*, the series*

$$\sum_{m=0}^{\infty} c_m P_m(y\Omega)$$

converges to an element of $\mathcal{F}_n(o_L, \mathbb{C}_p)$ *provided that* $|c_m|p^{m/eq^{n-1}(q-1)} \to 0$ *as* $m \to \infty$*. More generally, this series converges to an element of* $C^{an}(G, \mathbb{C}_p)$ *provided that there exists a real number* r*, with* $r > 1$*, such that* $|c_m|r^m \to 0$ *as* $m \to \infty$*.*

Theorems 2.3 and 3.6 together imply the existence of a pairing

$$\{\ ,\ \} : \mathcal{O}(\mathbf{B}/\mathbb{C}_p) \times C^{an}(G, \mathbb{C}_p) \to \mathbb{C}_p$$

that identifies $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ and the continuous dual of $C^{an}(G, \mathbb{C}_p)$, both equipped with their projective limit topologies. The following lemma gives some basic computational formulae for this pairing; we will use some of these in the proof of our main theorem in this section.

Lemma 4.6:

The following formulae hold for the pairing $\{\ ,\ \}$*, given* $F \in \mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ *and* $f \in C^{an}(G, \mathbb{C}_p)$*:*

1. $\{1, f\} = f(0)$;
2. $\{F_{at'_o}, f\} = f(a) - f(0)$ *for* $a \in o_L$;
3. $\{F, \kappa_z\} = F(z)$ *for* $z \in \mathbf{B}$;
4. $\{F_{at'_o}F, f\} = \{F, f(a + .) - f\}$ *for* $a \in o_L$;
5. $\{F, \kappa_z f\} = \{F(z +_{\mathcal{G}} .), f\}$ *for* $z \in \mathbf{B}$;
6. $\{F, f(a.)\} = \{F \circ [a], f\}$ *for* $a \in o_L$;
7. $\{F, f'\} = \{\Omega \log_{\mathcal{G}} \cdot F, f\}$;
8. $\{F, xf(x)\} = \{\Omega^{-1}\partial F, f\}$;
9. $\{F, P_m(.\Omega)\} = (1/m!)\frac{d^m F}{dZ^m}(0)$.

Proof: These properties follow from the definition of the Fourier transform and from the density of the subspace generated by the characters in $C^{an}(G, \mathbb{C}_p)$. For example to see property (5): if $\lambda'(f) := \lambda(\kappa_z f)$, then

$$F_{\lambda'}(z') = \lambda(\kappa_z \kappa_{z'}) = \lambda(\kappa_{z+_{\mathcal{G}}z'}) = F_\lambda(z +_{\mathcal{G}} z')\ .$$

For property (7), using (4) we have

$$\begin{aligned}\{F, f'\} &= \lim_{\epsilon \to 0} \epsilon^{-1}\{F, f(. + \epsilon) - f\} \\ &= \lim_{\epsilon \to 0} \epsilon^{-1}\{F_{\epsilon t'_o} F, f\} \\ &= \{\Omega \log_{\mathcal{G}} \cdot F, f\}\end{aligned}$$

using continuity and the fact that $F_{\epsilon t'_o} = \exp(\epsilon\Omega \log_{\mathcal{G}}) - 1$ for small ϵ. An analogous computation based on (5) gives (8). The last property (9) follows from Lemma 4.2.6 and (8).

We may now prove the main result of this section.

Theorem 4.7:

Any function $f \in C^{an}(G, \mathbb{C}_p)$ has a unique representation in the form

$$f = \sum_{m=0}^{\infty} c_m P_m(.\Omega)$$

as in Prop. 4.5; in this representation, $c_m = \{Z^m, f\}$.

Proof: Part (9) of Lemma 4.6, along with continuity, shows that if f has a representation in the given form then $c_m = \{Z^m, f\}$. The functions Z^m generate a dense subspace in $\mathcal{O}(\widehat{G}/\mathbb{C}_p)$, and so a function f with all $c_m = 0$ must be zero (for example all Dirac distributions pair to zero against f). This implies that this type of representation, if it exists, is unique. Suppose we show that, for any $f \in C^{an}(G, \mathbb{C}_p)$, there exists an $r > 1$ such that $|\{Z^m, f\}|\, r^m \to 0$ as $m \to \infty$. Then by Prop. 4.5, the series

$$\overline{f}(x) := \sum_{m=0}^{\infty} \{Z^m, f\} P_m(x\Omega)$$

converges to a locally analytic function and by Lemma 4.6, Part 9, we have $\{Z^m, f\} = \{Z^m, \overline{f}\}$ for all m. Therefore $\overline{f} = f$.

Thus we have reduced our main theorem to the claim that $|\{Z^m, f\}|\, r^m \to 0$ as $m \to \infty$ for some $r > 1$. The function f being locally analytic it belongs to one of the Banach spaces $\mathcal{F}_n(o_L, \mathbb{C}_p)$. Using the topological isomorphism between the Fréchet spaces $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ and $D(G, \mathbb{C}_p) = C^{an}(G, \mathbb{C}_p)'$, there is a rational number $s > 0$ such that the map $\mathcal{O}(\mathbf{B}/\mathbb{C}_p) \to \mathcal{F}_n(o_L, \mathbb{C}_p)'$ factors through the Tate algebra $\mathcal{O}(\mathbf{B}(p^{-s})/\mathbb{C}_p)$. If we choose another rational number s' so that $0 < s' < s$, then in the Tate algebra $\mathcal{O}(\mathbf{B}(p^{-s})/\mathbb{C}_p)$, the set of rigid functions $\{(Z/p^{-s'})^m\}_{m \geq 0}$ goes to zero and therefore so does $|\{Z^m, f\}|\, p^{s'm}$. This proves the existence of the desired expansion.

Remark: In [Ka1], Katz discusses what he calls "Gal-continuous" functions. These are continuous functions on G that satisfy (in our notation) $\sigma(f(x)) = f(\tau(\sigma)x)$ for all $\sigma \in G_L$. If $\{c_m\}$ is a sequence of elements of L such that $|c_m| \to 0$, then $f(x) := \sum_{m=0}^{\infty} c_m P_m(x\Omega)$ is continuous by Part (5) of Lemma 4.2, and by the Galois properties of Ω it is even Gal-continuous.

5. p-adic L-functions

In this section we will illustrate how the integration theory developed in this paper applies to yield p-adic L-functions for CM elliptic curves E at supersingular primes. In fact, our method allows us to apply the Coleman power series approach described in [dS] directly in the supersingular case. We will content ourselves with proving a supersingular analogue of a weak version of Theorem II.4.11 of [dS]; this should demonstrate sufficiently the nature of our construction, without requiring too much of a diversion into global arithmetic.

We should emphasize that the L-functions we will construct in the supersingular case come from locally analytic distributions on Galois groups, not measures. A character on the Galois group is integrable provided that its restriction to a small open subgroup is a power of $\overline{\varphi}$, where $\overline{\varphi}$ gives the representation on the dual Tate module of E. We therefore cannot make any immediate connection to Iwasawa module structure of, for example, elliptic units. This is an interesting problem for the future.

Our results are closely related to those of Boxall ([Box]). See the Remark after Prop. 5.2 for more discussion of the relationship.

Before discussing p-adic L-functions we will develop Fourier theory for the multiplicative group; this will be useful because the p-adic L-functions we construct arise as locally analytic distributions on Galois groups that are naturally isomorphic to multiplicative, rather than additive groups. Let H be $o_L^\times$ as L-analytic group and let H_1 be the subgroup $1 + \pi o_L$. Using the Teichmüller character ω, we have

$$H = H_1 \times k^\times$$

where k is the residue field of o_L. For $x \in H$, let $< x >$ be the projection of x to H_1.

As always, G is the additive group o_L. Let us assume that the absolute ramification index e of the field L satisfies $e < p - 1$. We define $\ell := \pi^{-1} \cdot \log$ so that $\ell : H_1 \xrightarrow{\cong} G$ is an L-analytic isomorphism.

This map induces an isomorphism between the distribution algebras $D(H_1, K)$ and $D(G, K)$. The group $\widehat{H}(\mathbb{C}_p)$ of locally L-analytic, $\mathbb{C}_p$-valued characters of H is isomorphic to a product of $q-1$ copies of the open unit disk $\mathbf{B}$ using the results of section 3, indexed by the (finite) character group of $k^\times$. For $z \in \mathbf{B}(\mathbb{C}_p)$, let ψ_z

be the corresponding character of H_1. Then for any distribution $\lambda \in D(H,K)$, and any character $\omega^i\psi_z$ with $z \in \mathbf{B}(\mathbb{C}_p)$, and $0 \le i \le q-1$, we have the "Mellin transform"

$$M_\lambda(z,\omega^i) = \lambda(\omega^i\psi_z) \ .$$

For each fixed value of the second variable, M_λ is a rigid function in $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$.

Now let us compare the Fourier transforms for G and H in a different way. The group $o_L^\times$, as an L-analytic manifold, is an open submanifold of o_L. If we have a distribution λ in $D(G,K)$ that vanishes on functions with support in πo_L, then λ gives a distribution on $H = o_L^\times \subset o_L$. It follows easily from Lemma 4.6.5 that λ is supported on H precisely when its Fourier transform F_λ satisfies

$$\sum_{[\pi](z)=0} F_\lambda(. +_{\mathcal{G}} z) = 0 \ .$$

We have the following result comparing the Fourier and Mellin transforms.

Proposition 5.1:

Let λ be a distribution in $D(G,\mathbb{C}_p)$ supported on H, let F_λ be its Fourier transform, and let M_λ be its Mellin transform; suppose that $n \in \mathbb{N}$ satisfies $n \equiv i$ (mod $q-1$); then

$$M_\lambda(\exp_{\mathcal{G}}(n\pi/\Omega),\omega^i) = \int_{o_L^\times} x^n d\lambda(x) = \Omega^{-n}(\partial^n F_\lambda(z)|_{z=0}) \ .$$

Note that the hypothesis $e < p-1$ guarantees that $M_\lambda(\exp_{\mathcal{G}}(x\pi/\Omega),\omega^i)$ is a (globally) analytic function of $x \in o_L$. Thus the left hand side of these equations gives a (globally) L-analytic interpolation of the values on the right side.

Proof: Let $z(n) = \exp_{\mathcal{G}}(n\pi/\Omega)$. By definition,

$$M_\lambda(z(n),\omega^i) = \lambda(\omega^i\psi_{z(n)}) \ .$$

Now

$$\begin{aligned} \psi_{z(n)}(< x >) &= \kappa_{z(n)}(\ell(< x >)) \\ &= t'_o([\ell(< x >)](z(n))) \\ &= \exp(\Omega\ell(< x >)\log_{\mathcal{G}}(z(n))) \end{aligned}$$

because $|[\ell(< x >)](z(n))| < p^{-1/e(q-1)}$ (by Lemma 3.4.b and the hypothesis $e < p-1$). But

$$\exp(\Omega\ell(< x >)\log_{\mathcal{G}}(z(n))) = \exp(n\log(< x >)) =< x >^n$$

so $(\omega^i \psi_{z(n)})(x) = x^n$. But

$$\lambda(x \mapsto x^n) = \Omega^{-n}(\partial^n F_\lambda(z)|_{z=0})$$

by Lemma 4.6.8/9.

Now we will embark on a digression into the theory of CM elliptic curves, following the notation and the logic of Chap. II in [dS]. Let $\mathbf{K}$ be an imaginary quadratic field, and let $\mathfrak{f}$ be an integral ideal of $\mathbf{K}$ such that the roots of unity in $\mathbf{K}$ are distinct mod $\mathfrak{f}$. Let p be a rational prime that is relatively prime to $6\mathfrak{f}$ and inert in $\mathbf{K}$. Let $\mathbf{F}$ be the ray class field $\mathbf{K}(\mathfrak{f})$ and let $\mathbf{F}_n := \mathbf{K}(p^n\mathfrak{f})$ and $\mathbf{F}_\infty := \bigcup_{n\in\mathbb{N}} \mathbf{F}_n$. Assume for technical reasons that will become clear in a moment that p as a prime of $\mathbf{K}$ *splits completely in* $\mathbf{F}$. Let $\wp$ be a prime above p in $\mathbf{F}$. The prime $\wp$ ramifies totally in $\mathbf{F}_\infty$; let F_∞ be the completion of $\mathbf{F}_\infty$ at the unique prime above $\wp$. Let o be the ring of integers in the local field $\mathbf{K}_p$.

Fix an elliptic curve E over $\mathbf{F}$ with CM by the ring of integers in $\mathbf{K}$ and with associated Hecke character of the form $\psi_{E/\mathbf{F}} = \varphi \circ N_{\mathbf{F}/\mathbf{K}}$, where φ is a Hecke character of $\mathbf{K}$ of type $(1,0)$ and conductor dividing $\mathfrak{f}$; we moreover assume that there is a complex period Ω_∞ so that the period lattice $\mathcal{L}$ of E is $\Omega_\infty\mathfrak{f}$. We view φ also as a character of $\Gamma_{\mathbf{K}} := Gal(\mathbf{F}_\infty/\mathbf{K})$; it is $\mathbf{K}_p^\times$-valued on the subgroup $\Gamma_{\mathbf{F}} = Gal(\mathbf{F}_\infty/\mathbf{F})$. If $\mathfrak{a}$ is an integral ideal of $\mathbf{K}$ such that the Artin symbol $\sigma_{\mathfrak{a}}$ belongs to $\Gamma_{\mathbf{F}}$, then $\sigma_{\mathfrak{a}}$ acts on the p-adic Tate module of E through multiplication by $\varphi(\mathfrak{a})$. We let $\overline{\varphi}$ be the Hecke character giving the action of $\Gamma_{\mathbf{F}}$ on the dual Tate module of E. The character φ gives us an isomorphism

$$\varphi : \Gamma_{\mathbf{F}} \to o^\times.$$

We use this isomorphism to equip $\Gamma_{\mathbf{F}}$ with an o-analytic structure. We let $\mathbf{N}$ denote the absolute norm.

Our assumption that p splits completely in $\mathbf{F}$ means that that the formal group $\widehat{E}_\wp$ of E at $\wp$ is a Lubin-Tate group over o of height two. (To handle general p, deShalit works with what he calls "relative" Lubin-Tate groups. Presumably one can generalize our results to this situation as well.) Furthermore, the field $\mathbf{F}_\infty$ contains all of the p-power torsion points of this formal group, as well as (by the Weil pairing) all of the p-power roots of unity. Thus our uniformization result holds over this field. Choose an o-generator t'_o of the (global) dual Tate module $\mathrm{Hom}(T_p(E), T_p(\mathbb{G}_m))$ defined over $\mathbf{F}_\infty$. Then the pairing $\{\ ,\ \}$ from section 4 looks like:

$$\mathcal{O}(\widehat{E}_\wp/F_\infty) \times C^{an}(o, F_\infty) \to F_\infty\ .$$

Now we introduce the machinery of Coleman power series and elliptic units. Let $\mathfrak{a}$ be an integral ideal relatively prime to $p\mathfrak{f}$ and let $\Theta(y; \mathcal{L}, \mathfrak{a})$ be the elliptic function from [dS] II.2.3 (10). Let $P(z)$ be the Taylor expansion of $\Theta(\Omega_\infty -$

$z; \mathcal{L}, \mathfrak{a})$ and let $Q_{\mathfrak{a}}(Z) := P(\log_{\widehat{E}_\wp}(Z))$. This power series belongs to $o[[Z]]$, as shown in [dS] Prop. II.4.9; note that this proposition is true for inert primes as well as split ones, as is clear from its proof. The power series $Q_{\mathfrak{a}}(Z)$ is the Coleman power series associated to a norm-compatible sequence of elliptic units, as deShalit explains.

Define

$$g_{\mathfrak{a}}(Z) = \log Q_{\mathfrak{a}}(Z) - \frac{1}{p^2} \sum_{\substack{z \in \widehat{E}_\wp \\ [p](z)=0}} \log Q_{\mathfrak{a}}(Z +_{\widehat{E}_\wp} z) \ .$$

Proposition 5.2:

The power series $g_{\mathfrak{a}}(Z) \in \mathcal{O}(\widehat{E}_\wp / F_\infty)$ is the Fourier transform of an F_∞-valued, locally analytic distribution on o supported on $o^\times$. By means of the isomorphism φ from $\Gamma_{\mathbf{F}}$ to $o^\times$, it defines a locally analytic distribution on $\Gamma_{\mathbf{F}}$ with the interpolation property

$$\Omega^m \{g_{\mathfrak{a}}(Z), \varphi^m\}$$

$$= -12(1 - \varphi(p)^m p^{-2}) \Omega_\infty^{-m} (m-1)! (\mathbf{N}(\mathfrak{a}) L_{\mathfrak{f}}(\overline{\varphi}^m, m, 1) - \varphi(\mathfrak{a})^m L_{\mathfrak{f}}(\overline{\varphi}^m, m, \mathfrak{a}))$$

for any $m \in \mathbb{N}$. Here $L_{\mathfrak{f}}(\overline{\varphi}, s, \mathfrak{c})$ denotes the "partial" Hecke L-function of conductor $\mathfrak{f}$, equal to $\sum_{\mathfrak{b}} \overline{\varphi}(\mathfrak{b}) \mathbf{N}(\mathfrak{b})^{-s}$ over ideals $\mathfrak{b}$ prime to $\mathfrak{f}$ and such that $(\mathfrak{b}, \mathbf{F}/\mathbf{K}) = (\mathfrak{c}, \mathbf{F}/\mathbf{K})$.

Proof: The first assertion follows easily from the formulae in Lemma 4.6. The interpolation property comes from the formula (Lemma 4.6 again):

$$\{g_{\mathfrak{a}}(Z), \varphi^m\} = \Omega^{-m} \partial^m g_{\mathfrak{a}}(Z)|_{Z=0} \ .$$

The rest of the computation is just a version of [dS] II.4.10. The invariant differential ∂ pulls back to d/dy on the complex uniformization of E, so

$$\Omega^m \{g_{\mathfrak{a}}(Z), \varphi^m\} = \left(\frac{d}{dy}\right)^m (\log \Theta(\Omega_\infty - y; \mathcal{L}, \mathfrak{a}) - p^{-2} \log \Theta(\Omega_\infty - y; p^{-1}\mathcal{L}, \mathfrak{a}))|_{y=0}$$

and the claimed formula then follows from the equivalent in our situation of [dS] II.4.7 (17), along with II.3.1 (7) and Prop. II.3.5.

Remark: From Prop. 5.1, we see that the Mellin transform $M_{\mathfrak{a}}$ of the distribution $\{g_{\mathfrak{a}}(Z), \cdot\}$ is an o-analytic function on o interpolating the special values $\Omega^{-m} \partial^m g_{\mathfrak{a}}(Z)|_{Z=0}$. This function (on $\mathbb{Z}_p$) was constructed by Boxall ([Box]). Other than the fact that our construction is arguably more natural, the principal new results here are that our function is $\mathbf{K}_p$-analytic on o rather than

$\mathbb{Q}_p$-analytic on $\mathbb{Z}_p$. The existence of such an analytic interpolating function implies congruences among the special values.

We may give the slightly larger Galois group $\Gamma_{\mathbf{K}}$ a locally analytic structure by transporting that of $\Gamma_{\mathbf{F}}$ to its finitely many cosets in $\Gamma_{\mathbf{K}}$. To extend the integration pairing to $\Gamma_{\mathbf{K}}$, recall that, along with E we have finitely many other elliptic curves E^σ as σ runs through $Gal(\mathbf{F}/\mathbf{K})$. We also have, for each $\sigma \in \Gamma_{\mathbf{K}}$, an isogeny $\iota(\sigma) : E \to E^\sigma$ as in [dS] Prop. II.1.5. If $\mathfrak{a}$ is an ideal prime to p, then the associated isogeny $\iota(\sigma_{\mathfrak{a}})$ has degree $\mathbf{N}(\mathfrak{a})$, which is prime to p and therefore induces an isomorphism between the formal groups $\widehat{E}_\wp$ and $\widehat{E^{\sigma_{\mathfrak{a}}}}_\wp$.

A typical locally analytic function $\overline{f}$ on $\Gamma_{\mathbf{K}}$ may be written

$$\overline{f}(\sigma) = f_i(\varphi(\sigma_i^{-1}\sigma)) \quad \text{when } \sigma \in \sigma_i\Gamma_{\mathbf{F}}$$

where $\mathfrak{c}_i$ is a collection of integral ideals of $\mathbf{K}$ so that the Artin symbols $\sigma_i = \sigma_{\mathfrak{c}_i}$ form the set $Gal(\mathbf{F}/\mathbf{K})$, and $f_i \in C^{an}(o, F_\infty)$, supported on $o^\times$.

Let $\mathcal{O}(\widehat{E}_\wp/F_\infty)^0$ denote the subspace of functions $F \in \mathcal{O}(\widehat{E}_\wp/F_\infty)$ satisfying $\sum_{[p](z)=0} F(. +_{\widehat{E}_\wp} z) = 0$. These are the distributions supported on $o^\times$. We define an extended integration pairing

$$\{\ ,\ \} : \bigoplus_{\sigma_{\mathfrak{a}} \in Gal(\mathbf{F}/\mathbf{K})} \mathcal{O}(\widehat{E^{\sigma_{\mathfrak{a}}}}_\wp/F_\infty)^0 \times C^{an}(\Gamma_{\mathbf{K}}, F_\infty) \longrightarrow F_\infty$$

by setting

$$\{\overline{h}, \overline{f}\} := \sum_i \{h_i \circ \iota(\sigma_i), f_i\} \ .$$

Lemma 5.3:

This pairing is well-defined (i.e., it is independent of the choice of coset representatives), and identifies the left hand space with the continuous dual of the right hand space.

Proof: The duality is clear; the key point is that the pairing is well defined. Suppose we replace σ_i with $\sigma_i\tau_{\mathfrak{b}}$, where $\tau_{\mathfrak{b}} \in \Gamma_{\mathbf{F}}$. Then $h_i \circ \iota(\sigma_i\tau_{\mathfrak{b}}) = h_i \circ \iota(\sigma_i) \circ [\varphi(\tau_{\mathfrak{b}})]$ (see [dS] II.4.5). The decomposition of $\overline{f}$ also changes, with f_i replaced by $f_i' = f_i(\varphi(\tau_{\mathfrak{b}})^{-1}.)$. Then, using Lemma 4.6 as usual, the pairing satisfies

$$\{h_i \circ \iota(\sigma_i) \circ [\varphi(\tau_{\mathfrak{b}})], f_i(\varphi(\tau_{\mathfrak{b}})^{-1}.)\} = \{h_i \circ \iota(\sigma_i), f_i\} \ .$$

Theorem 5.4: *(Compare [dS] Thm. II.4.11)*

Let $\overline{h} = \{h_i\}$ *where* $h_i := \sigma_i(g_{\mathfrak{a}})$ *with* $g_{\mathfrak{a}}$ *the (formal) elliptic function over* $\mathbf{F}$ *used for the construction of the partial L-function in Prop. 5.2. Let* ϵ *be any locally analytic character on* $\Gamma_{\mathbf{K}}$*, whose restriction to* $\Gamma_{\mathbf{F}}$ *is* φ^m *for some* $m \in \mathbb{N}$*. Then the locally analytic distribution* $\overline{h}$ *on* $\Gamma_{\mathbf{K}}$ *has the interpolation property*

$$\Omega^m\{\overline{h}, \epsilon\} = 12(m-1)!\,\Omega_\infty^{-m}(1-\epsilon(p)p^{-2})(\epsilon(\mathfrak{a}) - \mathbf{N}(\mathfrak{a}))L_{\mathfrak{f}}(\epsilon^{-1}, 0)\ .$$

Proof: The proof is a long computation very much in the spirit of [dS] Thm. II.4.11 (though we have cheated in the statement of the Theorem and avoided the case where p divides the conductor). Choose coset representatives $\sigma_i = \sigma_{\mathfrak{c}_i}$. The point of the computation is that ([dS] II.2.4 (ii) and II.4.5 (iv))

$$\sigma_i(g_{\mathfrak{a}}) \circ \iota(\sigma_i) = g_{\mathfrak{a}\mathfrak{c}_i} - \mathbf{N}(\mathfrak{a})g_{\mathfrak{c}_i}$$

and

$$f_i(a) = \epsilon(\sigma_i)a^m\ .$$

Then the partial terms in the pairing are

$$\epsilon(\sigma_i)\Omega^{-m}(\partial^m g_{\mathfrak{a}\mathfrak{c}_i} - \mathbf{N}(\mathfrak{a})\partial^m g_{\mathfrak{c}_i})\ .$$

These terms may then be evaluated using Prop. 5.2, and when the results are combined one obtains the statement of the theorem.

Remark: One can compute an interpolation result for more general locally analytic characters ϵ – explicitly, characters which restrict to φ^m on an open subgroup of $\Gamma_{\mathbf{F}}$ – by following the same line of argument as in [dS] II.4.11.

Appendix. p-adic periods of Lubin-Tate groups

In the analysis in section 3 of the behavior of the isomorphism ($\diamond\diamond$) relative to the affinoid coverings on $\widehat{G}$ and on $\mathbf{B}$ we needed rather exact information about the "period" Ω of the Lubin-Tate group $\mathcal{G}$. In this appendix, we apply results of Fontaine [Fon] to obtain this information.

All of the significant ideas in this section come from the article [Fon], and we follow the notation of that article with the following exceptions. We will use the letter X for the module of differentials $\Omega_{o_L}(o_{\overline{L}})$ (called Ω by Fontaine). We also do not distinguish between $\mathcal{G}$ as a formal group or p-divisible group thanks to [Tat] Prop. 1.

As before, we let $\mathcal{G}$ be the Lubin-Tate group over o associated to the uniformizer π and let $\mathcal{G}'$ be the dual p-divisible group. We denote by q and e respectively the number of elements in $o/\pi o$ and the ramification index of $L/\mathbb{Q}_p$. Furthermore, T and T' are the Tate modules of $\mathcal{G}$ and $\mathcal{G}'$ respectively, and $F_{t'}(Z) \in Zo_{\mathbb{C}_p}[[Z]]$ is the power series corresponding to $t' \in T'$ as in section 3. We let ω be the invariant differential on $\mathcal{G}$ such that $\omega = (1 + \ldots)dZ$. We define $\Omega_{t'}$ so that $F_{t'}(Z) = \Omega_{t'} Z + \cdots$.

We write $\mathcal{G}_n$ and $\mathcal{G}'_n$ for the group schemes of p^n torsion points on $\mathcal{G}$ and $\mathcal{G}'$ respectively. Let L_n be the finite extension field of L generated by the $\overline{L}$-points of $\mathcal{G}'_n$.

The various maps that are denoted by decorated forms of the letter ϕ are those defined in [Fon].

We remark that, in the case that $e \leq p-1$, the result of part (c) of the following Theorem was obtained by Boxall ([Box]) by power series computations.

Theorem:

a. For $t' \in T'$, we have

$$\phi^0_{\mathcal{G}'}(t') = \frac{dF_{t'}}{1 + F_{t'}} = \Omega_{t'}\omega \ ;$$

b. there is a sequence of elements $\Omega_{t'}(n) \in o_{L_n}$ for integers $n \geq 1$ such that $\Omega_{t'}(n+1) \equiv \Omega_{t'}(n) \pmod{p^n o_{L_{n+1}}}$ and such that the sequence of $\Omega_{t'}(n)$ converges in $\mathbb{C}_p$ to $\Omega_{t'}$;

c. let t'_o be any generator of the o module T' ; then the fundamental period $\Omega = \Omega_{t'_o}$ satisfies

$$|\Omega| = p^{-s}$$

where

$$s = \frac{1}{p-1} - \frac{1}{e(q-1)} \ .$$

Proof: We begin with parts (a) and (b). First of all, $F_{t'}(Z)$ being a formal group homomorphism from $\mathcal{G}$ to $\mathbb{G}_m$, the pullback of the invariant differential $dZ/(1+Z)$ on $\mathbb{G}_m$ must be a multiple of ω. Comparing leading coefficients shows that $dF_{t'}(Z)/(1 + F_{t'}(Z)) = \Omega_{t'}\omega$. Fontaine's map

$$\phi^0_{\mathcal{G}'} : o_{\mathbb{C}_p} \otimes_{\mathbb{Z}_p} T' \to t^*_{\mathcal{G}}(o_{\mathbb{C}_p})$$

as defined on p. 406 of [Fon], is the limit of maps

$$\phi^0_{\mathcal{G}'_n} : o_{\overline{L}} \otimes \mathcal{G}'_n(o_{\overline{L}}) \to t^*_{\mathcal{G}_n}(o_{\overline{L}})$$

defined on p. 396. To compute these maps, recall that the affine algebra of $\mathcal{G}_n$ is $R_n = o_L[[Z]]/J_n$ where J_n is the ideal generated by $[p^n](Z)$. The element t' is represented explicitly as $(t'(n))_n$ where the $t'(n)$ are a compatible sequence of homomorphisms $\mathcal{G}_n \to \mu_{p^n}$ over $o_{\mathbb{C}_p}$. Further, the element $t'(n)$ is given explicitly by the class $1 + F_{t'}(Z) + J_n$ in $o_{\mathbb{C}_p} \otimes R_n$. But each homomorphism $\mathcal{G}_n \to \mu_{p^n}$ is defined over o_{L_n}. Therefore $1 + F_{t'}(Z) \equiv g_n(Z) \pmod{J_n}$ for some $g_n \in o_{L_n} \otimes R_n$. Now on the one hand Fontaine's map is given by the formula

$$\phi^0_{\mathcal{G}'_n}(t'(n)) = \frac{dg_n}{g_n} \in t^*_{\mathcal{G}_n}(o_{L_n}) \ .$$

By Prop. 10 of [Fon] we know that

$$t^*_{\mathcal{G}_n}(o_{L_n}) = t^*_{\mathcal{G}}(o_{L_n})/p^n t^*_{\mathcal{G}}(o_{L_n}) \ .$$

Therefore we may choose $\Omega_{t'}(n) \in o_{L_n}$ so that

$$dg_n/g_n \equiv \Omega_{t'}(n)\omega \pmod{p^n t^*_{\mathcal{G}}(o_{L_n})} \ .$$

But both g_n and $1 + F_{t'}$ represent the same map over $o_{\mathbb{C}_p}$ from $\mathcal{G}_n$ to μ_{p^n}, and therefore we must have

$$\Omega_{t'}(n) \equiv \Omega_{t'} \pmod{p^n t^*_{\mathcal{G}}(o_{\mathbb{C}_p})} \ .$$

By definition $\phi^0_{\mathcal{G}'}(t')$ is the limit of the $\Omega_{t'}(n)\omega$, which we have just shown is $\Omega_{t'}\omega$.

Now consider the following commutative diagram, obtained from Prop. 8 of [Fon] by applying section 5.9 to pass to the inverse limit over multiplication by p :

$$\begin{array}{ccc}
o_{\mathbb{C}_p} \otimes_{\mathbf{Z}_p} T \ \times \ o_{\mathbb{C}_p} \otimes_{\mathbf{Z}_p} T' & \xrightarrow{\ \phi\ } & t^*_{\mathcal{G}'}(o_{\mathbb{C}_p}) \oplus t_{\mathcal{G}}(T_p(X)) \ \times \ t^*_{\mathcal{G}}(o_{\mathbb{C}_p}) \oplus t_{\mathcal{G}'}(T_p(X)) \\
\downarrow \theta & & \downarrow \nu \\
o_{\mathbb{C}_p} \otimes T_p(\mathbb{G}_m) & \xrightarrow{\qquad \xi \qquad} & T_p(X)
\end{array}$$

Here the map ϕ is

$$\phi = \phi_{\mathcal{G}, o_{\mathbb{C}_p}} \times \phi_{\mathcal{G}', o_{\mathbb{C}_p}}$$

as defined in Prop. 11 of [Fon], where it is shown to be injective, and to induce an isomorphism upon tensoring with $\mathbb{C}_p$. The vertical arrows are the natural pairings, and the lower horizontal arrow is induced by the map ξ of Thm. 1' of [Fon].

Each of the spaces $\mathbb{C}_p \otimes_{\mathbf{Z}_p} T$ and $\mathbb{C}_p \otimes_{\mathbf{Z}_p} T'$ decompose into a direct sum of one-dimensional eigenspaces corresponding to distinct embeddings of $L \hookrightarrow \mathbb{C}_p$. The map ϕ is o-linear and therefore must respect this decomposition. On the upper right, the o-actions on the spaces $t^*_{\mathcal{G}}(o_{\mathbb{C}_p})$ and $t_{\mathcal{G}}(T_p(X))$ are given by the given embedding $o \subseteq o_{\mathbb{C}_p}$ (while the o-actions on the corresponding spaces for $\mathcal{G}'$ are given by the other $[L : \mathbb{Q}_p] - 1$ embeddings of o in $o_{\mathbb{C}_p}$). Therefore the above diagram can be "reduced" to the following:

$$\begin{array}{ccc}
(o_{\mathbb{C}_p} \otimes_o T) \times \mathrm{Hom}_o(T, o_{\mathbb{C}_p}(1)) & \xrightarrow{\ \phi\ } & t_{\mathcal{G}}(T_p(X)) \times t^*_{\mathcal{G}}(o_{\mathbb{C}_p}) \\
\Big\downarrow \theta & & \Big\downarrow \nu \\
o_{\mathbb{C}_p} \otimes T_p(\mathbb{G}_m) & \xrightarrow{\qquad \xi \qquad} & T_p(X)
\end{array}$$

Choose a generator u of T, a generator ϵ of $T_p(\mathbb{G}_m)$, and let $f \in \mathrm{Hom}_o(T, o_{\mathbb{C}_p}(1))$ be the unique o-linear map such that $f(u) = \epsilon$. Trace the pairing (u, f) both ways through the square, using the explicit formulae for the maps involved from [Fon], and accounting for the fact that Fontaine writes $\mathbb{G}_m$ multiplicatively. If we write $\phi^0_{\mathcal{G}'}(f) = \Omega_f \omega$, then

$$\nu\phi(u, f) = \Omega_f u^* \omega$$

while

$$\xi\theta(u, f) = f(u)^* dZ/(1 + Z)\ .$$

Comparing these formulae with the explicit isomorphisms $\xi_{L,\mathcal{G}}$ and ξ_L defined in section 1 of [Fon], we see that the commutativity of the square means that

$$\Omega_f \xi_{L,\mathcal{G}}(u \otimes \omega) = \xi_L(f(u) \otimes dZ/(1 + Z))\ .$$

This fact, when combined with Thm. 1 and Cor. 1 of [Fon], tells us that

$$\Omega_f \mathfrak{a}_L = \mathfrak{a}_{L,\mathcal{G}}\ .$$

We conclude that

$$|\Omega_f| = p^{-r} \quad \text{with} \quad r = \frac{1}{p-1} - \frac{1}{e(q-1)} + \mathrm{ord}_p(\mathcal{D}_{L/\mathbb{Q}_p})$$

where $\mathcal{D}_{L/\mathbb{Q}_p}$ is the different of the extension $L/\mathbb{Q}_p$.

To complete the calculation let t'_0 be our chosen generator for the o-module T'. Some elementary linear algebra using properties of the different shows that there is a generator x of $\mathcal{D}_{L/\mathbb{Q}_p}$ such that we have

$$x t'_0 = f + f' \quad \text{in } o_{\mathbb{C}_p} \otimes_{\mathbb{Z}_p} T'$$

with f' vanishing on the eigenspace in $\mathbb{C}_p \otimes_o T$ corresponding to the given embedding $L \subseteq \mathbb{C}_p$. This means that $\phi^0_{\mathcal{G}'}(f') = 0$ so that

$$x\Omega_{t'_0} = \Omega_f\ .$$

In other words, the valuation of $\Omega = \Omega_{t'_0}$ is $\mathrm{ord}_p(\Omega_f) - \mathrm{ord}_p(x)$ as claimed.

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Peter Schneider
Mathematisches Institut
Westfälische Wilhelms-Universität Münster
Einsteinstr. 62
D-48149 Münster, Germany
pschnei@math.uni-muenster.de
http://www.uni-muenster.de/math/u/schneider

Jeremy Teitelbaum
Department of Mathematics, Statistics, and Computer Science (M/C 249)
University of Illinois at Chicago
851 S. Morgan St.
Chicago, IL 60607, USA
jeremy@uic.edu
http://raphael.math.uic.edu/~jeremy