

# Continuous representation theory of $p$ -adic Lie groups

Peter Schneider

**Abstract.** In this paper we give an overview over the basic features of the continuous representation theory of  $p$ -adic Lie groups as it has emerged during the last five years. The main motivation for developing such a theory is a possible extension of the local Langlands program to  $p$ -adic Galois representations. This is still very much in its infancy. But in the last section we will describe a first approximation to an extended Langlands functoriality principle for crystalline Galois representations.

**Mathematics Subject Classification (2000).** Primary 11S37, 22E50; Secondary 11F70.

**Keywords.** Lie groups, representation theory, Langlands functoriality.

## 1. Motivation

Throughout the paper we fix a finite extension  $L/\mathbb{Q}_p$  and let  $q$  denote the cardinality of its residue field. One object of major interest is the absolute Galois group  $\mathcal{G}_L := \text{Gal}(\bar{L}/L)$  where  $\bar{L}/L$  is an algebraic closure. One obtains a first very coarse idea of its structure by looking at the tower of fields  $L \subseteq L^{nr} \subseteq L^{tr} \subseteq \bar{L}$  where  $L^{nr}$ , resp.  $L^{tr}$ , denotes the maximal unramified, resp. tamely ramified, extension of  $L$ . Correspondingly we have the subgroups

$$\mathcal{P}_L := \text{Gal}(\bar{L}/L^{tr}) \subseteq \mathcal{I}_L := \text{Gal}(\bar{L}/L^{nr}) \subseteq \mathcal{G}_L .$$

All the complication of  $\mathcal{G}_L$  is contained in the pro- $p$ -group  $\mathcal{P}_L$ . The quotient  $\mathcal{I}_L/\mathcal{P}_L$  is pro-cyclic of a pro-order prime to  $p$ . The quotient  $\mathcal{G}_L/\mathcal{I}_L$  even has the natural Frobenius generator  $\phi$  which makes  $\mathcal{G}_L/\mathcal{I}_L$  naturally isomorphic to  $\hat{\mathbb{Z}}$ . In particular, this allows to introduce the Weil group

$$\mathcal{W}_L := \{g \in \mathcal{G}_L : g \equiv \phi^{\alpha(g)} \text{ mod } \mathcal{I}_L \text{ for some } \alpha(g) \in \mathbb{Z}\} ;$$

it is topologized by declaring the inertia subgroup  $\mathcal{I}_L$  to be open.

Throughout the paper  $K$  will denote the coefficient field of whatever kind of representation we like to consider. It always will have characteristic zero and, most of the time, will carry a topology. This introductory section is about representations of the Weil group  $\mathcal{W}_L$  by which we mean a continuous homomorphism

$$\rho : \mathcal{W}_L \longrightarrow GL(E)$$

where  $E$  is a finite dimensional  $K$ -vector space. In the case that  $K$  is an abstract field, i.e., that  $GL(E)$  carries the discrete topology, we will speak of a smooth representation. Of course, then  $\rho(\mathcal{I}_L)$  is a finite group. Even if we take  $K = \mathbb{C}$  to be the field of complex numbers with its natural topology we still have, since  $\mathcal{I}_L$  is totally disconnected, that  $\rho(\mathcal{I}_L)$  is a finite group.

The setting becomes richer if we choose a prime number  $\ell$  different from  $p$  and take for  $K = \overline{\mathbb{Q}_\ell}$  an algebraic closure of  $\mathbb{Q}_\ell$  with its  $\ell$ -adic topology. Then  $GL(E)$  contains an open subgroup which is pro- $\ell$ . This means that only the image  $\rho(\mathcal{P}_L)$  is finite. Nevertheless this setting still can be made “smooth” in the following way. By abuse of language we mean by a Weil-Deligne group representation a pair  $(\rho', N)$  consisting of a smooth representation  $\rho' : \mathcal{W}_L \rightarrow GL(E)$  in a finite dimensional  $K$ -vector space  $E$  and a (nilpotent) endomorphism  $N : E \rightarrow E$  such that

$$g \circ N \circ g^{-1} = q^{\alpha(g)} \cdot N \quad \text{for any } g \in \mathcal{W}_L .$$

The point is that by deriving the action (via  $\rho$ ) on  $E$  of the group  $\mathcal{I}_L / \ker(\rho)$  one obtains an endomorphism  $N$  of  $E$  and that dividing  $\rho$  through the exponential of  $N$  in an appropriate way results in a smooth representation  $\rho'$ . In fact, by Grothendieck’s abstract monodromy theorem, this construction sets up a natural bijection

$$\begin{array}{ccc} \text{isomorphism classes of continuous} & \longleftrightarrow & \text{isomorphism classes of Weil-Deligne} \\ \text{representations } \rho : \mathcal{W}_L \rightarrow GL(E) & & \text{group representations } (\rho', N). \end{array}$$

For a more detailed description we refer to [28]. The representation  $\rho$  is called Frobenius semisimple if the lifts in  $\mathcal{W}_L$  of the Frobenius  $\phi$  act semisimply on  $E$ . For the corresponding Weil-Deligne group representation  $(\rho', N)$  this amounts to the semisimplicity of the smooth representation  $\rho'$ .

In order to state the local Langlands correspondence we let  $G$  be the group of  $L$ -rational points of some connected reductive group over  $L$ . It naturally is a locally compact and totally disconnected group. Similarly as before a representation of  $G$  in a  $K$ -vector space  $V$  (but which here usually will be infinite dimensional) is called smooth if the corresponding map  $G \times V \rightarrow V$  is continuous with respect to the discrete topology on  $V$ . The local Langlands correspondence asserts ([15], [16]) the existence of a distinguished bijection

$$\begin{array}{ccc} \text{isomorphism classes of } n\text{-dim.} & & \text{isomorphism classes of irreducible} \\ \text{Frobenius semisimple continuous} & \longleftrightarrow & \text{smooth representation of} \\ \text{representations } \rho : \mathcal{W}_L \rightarrow GL(E) & & GL_n(L) \end{array}$$

for any  $n \geq 1$  (and where the coefficient field still is  $K = \overline{\mathbb{Q}_\ell}$ ). In fact, there is a much more general but conjectural local Langlands functoriality principle. The Langlands dual group  ${}^L G$  of  $G$  is a semidirect product

$${}^L G = {}^L G^\circ \rtimes \mathcal{G}_L$$

where  ${}^L G^\circ$  is the group of  $K$ -rational points of the connected reductive group over  $K$  whose root datum is dual to the root datum of  $G$  (over  $\bar{L}$ ). The functoriality

principle asserts that the set  $\Pi(G)$  of isomorphism classes of irreducible smooth representations of  $G$  has a distinguished partitioning into finite subsets  $\Pi_{\rho'}$  which are indexed by equivalence classes of certain Weil-Deligne group representations  $\rho'$  with values in  ${}^L G$ .

There is a particularly simple special case of this functoriality principle which is the unramified correspondence. An irreducible smooth representation of  $G$  is called unramified if it has a nonzero vector fixed by a good maximal compact subgroup. Suppose for simplicity that  $G$  is  $L$ -split. The Satake isomorphism for the Hecke algebra of this maximal compact subgroup produces from an unramified representation of  $G$  an orbit of unramified characters of a maximal split torus  $T$  in  $G$  and hence an orbit of points in  ${}^L T^\circ(K)$  where  ${}^L T^\circ \subseteq {}^L G^\circ$  is the dual torus. This means one has a natural bijection

$$\begin{array}{ccc} \text{isomorphism classes of unramified} & & \text{semisimple conjugacy} \\ \text{irreducible smooth representations} & \longleftrightarrow & \text{classes in } {}^L G^\circ. \\ \text{of } G & & \end{array}$$

Moreover the semisimple conjugacy class of an  $s \in {}^L G^\circ$  corresponds to the equivalence class of the unramified Weil-Deligne group representation  $\mathcal{W}_L \rightarrow \mathcal{W}_L/\mathcal{I}_L \rightarrow {}^L G^\circ$  which sends the Frobenius  $\phi$  to  $s$ . For more details about the functoriality conjecture we refer to [3].

In this paper we are interested in the case  $K = \bar{\mathbb{Q}}_p$ . Then there is no obvious restriction on the image  $\rho(\mathcal{P}_L)$  any more. This means that the  $p$ -adic representation theory of  $\mathcal{W}_L$  is drastically more complicated than the previous  $\ell$ -adic one. There is a natural functor, constructed by Fontaine ([13]), which associates with any continuous representation  $\rho : \mathcal{W}_L \rightarrow GL(E)$  a Weil-Deligne group representation  $Fon(\rho)$  in a free  $\hat{\mathbb{Q}}_p^{nr} \otimes_{\mathbb{Q}_p} K$ -module of finite rank; here  $\hat{\mathbb{Q}}_p^{nr}$  denotes the completion of  $\mathbb{Q}_p^{nr}$ . It should be viewed as a replacement for the monodromy theorem in the  $\ell$ -adic case. But its construction is much more involved. Moreover, for most  $\rho$  one in fact has  $Fon(\rho) = 0$ . The rich theory of  $p$ -adic Galois representations which has evolved during the last twenty years therefore is largely restricted to the so called potentially semistable ones, i.e., to those for which  $\rho$  and  $Fon(\rho)$  have the same rank. The author nevertheless is very much convinced that there is an extension of the Langlands functoriality principle which takes into account all  $p$ -adic representations of  $\mathcal{W}_L$ . Of course, it should be compatible with the traditional functoriality conjecture via the functor  $Fon$ .

Since the category of  $p$ -adic representations of  $\mathcal{W}_L$  is so much richer and more complicated than the category of  $\ell$ -adic representations it is clear that the author's conviction only has a chance if also on the side of the reductive group  $G$  we will be able to introduce a much richer but still reasonable category of representations of  $G$  than the category of smooth representations. The purpose of this paper is to report on such a construction which was developed in joint work with J. Teitelbaum. At the end we will come back to Galois representations and will briefly discuss a possible extension of the unramified correspondence.

## 2. Banach space representations

From now on we always assume  $K$  to be a finite extension of  $\mathbb{Q}_p$  and we let  $\mathfrak{o}$  denote the ring of integers in  $K$ . At first we let  $G$  be an arbitrary locally compact and totally disconnected group. Smoothness for a linear  $G$ -action on a  $K$ -vector space means continuity with respect to the discrete topology on the vector space. It therefore is a rather obvious idea to enrich the picture by considering continuous linear  $G$ -actions on some class of topological  $K$ -vector spaces.

A reasonable framework for topological  $K$ -vector spaces is provided by the notion of a locally convex topology. Such topologies on a  $K$ -vector space are defined by a family of nonarchimedean seminorms (cf. [21] §4). The open convex neighborhoods of the zero vector in a locally convex  $K$ -vector space are lattices, i.e.  $\mathfrak{o}$ -submodules which generate the vector space.

The most straightforward class of locally convex  $K$ -vector spaces is the class of  $K$ -Banach spaces. A  $K$ -Banach space is complete and its topology can be defined by a single nonarchimedean norm. A  $K$ -Banach space representation of  $G$  is a linear  $G$ -action on a  $K$ -Banach space  $V$  such that the corresponding map  $G \times V \rightarrow V$  is continuous. Unfortunately this definition is much too general in order to lead to a reasonable category. We mention only two pathological phenomena:

- There can exist nonzero  $G$ -equivariant continuous linear maps between two non-isomorphic topologically irreducible Banach space representations.
  - Already such a simple group as the additive group of  $p$ -adic integers  $G = \mathbb{Z}_p$  has infinite dimensional topologically irreducible Banach space representations ([8]).
- The challenge is to impose an additional “finiteness” condition leading to a category of representations which at the same time is rich enough and still is manageable.

To prepare the subsequent definition taken from [23] we point out the following. Let  $V$  be any  $K$ -Banach space representation of  $G$ . Given any compact open subgroup  $H \subseteq G$  and any open lattice  $M \subseteq V$  the  $\mathfrak{o}$ -submodule  $\bigcap_{g \in H} gM$  is an  $H$ -invariant open lattice in  $V$ . We also recall that an  $\mathfrak{o}$ -module  $N$  is called of cofinite type if its Pontrjagin dual  $\text{Hom}_{\mathfrak{o}}(N, K/\mathfrak{o})$  is a finitely generated  $\mathfrak{o}$ -module.

**Definition 2.1.** A  $K$ -Banach space representation  $V$  of  $G$  is called admissible if for any compact open subgroup  $H \subseteq G$ , for any bounded  $H$ -invariant open lattice  $M \subseteq V$ , and for any open subgroup  $H' \subseteq H$  the  $\mathfrak{o}$ -submodule  $(V/M)^{H'}$  of  $H'$ -invariant elements in the quotient  $V/M$  is of cofinite type.

We let  $\text{Ban}_G^{\mathfrak{o}}(K)$  denote the category of all admissible  $K$ -Banach space representations of  $G$  with continuous linear  $G$ -equivariant maps.

A first justification for this definition might be the following observation. We recall that a smooth representation of  $G$  is called admissible if the subspace of  $H'$ -fixed vectors, for any compact open subgroup  $H' \subseteq G$ , is finite dimensional. Let  $V$  be a  $K$ -Banach space representation of  $G$ , let  $H \subseteq G$  be a compact open subgroup, and let  $M \subseteq V$  be a bounded  $H$ -invariant open lattice. Then  $M/\pi M$ , where  $\pi$  is a prime element in  $\mathfrak{o}$ , evidently is a smooth representation of  $H$  over the residue field of  $K$ . If  $V$  is admissible then  $M/\pi M$  is an admissible smooth representation of  $H$ .

In which sense is this category  $\text{Ban}_G^a(K)$  manageable? In order to answer this question we have to assume that  $G$  is a locally  $\mathbb{Q}_p$ -analytic group, i. e., a  $p$ -adic Lie group. Moreover, since the definition of admissibility is in terms of compact open subgroups it suffices to discuss this issue for compact groups. In the following we therefore let  $H$  be a compact  $p$ -adic Lie group. The completed group ring of  $H$  is defined to be

$$o[[H]] := \varprojlim o[H/H']$$

where  $H'$  runs over all open normal subgroups of  $H$ . This is a compact linear-topological  $o$ -algebra. The  $H$ -action on any  $K$ -Banach space representation extends uniquely to a separately continuous  $o[[H]]$ -module structure. We have the following crucial fact due to Lazard ([18] V.2.2.4).

**Theorem 2.2.** *For any compact  $p$ -adic Lie group  $H$  the ring  $o[[H]]$  is noetherian.*

To understand how we will explore this fact let us look at the vector space  $C(H, K)$  of all  $K$ -valued continuous functions on  $H$ . This is a  $K$ -Banach space representation of  $H$  for the sup-norm and the left translation action of  $H$ . But it is difficult to say anything straightforward about the corresponding  $o[[H]]$ -module  $C(H, K)$ . Instead, let us pass to the continuous dual

$$D^c(H, K) := C(H, K)' .$$

First of all  $D^c(H, K)$  is a  $K$ -algebra with respect to the convolution product of continuous linear forms – the algebra of continuous distributions on  $H$ . Secondly, sending an element  $g \in H$  to the Dirac distribution  $\delta_g \in D^c(H, K)$  extends to an embedding of  $o$ -algebras

$$o[[H]] \hookrightarrow D^c(H, K)$$

whose image is a lattice so that, in fact,  $K \otimes_o o[[H]] \cong D^c(H, K)$ . Thirdly, via the latter isomorphism, the  $K \otimes_o o[[H]]$ -module structure on  $C(H, K)'$  induced by functoriality from the  $o[[H]]$ -module structure on  $C(H, K)$  simply corresponds to the action of  $D^c(H, K)$  on itself by multiplication. Quite generally, the continuous dual  $V'$  of a  $K$ -Banach space representation  $V$  of  $H$ , by functoriality, is a module over the  $K$ -algebra  $D^c(H, K)$ . The main result of [23] Thm. 3.5 now is the following.

**Theorem 2.3.** *Let  $\text{Mod}_{fg}(D^c(H, K))$  denote the category of finitely generated  $D^c(H, K)$ -modules; the functor*

$$\begin{aligned} \text{Ban}_H^a(K) &\xrightarrow{\sim} \text{Mod}_{fg}(D^c(H, K)) \\ V &\longmapsto V' \end{aligned}$$

*is an anti-equivalence of categories.*

By Thm. 2.2 the ring  $D^c(H, K)$  is noetherian. Hence with  $\text{Mod}_{fg}(D^c(H, K))$  also  $\text{Ban}_G^a(K)$  is an abelian category. We mention that the underlying vector space of the kernel, image, and cokernel of a morphism in  $\text{Ban}_G^a(K)$  is the kernel, image,

and cokernel, respectively, of the underlying linear map. We also emphasize that the above result completely algebraizes the theory of admissible  $K$ -Banach space representations.

As evidence for the category  $\text{Ban}_G^a(K)$  being rich enough we will discuss the continuous principal series of the group  $G$  of  $L$ -rational points of a connected reductive group over  $L$ . Let  $P \subseteq G$  be a parabolic subgroup and  $\chi : P \rightarrow K^\times$  be a continuous character. We put

$${}^c\text{Ind}_P^G(\chi) := \{f : G \rightarrow K \text{ continuous: } f(gb) = \chi(b)^{-1}f(g) \text{ for } g \in G, b \in P\}$$

on which  $G$  acts by left translations. If  $G_0 \subseteq G$  is a good maximal compact subgroup then we have the Iwasawa decomposition  $G = G_0P$ . Hence taking the supremum over  $G_0$  defines a norm on  ${}^c\text{Ind}_P^G(\chi)$ . A different choice of  $G_0$  leads to an equivalent norm.

**Proposition 2.4.**  *${}^c\text{Ind}_P^G(\chi)$  is an admissible  $K$ -Banach space representation of the reductive group  $G$ .*

*Proof.* We only indicate the argument for the admissibility. Restricting functions to  $G_0$  defines a topological embedding  ${}^c\text{Ind}_P^G(\chi) \rightarrow C(G_0, K)$ . Dually this exhibits the continuous dual  ${}^c\text{Ind}_P^G(\chi)'$  as a quotient of  $D^c(G_0, K)$ .  $\square$

It seems likely that the representation  ${}^c\text{Ind}_P^G(\chi)$  always has a finite composition series. We want to state a precise irreducibility conjecture. To avoid technicalities we assume that  $L \subseteq K$ , that  $G$  is  $L$ -split, semisimple, and simply connected, and that  $P$  is a Borel subgroup. Let  $T \subseteq P$  be a maximal  $L$ -split torus. Then  $\chi$  can be viewed as a continuous character  $\chi : T \rightarrow K^\times$ . Let  $X^*(T)$  resp.  $X_*(T)$ , denote as usual the group of rational characters, resp. cocharacters, of  $T$ . In  $X^*(T)$  we have the subsets  $\Delta \subseteq \Phi^+$  of simple and of positive roots with respect to  $P$ , respectively. For any  $\alpha \in \Phi^+$  there is the corresponding coroot  $\check{\alpha} \in X_*(T)$ . By our assumption on  $G$  any fundamental weight  $\omega_\alpha$ , for  $\alpha \in \Delta$ , and hence their sum  $\delta := \sum_{\alpha \in \Delta} \omega_\alpha$  lie in  $X^*(T)$ . The character  $\chi : T \rightarrow K^\times$  is called anti-dominant if  $\chi \delta \circ \check{\alpha} \neq (\ )^m$  for any integer  $m \geq 1$  and any  $\alpha \in \Phi^+$ . Here  $(\ )^m : L^\times \rightarrow K^\times$ , for any  $m \in \mathbb{Z}$ , is the continuous character sending  $a$  to  $a^m$ .

**Conjecture 2.5.** *The  $G$ -representation  ${}^c\text{Ind}_P^G(\chi)$  is topologically irreducible if  $\chi^{-1}$  is anti-dominant.*

**Proposition 2.6.** *Suppose that  $L = \mathbb{Q}_p$ ; we then have:*

- i. The above conjecture holds true for the group  $G = GL_2(\mathbb{Q}_p)$ ;*
- ii. if the anti-dominance condition for  $\chi^{-1}$  continues to hold after restriction to an arbitrary small open subgroup of  $\mathbb{Q}_p^\times$  then the continuous dual  ${}^c\text{Ind}_P^G(\chi)'$  is simple as a  $D^c(G_0, K)$ -module and  ${}^c\text{Ind}_P^G(\chi)$ , in particular, is topologically irreducible as a  $G_0$ -representation.*

The proof of this result is highly indirect. It requires corresponding facts for the locally analytic principal series (see section 3) and the density of analytic vectors in admissible Banach space representations (see section 4).

Breuil has introduced the notion of a unitary Banach space representation of  $G$  which means that the Banach space topology can be defined by a  $G$ -invariant norm. In recent work Berger/Breuil ([2]) and Colmez ([5]) construct, by very sophisticated methods, a series of topologically irreducible, admissible, and unitary Banach space representations of the group  $GL_2(\mathbb{Q}_p)$ . In fact, they put this series into correspondence with certain two dimensional Galois representations of  $\mathcal{G}_{\mathbb{Q}_p}$  called trianguline by Colmez.

We also point out that in a unitary Banach space representation  $V$  of  $G$  we find a  $G$ -invariant open lattice  $M \subseteq V$ . If  $V$  is admissible then  $M/\pi M$ , with  $\pi \in \mathfrak{o}$  a prime element, is an admissible smooth representation of  $G$  over the residue field of  $K$ . As to be expected, the theory described in this section therefore is closely related to the smooth representation theory of  $p$ -adic groups with characteristic  $p$  coefficients (cf. [29]).

### 3. Locally analytic representation

Many continuous representations of  $p$ -adic Lie groups in locally convex  $K$ -vector spaces which naturally arise in geometric situations are not Banach space representations. One basic example is the following. Let  $G = GL_2(L)$  and let  $\mathcal{X}_L := \mathbb{P}^1(\bar{L}) \setminus \mathbb{P}^1(L)$  be the  $p$ -adic upper half plane over  $L$ . The rigid analytic variety  $\mathcal{X}_L$  carries an obvious  $G$ -action. The vector space  $\Omega^1(\mathcal{X}_L)$  of global holomorphic 1-forms on  $\mathcal{X}_L$  is an  $L$ -Fréchet space with a natural continuous  $G$ -action. By a theorem of Morita ([19]) its continuous dual is isomorphic to the quotient  $C^{an}(\mathbb{P}^1(L), L)/L$  of the vector space  $C^{an}(\mathbb{P}^1(L), L)$  of  $L$ -valued locally  $L$ -analytic functions on the projective line  $\mathbb{P}^1(L)$  by the subspace of constant functions. Clearly,  $C^{an}(\mathbb{P}^1(L), L)$  is not a Banach space. In fact, its natural locally convex topology is rather complicated. But in return it has good properties like being reflexive.

This observation leads to the concept of a locally analytic representation of a locally  $L$ -analytic group  $G$ . For the rest of the paper we assume that  $L \subseteq K$ . We first remark that for any paracompact locally  $L$ -analytic manifold  $X$  and any Hausdorff locally convex  $K$ -vector space  $V$  the  $K$ -vector space  $C^{an}(X, V)$  of  $V$ -valued locally analytic functions on  $X$  is well defined. It carries a natural Hausdorff locally convex topology for whose rather technical construction we refer to [12].

**Definition 3.1.** A locally analytic representation  $V$  of  $G$  (over  $K$ ) is a barrelled locally convex Hausdorff  $K$ -vector space  $V$  equipped with a  $G$ -action by continuous linear endomorphisms such that, for each  $v \in V$ , the map  $g \mapsto gv$  lies in  $C^{an}(G, V)$ .

The requirement that  $V$  is barrelled (i.e., that each closed lattice in  $V$  is open) is a convenient (and mild) technical restriction which makes applicable the Banach-Steinhaus theorem ([21] Prop. 6.15). It implies that the map  $G \times V \rightarrow V$  describing the  $G$ -action is continuous and that the Lie algebra  $\mathfrak{g}$  of  $G$  acts continuously on  $V$ .

by

$$\begin{aligned} \mathfrak{g} \times V &\rightarrow V \\ (\mathfrak{x}, v) &\mapsto \mathfrak{x}v := \frac{d}{dt} \exp(t\mathfrak{x})v|_{t=0} . \end{aligned}$$

The next step is, as in the previous section, to pass from the  $G$ -action to a module structure. The strong dual

$$D(G, K) := C^{an}(G, K)'_b$$

is called the locally convex vector space of  $K$ -valued locally analytic distributions on  $G$ .

**Proposition 3.2.** *i. The convolution on  $D(G, K)$  is well defined, is separately continuous, and makes  $D(G, K)$  into an associative  $K$ -algebra with the Dirac distribution  $\delta_1$  in  $1 \in G$  as the unit element;*

*ii. the map*

$$\begin{aligned} \mathfrak{g} &\longrightarrow D(G, K) \\ \mathfrak{x} &\longmapsto [f \mapsto ((-\mathfrak{x})f)(1)] \end{aligned}$$

*extends to a monomorphism of  $K$ -algebras  $U(\mathfrak{g}) \otimes_L K \longrightarrow D(G, K)$  where  $U(\mathfrak{g})$  denotes the universal enveloping algebra of  $\mathfrak{g}$ ;*

*iii. the  $G$ -action on any locally analytic representation  $V$  of  $G$  extends uniquely to a separately continuous action of the algebra  $D(G, K)$  on  $V$ .*

*Proof.* i. [11] 4.4.1 and 4.4.4 or [26] Remark A.1. ii. [24] p. 450. iii. [24] Prop. 3.2.  $\square$

It is important to notice ([24] Lemma 2.1) that for a compact group  $G$  the locally convex topology on  $C^{an}(G, K)$  is of compact type ([24] §1 and [21] §16). In particular,  $D(G, K)$  then is a Fréchet algebra. Locally convex topologies of compact type are rather complicated but have the remarkable property that they can be characterized by their strong dual being nuclear and Fréchet. This leads to the following result ([24] Cor. 3.4).

**Proposition 3.3.** *If  $G$  is compact then the functor*

$$\begin{array}{ccc} \text{locally analytic representations of } G & & \text{continuous } D(G, K)\text{-modules on} \\ \text{on } K\text{-vector spaces of compact type} & \xrightarrow{\sim} & \text{nuclear Fréchet spaces with} \\ \text{with continuous linear } G\text{-maps} & & \text{continuous } D(G, K)\text{-module maps} \end{array}$$

*which sends  $V$  to its strong dual  $V'_b$  is an anti-equivalence of categories.*

This last result means that in the context of locally analytic representations we now have singled out the correct type of locally convex topology to which we should restrict our attention. But we are still lacking the equivalent of the algebraic finiteness condition which we have imposed in the last section. The situation is



considerably complicated by the fact that even for compact  $G$  the algebra  $D(G, K)$  is far from being noetherian. As a remedy for this we develop in [25] §3 the following axiomatic framework.

Suppose that  $A$  is a  $K$ -Fréchet algebra. For any continuous algebra seminorm  $q$  on  $A$  the completion  $A_q$  of  $A$  with respect to  $q$  is a  $K$ -Banach algebra. For any two such seminorms  $q' \leq q$  the identity on  $A$  extends to a continuous  $K$ -algebra homomorphism  $\phi_q^{q'} : A_q \rightarrow A_{q'}$ .

**Definition 3.4.** A  $K$ -Fréchet algebra  $A$  is called a Fréchet-Stein algebra if there is a sequence  $q_1 \leq \dots \leq q_n \leq \dots$  of continuous algebra seminorms on  $A$  which define the Fréchet topology and such that

- (i)  $A_{q_n}$  is (left) noetherian, and
- (ii)  $A_{q_n}$  is flat as a right  $A_{q_{n+1}}$ -module (via  $\phi_{q_{n+1}}^{q_n}$ )

for any  $n \in \mathbb{N}$ .

Suppose that  $A$  is a Fréchet-Stein algebra as in the above definition. We have the obvious isomorphism of Fréchet algebras

$$A \xrightarrow{\cong} \varprojlim_n A_{q_n} .$$

**Definition 3.5.** A (left)  $A$ -module  $N$  is called coadmissible if

- (i)  $A_{q_n} \otimes_A N$  is finitely generated over  $A_{q_n}$  for any  $n \in \mathbb{N}$ , and
- (ii) the natural map  $N \xrightarrow{\cong} \varprojlim_n (A_{q_n} \otimes_A N)$  is an isomorphism.

We let  $\mathcal{C}_A$  denote the full subcategory of all coadmissible modules in the category  $\text{Mod}(A)$  of all left  $A$ -modules. By a cofinality argument it is independent of the particular choice of the sequence of seminorms  $q_n$ .

**Proposition 3.6.**  $\mathcal{C}_A$  is an abelian subcategory of  $\text{Mod}(A)$  containing all finitely presented  $A$ -modules.

Over a noetherian Banach algebra every finitely generated module carries a unique Banach space topology which makes the module structure continuous. Using its representation as a projective limit in Def. 3.5 (ii) we therefore see that any coadmissible  $A$ -module carries a distinguished Fréchet topology which will be called its canonical topology.

In [25] Thm. 5.1 we give the following justification for introducing these definitions.

**Theorem 3.7.** For any compact locally  $L$ -analytic group  $G$  the Fréchet algebra  $D(G, K)$  is a  $K$ -Fréchet-Stein algebra.

If  $G_{\mathbb{Q}_p}$  denotes the locally  $\mathbb{Q}_p$ -analytic group which underlies  $G$  then one has a natural continuous surjection

$$D(G_{\mathbb{Q}_p}, K) \longrightarrow D(G, K) .$$

By a general argument with Fréchet-Stein algebras ([25] Prop. 3.7) this allows to reduce the proof of the theorem to the case  $L = \mathbb{Q}_p$ . Furthermore it is easy to see that it suffices to prove the theorem for a conveniently chosen open normal subgroup of  $G$ . To avoid technicalities we assume in the following that  $p \neq 2$ . Let  $\omega_p$  denote the additive  $p$ -adic valuation on  $\mathbb{Z}_p$ . Lazard in [18] III.2.1.2 has introduced the notion of a  $p$ -valuation on a group  $H$  which is a real valued function  $\omega : H \setminus \{1\} \longrightarrow (1/(p-1), \infty)$  such that

$$\begin{aligned} \omega(gh^{-1}) &\geq \min(\omega(g), \omega(h)) , \\ \omega(g^{-1}h^{-1}gh) &\geq \omega(g) + \omega(h) , \text{ and} \\ \omega(g^p) &= \omega(g) + 1 . \end{aligned}$$

Our proof of the above theorem very much relies on the techniques and results in [18]. First of all we may assume, by the above reductions and [9] Cor. 8.34 (ii), that  $G$  is a compact  $p$ -adic Lie group which has an (ordered) set of topological generators  $h_1, \dots, h_d$  such that:

(i) The map

$$\begin{aligned} \mathbb{Z}_p^d &\xrightarrow{\sim} G \\ (x_1, \dots, x_d) &\longmapsto h_1^{x_1} \cdot \dots \cdot h_d^{x_d} \end{aligned}$$

is a global chart of the manifold  $G$ .

(ii) The function

$$\omega(h_1^{x_1} \cdot \dots \cdot h_d^{x_d}) := \min_{1 \leq i \leq d} (1 + \omega_p(x_i))$$

is a  $p$ -valuation on  $G$ .

(iii) Every  $g \in G$  such that  $\omega(g) \geq 2$  is a  $p$ -th power in  $G$ .

We define  $b_i := \delta_{h_i} - 1$  and, for any multi-index  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ ,

$$\mathbf{b}^\alpha := b_1^{\alpha_1} \cdot \dots \cdot b_d^{\alpha_d} \in D(G, K)$$

(which does depend on the ordering of the generators  $h_i$ !). Using the global chart in (i) we may identify  $D(G, K)$  as a Fréchet space with  $D(\mathbb{Z}_p^d, K)$ . The latter, by Amice's  $p$ -adic Fourier isomorphism ([1]), is naturally isomorphic to the ring of power series with coefficients in  $K$  converging on the open unit polydisk. It follows that any distribution  $\lambda \in D(G, K)$  has a unique convergent expansion

$$\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha$$

where the set  $\{|d_\alpha|r^{\alpha_1+\dots+\alpha_d}\}_{\alpha \in \mathbb{N}_0^d}$ , for any  $0 < r < 1$ , is bounded. Moreover, the family of norms

$$\|\lambda\|_r := \sup_{\alpha \in \mathbb{N}_0^d} |d_\alpha|r^{\alpha_1+\dots+\alpha_d}$$

defines the Fréchet topology on  $D(G, K)$ . Lazard in [18] investigates the norm  $\|\cdot\|_{\frac{1}{p}}$ .

**Lemma 3.8.** *For each  $\frac{1}{p} \leq r < 1$  the norm  $\|\cdot\|_r$  is submultiplicative.*

The completion  $D_r(G, K)$  of  $D(G, K)$  with respect to the norm  $\|\cdot\|_r$  for  $\frac{1}{p} \leq r < 1$ , is a  $K$ -Banach algebra and

$$D(G, K) = \varprojlim_{\frac{1}{p} \leq r < 1} D_r(G, K).$$

**Proposition 3.9.** *For  $\frac{1}{p} < r' \leq r < 1$  in  $p^\mathbb{Q}$  we have:*

- i.  $D_r(G, K)$  is noetherian with multiplicative norm  $\|\cdot\|_r$ ;*
- ii. the homomorphism  $D_r(G, K) \longrightarrow D_{r'}(G, K)$  is flat.*

This is proved in [25], following the model for  $\|\cdot\|_{\frac{1}{p}}$  in [18], by explicitly computing the associated graded ring for the filtration on  $D_r(G, K)$  defined by the norm  $\|\cdot\|_r$ . Philosophically this technique means to view the noncommutative ring  $D_r(G, K)$  as a deformation quantization of the commutative ring  $D_r(\mathbb{Z}_p^d, K)$  which allows to transfer many ring theoretic properties from the latter to the former.

For later purposes we mention the following important additional result in [14] §1.4 Cor. 2.

**Theorem 3.10.** *(Frommer) Suppose that  $G$  is as above and that  $\frac{1}{p} < r < 1$  in  $p^\mathbb{Q}$ ; then  $D_r(G, K)$  is finitely generated free as a module over the closure of the universal enveloping algebra  $U(\mathfrak{g}) \otimes_L K$ .*

By [17] Thm. 1.4.2 this result continues to hold for  $L \neq \mathbb{Q}_p$  but the precise conditions on the compact locally  $L$ -analytic group  $G$  are a little too technical to be formulated here. Having Thm. 3.7 at our disposal we now make the following definition where  $G$  again is a general locally  $L$ -analytic group.

**Definition 3.11.** A locally analytic representation of  $G$  on a  $K$ -vector space of compact type  $V$  is called admissible if its strong dual  $V'_b$  as a  $D(H, K)$ -module, for some (or equivalently any) compact open subgroup  $H \subseteq G$ , is coadmissible with its canonical topology.

Using Prop. 3.6 we obtain the following facts about the category  $\text{Rep}_G^a(K)$  of all admissible locally analytic representations of  $G$  ([25] Prop. 6.4).

- Proposition 3.12.**
- i.  $\text{Rep}_G^a(K)$  is an abelian category; kernel and image of a morphism in  $\text{Rep}_G^a(K)$  are the algebraic kernel and image with the subspace topology;*
  - ii. any morphism in  $\text{Rep}_G^a(K)$  is strict and has closed image;*
  - iii. the category  $\text{Rep}_G^a(K)$  is closed with respect to the passage to closed  $G$ -invariant subspaces.*

A first evidence that this category  $\text{Rep}_G^a(K)$  is not too small is provided by the following result ([25] Thm. 6.6). Let  $\text{Rep}_G^{\infty,a}(K)$  denote the abelian category of admissible smooth representations of  $G$  in  $K$ -vector spaces as recalled in section 2.

**Theorem 3.13.** *The functor  $\text{Rep}_G^{\infty,a}(K) \longrightarrow \text{Rep}_G^a(K)$  of equipping an admissible smooth representation with the finest locally convex topology is a fully faithful embedding; its image is characterized by the condition that the Lie algebra  $\mathfrak{g}$  acts trivially (i.e.,  $\mathfrak{g}V = 0$ ).*

This says that the representation theory appearing in the Langlands functoriality conjecture is fully contained in the new admissible locally analytic theory. It is rather obvious that in case  $G$  is an algebraic group also every rational representation of  $G$  is admissible locally analytic. Similarly as in the previous section there is a locally analytic principal series for any connected reductive group  $G$  over  $L$ . Let  $P \subseteq G$  be a parabolic subgroup and  $\chi : P \longrightarrow K^\times$  be a locally  $L$ -analytic character. Then

$$\text{Ind}_P^G(\chi) := \{f : G \longrightarrow K \text{ locally analytic: } f(gb) = \chi(b)^{-1}f(g) \text{ for } g \in G, b \in P\}$$

with  $G$  acting by left translations is an admissible locally analytic representation of  $G$ . This is proved quite similarly as in the Banach space case. Again we expect that  $\text{Ind}_P^G(\chi)$  always has a finite composition series. But there are important differences. For example, if  $\chi$  is locally constant then the smooth induction  ${}^\infty\text{Ind}_P^G(\chi)$  is a proper closed subspace of  $\text{Ind}_P^G(\chi)$  but is dense in  ${}^c\text{Ind}_P^G(\chi)$ . At the moment the deepest result about the locally analytic principal series is the following. If  $\mathfrak{p}$  denotes the Lie algebra of  $P$  then we have the derived character  $d\chi : \mathfrak{p} \longrightarrow K$  and we may form the Verma module  $\mathcal{V}_{d\chi} := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} K_{d\chi}$  of  $U(\mathfrak{g}) \otimes_L K$ .

**Theorem 3.14.** *(Frommer) Suppose that  $L = \mathbb{Q}_p$  and that  $G$  is  $\mathbb{Q}_p$ -split; if  $\mathcal{V}_{-d\chi}$  is a simple module for  $U(\mathfrak{g}) \otimes_L K$  then  $\text{Ind}_P^G(\chi)$  is topologically irreducible as a representation of  $G$*

It is his Thm. 3.10 which allows Frommer to make this close connection to Verma modules. In view of Kohlhaase's generalization of Thm. 3.10 it seems very likely that Thm. 3.14 holds for any  $L$ -split group  $G$  where  $L$  is arbitrary. Without recalling the details we mention that the simplicity of Verma modules is decided by anti-dominance properties of the inducing character. In [22] §4 we determine, in the case of the group  $G = SL_2(\mathbb{Q}_p)$ , a complete composition series for the reducible locally analytic principal series; its length is two or three.

We want to briefly mention three further results in the locally analytic theory. Extending Harish Chandra's computation of the center of  $U(\mathfrak{g})$  Kohlhaase determines in [17] the center of  $D(G, K)$ . His result is particularly simple to formulate in the following case ([17] Thm.s 2.1.6 and 2.4.2).

**Theorem 3.15.** *(Kohlhaase) Suppose that  $G$  is an  $L$ -split connected reductive group with maximal  $L$ -split torus  $T$  and corresponding Weyl group  $W$ . There is a canonical isomorphism*

$$\text{center of } D(G, K) \cong D(T, K)_Z^W$$

where the right hand side denotes the subalgebra of  $D(T, K)$  consisting of all  $W$ -invariant distributions supported on the center  $Z$  of  $G$ . Furthermore, if in addition  $Z = \{1\}$  then  $D(T, K)_Z^W$  is isomorphic to the ring of entire functions on the rigid analytic affine space over  $K$  of dimension equal to the rank of  $T$ .

A very basic construction in representation theory is the construction of the contragredient representation. But in the present context the strong dual  $V'_b$  of a  $K$ -vector space of compact type  $V$  is a Fréchet space and hence rarely is again of compact type. The naive method to define the contragredient representation therefore does not work in the locally analytic theory. Perhaps this is not too surprising since we already have used up, so to speak, the strong dual to define the notion of admissibility. In [26] we construct in case  $L = \mathbb{Q}_p$ , as a replacement for the contragredient, a natural auto-antiequivalence of a certain derived category of  $\text{Rep}_G^a(K)$ . This is based on the fact that the rings  $D_r(G, K)$ , for a  $p$ -valued Lie group  $G$  as discussed above, are Auslander regular, hence that the category  $\mathcal{C}_G$  in an appropriate sense is Auslander regular, and it then uses the derived functor of the functor  $\text{Hom}_{D(G, K)}(\cdot, D(G, K))$  on  $\mathcal{C}_G$ . The restriction to the case  $L = \mathbb{Q}_p$  will be removed in the forthcoming thesis of T. Schmidt.

As exemplified by the various principal series, a very important method to construct representations is, in the case of a reductive group  $G$ , the induction from the Levi quotient of a parabolic subgroup. In the smooth theory this functor, called parabolic induction, has an adjoint functor in the opposite direction, called parabolic restriction or Jacquet functor. In [10] Emerton constructs by rather sophisticated techniques a generalization of the Jacquet functor to locally analytic representations. Although no longer adjoint to parabolic induction it is doubtlessly an important construction which must be investigated further.

## 4. Analytic vectors

In order to describe the connection between Banach space and locally analytic representations we let  $V$  be a  $K$ -Banach space representation of the locally  $L$ -analytic group  $G$ .

**Definition 4.1.** A vector  $v \in V$  is called analytic if the (continuous) map  $\rho_v(g) := g^{-1}v$  lies in  $C^{an}(G, V)$ .

Obviously  $V_{an} := \{v \in V : v \text{ is analytic}\}$  is a  $G$ -invariant subspace of  $V$ . But we always equip  $V_{an}$  with the subspace topology with respect to the  $G$ -equivariant embedding

$$\begin{aligned} V_{an} &\longrightarrow C^{an}(G, V) \\ v &\longmapsto \rho_v . \end{aligned}$$

One checks that  $V_{an}$  is closed in  $C^{an}(G, V)$ . Of course, in this generality the subspace  $V_{an}$  might very well be zero. But in [25] Thm. 7.1 the following is shown.

**Theorem 4.2.** *Suppose that  $L = \mathbb{Q}_p$ ; if  $V$  is admissible then  $V_{an}$  is dense in  $V$  and is an admissible locally analytic representation of  $G$ ; moreover the functor*

$$\begin{aligned} \text{Ban}_G^a(K) &\longrightarrow \text{Rep}_G^a(K) \\ V &\longmapsto V_{an} \end{aligned}$$

*is exact.*

The key reason for this result is the following purely algebraic fact ([25] Thm. 5.2).

**Theorem 4.3.** *Suppose that  $L = \mathbb{Q}_p$  and that  $G$  is compact; then the natural ring homomorphism*

$$D^c(G, K) \longrightarrow D(G, K)$$

*is faithfully flat.*

In the case of the principal series for a locally  $L$ -analytic character  $\chi$  we, of course, have

$${}^c \text{Ind}_P^G(\chi)_{an} = \text{Ind}_P^G(\chi) .$$

It is a remarkable fact that  ${}^c \text{Ind}_P^G(\chi)$  and  $\text{Ind}_P^G(\chi)$  can have different length. In particular, the former can be topologically irreducible and the latter not.

## 5. Unramified $p$ -adic functoriality

For the simplicity of the presentation we assume in this section that the base field is  $L = \mathbb{Q}_p$ . As before  $K/\mathbb{Q}_p$  is a finite extension of which we assume that it contains a square root  $p^{1/2}$  of  $p$ . The group  $G$  is assumed to be  $\mathbb{Q}_p$ -split. We let  $T \subseteq G$  be a maximal split torus and we fix a maximal compact subgroup  $U \subseteq G$  which is special for  $T$ . The Satake-Hecke algebra  $\mathcal{H}(G, 1_U)$  is the convolution algebra of  $K$ -valued  $U$ -bi-invariant compactly supported functions on  $G$ . By the Satake isomorphism this algebra is commutative and its characters  $\zeta$  into  $\bar{K}$  are in natural bijection with the semisimple conjugacy classes  $s(\zeta)$  in  ${}^L G^\circ(\bar{K})$ . Representation theoretically the Satake-Hecke algebra can be described as the algebra of endomorphisms of the smooth  $G$ -representation  $\text{ind}_U^G(1_U)$  over  $K$  obtained by compact induction from the trivial representation  $1_U$  of  $U$ . Any character  $\zeta$  of  $\mathcal{H}(G, 1_U)$  therefore gives rise, by specialization, to the smooth  $G$ -representation

$$\text{ind}_U^G(1_U) \otimes_{\mathcal{H}(G, 1_U)} \bar{K}_\zeta$$

which has a unique irreducible quotient  $V_\zeta$ . The correspondence  $V_\zeta \longleftrightarrow s(\zeta)$  is the unramified (smooth) Langlands functoriality we have alluded to already in the first section. We also repeat that  $s(\zeta)$  should be viewed as the Weil-Deligne group representation  $\mathcal{W}_{\mathbb{Q}_p} \rightarrow \mathcal{W}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p} \rightarrow {}^L G^\circ(\bar{K})$  which sends the Frobenius  $\phi$  to  $s(\zeta)$  (and with  $N = 0$ ).

We now broaden the picture by bringing in an irreducible  $\mathbb{Q}_p$ -rational representation  $\sigma$  of  $G$  of highest weight  $\xi \in X^*(T)$ . The corresponding Satake-Hecke algebra  $\mathcal{H}(G, \sigma_U)$  is the convolution algebra over  $K$  of all compactly supported functions  $\psi : G \rightarrow \text{End}_K(\sigma)$  satisfying

$$\psi(u_1 g u_2) = \sigma(u_1) \circ \psi(g) \circ \sigma(u_2) \quad \text{for any } u_1, u_2 \in U \text{ and } g \in G .$$

Again the algebra  $\mathcal{H}(G, \sigma_U)$  can naturally be identified with the algebra of endomorphisms of the compact induction  $\text{ind}_U^G(\sigma_U)$  of the restriction  $\sigma_U := \sigma|_U$ . In fact, since  $\sigma$  is a representation of the full group  $G$  the algebras  $\mathcal{H}(G, \sigma_U)$  and  $\mathcal{H}(G, 1_U)$  are isomorphic. But fixing once and for all a  $U$ -invariant norm  $\| \cdot \|$  on the  $K$ -vector space which underlies  $\sigma$  we may equip  $\mathcal{H}(G, \sigma_U)$  with the sup-norm

$$\| \psi \|_\xi := \sup_{g \in G} \| \psi(g) \|$$

where on the right hand side  $\| \cdot \|$  refers to the operator norm on  $\text{End}_K(\sigma)$ . Since  $\| \cdot \|_\xi$  obviously is submultiplicative the algebra  $\mathcal{H}(G, \sigma_U)$  gives rise, by completion with respect to  $\| \cdot \|_\xi$ , to a  $K$ -Banach algebra  $\mathcal{B}(G, \sigma_U)$ . Clearly,  $\mathcal{B}(G, \sigma_U)$  is very far from being isometrically isomorphic to  $\mathcal{B}(G, 1_U)$ . Correspondingly we have a sup-norm on  $\text{ind}_U^G(\sigma_U)$  which by completion leads to a unitary Banach space representation  $B_U^G(\sigma_U)$  of  $G$ .

**Lemma 5.1.**  *$\mathcal{B}(G, \sigma_U)$  is isometrically isomorphic to the algebra of continuous  $G$ -equivariant endomorphisms of the Banach space  $B_U^G(\sigma_U)$ .*

*Proof.* [27] Lemma 1.3. □

As the completion of a commutative algebra  $\mathcal{B}(G, \sigma_U)$  of course is commutative as well. For any  $K$ -valued (continuous) character  $\zeta$  of  $\mathcal{B}(G, \sigma_U)$  we obtain, by specialization, the unitary Banach space representation

$$B_{\xi, \zeta} := B_U^G(\sigma_U) \hat{\otimes}_{\mathcal{B}(G, \sigma_U)} K_\zeta$$

of  $G$  (where  $\hat{\otimes}$  denotes the completed tensor product). Unfortunately the following conjecture seems to be a very difficult problem.

**Conjecture 5.2.**  *$B_{\xi, \zeta}$  always is nonzero.*

On the other hand it is not to be expected that the Banach space representations  $B_{\xi, \zeta}$  are admissible in general. One of the main results in [27] is the explicit computation of the Banach algebra  $\mathcal{B}(G, \sigma_U)$ . For this we let  $\omega_p : K^\times \rightarrow \mathbb{R}$  denote the unique additive valuation such that  $\omega_p(p) = 1$  and we introduce the map

$$\text{val} : {}^L T^\circ(K) = \text{Hom}(T/U \cap T, K^\times) \xrightarrow{\omega_p \circ} \text{Hom}(T/U \cap T, \mathbb{R}) =: V_{\mathbb{R}} .$$

We note that via the isomorphism

$$\begin{aligned} X^*(T) \otimes \mathbb{R} &\xrightarrow{\cong} V_{\mathbb{R}} \\ \chi \otimes a &\longmapsto a \cdot \omega_p \circ \chi \end{aligned}$$

we may view  $V_{\mathbb{R}}$  as the root space of  $G$  with respect to  $T$ . In particular we may consider the highest weight  $\xi$  as well as half the sum of the positive roots  $\eta$  as elements of  $V_{\mathbb{R}}$ . Let  $\leq$  denote the usual partial order on  $V_{\mathbb{R}}$ . Finally let  $W$  be, as before, the Weyl group of  $T$  and let  $z^{\text{dom}}$ , for any point  $z \in V_{\mathbb{R}}$ , be the unique dominant point in the  $W$ -orbit of  $z$ . We put

$${}^L T_{\xi, \text{norm}}^{\circ} := \{\zeta \in {}^L T^{\circ} : \text{val}(\zeta)^{\text{dom}} \leq \eta + \xi\} .$$

**Theorem 5.3.** *i.  ${}^L T_{\xi, \text{norm}}^{\circ}$  is an open  $K$ -affinoid subdomain of the dual torus  ${}^L T^{\circ}$  which is preserved by the action of  $W$ ;*  
*ii. the Banach algebra  $\mathcal{B}(G, \sigma_U)$  is naturally isomorphic to the ring of analytic functions on the quotient affinoid  $W \backslash {}^L T_{\xi, \text{norm}}^{\circ}$ .*

*Proof.* [27] Prop. 2.4, Lemma 2.7, and the discussion before the Remark in §6.  $\square$

For any given highest weight  $\xi$  the parameter space for our family of Banach space representations  $B_{\xi, \zeta}$  therefore is  $W \backslash {}^L T_{\xi, \text{norm}}^{\circ}$ . We emphasize that the pair  $(\xi, \zeta)$  should be viewed as consisting of a  $K$ -rational cocharacter  $\xi \in X_*({}^L G^{\circ})$  and a semisimple conjugacy class  $\zeta$  in  ${}^L G^{\circ}$ .

In a second step we have to recognize this parameter space on the Galois side. This has its origin in a fundamental theorem about  $p$ -adic Galois representations ([6]) which asserts the existence of an equivalence of categories

$$\begin{array}{ccc} \text{Fon} : & K\text{-linear crystalline represen-} & \xrightarrow{\sim} & \text{weakly admissible} \\ & \text{tations of } \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) & & \text{filtered } K\text{-isocrystals.} \end{array}$$

A filtered  $K$ -isocrystal is a triple  $(D, \varphi, \text{Fil } D)$  consisting of a finite dimensional  $K$ -vector space  $D$ , a  $K$ -linear automorphism  $\varphi$  of  $D$  - the ‘‘Frobenius’’ -, and an exhaustive and separated decreasing filtration  $\text{Fil } D = (\text{Fil}^i D)_{i \in \mathbb{Z}}$  on  $D$  by  $K$ -subspaces. Note that the pair  $(D, \varphi)$  can be viewed as a Weil-Deligne group representation  $\mathcal{W}_{\mathbb{Q}_p} \rightarrow \mathcal{W}_{\mathbb{Q}_p}/\mathcal{I}_{\mathbb{Q}_p} \rightarrow GL(D)$  sending  $\phi$  to  $\varphi$ . Let  $\text{FIC}_K$  denote the additive tensor category of filtered  $K$ -isocrystals. Weak admissibility is a certain condition on the relation between the filtration  $\text{Fil } D$  and the eigenvalues of the Frobenius  $\varphi$  which we will not recall here.

Let  $\text{REP}_K({}^L G^{\circ})$  denote the Tannakian category of all  $K$ -rational representations of  ${}^L G^{\circ}$ . Consider now any pair  $(\nu, b)$  consisting of a  $K$ -rational cocharacter  $\nu \in X_*({}^L G^{\circ})$  and an element  $b \in {}^L G^{\circ}$ . We then have the tensor functor

$$\begin{aligned} I_{(\nu, b)} : \text{REP}_K({}^L G^{\circ}) &\longrightarrow \text{FIC}_K \\ (\tau, D) &\longmapsto (D, \tau(b), \text{Fil}_{\tau \circ \nu}^i D) \end{aligned}$$

with the filtration

$$\text{Fil}_{\tau \circ \nu}^i D := \bigoplus_{j \geq i} D_j$$

defined by the weight spaces  $D_j$  of the cocharacter  $\tau \circ \nu$ . Borrowing a terminology from [20] Chap. 1 we make the following definition.



**Definition 5.4.** The pair  $(\nu, b)$  is called weakly admissible if the filtered  $K$ -isocrystal  $I_{(\nu, b)}(\tau, D)$ , for any  $(\tau, D)$  in  $\text{REP}_K({}^L G^\circ)$ , is weakly admissible.

Suppose that  $(\nu, b)$  is weakly admissible. Then we may compose  $I_{(\nu, b)}$  with the inverse of the functor  $Fon$  and obtain a faithful tensor functor

$$\Gamma_{(\nu, b)} : \text{REP}_K({}^L G^\circ) \longrightarrow \text{Rep}_K^{\text{con}}(\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p))$$

into the Tannakian category of all finite dimensional  $K$ -linear continuous representations of  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ . By the general formalism of neutral Tannakian categories ([7]) the functor  $\Gamma_{(\nu, b)}$  gives rise to a continuous homomorphism of groups

$$\gamma_{\nu, b} := \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow {}^L G^\circ(\bar{K})$$

which is unique up to conjugation in  ${}^L G^\circ(\bar{K})$ . Hence any weakly admissible pair  $(\nu, b)$  determines an isomorphism class of ‘‘Galois parameters’’  $\gamma_{\nu, b}$ . The connection to our parameter space from the first step is provided by [27] Prop. 6.1 as follows.

**Theorem 5.5.** *Suppose that  $\eta \in X^*(T)$ , let  $\xi \in X^*(T)$  be dominant, and let  $\zeta \in {}^L T^\circ(K)$ ; then there exists a weakly admissible pair  $(\nu, b)$  such that  $\nu$  lies in the  ${}^L G^\circ(K)$ -orbit of  $\xi\eta$  and  $b$  has semisimple part  $\zeta$  if and only if  $\zeta \in {}^L T_{\xi, \text{norm}}^\circ(K)$ .*

We see that, given a pair  $(\xi, \zeta)$  with  $\xi \in X^*(T)$  dominant and  $\zeta \in {}^L T_{\xi, \text{norm}}^\circ(K)$  and assuming that  $\eta \in X^*(T)$ , we have on the one hand the conjecturally nonzero unitary Banach space representation  $B_{\xi, \zeta}$  of  $G$ . On the other hand we have the Galois parameters  $\gamma_{\nu, b}$  into  ${}^L G^\circ(\bar{K})$  for all weakly admissible  $(\nu, b)$  such that  $\nu$  is conjugate to  $\xi\eta$  and  $b$  has semisimple part  $\zeta$ . This is the basis for our belief that these Galois parameters  $\gamma_{\nu, b}$  essentially classify the topologically irreducible ‘‘quotient’’ representations of  $B_{\xi, \zeta}$  (the quotation marks indicate that we want to allow for the possibility that the quotient map only has dense image). This would constitute an unramified  $p$ -adic Langlands functoriality principle. The technical assumption that  $\eta \in X^*(T)$  is satisfied if  $G$  is semisimple and simply connected. But it is interesting to realize that it can be altogether avoided by working with a modification of the Galois group  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ . By local class field theory the group  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$  has a (up to isomorphism) unique nontrivial central extension of the form

$$1 \longrightarrow \{\pm 1\} \longrightarrow \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)^{(2)} \longrightarrow \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow 1 .$$

We now impose on our coefficient field  $K$  the slightly stronger condition that  $\mathbb{Q}_p^\times \subseteq (K^\times)^2$ . If  $\varepsilon : \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow \mathbb{Z}_p^\times$  denotes the cyclotomic character then we have a cartesian square

$$\begin{array}{ccc} \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)^{(2)} & \longrightarrow & \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \\ \varepsilon_2 \downarrow & & \downarrow \varepsilon \\ K^\times & \xrightarrow{(\cdot)^2} & K^\times . \end{array}$$

The important point is that on  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)^{(2)}$  the cyclotomic character  $\varepsilon$  has the square root  $\varepsilon_2$ . If  $\mathbb{D}$  denotes the protorus with character group  $\mathbb{Q}$  then  $\eta$  always can be viewed as a  $K$ -rational cocharacter  $\eta : \mathbb{D} \rightarrow {}^L T^\circ$  such that  $\eta^2 \in X_*({}^L T^\circ)$ .

The notion of weak admissibility extends to filtered  $K$ -isocrystals where the filtration is indexed by  $\frac{1}{2}\mathbb{Z}$ . As a consequence we may define weak admissibility for any pair  $(\nu, b)$  where  $\nu : \mathbb{D} \rightarrow {}^L G^\circ$  is a  $K$ -rational cocharacter such that  $\nu^2 \in X_*({}^L G^\circ)$  and  $b \in {}^L G^\circ$ . Theorem 5.5 without the restriction on  $\eta$  remains true in this more general setting ([27] end of §6). Moreover, it is shown in [4] that the above construction of Galois parameters extends in the sense that any weakly admissible pair  $(\nu, b)$  (of this more general kind) such that  $\nu$  is conjugate to  $\xi\eta$  gives rise to an isomorphism class of ‘‘Galois parameters’’

$$\gamma_{\nu, b} : \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)^{(2)} \longrightarrow {}^L G^\circ(\bar{K}) .$$

The picture can be made somewhat more precise in the case of the group  $GL_{d+1}(\mathbb{Q}_p)$ . But first we remark that the reason for our assumption that  $p^{1/2} \in K$  is to make the affinoid  ${}^L T_{\xi, \text{norm}}^\circ$  functorial on the category  $\text{REP}_K({}^L G^\circ)$ . Even for an arbitrary group  $G$  the Satake isomorphism can be renormalized in such a way that it is defined over any finite extension  $K/\mathbb{Q}_p$ . The Banach algebra  $\mathcal{B}(G, \sigma_U)$  then becomes isomorphic to the ring of analytic functions on the quotient by  $W$  of the affinoid

$${}^L T_\xi^\circ := \{ \zeta \in {}^L T^\circ : (\text{val}(\zeta) + \eta)^{\text{dom}} \leq \eta + \xi \} .$$

Of course, the  $W$ -action now is a twisted version of the natural action. If the derived group of  $G$  is simply connected then one can go one step further and make, in addition, the point  $\eta$  integral. In the following we describe this in the case of the general linear group.

For the rest of this section  $K/\mathbb{Q}_p$  is an arbitrary finite extension, and we let  $G = GL_{d+1}(\mathbb{Q}_p)$ . We also let  $U := GL_{d+1}(\mathbb{Z}_p)$  and  $T$  be the torus of diagonal matrices. Our preferred choice of positive roots corresponds to the Borel subgroup of lower triangular matrices. For any  $1 \leq i \leq d+1$  we let  $\lambda_i \in T/U \cap T$  be the coset of the diagonal matrix having  $p$  at the place  $i$  and 1 elsewhere. We make the identification

$$\begin{aligned} V_{\mathbb{R}} = \text{Hom}(T/U \cap T, \mathbb{R}) &\longrightarrow \mathbb{R}^{d+1} \\ z &\longmapsto (z_1, \dots, z_{d+1}) \text{ with } z_i := z(\lambda_i) . \end{aligned}$$

The dominant weight  $\xi \in X^*(T)$  is given by

$$\begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_{d+1} \end{pmatrix} \longmapsto \prod_{i=1}^{d+1} g_i^{a_i}$$

for an increasing sequence  $a_1 \leq \dots \leq a_{d+1}$  of integers. In fact,  $(a_1, \dots, a_{d+1})$  is the point in  $\mathbb{R}^{d+1}$  which corresponds to  $\xi$  under the above identification. Our other point  $\eta$  corresponds to

$$\frac{1}{2}(-d, -(d-2), \dots, d-2, d) .$$

We now note that the point

$$\tilde{\eta} := (0, \dots, d) = \eta + \frac{1}{2}(d, \dots, d)$$

is integral with a correcting summand which is fixed by  $W$ . Hence we may rewrite the definition of  ${}^L T_\xi^\circ$  as

$${}^L T_\xi^\circ = \{\zeta \in {}^L T^\circ : (\text{val}(\zeta) + \tilde{\eta})^{\text{dom}} \leq \tilde{\eta} + \xi\}.$$

Finally, for  ${}^L T^\circ(K) = \text{Hom}(T/U \cap T, K^\times)$  we use the coordinates

$$\begin{aligned} {}^L T^\circ &\longrightarrow (K^\times)^{d+1} \\ \zeta &\longmapsto (\zeta_1, \dots, \zeta_{d+1}) \text{ with } \zeta_i := p^{i-1} \zeta(\lambda_i). \end{aligned}$$

With these identifications our map  $\text{val}$  corresponds to the map

$$\begin{aligned} (K^\times)^{d+1} &\longrightarrow \mathbb{R}^{d+1} \\ (\zeta_1, \dots, \zeta_{d+1}) &\longmapsto (\omega_p(\zeta_1), \dots, \omega_p(\zeta_{d+1})) - (0, \dots, d) \end{aligned}$$

and  ${}^L T_\xi^\circ$  corresponds to the subdomain

$$\{(\zeta_1, \dots, \zeta_{d+1}) \in (K^\times)^{d+1} : (\omega_p(\zeta_1), \dots, \omega_p(\zeta_{d+1}))^{\text{dom}} \leq (a_1, a_2 + 1, \dots, a_{d+1} + d)\}$$

where now  $(\cdot)^{\text{dom}}$  simply means rearrangement in increasing order. Theorem 5.5 in this case amounts to the following.

**Proposition 5.6.** *For any  $(\zeta_1, \dots, \zeta_{d+1}) \in (K^\times)^{d+1}$  the following are equivalent:*

- i. There is a weakly admissible filtered  $K$ -isocrystal of the form  $(K^{d+1}, \varphi, \text{Fil } K^{d+1})$  such that  $\zeta_1, \dots, \zeta_{d+1}$  are the eigenvalues of  $\varphi$  and  $(a_1, a_2 + 1, \dots, a_{d+1} + d)$  are the break points of the filtration  $\text{Fil } K^{d+1}$ ;*
- ii.  $(\omega_p(\zeta_1), \dots, \omega_p(\zeta_{d+1}))^{\text{dom}} \leq (a_1, a_2 + 1, \dots, a_{d+1} + d)$ .*

For any  $K$ -linear crystalline representation  $\rho$  of  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$  we call the break points of the filtration on  $\text{Fon}(\rho)$  the Hodge-Tate coweights of  $\rho$ . Moreover, we say that  $\rho$  is  $K$ -split if all eigenvalues of the Frobenius on  $\text{Fon}(\rho)$  are contained in  $K$ . Using the Colmez-Fontaine equivalence of categories we deduce from Prop. 5.6 the existence of a natural map

$$\begin{array}{l} \text{set of isomorphism classes of } (d+1)\text{-dim.} \\ K\text{-split crystalline representations of} \\ \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \text{ all of whose Hodge-Tate} \\ \text{coweights have multiplicity one} \end{array} \longrightarrow \bigcup_{\sigma} \text{set of } K\text{-valued} \\ \text{characters of } \mathcal{B}(G, \sigma_U).$$

In the limit with respect to  $K$  this map is surjective. Our earlier speculation means that the fiber in a point  $(\xi, \zeta)$  in the right hand side should essentially parametrize the topologically irreducible “quotients” of  $B_{\xi, \zeta}$ .

For the group  $G = GL_2(\mathbb{Q}_p)$  the above picture was the original and basic insight of Breuil. The drastic simplification which occurs in this case is that the fibers

of the above map have at most two elements and, in fact, only one element most of the time (whereas these fibers are infinite in general). Later Breuil and Berger were able in [2] to actually prove that the Banach space representations  $B_{\xi,\zeta}$  in the case where the corresponding two dimensional crystalline Galois representation is irreducible indeed are nonzero, topologically irreducible, and admissible.

We finish by remarking that the content of this section can be developed for any base field  $L$  finite over  $\mathbb{Q}_p$ . We refer to [22] and [4] for the details.

## References

- [1] Amice Y., Duals. Proc. Conf. on  $p$ -Adic Analysis, Nijmegen 1978, pp. 1–15.
- [2] Berger L., Breuil C., Représentations cristallines irréductibles de  $GL_2(\mathbb{Q}_p)$ . Preprint 2005.
- [3] Borel A., Automorphic  $L$ -Functions. In Automorphic Forms, Representations, and  $L$ -Functions (eds. Borel, Casselmann). Proc. Symp. Pure Math. 33 (2), pp. 27–61. American Math. Soc. 1979.
- [4] Breuil C., Schneider P., First steps towards  $p$ -adic Langlands functoriality. Preprint 2006.
- [5] Colmez P., Série principale unitaire pour  $GL_2(\mathbb{Q}_p)$  et représentations triangulines de dimension 2. Preprint 2005.
- [6] Colmez P., Fontaine J.-M., Construction des représentations semistable. Invent. math. 140 (2000), 1–43.
- [7] Deligne P., Milne J. S., Tannakian categories. In Hodge Cycles, Motives, and Shimura Varieties (eds. Deligne, Milne, Ogus, Shih). Lecture Notes in Math. 900, pp. 101–228. Berlin-Heidelberg-New York: Springer 1982.
- [8] Diarra B., Sur quelques représentations  $p$ -adiques de  $\mathbb{Z}_p$ . Indagationes Math. 41 (1979), 481–493.
- [9] Dixon J.D., du Sautoy M.P.F., Mann A., Segal D., Analytic Pro- $p$ -Groups. Cambridge Univ. Press 1999.
- [10] Emerton M., Jacquet modules of locally analytic representations of  $p$ -adic reductive groups I: Definitions and first properties. To appear in Ann. Sci. ENS.
- [11] Féaux de Lacroix C. T.,  $p$ -adische Distributionen. Diplomarbeit, Köln 1992.
- [12] Féaux de Lacroix C. T., Einige Resultate über die topologischen Darstellungen  $p$ -adischer Liegruppen auf unendlich dimensionalen Vektorräumen über einem  $p$ -adischen Körper. Thesis, Köln 1997, Schriftenreihe Math. Inst. Univ. Münster, 3. Serie, Heft 23 (1999), pp. 1–111.
- [13] Fontaine J.-M., Représentations  $\ell$ -adiques potentiellement semi-stables. In Périodes  $p$ -adiques. Astérisque 223 (1994), 321–347.
- [14] Frommer H., The locally analytic principal series of split reductive groups. Preprintreihe SFB 478 available at <http://wwwmath1.uni-muenster.de/sfb/about/publ/>. Münster 2003.
- [15] Harris M., Taylor R., The geometry and cohomology of some simple Shimura varieties. Annals of Math. Studies 151. Princeton Univ. Press 2001.

- [16] Henniart G., Une preuve simple des conjectures de Langlands pour  $GL(n)$  sur un corps  $p$ -adique. *Invent. math.* 139 (2000), 439–455.
- [17] Kohlhaase J., Invariant distributions on  $p$ -adic analytic groups. Thesis, Münster 2005.
- [18] Lazard M., Groupes analytiques  $p$ -adique. *Publ. Math. IHES* 26 (1965), 389–603.
- [19] Morita Y., Analytic representations of  $SL_2$  over a  $p$ -adic number field, II. In *Automorphic Forms of Several Variables* (eds. Satake, Morita), Taniguchi Symp. Katata 1983, *Progress in Math.* 46, pp. 282–297. Boston-Basel-Stuttgart: Birkhäuser 1984.
- [20] Rapoport M., Zink T., *Period Spaces for  $p$ -divisible Groups*. *Annals Math. Studies* 141. Princeton Univ. Press 1996.
- [21] Schneider P., *Nonarchimedean Functional Analysis*. Berlin-Heidelberg-New York: Springer 2002.
- [22] Schneider P., Teitelbaum J.,  $U(\mathfrak{g})$ -finite locally analytic representations. *Representation Theory* 5 (2001), 111–128.
- [23] Schneider P., Teitelbaum J., Banach space representations and Iwasawa theory. *Israel J. Math.* 127 (2002), 359–380.
- [24] Schneider P., Teitelbaum J., Locally analytic distributions and  $p$ -adic representation theory, with applications to  $GL_2$ . *J. AMS* 15 (2002), 443–468.
- [25] Schneider P., Teitelbaum J., Algebras of  $p$ -adic distributions and admissible representations. *Invent. math.* 153 (2003), 145–196.
- [26] Schneider P., Teitelbaum J., Duality for admissible locally analytic representations. *Representation Theory* 9 (2005), 297–326.
- [27] Schneider P., Teitelbaum J., Banach-Hecke algebras and  $p$ -adic Galois representations. Preprint 2005.
- [28] Tate, J., Number theoretic background. In *Automorphic Forms, Representations, and  $L$ -Functions* (eds. Borel, Casselman). *Proc. Symp. Pure Math.* 33 (2), pp. 3–26. American Math. Soc. 1979.
- [29] Vigneras M.-F., Modular representations of  $p$ -adic groups and of affine Hecke algebras. *Proc. International Congress of Mathematicians, Beijing 2002, Vol. II*, pp. 667–677. Beijing: Higher Education Press 2002.

Peter Schneider  
Westfälische Wilhelmsuniversität Münster  
Mathematisches Institut  
Einsteinstraße 62  
48149 Münster  
E-mail: pschnei@math.uni-muenster.de