

**Locally analytic distributions and p -adic representation theory,
with applications to GL_2**

P. Schneider, J. Teitelbaum

Let L/\mathbb{Q}_p be a finite extension, and let G be a locally L -analytic group such as the L -points of an algebraic group over L . The categories of smooth and of finite-dimensional algebraic representations of G may be viewed as subcategories of the category of continuous representations of G in locally convex L -vector spaces. This larger category contains many interesting new objects, such as the action of G on global sections of equivariant vector bundles over p -adic symmetric spaces and other representations arising from the theory of p -adic uniformization as studied for example in [ST]. A workable theory of continuous representations of G in L -vector spaces offers the opportunity to unify these disparate examples in a single theoretical framework.

There are a number of technical obstacles to developing a reasonable theory of such representations. For example, there are no unitary representations over L , and continuous, or even locally analytic, functions on G are not integrable against Haar measure. As a result, even for compact groups one is forced to consider representations of G in fairly general locally convex vector spaces. In such situations one encounters a range of pathologies apparent even in the theory of representations of real Lie groups in Banach spaces. For this reason one must formulate some type of “finiteness” or “admissibility” condition in order to have a manageable theory.

In this paper we introduce a restricted category of continuous representations of locally L -analytic groups G in locally convex K -vector spaces, where K is a spherically complete non-archimedean extension field of L . We call the objects of this category “strongly admissible” representations and we establish some of their basic properties. Most importantly we show that (at least when G is compact) the category of strongly admissible representations in our sense can be algebraized; it is faithfully full (anti)-embedded into the category of modules over the locally analytic distribution algebra $D(G, K)$ of G over K . We may then replace the topological notion of irreducibility with the algebraic property of simplicity as $D(G, K)$ -module. Our hope is that our definition of strongly admissible representation may be used as a foundation for a general theory of continuous K -valued representations of locally L -analytic groups.

As an application of our theory, we prove the topological irreducibility of generic members of the p -adic principal series of $GL_2(\mathbb{Q}_p)$. This result was claimed by Morita for general L , not only \mathbb{Q}_p , in [Mor] Thm. 1(i), but his method to deduce irreducibility from something weaker that he calls local irreducibility is flawed. We believe that this failure is in fact an unavoidable consequence of the above mentioned pathologies of Banach space representations. Our completely different method is based instead on our algebraization theory and leads to the much

stronger result of the algebraic simplicity of the corresponding $D(GL_2(\mathbb{Q}_p), K)$ -modules. Morita also described the intertwining operators between the various p -adic principal series representations ([Mor] Thm. 2); again we use our methods to establish a slightly stronger version of this.

We rely heavily on two comparatively old results. The first one is Amice's Fourier isomorphism in [Am2] between $D(\mathbb{Z}_p, K)$ and the ring of power series over K convergent on the open p -adic unit disk. The second one is Lazard's result in [Laz] that in the latter ring every divisor is principal. The structure of the ring $D(o_L)$, for the ring of integers o_L in a general finite extension L of \mathbb{Q}_p , is unclear at present, and for this reason we obtain our irreducibility theorem only for \mathbb{Q}_p .

We make extensive use of results from p -adic functional analysis. Where possible, we give references to both the classical and the p -adic literature. In some cases, however, we have not found suitable p -adic references and we have given references to the needed result in the classical case. These classical results are general properties of Fréchet spaces and their p -adic analogues are not difficult to obtain (see [NFA] for a systematic presentation of p -adic methods).

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1. Vector spaces of compact type

In this section, we introduce a special class of topological vector spaces. This class arises naturally when considering locally analytic representations, and it enjoys good properties with respect to passage to subspaces, quotient spaces, and dual spaces.

We fix a spherically complete nonarchimedean field K and let o_K denote its ring of integers. Define $\mathcal{LC}(K)$ to be the category of locally convex K -vector spaces. We recall that a topological K -vector space is called locally convex if it has a fundamental system of open 0-neighbourhoods consisting of o_K -submodules. If V and W are objects in $\mathcal{LC}(K)$, we denote by $\mathcal{L}(V, W)$ the space of continuous linear maps from V to W . Following traditional usage, we write $\mathcal{L}_s(V, W)$ and $\mathcal{L}_b(V, W)$ for the vector space $\mathcal{L}(V, W)$ equipped with its weak and strong topologies respectively. For any V in $\mathcal{LC}(K)$, we denote by V' the K -vector space $\mathcal{L}(V, K)$ and write V'_s for $\mathcal{L}_s(V, K)$ and V'_b for $\mathcal{L}_b(V, K)$. Both of these dual spaces are Hausdorff. An o_K -submodule A of a K -vector space V is called a *lattice* if it K -linearly generates V . Any open o_K -submodule of a locally convex K -vector space is a lattice. If each closed lattice in V is open then the locally convex K -vector space V is called *barrelled*. This very basic property puts the Banach-Steinhaus theorem into force ([B-TVS] III.25). We recall a few other concepts before we introduce the notion in the title of this section.

Definition:

Let V and W be Hausdorff locally convex K -vector spaces.

1. A subset $B \subseteq V$ is called compactoid if for any open lattice $A \subseteq V$ there are finitely many vectors $v_1, \dots, v_m \in V$ such that $B \subseteq A + o_K v_1 + \dots + o_K v_m$.

2. A bounded o_K -submodule $B \subseteq V$ is called c-compact if it is compactoid and complete.

3. A continuous linear map $f : V \rightarrow W$ is called compact if there is an open lattice A in V such that the closure of $f(A)$ in W is bounded and c-compact.

4. V is said to be of compact type if it is the locally convex inductive limit of a sequence

$$V_1 \xrightarrow{\iota_1} V_2 \xrightarrow{\iota_2} V_3 \longrightarrow \dots$$

of locally convex Hausdorff K -vector spaces V_n , for $n \in \mathbb{N}$, with injective compact linear maps ι_n .

In part 4 of the above Definition one can, in fact, suppose that the V_n are Banach spaces (compare [GKPS] 3.1.4 or [Kom] Lemma 2).

Theorem 1.1:

Any space V of compact type is Hausdorff, complete, bornological, and reflexive; its strong dual is a Fréchet space and satisfies $V'_b = \varprojlim (V'_n)_b$.

Proof: [GKPS] 3.1.7 or [Kom] Thm. 6' and Thm. 12 or [NFA] 16.9 and 16.10 (compare also [B-TVS] III.6 Prop.7).

Let $\mathcal{LC}_c(K)$ be the full subcategory of $\mathcal{LC}(K)$ consisting of spaces of compact type. We note that any reflexive space is barrelled ([NFA] 15.4).

Proposition 1.2:

i. Let V be a space of compact type, and let $U \subseteq V$ be a closed vector subspace. Then U and V/U are also of compact type, and the maps in the exact sequence of vector spaces

$$0 \rightarrow (V/U)'_b \rightarrow V'_b \rightarrow U'_b \rightarrow 0$$

are strict.

ii. The category $\mathcal{LC}_c(K)$ is closed under countable locally convex direct sums (and therefore also finite products.)

Proof: i. [Kom] Thm. 8 and Thm. 7', [GKPS] 3.1.16, and [B-TVS] IV.29 Cor. 1 and 2. ii. [Kom] Thm. 9 and 10.

The duals of spaces of compact type have a natural characterization.

Theorem 1.3:

A K -Fréchet space V is the strong dual of a space of compact type if and only if V is nuclear.

Proof: [Kom] Thm. 17 or [GKPS] 2.5.8, 3.1.7, and 3.1.13.

Let $\underline{\mathcal{L}}\mathcal{C}_{nF}(K)$ be the full subcategory of $\underline{\mathcal{L}}\mathcal{C}(K)$ consisting of nuclear Fréchet spaces.

Corollary 1.4:

The functor

$$\begin{aligned} \underline{\mathcal{L}}\mathcal{C}_c(K) &\longrightarrow \underline{\mathcal{L}}\mathcal{C}_{nF}(K) \\ V &\longmapsto V'_b \end{aligned}$$

is an anti-equivalence of categories. For any two V and W in $\underline{\mathcal{L}}\mathcal{C}_c(K)$, the natural linear map $\mathcal{L}_b(V, W) \rightarrow \mathcal{L}_b(W'_b, V'_b)$ is a topological isomorphism.

Proof: By reflexivity, applying the operation $V \mapsto V'_b$ twice yields the identity functor. The transpose operation gives a continuous map from $\mathcal{L}_b(V, W)$ to $\mathcal{L}_b(W'_b, V'_b)$, which, by reflexivity, has a continuous inverse.

We conclude with a generalization of reflexivity for spaces of compact type. We will apply this result in our study of distributions in the next section.

Suppose that V is of compact type, with defining sequence of Banach spaces V_n , and that W is a Banach space. An element $v \otimes w$ of $V_n \otimes W$ defines an element of $\mathcal{L}((V_n)'_b, W)$ by $\ell \mapsto \ell(v)w$. $\mathcal{L}_b((V_n)'_b, W)$ being again a Banach space, we obtain a map $V_n \otimes W \rightarrow \mathcal{L}_b((V_n)'_b, W)$.

Composing with the natural map $V'_b = \varprojlim (V_n)'_b \rightarrow (V_n)'_b$ yields a map

$$C : \varinjlim (V_n \hat{\otimes} W) \rightarrow \mathcal{L}(V'_b, W).$$

Proposition 1.5:

The map C is a linear isomorphism.

Proof: For simplicity we denote the strong or Banach space dual of a Banach space U simply by U' . Because V is of compact type, each transition map $\iota_n : V_n \rightarrow V_{n+1}$ is compact. It follows that the map $\iota''_n : V''_n \rightarrow V''_{n+1}$ factors through $V_{n+1} \subseteq V''_{n+1}$ (see [Kom] Lemma 1 or [NFA] 16.4). Consequently, the inductive sequences

$$V_1 \hat{\otimes} W \xrightarrow{\iota_1} V_2 \hat{\otimes} W \xrightarrow{\iota_2} \dots$$

and

$$(1) \quad V_1'' \hat{\otimes} W \xrightarrow{i_1''} V_2'' \hat{\otimes} W \xrightarrow{i_2''} \dots$$

have the same limit. Consider now the inductive sequence

$$(2) \quad \mathcal{L}(V_1', W) \xrightarrow{j_1} \mathcal{L}(V_2', W) \xrightarrow{j_2} \dots$$

The transition maps in this sequence are given by $j_n(f) = f \circ i_n'$. Since the v_n are compact, so are the v_n' (loc. cit.), and therefore the image of $\mathcal{L}(V_n', W)$ under j_n lies in the subspace of compact operators in $\mathcal{L}(V_{n+1}', W)$. By [Gru] Prop. 3.3.1, 5.2.2, and 5.3.3, this space of compact operators is precisely $V_{n+1}'' \hat{\otimes} W$. As a result, the inductive sequences (1) and (2) have the same limit. Now, because W is a Banach space, any continuous linear map $\varprojlim V_n' \rightarrow W$ must factor through one of the V_n' . Therefore, we have a linear isomorphism

$$\varinjlim \mathcal{L}(V_n', W) \xrightarrow{\sim} \mathcal{L}(\varprojlim V_n', W) .$$

By Thm. 1.1, $\varprojlim V_n' = V_b'$. Tracing through the steps in this isomorphism shows that the map is as described in the above statement.

2. Distributions

In this section, we discuss the space of distributions on a locally analytic manifold. We construct a general “integration” map allowing us to apply distributions to vector valued locally analytic functions. We review the convolution product of distributions on a locally analytic group. Finally, we show that entire power series in Lie algebra elements converge to distributions on the group.

We fix fields $\mathbb{Q}_p \subseteq L \subseteq K$ such that L/\mathbb{Q}_p is finite and K is spherically complete with respect to a nonarchimedean absolute value $|\cdot|$ extending the one on L . We let \mathcal{O}_L denote the ring of integers in L . In the following L plays the role of the base field whereas K will be the coefficient field.

Let M be a d -dimensional locally L -analytic manifold ([B-VAR] §5.1 and [Fe1] §3.1). We always assume that M is strictly paracompact (and hence locally compact); this means that any open covering of the topological space M can be refined into a covering by pairwise disjoint open subsets; e.g., if M is compact then it is strictly paracompact ([Fe1] §3.2).

Let V be a Hausdorff locally convex K -vector space. In this situation the locally convex K -vector space $C^{an}(M, V)$ of all V -valued locally analytic functions on M is defined ([Fe2] 2.1.10). We briefly recall this construction. A V -index \mathcal{I} on M is a family of triples

$$\{(D_i, \phi_i, V_i)\}_{i \in I}$$

where the D_i are pairwise disjoint open subsets of M which cover M , each $\phi_i : D_i \rightarrow L^d$ is a chart of the manifold M whose image is an affinoid ball, and $V_i \hookrightarrow V$ is a BH -space of V , i.e., an injective continuous linear map from a K -Banach space V_i into V . We form the locally convex direct product

$$\mathcal{F}_{\mathcal{I}}(V) := \prod_{i \in I} \mathcal{F}_{\phi_i}(V_i)$$

of the K -Banach spaces

$$\mathcal{F}_{\phi_i}(V_i) := \text{all functions } f : D_i \rightarrow V_i \text{ such that } f \circ \phi_i^{-1} \text{ is a } V_i\text{-valued holomorphic function on the affinoid ball } \phi_i(D_i).$$

Denoting by $\mathcal{O}(\phi_i(D_i))$ the K -Banach algebra of all power series in d variables with coefficients in K converging on all points of the affinoid $\phi_i(D_i)$ defined over an algebraic closure of K we have

$$\mathcal{F}_{\phi_i}(V_i) \cong \mathcal{O}(\phi_i(D_i)) \hat{\otimes}_K V_i .$$

The V -indices on M form a directed set on which $\mathcal{F}_{\mathcal{I}}(V)$ is a direct system of locally convex K -vector spaces. We then may form the locally convex inductive limit

$$C^{an}(M, V) := \varinjlim_{\mathcal{I}} \mathcal{F}_{\mathcal{I}}(V) .$$

The formation of this space is compatible with disjoint coverings in the following sense ([Fe2] 2.2.4): Whenever $M = \bigcup_{j \in J} M_j$ is an open covering by pairwise disjoint subsets M_j then one has the natural topological isomorphism

$$C^{an}(M, V) = \prod_{j \in J} C^{an}(M_j, V) .$$

Definition:

The strong dual $D(M, K) := C^{an}(M, K)'_b$ of $C^{an}(M, K)$ is called the locally convex vector space of K -valued distributions on M .

Lemma 2.1:

If M is compact then $C^{an}(M, K)$ is of compact type; in particular, $D(M, K)$ is the projective limit over the K -indices on M of the strong dual spaces $\mathcal{F}_{\mathcal{I}}(K)'_b$ and is a K -Fréchet space.

Proof: Apply [ST] Lemma 1.5 (in fact, a straightforward generalization to the case of a spherically complete K) to the transition maps in the direct system $\mathcal{F}_{\mathcal{I}}(K)$ or use [Fe2] 2.3.2.

For a general M we may choose a covering $M = \bigcup_{j \in J} M_j$ by pairwise disjoint compact open subsets and obtain a topological isomorphism

$$D(M, K) = \bigoplus_{j \in J} D(M_j, K) .$$

This observation together with the above Lemma shows that the strong topology on $D(M, K)$ coincides with the topology considered in [Fe1] 2.5.5 and 3.4.1.

The only completely obvious elements in $D(M, K)$ are the Dirac distributions δ_x , for $x \in M$, defined by $\delta_x(f) := f(x)$.

Theorem 2.2: (Integration)

If V is the union of countably many BH -spaces then the map

$$\begin{aligned} I^{-1} : \mathcal{L}(D(M, K), V) &\xrightarrow{\cong} C^{an}(M, V) \\ A &\longmapsto [x \longmapsto A(\delta_x)] \end{aligned}$$

is a well defined K -linear isomorphism.

Proof: Clearly the map in the assertion is compatible with disjoint open coverings of M . We therefore may assume that M is compact so that $D(M, K)$ is a Fréchet space. On the other hand, since the sum of two BH -spaces again is a BH -space we find, by our assumption on V , an increasing sequence $V_1 \subseteq V_2 \subseteq \dots$ of BH -spaces of V such that $V = \bigcup_{n \in \mathbf{N}} V_n$. By the closed graph theorem ([B-TVS] I. 20 Prop. 1) any continuous linear map from the Fréchet space $D(M, K)$ into V factors through some V_n . In other words we have

$$\mathcal{L}(D(M, K), V) = \varinjlim_{n \in \mathbf{N}} \mathcal{L}(D(M, K), V_n) .$$

Moreover, by the same closed graph theorem any BH -space of V is contained in some V_n . Since M is compact (so that in the definition of $C^{an}(M, V)$ we need to consider only V -indices whose underlying covering of M is finite) this means that

$$C^{an}(M, V) = \varinjlim_{n \in \mathbf{N}} C^{an}(M, V_n) .$$

Hence we are further reduced to the case that V is a Banach space. But then we are precisely in the situation of Prop. 1.5. The only additional observation to make is that the map

$$\begin{aligned} I : \mathcal{O}(\phi_i(D_i)) \hat{\otimes}_K V &\longrightarrow \mathcal{L}(\mathcal{O}(\phi_i(D_i))'_b, V) \\ f \otimes v &\longmapsto [\lambda \longmapsto \lambda(f)v] \end{aligned}$$

considered there satisfies $I(f \otimes v)(\delta_x) = f(x) \cdot v$ for $x \in M$.

Since any locally analytic function on a compact manifold factors through a BH -space the arguments in the above proof show that for an arbitrary V we still have a natural map

$$I : C^{an}(M, V) \longrightarrow \mathcal{L}(D(M, K), V)$$

such that $I(f)(\delta_x) = f(x)$ for any $f \in C^{an}(M, K)$ and $x \in M$. This map I should be considered as integration: Given a locally analytic function $f : M \longrightarrow V$ and a distribution λ on M we may formally write

$$I(f)(\lambda) = \int_M f(x) d\lambda(x).$$

The case where $M := G$ is a locally L -analytic group has additional features which we now want to discuss. First of all we note that any such G is strictly paracompact ([Fe1] 3.2.7). According to [Fe1] 4.1.6 there is a unique separately continuous K -bilinear map

$$\hat{\otimes} : D(G, K) \times D(G, K) \longrightarrow D(G \times G, K)$$

with the property

$$(\lambda \hat{\otimes} \mu)(f_1 \times f_2) = \lambda(f_1) \cdot \mu(f_2)$$

for any $\lambda, \mu \in D(G, K)$, $f_i \in C^{an}(G, K)$, and $(f_1 \times f_2)(g_1, g_2) := f_1(g_1) \cdot f_2(g_2)$. If G is compact then $\hat{\otimes}$ is (jointly) continuous by [Fe1] 4.2.1 (the separate continuity in general is an immediate consequence of that). The group multiplication $m : G \times G \longrightarrow G$ induces by functoriality a continuous linear map

$$m_* : D(G \times G, K) \longrightarrow D(G, K).$$

We define the convolution $*$ on $D(G, K)$ by

$$* := m_* \circ \hat{\otimes},$$

i.e.,

$$(\lambda * \mu)(f) = (\lambda \hat{\otimes} \mu)(f \circ m) \text{ for } \lambda, \mu \in D(G, K) \text{ and } f \in C^{an}(G, K).$$

Proposition 2.3:

$(D(G, K), *)$ is an associative K -algebra with the Dirac distribution δ_1 in $1 \in G$ as the unit element; the convolution $*$ is separately continuous. If G is compact then $D(G, K)$ is a Fréchet algebra.

Proof: [Fe1] 4.4.1 and 4.4.4.

The selfmap $g \mapsto g^{-1}$ of G induces an anti-automorphism of $D(G, K)$. For this reason we never have to consider right modules for $D(G, K)$ and we will simply speak of modules when we mean left modules.

One method to explicitly construct elements in $D(G, K)$ is through the Lie algebra \mathfrak{g} of G . Let C_1^ω be the “stalk” of $C^{an}(G, K)$ at the identity element, as defined in [Fe2] §2.3.1. This is the locally convex inductive limit of the spaces $C^{an}(Y, K)$ as Y runs through the family of compact open neighborhoods of the identity in G . Note that each map $C^{an}(Y, K) \rightarrow C_1^\omega$ is surjective.

For each compact open subgroup $H \subseteq G$ the Lie algebra \mathfrak{g} of G acts on $C^{an}(H, K)$ by continuous endomorphisms defined by

$$(\mathfrak{r}f)(g) := \frac{d}{dt} f(\exp(-t\mathfrak{r})g)|_{t=0}$$

or written in a more invariant way, by using the left translation action of the group H on $C^{an}(H, K)$,

$$\mathfrak{r}f := \lim_{t \rightarrow 0} \frac{\exp(t\mathfrak{r})f - f}{t}$$

for $\mathfrak{r} \in \mathfrak{g}$ where $\exp : \mathfrak{g} \rightarrow G$ denotes the exponential map defined locally around 0 ([Fe2] 3.1.2 and 3.3.4). This extends, by the universal property, to a left action of the universal enveloping algebra $U(\mathfrak{g})$ on each $C^{an}(H, K)$ (and hence on C_1^ω) by continuous endomorphisms. In particular, an element \mathfrak{z} of $U(\mathfrak{g})$ gives a continuous linear form on any $C^{an}(H, K)$ (and hence on C_1^ω) by the rule

$$f \mapsto (\mathfrak{z}(f))(1) .$$

where $\mathfrak{z} \mapsto \mathfrak{j}$ is the unique anti-automorphism of $U(\mathfrak{g})$ which extends the multiplication by -1 on \mathfrak{g} ([B-GAL] Chap.I,§2.4). By [Fe2] 4.7.4 this induces an injection $U(\mathfrak{g}) \rightarrow (C_1^\omega)'$. Moreover, the continuous surjection $C^{an}(G, K) \rightarrow C_1^\omega$

yields a continuous injection $(C_1^\omega)'_b \rightarrow D(G, K)$. Altogether we have the chain of inclusions

$$U(\mathfrak{g}) \otimes_L K \hookrightarrow (C_1^\omega)' \hookrightarrow D(G, K) .$$

The space C_1^ω can be represented explicitly. Fix an ordered basis $\mathfrak{x}_1, \dots, \mathfrak{x}_d$ of \mathfrak{g} . The corresponding ‘‘canonical system of coordinates of the second kind’’ ([B-GAL] Chap.III,§4.3) is the map

$$\begin{aligned} \text{small ball around } 0 \text{ in } L^d &\longrightarrow G \\ (t_1, \dots, t_d) &\longmapsto \exp(t_1 \mathfrak{x}_1) \cdots \exp(t_d \mathfrak{x}_d) . \end{aligned}$$

This chart gives us, in a neighborhood of 1, and for a multi-index \underline{n} , the monomial functions

$$T^{\underline{n}}(\exp(t_1 \mathfrak{x}_1) \cdots \exp(t_d \mathfrak{x}_d)) := t_1^{n_1} \cdots t_d^{n_d} .$$

Lemma 2.4:

The space C_1^ω is isomorphic to the space of all power series

$$C_1^\omega = \left\{ \sum_{\underline{n}} a_{\underline{n}} T^{\underline{n}} : a_{\underline{n}} \in K \text{ and } |a_{\underline{n}}| r^{|\underline{n}|} \rightarrow 0 \text{ as } |\underline{n}| \rightarrow \infty \text{ for some } r > 0 \right\};$$

its dual is the space of all power series

$$(C_1^\omega)' = \left\{ \sum_{\underline{n}} b_{\underline{n}} Z^{\underline{n}} : b_{\underline{n}} \in K \text{ and } \sup_{\underline{n}} |b_{\underline{n}}| r^{-|\underline{n}|} < \infty \text{ for any } r > 0 \right\};$$

the strong topology on $(C_1^\omega)'_b$ is defined by the family of norms

$$\left\| \sum_{\underline{n}} b_{\underline{n}} Z^{\underline{n}} \right\|_r := \sup_{\underline{n}} |b_{\underline{n}}| r^{-|\underline{n}|} \quad \text{for } r > 0 ;$$

finally, the map $U(\mathfrak{g}) \rightarrow (C_1^\omega)'$ is given explicitly by

$$\mathfrak{x}_1^{n_1} \cdots \mathfrak{x}_d^{n_d} \mapsto (-1)^{|\underline{n}|} \underline{n}! Z^{\underline{n}} .$$

Proof: By definition, C_1^ω consists of functions represented by power series with a non-zero radius of convergence around 1. This is precisely the condition we specify here. The description of the dual follows easily, with $Z^{\underline{n}}$ dual to $T^{\underline{n}}$. The map on $U(\mathfrak{g})$ comes from the discussion after Lemma 4.7.2 in [Fe2] which shows that

$$(((\mathfrak{x}_1^{i_1} \cdots \mathfrak{x}_d^{i_d}) \cdot) T^{\underline{n}})(1) = (-1)^{|\underline{n}|} \underline{n}! \delta_{\underline{i}, \underline{n}}$$

where

$$|\underline{n}| = \sum_{j=1}^d n_j \quad \text{and} \quad \underline{n}! = n_1! n_2! \cdots n_d!$$

and $\delta_{i,\underline{n}}$ is Kronecker's symbol.

Lemma 2.5:

Let $\{b_{\underline{n}}\}_{\underline{n}}$ be a set of elements of K such that $\sup_{\underline{n}} |b_{\underline{n}}| r^{-|\underline{n}|} < \infty$ for any $r > 0$; then the series

$$\sum_{\underline{n}} b_{\underline{n}} \mathfrak{x}_1^{n_1} \cdots \mathfrak{x}_d^{n_d}$$

converges in $(C_1^\omega)'_b$ (that is, its sequence of partial sums converges to an element of $(C_1^\omega)'_b$).

Proof: Given the previous Lemma, it suffices to show that the series $\sum_{\underline{n}} b_{\underline{n}} Z^{\underline{n}}$ is the limit of its partial sums. We compute:

$$\begin{aligned} \left\| \sum_{\underline{n}} b_{\underline{n}} Z^{\underline{n}} - \sum_{|\underline{n}| \leq n} b_{\underline{n}} Z^{\underline{n}} \right\|_r &= \sup_{|\underline{n}| > n} |b_{\underline{n}}| r^{-|\underline{n}|} \\ &= \sup_{|\underline{n}| > n} |b_{\underline{n}}| \left(\frac{r}{2}\right)^{-|\underline{n}|} \left(\frac{1}{2}\right)^{|\underline{n}|} \\ &\leq \left(\frac{1}{2}\right)^n \left\| \sum_{\underline{n}} b_{\underline{n}} Z^{\underline{n}} \right\|_{r/2} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Corollary 2.6:

Given any set of coefficients $\{b_{\underline{n}}\}_{\underline{n}}$ as in the Lemma, the series

$$\sum_{\underline{n}} b_{\underline{n}} \mathfrak{x}_1^{n_1} \cdots \mathfrak{x}_d^{n_d}$$

converges in $D(G, K)$.

Proof: This follows from the continuity of the map $(C_1^\omega)'_b \rightarrow D(G, K)$.

In somewhat vague terms this Corollary says that any entire power series over K in the $\mathfrak{x}_1, \dots, \mathfrak{x}_d$ converges in $D(G, K)$.

It is useful to have various other interpretations of the above inclusion of \mathfrak{g} into $D(G, K)$. The \mathfrak{g} -action on $C^{an}(G, K)$ induces a \mathfrak{g} -action on $D(G, K)$ by $(\mathfrak{r}\lambda)(f) := \lambda((- \mathfrak{r})f)$. In particular $\mathfrak{r}\delta_1$ is the image of \mathfrak{r} in $D(G, K)$. One easily computes

$$(\mathfrak{r}\lambda)(f) = \lim_{t \rightarrow 0} \frac{\exp(t\mathfrak{r})\lambda - \lambda}{t}(f) .$$

This means that in $C^{an}(G, K)'_s$ we have the limit formula

$$\mathfrak{r}\lambda = \lim_{t \rightarrow 0} \left(\frac{\delta_{\exp(t\mathfrak{r})} - \delta_1}{t} * \lambda \right) .$$

As a direct product of spaces of compact type the space $C^{an}(G, K)$ is reflexive. For such a space $C^{an}(G, K)'_s$ coincides with $C^{an}(G, K)'_b$ given its weakened topology, and any bounded subset in $C^{an}(G, K)'_s$ (e.g., a convergent sequence in this space together with its limit) is contained in a bounded and c -compact o_K -submodule of $C^{an}(G, K)'_b = D(G, K)$ ([Ti1] III §3 and §4 and [Ti2] Thm. 2 or [NFA] 14.2, 15.3, and 15.4). Moreover, on submodules of the latter type the strong topology coincides with the weak topology ([Gru] 5.3.4 or [NFA] 14.5). It follows that the above limit formula even holds in $D(G, K)$. But the convolution in $D(G, K)$ is separately continuous. We therefore obtain

$$\mathfrak{r}\lambda = \left(\lim_{t \rightarrow 0} \frac{\delta_{\exp(t\mathfrak{r})} - \delta_1}{t} \right) * \lambda = (\mathfrak{r}\delta_1) * \lambda \text{ in } D(G, K) .$$

3. Locally analytic G -representations

In this section, we show that locally analytic G -representations on K -vector spaces V of compact type are equivalent to a certain class of modules for the ring $D(G, K)$. We let $\mathbf{Q}_p \subseteq L \subseteq K$ be fields as in the previous section and we let G be a d -dimensional locally L -analytic group with Lie algebra \mathfrak{g} .

Definition:

A locally analytic G -representation V (over K) is a barrelled locally convex Hausdorff K -vector space V equipped with a G -action by continuous linear endomorphisms such that, for each $v \in V$, the orbit map $\rho_v(g) := gv$ is a V -valued locally analytic function on G .

By the Banach-Steinhaus Theorem, the map $(g, v) \rightarrow gv$ is continuous, as is the Lie algebra action

$$v \rightarrow \mathfrak{r}v := \frac{d}{dt} \exp(t\mathfrak{r})v|_{t=0}$$

for $\mathfrak{x} \in \mathfrak{g}$ and $v \in V$. This \mathfrak{g} -action extends to an action of the universal enveloping algebra $U(\mathfrak{g})$ on V by continuous linear endomorphisms. Taylor's formula (compare [Fe2] 3.1.4) says that, for each fixed $v \in V$ there is a sufficiently small neighborhood U of 0 in \mathfrak{g} such that, for $\mathfrak{x} \in U$, we have a convergent expansion

$$\exp(\mathfrak{x})v = \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{x}^n v .$$

Now suppose that V is a locally analytic representation of G . We obtain a candidate for a $D(G, K)$ -module structure on V by defining

$$\begin{aligned} D(G, K) \times V &\longrightarrow V \\ (\mu, v) &\longmapsto \mu * v = I(\rho_v)(\mu) \end{aligned}$$

where I is the integration map from the previous section. For a Dirac distribution δ_g , we have $\delta_g * v = gv$.

Lemma 3.1:

The Dirac distributions generate a dense subspace in $D(G, K)$.

Proof: If $H \subseteq G$ is a compact open subgroup then we have

$$D(G, K) = \bigoplus_{g \in G/H} \delta_g * D(H, K) .$$

This observation shows that we may assume G to be compact. Let Δ be the closure in $D(G, K)$ of the subspace generated by the Dirac distributions. Let ℓ be a linear form on $D(G, K)$ vanishing on Δ . By reflexivity of $D(G, K)$, such a linear form must be given by a locally analytic function f on G , and to say ℓ vanishes on Δ is to say that f is identically zero on G . By the Hahn-Banach Theorem, we conclude that $\Delta = D(G)$.

Proposition 3.2:

*The map $(\mu, v) \rightarrow \mu * v$ is separately continuous in μ and v , and V is a module over $D(G, K)$; this $D(G, K)$ -module structure extends the action of $U(\mathfrak{g})$ on V . Any continuous linear G -map between locally analytic G -representations is a $D(G, K)$ -module homomorphism.*

Proof: The continuity in μ is clear by construction. To show the continuity in v we use the previous Lemma. If G is compact then $D(G, K)$ is metrisable and any $\mu \in D(G, K)$ is approximated by a sequence $\mu = \lim_{n \rightarrow \infty} \mu_n$ where each μ_n is a linear combination of Dirac distributions. This remains true for an arbitrary G

by the same observation as in the previous proof. Now, each μ_n is a continuous linear map from V to V , and μ is their pointwise limit. Since V is barrelled, we see from the Banach-Steinhaus theorem that μ itself is a continuous linear map from V to V . To show that V is in fact a $D(G, K)$ -module, we must show that, for fixed v , $\mu' * (\mu * v) = (\mu' * \mu) * v$. This identity clearly holds for Dirac distributions and it follows for general μ by continuity. The argument for the last assertion is similar. Finally, since the action of \mathfrak{g} and $U(\mathfrak{g})$ is expressible as a limit of actions by Dirac distributions, the remaining claim, too, follows from continuity.

If V is a barrelled Hausdorff space which is the union of countably many BH-spaces then the above Proposition has a converse: As a consequence of Theorem 2.2 any separately continuous $D(G, K)$ -module structure on V comes from a locally analytic G -representation on V in the way described above. In other words, the category of locally analytic G -representations on such spaces is equivalent to the category of separately continuous $D(G, K)$ -module structures on them.

Corollary 3.3:

The functor

$$\begin{array}{ccc}
 \begin{array}{l} \text{locally analytic } G\text{-representations} \\ \text{on } K\text{-vector spaces of compact type} \\ \text{with continuous linear } G\text{-maps} \end{array} & \rightarrow & \begin{array}{l} \text{separately continuous } D(G, K)\text{-} \\ \text{modules on nuclear Fréchet} \\ \text{spaces with continuous} \\ D(G, K)\text{-module maps} \end{array} \\
 V & \mapsto & V'_b
 \end{array}$$

is an anti-equivalence of categories.

Proof: This follows from the above discussion together with Cor. 1.4 once we show that V carries a separately continuous $D(G, K)$ -module structure if and only if V'_b does. Indeed, the separate continuity of the pairing $D(G, K) \times V \rightarrow V$ is equivalent to the assertion that the map $D(G, K) \rightarrow \mathcal{L}_s(V, V)$ is continuous. It follows from ([B-TVS] III.31 Prop.6) that this map remains continuous when $\mathcal{L}_b(V, V)$ is given its strong topology. Applying Cor. 1.4, we obtain a continuous map from $D(G, K)$ into $\mathcal{L}_b(V'_b, V'_b)$, so that V'_b carries a separately continuous $D(G, K)$ -module structure. By the reflexivity of V this argument works the same way in the opposite direction.

If the group G is compact then $D(G, K)$ is a Fréchet space. Since any separately continuous bilinear map between Fréchet spaces is jointly continuous ([B-TVS] III.30 Cor. 1) we may reformulate in this case the last assertion as follows.

Corollary 3.4:

If G is compact then the functor

$$\begin{array}{ccc}
 \text{locally analytic } G\text{-representations} & & \text{continuous } D(G, K)\text{-modules on} \\
 \text{on } K\text{-vector spaces of compact type} & \rightarrow & \text{nuclear Fréchet spaces with con-} \\
 \text{with continuous linear } G\text{-maps} & & \text{tinuous } D(G, K)\text{-module maps} \\
 V & \mapsto & V'_b
 \end{array}$$

is an anti-equivalence of categories.

Let $\mathcal{M}_K(G)$ be the category of all left $D(G, K)$ -modules in the purely algebraic sense. The full subcategory of $\mathcal{M}_K(G)$ generated by the objects which are countably generated (i.e., such that there exists a surjective $D(G, K)$ -module map $\bigoplus_{\mathbb{N}} D(G, K) \rightarrow M$) will be denoted by $\mathcal{M}_K^c(G)$.

Definition:

Assume G to be compact; a locally analytic G -representation V is called strongly admissible if V is of compact type and V'_b as a $D(G, K)$ -module is countably generated.

The condition imposed in the above definition looks weaker than it actually is. A straightforward Baire category theory argument shows that the dual $D(G, K)$ -module $M := V'_b$ of a strongly admissible G -representation V necessarily is finitely generated: Write $M = \bigcup_{n \in \mathbb{N}} M_n$ as a union of an increasing sequence $M_1 \subseteq M_2 \subseteq \dots$ of finitely generated $D(G, K)$ -submodules M_n . Since M is a Fréchet space Baire's theorem says that M_n , for some $n \in \mathbb{N}$, is non-meagre. Now choose generators m_1, \dots, m_r for M_n and consider the continuous map $\alpha : D(G, K)^r \rightarrow M$ defined by $\alpha(\mu_1, \dots, \mu_r) = \mu_1 m_1 + \dots + \mu_r m_r$. Its image is the non-meagre submodule M_n . The open mapping theorem ([B-TVS] I.17 Thm.1) then implies that α is surjective onto M .

For a compact group G we let $\text{Rep}_K^{\text{adm}}(G)$ denote the category of all strongly admissible G -representations with continuous linear G -maps.

Lemma 3.5:

Assume G to be compact; the category $\text{Rep}_K^{\text{adm}}(G)$ is closed with respect to the passage to closed G -invariant subspaces.

Proof: Let $U \subseteq V$ be a closed G -invariant subspace. By Prop. 1.2.i, U is of compact type. Using [Fe2] 1.2.3 and the fact that the coefficients of the power series expansions of the orbit maps ρ_v are limits of linear combinations of values it easily follows that U is a locally analytic G -representation. Finally, looking

at the exact sequence in Prop. 1.2.i we see that U' is a quotient module of the countably generated $D(G, K)$ -module V' and hence is countably generated as well.

Later on we will make use of the following simple consequence of this Lemma: Any V in $\text{Rep}_K^{\text{adm}}(G)$ which is such that V' is an (algebraically) simple $D(G, K)$ -module must be topologically irreducible as a G -representation.

Proposition 3.6:

If G is compact then the contravariant functor

$$\begin{array}{ccc} \text{Rep}_K^{\text{adm}}(G) & \rightarrow & \mathcal{M}_K^c(G) \\ V & \mapsto & V' \end{array}$$

is a fully faithful embedding.

Proof: Let M and N be $D(G, K)$ -modules in the image of the functor (in fact, N can be any separately continuous $D(G, K)$ -module). We have to show that any abstract $D(G, K)$ -module homomorphism $\alpha : M \rightarrow N$ is continuous. Choose generators $(m_i)_{i \in \mathbf{N}}$ for M , and let

$$\beta : \bigoplus_{\mathbf{N}} D(G, K) \rightarrow M$$

be the map $\beta((\mu_i)) = \sum_{\mathbf{N}} \mu_i m_i$. By ([B-TVS] II.34, Cor. to Prop.10), this map is strict; in other words, M carries the quotient topology with respect to the map β . At the same time, we have a map

$$\gamma : \bigoplus_{\mathbf{N}} D(G, K) \rightarrow N$$

given by $\gamma((\mu_i)) = \sum_{\mathbf{N}} \mu_i \alpha(m_i)$. Since the $D(G, K)$ -action on N is continuous, this map is continuous, and by construction it factors through M , where it induces the original map α . By the universal property of the quotient topology, α is continuous.

This result can be seen as a first step towards algebraizing the theory of strongly admissible G -representations.

A further important algebraic concept is concerned with the action of the centre $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$.

Proposition 3.7.:

If G is an open subgroup of the group of L -rational points of a connected algebraic L -group \mathbf{G} then $Z(\mathfrak{g}) \otimes_L K$ lies in the centre of $D(G, K)$.

Proof: Since $\mathfrak{g} = \text{Lie}(G) = \text{Lie}(\mathbf{G})$ and $U(\mathfrak{g}) \subseteq D(G, K) \subseteq D(\mathbf{G}(L), K)$ we may assume that $G = \mathbf{G}(L)$. By Lemma 3.1 it suffices to show that $\delta_g * \mathfrak{z} * \delta_{g^{-1}} = \mathfrak{z}$ for any $g \in \mathbf{G}(L)$ and any $\mathfrak{z} \in Z(\mathfrak{g})$. This amounts to ([B-GAL] III §3.12 Cor. 2) the adjoint action of $\mathbf{G}(L)$ on $U(\mathfrak{g})$ fixing the centre $Z(\mathfrak{g})$ which is a consequence of the connectedness of \mathbf{G} ([DG] II §6.1.5).

Let us assume for the rest of this section that G satisfies the assumption of the above Proposition.

Definition:

1. A $D(G, K)$ -module M is called $Z(\mathfrak{g})$ -locally finite if $Z(\mathfrak{g})m$, for each $m \in M$, is finite dimensional over K .
2. A locally analytic G -representation V is called $Z(\mathfrak{g})$ -locally cofinite if the $D(G, K)$ -module V' is $Z(\mathfrak{g})$ -locally finite.

Clearly, a finitely generated $D(G, K)$ -module M which is $Z(\mathfrak{g})$ -locally finite actually is $Z(\mathfrak{g})$ -finite in the sense that the annihilator ideal

$$\mathfrak{a}_M := \{\mathfrak{z} \in Z(\mathfrak{g}) \otimes_L K : \mathfrak{z}M = 0\}$$

is of finite codimension in $Z(\mathfrak{g}) \otimes_L K$. If M is a $Z(\mathfrak{g})$ -locally finite and simple $D(G, K)$ -module then \mathfrak{a}_M is a maximal ideal in $Z(\mathfrak{g}) \otimes_L K$: By Schur's lemma $\text{End}_{D(G, K)}(M)$ is a skew field which implies that $Z(\mathfrak{g}) \otimes_L K / \mathfrak{a}_M$ is an integral domain. But being of finite dimension $Z(\mathfrak{g}) \otimes_L K / \mathfrak{a}_M$ then has to be a field. In this case the action of $Z(\mathfrak{g})$ on M therefore is given by a character χ_M of $Z(\mathfrak{g})$ into the multiplicative group of the finite field extension $Z(\mathfrak{g}) \otimes_L K / \mathfrak{a}_M$ of K . This χ_M is called the *infinitesimal character* of M . If K is algebraically closed then the infinitesimal character is a character $\chi_M : Z(\mathfrak{g}) \rightarrow K^\times$. Assume that K is algebraically closed, G is compact, and M is a simple $D(G, K)$ -module which is the dual $M = V'$ of a strongly admissible and $Z(\mathfrak{g})$ -locally finite G -representation V then $Z(\mathfrak{g})$ acts on V through the character $\dot{\chi}_M : Z(\mathfrak{g}) \rightarrow K^\times$ defined by $\dot{\chi}_M(\mathfrak{z}) := \chi_M(\mathfrak{z})$.

4. The one dimensional case

As the first important example we will discuss in this section the case of the group $G = \mathbf{Z}_p$. Hence in the following $L = \mathbf{Q}_p$, and G always denotes the one dimensional additive group \mathbf{Z}_p . The spherically complete field K/\mathbf{Q}_p is assumed to be contained in \mathbf{C}_p .

In order to determine the structure of the ring $D(\mathbb{Z}_p, K)$ we use the theory of the Fourier transform. The character group

$$\hat{G} := \text{Hom}_{an}(G, \mathbb{C}_p^\times)$$

of $G = \mathbb{Z}_p$ is defined to be the group of all locally \mathbb{C}_p -analytic group homomorphisms $\kappa : G \rightarrow \mathbb{C}_p^\times$. If X denotes the rigid analytic open unit disk over K then it is easy to see that

$$\begin{aligned} \hat{G} &\longrightarrow X(\mathbb{C}_p) \\ \kappa &\longmapsto \kappa(1) - 1 \end{aligned}$$

is a well defined injective map. It is in fact bijective since, for any $z \in X(\mathbb{C}_p)$ and any $a \in \mathbb{Z}_p$ the series

$$\kappa_z(a) := (1+z)^a := \sum_{n \geq 0} \binom{a}{n} z^n$$

converges and defines a locally \mathbb{C}_p -analytic character κ_z of \mathbb{Z}_p with $\kappa_z(1) - 1 = z$. Actually κ_z , for $z \in X(K)$, is locally K -analytic so that we have the embedding

$$\begin{aligned} X(K) &\hookrightarrow C^{an}(\mathbb{Z}_p, K) \\ z &\mapsto \kappa_z \end{aligned} .$$

Hence any linear form $\lambda \in D(\mathbb{Z}_p, K)$ gives rise to the function

$$F_\lambda(z) := \lambda(\kappa_z)$$

on $X(K)$ which is called the Fourier transform of λ . In fact, F_λ is a holomorphic function on X . Let $\mathcal{O}(X)$ denote the ring of all K -holomorphic functions on X . We recall that

$$\begin{aligned} \mathcal{O}(X) = & \text{all power series } F(T) = \sum_{n \geq 0} a_n T^n \text{ with } a_n \in K \\ & \text{which converge on } X(\mathbb{C}_p) . \end{aligned}$$

For any $r \in p^\mathbb{Q}$ let X_r denote the K -affinoid disk of radius r around zero. The ring $\mathcal{O}(X_r)$ of all K -holomorphic functions on X_r is a K -Banach algebra with respect to the multiplicative spectral norm. Since

$$\mathcal{O}(X) = \lim_{\longleftarrow r < 1} \mathcal{O}(X_r) = \bigcap_{r < 1} \mathcal{O}(X_r)$$

the ring $\mathcal{O}(X)$ is an integral domain and is, in a natural way, a K -Fréchet algebra.

Theorem 4.1: (Amice)

The Fourier transform

$$\begin{array}{ccc} D(\mathbb{Z}_p, K) & \xrightarrow{\cong} & \mathcal{O}(X) \\ \lambda & \mapsto & F_\lambda \end{array}$$

is an isomorphism of K -Fréchet algebras.

Proof: [Am2] 1.3 and 2.3.4 (based on [Am1]); compare [Sch] for a concise write-up of this proof.

By work of Lazard the structure of the ring $\mathcal{O}(X)$ is well known. We recall that an effective divisor on X_r is a map $D : X_r \rightarrow \mathbb{Z}_{\geq 0}$ with finite support. The effective divisors are partially ordered by

$$D \leq D' \text{ iff } D(x) \leq D'(x) \text{ for any } x \in X_r .$$

Moreover, for any $F \in \mathcal{O}(X_r)$ there is the associated divisor of zeros (F) on X_r .

We define

$$\text{Div}^+(X) := \text{all maps } D : X \rightarrow \mathbb{Z}_{\geq 0} \text{ such that, for any } r < 1, \\ \text{the restriction } D|_{X_r} \text{ has finite support}$$

to be the partially ordered set of effective divisors on X and we obtain the divisor map

$$\begin{array}{ccc} \mathcal{O}(X) \setminus \{0\} & \longrightarrow & \text{Div}^+(X) \\ F & \longmapsto & (F) \end{array}$$

as the projective limit of the divisor maps for the X_r with $r < 1$. It is clear that, for any $0 \neq F, F' \in \mathcal{O}(X)$, we have

$$F|F' \text{ if and only if } (F) \leq (F') ;$$

in particular:

$$(F) = (F') \text{ if and only if } F = uF' \text{ for some } u \in \mathcal{O}(X)^\times .$$

Theorem 4.2: (Lazard)

i. The divisor map $\mathcal{O}(X) \setminus \{0\} \rightarrow \text{Div}^+(X)$ is surjective;

ii. the map

$$\begin{array}{ccc} \text{Div}^+(X) & \xrightarrow{\sim} & \text{all nonzero closed ideals in } \mathcal{O}(X) \\ D & \longmapsto & I_D := \{f \in \mathcal{O}(X) : (f) \geq D\} \cup \{0\} \end{array}$$

is a well defined bijection;

iii. the closure of a nonzero ideal $I \subseteq \mathcal{O}(X)$ is the ideal $I_{D(I)}$ for the divisor $D(I)(x) := \min_{0 \neq f \in I} (f)(x)$;

iv. in $\mathcal{O}(X)$ the three families of closed ideals, finitely generated ideals, and principal ideals coincide.

Proof: [Laz] Thm. 7.2, Prop. 7.10, (7.3), and Prop. 8.11 (again we refer to [Sch] for a concise write-up).

We record the following facts for later use.

Lemma 4.3:

The translation action $\lambda \mapsto {}^b\lambda$, for $b \in \mathbb{Z}_p$, of \mathbb{Z}_p on $D(\mathbb{Z}_p, K)$ corresponds under the Fourier transform to

$$F_{{}^b\lambda}(T) = (1 + T)^b F_\lambda(T) .$$

If $\mathfrak{r}_1 \in \text{Lie}(\mathbb{Z}_p) = \mathbb{Q}_p$ corresponds to $1 \in \mathbb{Q}_p$ then the Lie algebra action $\lambda \mapsto \mathfrak{r}_1\lambda$ on $D(\mathbb{Z}_p, K)$ corresponds to

$$F_{\mathfrak{r}_1\lambda}(T) = \log(1 + T) F_\lambda(T) .$$

For $b \in \mathbb{Z}_p$, the multiplication by b on \mathbb{Z}_p induces an operator m_{b*} on $D(\mathbb{Z}_p, K)$ which corresponds to

$$F_{m_{b*}\lambda}(T) = F_\lambda((1 + T)^b - 1) .$$

Finally, the operator $\Theta(f) := af(a)$ on $C^{an}(\mathbb{Z}_p, K)$ induces the map

$$F_{\lambda \circ \Theta}(T) = (1 + T) \frac{dF_\lambda(T)}{dT} .$$

Proof: These are easy computations (see also [Am2] 2.3).

Having determined the structure of the ring $D(\mathbb{Z}_p, K)$ we now turn to a discussion of the module theory for this ring. More specifically we are aiming at some information about the image of our fully faithful embedding

$$\begin{array}{ccc} \text{Rep}_K^{adm}(\mathbb{Z}_p) & \rightarrow & \mathcal{M}_K^c(\mathbb{Z}_p) \\ V & \mapsto & V' \end{array}$$

in Prop. 3.6.

Proposition 4.4:

Let M be a free $D(\mathbb{Z}_p, K)$ -module of finite rank equipped with the obvious direct product topology; then any finitely generated $D(\mathbb{Z}_p, K)$ -submodule N of M is closed.

Proof: The ring $D(\mathbb{Z}_p, K) = \mathcal{O}(X)$ is an adequate ring in the sense of [Hel] §2 since:

- it is an integral domain;
- any finitely generated ideal is principal (by Thm. 4.2.iv);
- for any two elements $F \neq 0$ and F' in $\mathcal{O}(X)$ the "relatively prime part R of F w.r.t. F' " exists: Define the divisor D by

$$D(x) := \begin{cases} (F)(x) & \text{if } (F')(x) = 0, \\ 0 & \text{otherwise} \end{cases}$$

and let R be such that $(R) = D$ (by Thm. 4.2.i).

The main result Thm. 3 in [Hel] then says that the elementary divisor theorem holds over $\mathcal{O}(X)$. This means that there is an isomorphism $M \cong \mathcal{O}(X)^m$ under which N corresponds to a submodule of the form $F_1\mathcal{O}(X) \oplus \dots \oplus F_n\mathcal{O}(X)$ with $n \leq m$ and $F_i \in \mathcal{O}(X)$. Since, by Thm. 4.2.iv, principal ideals are closed in $\mathcal{O}(X)$ it follows that N is closed in M .

Let now M be any finitely presented $D(\mathbb{Z}_p, K)$ -module. If we choose a finite presentation

$$D(\mathbb{Z}_p, K)^n \rightarrow D(\mathbb{Z}_p, K)^m \rightarrow M \rightarrow 0$$

then as a consequence of the Proposition the quotient topology on M is Hausdorff. Hence M with this topology is a nuclear Fréchet space with a continuous $D(\mathbb{Z}_p, K)$ -action. Moreover, this quotient topology is independent of the chosen presentation. We therefore see that any $D(\mathbb{Z}_p, K)$ -module of finite presentation M carries a natural Fréchet topology such that $M \cong V'_b$ for some strongly admissible \mathbb{Z}_p -representation V .

By Lazard's theorem the simple $D(\mathbb{Z}_p, K)$ -modules of finite presentation are (up to isomorphism) those of the form $D(\mathbb{Z}_p, K)/\mathfrak{m}$ for some closed maximal ideal $\mathfrak{m} \subseteq D(\mathbb{Z}_p, K)$ and they all are finite dimensional as K -vector spaces. Any such is the dual of an irreducible strongly admissible \mathbb{Z}_p -representation.

Remark:

A topologically irreducible locally analytic \mathbb{Z}_p -representation V is finite dimensional (and irreducible) if and only if it is $Z(\mathbb{Q}_p)$ -locally cofinite.

Proof: The other implication being trivial we assume that V is $Z(\mathbb{Q}_p)$ -locally cofinite. Fix a nonzero $\ell \in V'$. Then the closed ideal $I := \{F \in \mathcal{O}(X) : F*\ell = 0\}$ in $\mathcal{O}(X) = D(\mathbb{Z}_p, K)$ is nonzero and proper. Choose a point $x_0 \in X$ such that $D(I)(x_0) > 0$. The closed ideal $J \subseteq \mathcal{O}(X)$ such that $D(J)(x) = D(I)(x)$ for $x \neq x_0$ and $D(J)(x_0) = D(I)(x_0) - 1$ has the property that J/I is finite dimensional. Hence $\{v \in V : \ell'(v) = 0 \text{ for any } \ell' \in J * \ell\}$ is a $D(\mathbb{Z}_p, K)$ -invariant proper closed subspace of finite codimension in V . By the topological irreducibility of V such a subspace must be zero. Hence V is finite dimensional.

5. The principal series of the Iwahori subgroup in $GL_2(\mathbb{Q}_p)$

In this section we let $G = B$ be the Iwahori subgroup of $GL_2(\mathbb{Q}_p)$. This is the subgroup of $GL_2(\mathbb{Z}_p)$ consisting of all matrices which are lower triangular mod p . As before, K/\mathbb{Q}_p is a spherically complete extension field contained in \mathbb{C}_p . Let P_o^- , resp. T_o , denote the subgroup of B of all upper triangular, resp. diagonal, matrices.

We fix a K -valued locally analytic character

$$\chi : T_o \rightarrow K^\times$$

and we define $c(\chi) \in K$ so that

$$\chi\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) = \exp(c(\chi) \log(t))$$

for t sufficiently close to 1 in \mathbb{Z}_p .

Our aim is to study the locally analytically induced B -representation

$$\text{Ind}_{P_o^-}^B(\chi) := \{f \in C^{an}(B, K) : f(gp) = \chi(p^{-1})f(g) \text{ for any } g \in B, p \in P_o^-\}$$

where B acts by left translation.

Proposition 5.1:

$\text{Ind}_{P_o^-}^B(\chi)$ is a locally analytic B -representation; the underlying vector space is topologically isomorphic to $C^{an}(\mathbb{Z}_p, K)$ and in particular is of compact type.

Proof: That $\text{Ind}_{P_o^-}^B(\chi)$ is locally analytic follows from [Fe2] 4.1.5 and the fact that B/P_o^- is compact. The map

$$\begin{aligned} \iota : \mathbb{Z}_p &\longrightarrow B \\ b &\longmapsto \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \end{aligned}$$

is a section of the projection map $B \rightarrow B/P_o^- = \mathbf{Z}_p$, and applying [Fe2] 4.3.1, we see that pullback by this section yields the asserted isomorphism of topological vector spaces. The last claim then follows from Lemma 2.1.

Definition:

Let $M_\chi^- := \text{Ind}_{P_o^-}^B(\chi)'_b$ be the $D(B, K)$ -module obtained from $\text{Ind}_{P_o^-}^B(\chi)$ by applying the duality functor of Cor. 3.4.

As a vector space, the module M_χ^- is isomorphic to $D(\mathbf{Z}_p, K)$, and the $D(\mathbf{Z}_p, K)$ -module structure on M_χ^- coming from the ring inclusion

$$\iota_* : D(\mathbf{Z}_p, K) \rightarrow D(B, K)$$

is simply the ring multiplication. Thus M_χ^- is a copy of $D(\mathbf{Z}_p, K)$ with the additional structure corresponding to the action of $D(B, K)$. Before we calculate this additional structure explicitly we remark that this observation already implies the strong admissibility of the B -representation $\text{Ind}_{P_o^-}^B(\chi)$.

Lemma 5.2:

Under the identification $\text{Ind}_{P_o^-}^B(\chi) = C^{an}(\mathbf{Z}_p, K)$ we have:

i. The translation action of $\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \in B$ is given by the formula

$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} f = \chi\left(\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\right) \cdot (m_{b^{-1}}^* f) ;$$

ii. the translation action of $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in B$ is given by

$$\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f\right)(a) = \chi\left(\begin{pmatrix} (1-ab)^{-1} & 0 \\ 0 & (1-ab) \end{pmatrix}\right) f\left(\frac{a}{1-ab}\right) ;$$

iii. the Lie algebra element $\mathbf{u}^+ := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{gl}_2(\mathbf{Q}_p) = \text{Lie}(B)$ acts by

$$(\mathbf{u}^+ f)(a) = -\frac{df(a)}{da} ;$$

iv. the Lie algebra element $\mathbf{u}^- := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \text{Lie}(B)$ acts by

$$\mathbf{u}^- f = -c(\chi)\Theta(f) - \Theta^2(\mathbf{u}^+ f) ;$$

here m_{b-1}^* and Θ are the operators from Lemma 4.3.

Proof: An elementary computation.

Lemma 5.3:

Let

$$c_i^{(n)} = \frac{n!}{i!} \binom{c(\chi) - i}{n - i} \quad \text{for } 0 \leq i \leq n ;$$

then, after making the identification $M_\chi^- = D(\mathbb{Z}_p, K)$ and applying the Fourier transform, the action of $(\mathbf{u}^-)^m \in U(\text{Lie}(B))$, for each $m \geq 0$, satisfies

$$F_{(\mathbf{u}^-)^m \lambda} = \sum_{i=0}^m (-1)^i c_i^{(m)} (\log(1+T))^i \Delta^{m+i} F_\lambda$$

where the operator Δ is defined by $\Delta F := (1+T) \frac{dF(T)}{dT}$. Moreover, the translation action of $\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \in B$ satisfies

$$F_{\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \lambda}(T) = \chi\left(\begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix}\right) \cdot F_\lambda((1+T)^b - 1) .$$

Proof: Using the formulas

$$c_i^{(m)} (c(\chi) - m - i) + c_{i-1}^{(m)} = c_i^{(m+1)}$$

and

$$\mathbf{u}^+ \Theta^m = -m \Theta^{m-1} + \Theta^m \mathbf{u}^+$$

as well as Lemma 5.2.iv one obtains by induction the identity

$$(\mathbf{u}^-)^m = (-1)^m \sum_{i=0}^m c_i^{(m)} \Theta^{m+i} \circ (\mathbf{u}^+)^i$$

in $\mathcal{L}(C^{an}(\mathbb{Z}_p, K), C^{an}(\mathbb{Z}_p, K))$. By Lemma 4.3 this transposes into the first of the asserted identities. The second one is a direct consequence of Lemma 5.2 and Lemma 4.3.

We now come to the main result of this section.

Theorem 5.4:

If $c(\chi) \notin \mathbb{N}_0$, then M_χ^- is an (algebraically) simple $D(B, K)$ -module.

Proof: As we pointed out above, M_χ^- is isomorphic to $D(\mathbb{Z}_p, K)$, and, under this isomorphism, the subalgebra $D(\mathbb{Z}_p, K) \subseteq D(B, K)$ acts via the usual multiplication action of $D(\mathbb{Z}_p, K)$ on itself. Let I be a nonzero $D(B, K)$ -submodule of M_χ^- ; then I corresponds to a nonzero $D(B, K)$ -invariant ideal in $D(\mathbb{Z}_p, K)$. Thus we may apply Lazard's theory of divisors to study I . (Note, however, that we do NOT assume that I is closed.)

In a first step we claim that the ideal I is generated by elements F such that (F) is supported on the set

$$\mu_\infty := \{z \in \mathbb{C}_p : (1+z)^{p^n} = 1 \text{ for some } n \in \mathbb{N}\}.$$

To see this claim, choose a nonzero $F \in I$, let S be the set of zeros of F and let $S_0 := S \setminus S \cap \mu_\infty$. For any pair $z, z' \in S_0$, there is at most one $u \in \mathbb{Z}_p^\times$ such that

$$u_*(z) := (1+z)^u - 1 = z'.$$

Because \mathbb{Z}_p^\times is uncountable, and S_0 is countable, there must be a $u \in \mathbb{Z}_p^\times$ so that $u_*(S_0) \cap S_0 = \emptyset$. As a result of the second part of the previous Lemma, the common support of the divisors (F) and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right) F$ is contained in μ_∞ . By

Thm. 4.2 the ideal $\langle F, \left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right) F \rangle$ is generated by an element whose divisor is supported on μ_∞ . Since $F \in I$ was arbitrary, and I is translation invariant, I must be generated by elements whose divisors are supported on μ_∞ .

If, as before X denotes the open unit disk considered as a rigid analytic variety over K we let the sequence of points x_0, x_1, \dots in X correspond to the orbits of the absolute Galois group of K in the set μ_∞ .

We now begin with a nonzero element $F \in I$ with the following property:

- (*) There is a sequence of positive integers $0 < m_0 < m_1 < \dots$ such that the divisor (F) is supported on x_0, x_1, \dots and, for each integer $k \geq 0$, the multiplicity of (F) in x_k is m_k .

The existence of such an F follows from the above first step, and from the principality of all divisors (see Thm. 4.2.i). We will now use the action of \mathfrak{u}^- to perturb the element F , thereby obtaining a new element $F^c \in I$ whose divisor is supported entirely off the set μ_∞ . By Lazard's theory, F and F^c together generate $D(\mathbb{Z}_p, K)$, and we will conclude that $I = D(\mathbb{Z}_p, K)$. This perturbation will be carried out by induction. Choose a function $\sigma : \mathbb{N}_0 \rightarrow p^\mathbb{Q}$ such that, for all real numbers $r > 0$, $\sigma(n)r^n \rightarrow 0$ as $n \rightarrow \infty$. First, observe that by the first part of the previous Lemma

$$(\mathfrak{u}^-)^{m_k} F \equiv \left[\prod_{i=0}^{m_k-1} (c(\chi) - i) \right] \cdot \left((1+T) \frac{d}{dT} \right)^{m_k} F \pmod{\log(1+T)\mathcal{O}(X)}.$$

Because the divisor $(\log(1 + T))$ has multiplicity one in each x_k , we see that the divisor of $(\mathbf{u}^-)^{m_k} F$ has multiplicity zero in x_k but has positive multiplicity in each x_j for $j > k$ (because of $m_j > m_k$). Now suppose that we have found elements $b_0, \dots, b_{k-1} \in K$ so that $|b_i| < \sigma(i)$ for $0 \leq i < k$ and the divisor of

$$F_{k-1} := (b_0(\mathbf{u}^-)^{m_0} + \dots + b_{k-1}(\mathbf{u}^-)^{m_{k-1}})F$$

has zero multiplicity in x_0, \dots, x_{k-1} . Note, however, that (F_{k-1}) must have positive multiplicity in x_k , while $(\mathbf{u}^-)^{m_k} F$ does not. By avoiding finitely many special cases, we may choose $b_k \in K$ so that $|b_k| < \sigma(k)$ and the divisor of $F_{k-1} + b_k(\mathbf{u}^-)^{m_k} F$ has zero multiplicity in x_0, \dots, x_k . Having constructed the b_k in this way for all $k \geq 0$ our choice of σ , together with Cor. 2.6, tells us that the element

$$\mathbf{u}_\infty := \sum_{k=0}^{\infty} b_k(\mathbf{u}^-)^{m_k} \in D(B, K)$$

is well defined so that we have $F^c := \mathbf{u}_\infty F \in I$. Since, by construction, the divisor of $(b_0(\mathbf{u}^-)^{m_0} + \dots + b_k(\mathbf{u}^-)^{m_k})F$ has zero multiplicity in x_k whereas $\sum_{j>k} b_j(\mathbf{u}^-)^{m_j} F$ has positive multiplicity it follows that (F^c) is supported completely outside of μ_∞ . Hence we see that $I = D(\mathbf{Z}_p, K)$. This proves that M_{χ^-} is simple.

In order to formulate the next result we introduce the character ϵ defined by $\epsilon\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right) := t_2/t_1$.

Proposition 5.5:

For any two K -valued locally analytic characters χ and χ' of T_\circ the vector space $\text{Hom}_{D(B, K)}(M_{\chi^-}, M_{\chi'^-})$ of all $D(B, K)$ -module homomorphisms is 1-dimensional if either $\chi = \chi'$ or $c(\chi) \in \mathbb{N}_0$ and $\chi' = \epsilon^{-1-c(\chi)}\chi$ and is zero otherwise.

Proof: By Prop. 3.6 we may dualize back and determine the B -equivariant continuous linear maps from $\text{Ind}_{P_\circ^-}^B(\chi)$ to $\text{Ind}_{P_\circ^-}^B(\chi')$. By Frobenius reciprocity ([Fe2] 4.2.6) this is equivalent to determining the space of all P_\circ^- -equivariant continuous linear maps from $\text{Ind}_{P_\circ^-}^B(\chi)$ to $K_{\chi'}$ where the latter denotes the 1-dimensional representation of P_\circ^- given by χ' . Making our usual identification $M_{\chi^-} \cong D(\mathbf{Z}_p, K)$ this identifies with the space

$$\mathcal{J} := \{F \in D(\mathbf{Z}_p, K) : \chi'(g) \cdot gF = F \text{ for any } g \in P_\circ^-\} .$$

We study this defining condition separately for three different kinds of elements g in P_\circ^- .

1. Any $g = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$ acts on $\text{Ind}_{P_o^-}^B(\chi)$ by multiplication with $\chi(g)$ and hence on $D(\mathbf{Z}_p, K)$ by multiplication with $\chi^{-1}(g)$. For \mathcal{J} to be nonzero we therefore must have

$$(1) \quad \chi | \mathbf{Z}_p^\times \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \chi' | \mathbf{Z}_p^\times \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

2. According to Lemma 5.3 any $g = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ acts on $D(\mathbf{Z}_p, K)$ via

$$(gF)(T) = \chi \left(\begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix} \right) \cdot F((1+T)^b - 1) .$$

Any nonzero $F \in \mathcal{J}$ therefore satisfies

$$(2) \quad F(T) = \frac{\chi'}{\chi} \left(\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \right) \cdot F((1+T)^b - 1) .$$

Applying the operator $\frac{d}{db}|_{b=1}$ to this equation we obtain the differential equation

$$(3) \quad \frac{c(\chi') - c(\chi)}{2} F(T) + (1+T) \log(1+T) \frac{dF(T)}{dT} = 0 .$$

Here we have used that

$$2 \frac{d}{db} \left(\frac{\chi'}{\chi} \left(\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \right) \right) |_{b=1} = \frac{d}{db} \left(\frac{\chi'}{\chi} \left(\begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix} \right) \right) |_{b=1} = c(\chi') - c(\chi) .$$

The differential equation (3) has a nonzero solution in $D(\mathbf{Z}_p, K)$ if and only if

$$(4) \quad c(\chi) - c(\chi') \in 2\mathbb{N}_0 .$$

In order to see this put $c := (c(\chi) - c(\chi'))/2$ and assume that $F(T) = \sum_{n \geq 0} a_n T^n$ is a solution of (3). By inserting F into (3) and comparing coefficients we obtain

$$ca_0 = 0 \quad \text{and} \quad (n-c)a_n = \sum_{i=1}^{n-1} (-1)^i \frac{n-i}{i(i+1)} a_{n-i} \quad \text{for } n \geq 1 .$$

From this one easily deduces by induction:

- in case $c \notin \mathbb{N}_0$ that $a_n = 0$ for all $n \geq 0$, i.e., that $F = 0$;
- in case $c \in \mathbb{N}_0$ that $a_n = 0$ for $n < c$ and that a_n for $n > c$ can be recursively computed from a_c , i.e., that up to scalars there is at most one F .

In the latter case one verifies that

$$F(T) := [\log(1+T)]^{(c(\chi) - c(\chi'))/2}$$

indeed is a solution of (3). One immediately checks that this solution satisfies the equation (2) if and only if

$$(5) \quad \chi' \left(\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \right) = b^{(c(\chi')-c(\chi))/2} \cdot \chi \left(\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \right) \quad \text{for any } b \in \mathbf{Z}_p^\times .$$

Note that the conditions (1), (4) and (5) together are equivalent to the requirement that

$$(6) \quad \chi' = \epsilon^{-m} \chi \quad \text{for some } m \in \mathbb{N}_0 .$$

3. Assuming that this latter identity holds we finally look at the action of $g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. We again consider first the derived action of u^- . By Lemma 5.3 the solution $F(T) := (\log(1+T))^m$ must satisfy the equation

$$c(\chi)\Delta F = \log(1+T)\Delta^2 F$$

if it lies in \mathcal{J} . But this means that either $m = 0$ or $c(\chi) = m - 1 \geq 0$. Because of (6) the former implies $\chi' = \chi$ and the latter $\chi' = \epsilon^{-1-c(\chi)}\chi$ with $c(\chi) \in \mathbb{N}_0$. It remains to show that in the latter case $(\log(1+T))^m$ indeed is invariant under any $g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. This we do by looking at the distribution

$$\begin{aligned} C^{an}(\mathbf{Z}_p, K) &\longrightarrow K \\ f(a) &\longmapsto \left(\left(\frac{d}{da} \right)^m f \right)(0) \end{aligned}$$

which by the second assertion in Lemma 4.3 and Lemma 5.2.iii corresponds to $(\log(1+T))^m$ under the Fourier transform. By induction we deduce from Lemma 5.2.ii the formula

$$\left(\left(\frac{d}{da} \right)^m (gf) \right)(a) = \sum_{i=0}^m (-1)^{m+i} c_i^{(m)} \frac{b^{m-i}}{(1-ab)^{m+i}} \left(g \left(\left(\frac{d}{da} \right)^i f \right) \right)(a)$$

with the $c_i^{(m)}$ as defined in Lemma 5.3. But since $c(\chi) = m - 1$ we actually have $c_i^{(m)} = 0$ for $i < m$. Setting $a = 0$ and using Lemma 5.2.ii again we therefore obtain

$$\left(\left(\frac{d}{da} \right)^m (gf) \right)(0) = \left(\left(\frac{d}{da} \right)^m f \right)(0) .$$

Letting $P_\circ^+ \subseteq B$ denote the subgroup of lower triangular matrices we may analogously consider the B -representation by left translation on the induction $\text{Ind}_{P_\circ^+}^B(\chi)$ from P_\circ^+ . It is strongly admissible and, setting $M_\chi^+ := \text{Ind}_{P_\circ^+}^B(\chi)'_b$,

we have the following result whose proof is completely parallel to the proofs of Thm. 5.4 and Prop. 5.5 and is therefore omitted.

Theorem 5.6:

- i. If $c(\chi) \notin -\mathbb{N}_0$ then M_χ^+ is an (algebraically) simple $D(B, K)$ -module;
- ii. for any two χ and χ' the vector space $\text{Hom}_{D(B, K)}(M_{\chi'}^+, M_\chi^+)$ is 1-dimensional if either $\chi' = \chi$ or $c(\chi) \in -\mathbb{N}_0$ and $\chi' = \epsilon^{1-c(\chi)}\chi$ and is zero otherwise.

Corollary 5.7:

If $c(\chi) \notin \pm\mathbb{N}_0$ then $\text{Ind}_{P_\mp}^B(\chi)$ is topologically irreducible as a B -representation.

Proposition 5.8:

$$\text{Hom}_{D(B, K)}(M_\chi^+, M_{\chi'}^-) = \text{Hom}_{D(B, K)}(M_\chi^-, M_{\chi'}^+) = 0 \quad \text{for any } \chi \text{ and } \chi' .$$

Proof: We only discuss the vanishing of the first space, the other case being completely analogous. By the same reasoning as at the beginning of the proof of Prop. 5.5 we have to show that there are no nontrivial P_\circ^+ -equivariant continuous linear maps from $\text{Ind}_{P_\circ^-}^B(\chi')$ to K_χ . Each such is a continuous linear form on $\text{Ind}_{P_\circ^-}^B(\chi')$ which is invariant under the unipotent subgroup $\iota(\mathbb{Z}_p)$ of P_\circ^+ . Using Prop. 5.1 it is sufficient to show that there is no nonzero distribution in $D(\mathbb{Z}_p, K)$ which is \mathbb{Z}_p -invariant. But this is the well known nonexistence of a "p-adic Haar measure" on \mathbb{Z}_p (one way to see this is to use the first assertion in Lemma 4.3).

Define the elements $\mathfrak{e} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathfrak{h} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\mathfrak{gl}_2(\mathbb{Q}_p)$ and

$$\mathfrak{c} := \frac{1}{2}\mathfrak{h}^2 + \mathfrak{u}^+\mathfrak{u}^- + \mathfrak{u}^-\mathfrak{u}^+ \in U(\mathfrak{gl}_2(\mathbb{Q}_p)) .$$

It is well known that the centre of $U(\mathfrak{gl}_2(\mathbb{Q}_p))$ is the polynomial ring

$$Z(\mathfrak{gl}_2(\mathbb{Q}_p)) = \mathbb{Q}_p[\mathfrak{e}, \mathfrak{c}]$$

in the two variables $\mathfrak{e}, \mathfrak{c}$. Using the formulas iii. and iv. in Lemma 5.2 and the corresponding formula

$$(\mathfrak{h}f)(a) = -c(\chi)f(a) + 2a \frac{df(a)}{da}$$

one easily checks that \mathfrak{c} acts on M_χ^\pm through multiplication by the scalar

$$\left(\frac{1}{2}c(\chi) \mp 1\right)c(\chi) .$$

In particular, M_χ^\pm is $Z(\mathfrak{gl}_2(\mathbb{Q}_p))$ -finite.

6. The principal series of $GL_2(\mathbb{Z}_p)$ and $GL_2(\mathbb{Q}_p)$

The results of the preceding section will enable us in this section to analyze completely the irreducibility properties of the principal series of the group $GL_2(\mathbb{Q}_p)$.

We let $G := GL_2(\mathbb{Q}_p)$ and $G_o := GL_2(\mathbb{Z}_p)$. Furthermore, P denotes the Borel subgroup of lower triangular matrices in G , T the subgroup of diagonal matrices, and B as before the Iwahori subgroup in G_o . The field K is as before.

This time we fix a K -valued locally analytic character

$$\chi : T \rightarrow K^\times$$

and we again define $c(\chi) \in K$ so that

$$\chi\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) = \exp(c(\chi) \log(t))$$

for t sufficiently close to 1 in \mathbb{Z}_p . The corresponding principal series representation is $\text{Ind}_P^G(\chi)$ with G acting, as always, by left translation. From the Iwasawa decomposition $G = G_o P$ and [Fe2] 4.1.4 we obtain that the obvious restriction map

$$\text{Ind}_P^G(\chi) \xrightarrow{\simeq} \text{Ind}_{P \cap G_o}^{G_o}(\chi)$$

is a G_o -equivariant topological isomorphism. (For simplicity we denote by χ also the restriction of the original χ to various subgroups of T .) Using the Bruhat decomposition

$$G_o = B \dot{\cup} Bw(G_o \cap P)$$

with $w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G_o$ and by [Fe2] 2.2.4 and 4.1.4 we may further decompose into

$$\text{Ind}_{P \cap G_o}^{G_o}(\chi) = \text{Ind}_{P_o^+}^B(\chi) \oplus \text{Ind}_{P_o^-}^B(w\chi) .$$

This is a B -equivariant decomposition of topological vector spaces. First of all we see that, as a consequence of the previous section, $\text{Ind}_{P \cap G_o}^{G_o}(\chi)$ is a strongly admissible G_o -representation.

Definition:

Let $M_\chi := \text{Ind}_P^G(\chi)'_b$ be the $D(G, K)$ -module obtained from $\text{Ind}_P^G(\chi)$ by applying the duality functor of Cor. 3.3.

Theorem 6.1:

If $c(\chi) \notin -\mathbb{N}_0$, then M_χ is an (algebraically) simple $D(G_o, K)$ -module. In particular, $\text{Ind}_P^G(\chi)$ is a topologically irreducible G_o - and a fortiori G -representation.

Proof: As a $D(B, K)$ -module M_χ is isomorphic to the direct sum

$$M_\chi = M_\chi^+ \oplus M_{w\chi}^-$$

of the two modules M_χ^+ and $M_{w\chi}^-$ which by Thm. 5.4, Thm. 5.6.i, and Prop. 5.8 are simple and nonisomorphic under our assumption on $c(\chi)$. Hence these two are the only nonzero proper $D(B, K)$ -submodules of M_χ . Since the action of w mixes the two summands it follows that M_χ has to be simple as a $D(G_o, K)$ -module.

Proposition 6.2:

Let χ and χ' be any two K -valued locally analytic characters of T ; we then have:

- i. $\text{Hom}_{D(G_o, K)}(M_{\chi'}, M_\chi)$ is 1-dimensional if either $\chi'|_{T_o} = \chi|_{T_o}$ or $c(\chi) \in -\mathbb{N}_0$ and $\chi'|_{T_o} = (\epsilon^{1-c(\chi)}\chi)|_{T_o}$ and is zero otherwise;
- ii. $\text{Hom}_{D(G, K)}(M_{\chi'}, M_\chi)$ is 1-dimensional if either $\chi' = \chi$ or $c(\chi) \in -\mathbb{N}_0$ and $\chi' = \epsilon^{1-c(\chi)}\chi$ and is zero otherwise.

Proof: It is immediate from 5.5, 5.6, and 5.8 that the vector spaces $\text{Hom}_{D(G, K)}(M_{\chi'}, M_\chi) \subseteq \text{Hom}_{D(G_o, K)}(M_{\chi'}, M_\chi)$ are at most 1-dimensional and are zero except possibly in the two cases where either $\chi'|_{T_o} = \chi|_{T_o}$ or $c(\chi) \in -\mathbb{N}_0$ and $\chi'|_{T_o} = (\epsilon^{1-c(\chi)}\chi)|_{T_o}$.

Let us first look at the case where $\chi'|_{T_o} = \chi|_{T_o}$. Then $\text{Hom}_{D(G_o, K)}(M_{\chi'}, M_\chi)$ is 1-dimensional since it contains the dual of the composed isomorphism

$$J : \text{Ind}_P^G(\chi) \xrightarrow{\cong} \text{Ind}_{P \cap G_o}^{G_o}(\chi) = \text{Ind}_{P \cap G_o}^{G_o}(\chi') \xleftarrow{\cong} \text{Ind}_P^G(\chi') .$$

In order to check when this map J even is G -equivariant fix a function $f \in \text{Ind}_P^G(\chi)$ such that $f(1) \neq 0$ and let $h \in T$ be any diagonal matrix. We compute

$$J(hf)(1) = hf(1) = f(h^{-1}) = \chi(h) \cdot f(1)$$

and

$$h(J(f))(1) = J(f)(h^{-1}) = \chi'(h) \cdot J(f)(1) = \chi'(h) \cdot f(1) .$$

It follows that J is G -equivariant (i.e., that the dual of J lies in $\text{Hom}_{D(G_\circ, K)}(M_{\chi'}, M_\chi)$) if and only if $\chi' = \chi$.

The case where $m := 1 - c(\chi) \in \mathbb{N}$ and $\chi'|_{T_\circ} = (\epsilon^m \chi)|_{T_\circ}$ is slightly more complicated. We begin by exhibiting under the stronger assumption that $\chi' = \epsilon^m \chi$ a nonzero G -equivariant continuous linear map

$$I : \text{Ind}_P^G(\chi) \longrightarrow \text{Ind}_P^G(\chi').$$

We observe that $U(\mathfrak{gl}_2(\mathbb{Q}_p))$ acts continuously on $C^{an}(G, K)$ by left invariant differential operators which, solely for the purposes of this proof, we denote by $(\mathfrak{z}, f) \mapsto \mathfrak{z}f$; for $\mathfrak{x} \in \mathfrak{gl}_2(\mathbb{Q}_p)$ the formula is

$$(\mathfrak{x}f)(g) := \frac{d}{dt} f(g \exp(t\mathfrak{x}))|_{t=0}.$$

We claim that the operator on $C^{an}(G, K)$ corresponding to the element $\mathfrak{z} := (\mathfrak{u}^-)^m$ restricts to a linear map I as above, i.e., that the map

$$\begin{aligned} I : \text{Ind}_P^G(\chi) &\longrightarrow \text{Ind}_P^G(\chi') \\ f &\longmapsto (\mathfrak{u}^-)^m f \end{aligned}$$

is well defined. For this we have to show that, given an $f \in \text{Ind}_P^G(\chi)$, we have

$$(\mathfrak{z}f)(gp) = \chi'(p^{-1}) \cdot (\mathfrak{z}f)(g) \quad \text{for any } g \in G, p \in P.$$

We check this separately for diagonal matrices and for lower triangular unipotent matrices. First let h be a diagonal matrix. Then

$$(\mathfrak{z}f)(gh) = \chi(h^{-1})\epsilon^m(h^{-1}) \cdot (\mathfrak{z}f)(g) = \chi'(h^{-1}) \cdot (\mathfrak{z}f)(g)$$

as required, using the easily checked fact that $\text{Ad}(h)(\mathfrak{u}^-) = \epsilon(h^{-1})\mathfrak{u}^-$. Next we consider a lower triangular unipotent matrix

$$u = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \exp(\mathfrak{a}\mathfrak{u}^+).$$

Since f is right u -invariant we have

$$(\mathfrak{z}f)(gu) = ((\text{Ad}(u)\mathfrak{z})f)(g).$$

Now

$$\begin{aligned} \text{Ad}(u)\mathfrak{z} &= \text{Ad}(\exp(\mathfrak{a}\mathfrak{u}^+))\mathfrak{z} = \exp(\text{ad}(\mathfrak{a}\mathfrak{u}^+))\mathfrak{z} \\ &= \mathfrak{z} + a\text{ad}(\mathfrak{u}^+)\mathfrak{z} + (a^2/2)\text{ad}(\mathfrak{u}^+)(\text{ad}(\mathfrak{u}^+)\mathfrak{z}) + \dots \end{aligned}$$

where the “series” in this expression is actually finite. Using the elementary identity

$$\text{ad}(\mathfrak{u}^+)(\mathfrak{z}) = -m \cdot (\mathfrak{u}^-)^{m-1}(m-1 + \mathfrak{h})$$

in $U(\mathfrak{sl}_2(\mathbb{Q}_p))$ we see that

$$[\text{ad}(\mathfrak{u}^+)]^j(\mathfrak{z}) \in U(\mathfrak{gl}_2(\mathbb{Q}_p))(m-1 + \mathfrak{h}) + U(\mathfrak{gl}_2(\mathbb{Q}_p))\mathfrak{u}^+$$

for any $j \geq 1$. Hence

$$\text{Ad}(u)(\mathfrak{z}) \in \mathfrak{z} + U(\mathfrak{gl}_2(\mathbb{Q}_p))(m-1 + \mathfrak{h}) + U(\mathfrak{gl}_2(\mathbb{Q}_p))\mathfrak{u}^+ .$$

But $\mathfrak{u}^+f = 0$ and $\mathfrak{h}f = c(\chi)f = (1-m)f$. Consequently $(\mathfrak{z}f)(gu) = (\mathfrak{z}f)(g)$. The map I therefore is well defined; it is G -equivariant and continuous by construction. Finally, it is nonzero since it is easily seen to restrict on $\text{Ind}_{P_o^-}^B(w\chi)$ to the nonzero map considered in the proof of Prop. 5.5.

If we only have $\chi'|T_o = (\epsilon^m\chi)|T_o$ then clearly the same arguments show that the composed map

$$I_o : \text{Ind}_P^G(\chi) \xrightarrow{\cong} \text{Ind}_{P \cap G_o}^{G_o}(\chi) \xrightarrow{(\mathfrak{u}^-)^m} \text{Ind}_{P \cap G_o}^{G_o}(\chi') \xleftarrow{\cong} \text{Ind}_P^G(\chi')$$

is a well defined nonzero G_o -equivariant continuous linear map. Hence the space $\text{Hom}_{D(G_o, K)}(M_{\chi'}, M_{\chi})$ is 1-dimensional in this case. To finish the proof it remains to show that I_o is not G -equivariant if $\chi' \neq \epsilon^m\chi$. Fix a function $f \in \text{Ind}_P^G(\chi)$ such that $((\mathfrak{u}^-)^m f)(1) \neq 0$ and let $h \in T$ be such that $\chi'(h) \neq (\epsilon^m\chi)(h)$. Using $\text{Ad}(h^{-1})\mathfrak{u}^- = \epsilon(h)\mathfrak{u}^-$ we compute

$$\begin{aligned} I_o(hf)(1) &= [(\mathfrak{u}^-)^m(hf)](1) = \chi(h) \cdot [(\text{Ad}(h^{-1})\mathfrak{u}^-)^m f](1) \\ &= \chi(h)\epsilon(h)^m [(\mathfrak{u}^-)^m f](1) \end{aligned}$$

and

$$h(I_o(f))(1) = I_o(f)(h^{-1}) = \chi'(h) \cdot I_o(f)(1) = \chi'(h) \cdot [(\mathfrak{u}^-)^m f](1) .$$

We see that I_o does not commute with the action of h .

Remarks:

1. (Compare [Jan] II.2 and II.8.23.) Consider the algebraic character

$$\chi\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right) = t_1^{m_1} t_2^{m_2}$$

with $m_i \in \mathbb{Z}$ of T . Then χ is dominant (for the opposite Borel subgroup P^-) if and only if $c(\chi) = m_2 - m_1 \leq 0$. In this case the K -rational representation of GL_2 of highest weight χ (w.r.t. P^-) can be described as the algebraic induction $\text{ind}_P^G(\chi)$ which obviously is a finite dimensional G -invariant subspace of $\text{Ind}_P^G(\chi)$; in particular, M_χ has the finite dimensional $D(G, K)$ -module quotient $\text{ind}_P^G((w\chi)^{-1})$.

2. Consider the case $c(\chi) \in -\mathbb{N}_0$. By Thm. 6.1 and Prop. 6.2 the simple $D(G, K)$ -module $M_{\epsilon^{1-c(\chi)}\chi}$ maps isomorphically onto a simple $D(G, K)$ -submodule $M_\chi^{(0)}$ in M_χ . Morita shows in [Mor] that the quotient $M_\chi/M_\chi^{(0)}$ is dual to the G -invariant closed subspace of “locally polynomial” functions in $\text{Ind}_P^G(\chi)$ (see [Mor] 2-2 for the definition of this subspace.) Because the space of locally polynomial functions is a direct limit of finite dimensional spaces, it acquires the finest locally convex topology, and so $M_\chi/M_\chi^{(0)}$ is its full algebraic dual. Morita analyzes the space of locally polynomial functions in [Mor] §7 and shows that it is either simple as a $(\mathfrak{gl}_2(\mathbb{Q}_p), G)$ module or has a two step filtration with simple $(\mathfrak{gl}_2(\mathbb{Q}_p), G)$ -module subquotients (the bottom one being the algebraic induction discussed in the first remark). A fortiori, these subquotients are simple as $D(G, K)$ modules. The subquotients of the corresponding one or two-step dual filtration on $M_\chi/M_\chi^{(0)}$, are not in general simple modules. We also remark that the map I from the proof of Prop. 6.2 appears in a less conceptual form already in [Mor] 6-2 and is attributed there to Casselman.

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Peter Schneider
Mathematisches Institut
Westfälische Wilhelms-Universität Münster
Einsteinstr. 62
D-48149 Münster, Germany
pschnei@math.uni-muenster.de
<http://www.uni-muenster.de/math/u/schneider>

Jeremy Teitelbaum
Department of Mathematics, Statistics, and Computer Science (M/C 249)
University of Illinois at Chicago
851 S. Morgan St.
Chicago, IL 60607, USA
jeremy@uic.edu
<http://raphael.math.uic.edu/~jeremy>