THE MODULAR PRO-P IWAHORI-HECKE Ext-ALGEBRA

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Dedicated to J. Bernstein on the occasion of his 72nd birthday

Contents

1. Introduction

Let $\mathfrak{F}$ be a locally compact nonarchimedean field with residue characteristic p, and let G be the group of $\mathfrak{F}$-rational points of a connected reductive group $\mathbf{G}$ over $\mathfrak{F}$. We suppose that $\mathbf{G}$ is $\mathfrak{F}$-split in this article.

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Let k be a field and let $\mathrm{Mod}(G)$ denote the category of all smooth representations of G in k-vector spaces. When $k = \mathbb{C}$, by a theorem of Bernstein [Ber] Cor. 3.9.ii, one of the blocks of $\mathrm{Mod}(G)$ is equivalent to the category of modules over the Iwahori-Hecke algebra of G. This block is the subcategory of all representations which are generated by their Iwahori-fixed vectors. It does not contain any supercuspidal representation of G.

When k has characteristic p, it is natural to consider the Hecke algebra H of the pro-p-Iwahori subgroup $I \subset G$. In this case the natural left exact functor

$$\begin{aligned} \mathfrak{h} : \mathrm{Mod}(G) &\longrightarrow \mathrm{Mod}(H) \\ V &\longmapsto V^I = \mathrm{Hom}_{k[G]}(\mathbf{X}, V) \end{aligned}$$

sends a nonzero representation onto a nonzero module. Its left adjoint is

$$\begin{aligned} \mathfrak{t} : \mathrm{Mod}(H) &\longrightarrow \mathrm{Mod}^I(G) \subseteq \mathrm{Mod}(G) \\ M &\longmapsto \mathbf{X} \otimes_H M \ . \end{aligned}$$

Here $\mathbf{X}$ denotes the space of k-valued functions with compact support on G/I with the natural left action of G. The functor $\mathfrak{t}$ has values in the category $\mathrm{Mod}^I(G)$ of all smooth k-representations of G generated by their I-fixed vectors. This category, which a priori has no reason to be an abelian subcategory of $\mathrm{Mod}(G)$, contains all irreducible representations including the supercuspidal ones. But in general $\mathfrak{h}$ and $\mathfrak{t}$ are not quasi-inverse equivalences of categories and little is known about $\mathrm{Mod}^I(G)$ and $\mathrm{Mod}(G)$ unless $G = \mathrm{GL}_2(\mathbb{Q}_p)$ or $G = \mathrm{SL}_2(\mathbb{Q}_p)$ ([Koz1], [Oll1], [OS2], [Pas]).

From now on we assume k has characteristic p. The functor $\mathfrak{h}$, although left exact, is not right exact since p divides the pro-order of I. It is therefore natural to consider the derived functor. In [SDGA] the following result is shown: When $\mathfrak{F}$ is a finite extension of $\mathbb{Q}_p$ and I is a torsionfree pro-p-group, there exists a derived version of the functor $\mathfrak{h}$ and $\mathfrak{t}$ providing an equivalence between the derived category of smooth representations of G in k-vector spaces and the derived category of differential graded modules over a certain differential graded pro-p Iwahori-Hecke algebra $H^\bullet$.

The current article is largely motivated by this theorem. The derived categories involved are not understood, and in fact the Hecke differential graded algebra $H^\bullet$ itself has no concrete description yet. We provide here our first results on the structure of its cohomology algebra $\mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})$:

- We describe explicitly the product in $\mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})$ (Proposition 5.3).
- We deduce the existence of an involutive anti-automorphism of $\mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})$ as a graded Ext-algebra (Proposition 6.1).
- When I is a Poincaré group of dimension d, the Ext algebra is supported in degrees 0 to d and we establish a duality theorem between its i^{th} and $d - i^{\mathrm{th}}$ pieces (Proposition 7.18).
- Under the same hypothesis (and assuming that $\mathbf{G}$ is almost simple and simply connected), we compute $\mathrm{Ext}^d_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})$ as an H-module on the left and on the right (Corollary 8.7). We prove that it is a direct sum of the trivial character, and of supersingular modules.

We hope that these results illustrate that the Ext-algebra $\mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})$ is a natural object whose structure is rich and interesting in itself, even beyond its link to the representation theory of p-adic reductive groups. As a derived version of the Hecke algebra of the

I-equivariant functions on an (almost) affine flag variety, we suspect that it will contribute to relating the mod p Langlands program to methods that appear in the study of geometric Langlands.

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2. Notations and preliminaries

Throughout the paper we fix a locally compact nonarchimedean field $\mathfrak{F}$ (for now of any characteristic) with ring of integers $\mathfrak{O}$, its maximal ideal $\mathfrak{M}$, and a prime element π. The residue field $\mathfrak{O}/\pi\mathfrak{O}$ of $\mathfrak{F}$ is $\mathbb{F}_q$ for some power $q = p^f$ of the residue characteristic p. We choose the valuation $\mathrm{val}_{\mathfrak{F}}$ on $\mathfrak{F}$ normalized by $\mathrm{val}_{\mathfrak{F}}(\pi) = 1$ We let $G := \mathbf{G}(\mathfrak{F})$ be the group of $\mathfrak{F}$-rational points of a connected reductive group $\mathbf{G}$ over $\mathfrak{F}$ which we always assume to be $\mathfrak{F}$-split.

We fix an $\mathfrak{F}$-split maximal torus $\mathbf{T}$ in $\mathbf{G}$, put $T := \mathbf{T}(\mathfrak{F})$, and let T^0 denote the maximal compact subgroup of T and T^1 the pro-p Sylow subgroup of T^0. We also fix a chamber C in the apartment of the semisimple Bruhat-Tits building $\mathscr{X}$ of G which corresponds to $\mathbf{T}$. The stabilizer $\mathcal{P}^{\dagger}_C$ of C contains an Iwahori subgroup J. Its pro-p Sylow subgroup I is called the pro-p Iwahori subgroup and is the main player in this paper. We have $T \cap J = T^0$ and $T \cap I = T^1$. If $N(T)$ is the normalizer of T in G, then we define the group $\widetilde{W} := N(T)/T^1$. In particular, it contains T^0/T^1. The quotient $W := N(T)/T^0 \cong \widetilde{W}/(T^0/T^1)$ is the extended affine Weyl group. The finite Weyl group is $W_0 := N(T)/T$. For any compact open subset $A \subseteq G$ we let char_A denote the characteristic function of A.

The coefficient field for all representations in this paper is an arbitrary field k of characteristic $p > 0$. For any open subgroup $U \subseteq G$ we let $\mathrm{Mod}(U)$ denote the abelian category of smooth representations of U in k-vector spaces. As usual, $K(U)$ denotes the homotopy category of unbounded (cohomological) complexes in $\mathrm{Mod}(U)$ and $D(U)$ the corresponding derived category.

2.1. Elements of Bruhat-Tits theory.

We consider the root data associated to the choice of the maximal $\mathfrak{F}$-split torus $\mathbf{T}$ and record in this section the notations and properties we will need. We follow the exposition of [OS1] §4.1–§4.4 which refers mainly to [SchSt] I.1. Further references are given in [OS1].

2.1.1. The root datum $(\Phi, X^*(T), \check{\Phi}, X_*(T))$ is reduced because the group $\mathbf{G}$ is $\mathfrak{F}$-split. Recall that $X^*(T)$ and $X_*(T)$ denote respectively the group of algebraic characters and cocharacters of T. Similarly, let $X^*(Z)$ and $X_*(Z)$ denote respectively the group of algebraic characters and cocharacters of the connected center Z of G. The standard apartment $\mathscr{A}$ attached to T in the semisimple building $\mathscr{X}$ of G is denoted by $\mathscr{A}$. It can be seen as the vector space

$$\mathbb{R} \otimes_{\mathbb{Z}} (X_*(T)/X_*(\mathrm{Z}))$$

considered as an affine space on itself. We fix a hyperspecial vertex of the chamber C and, for simplicity, choose it to be the zero point in $\mathscr{A}$. Denote by $\langle\, .\,,.\,\rangle : X_*(T) \times X^*(T) \to \mathbb{Z}$ the natural perfect pairing, as well as its $\mathbb{R}$-linear extension. Each root $\alpha \in \Phi$ defines a function $x \mapsto \alpha(x)$ on $\mathscr{A}$. For any subset Y of $\mathscr{A}$, we write $\alpha(Y) \geq 0$ if α takes nonnegative values on Y. To α is also associated a coroot $\check{\alpha} \in \check{\Phi}$ such that $\langle \check{\alpha}, \alpha \rangle = 2$ and a reflection on $\mathscr{A}$ defined by

$$s_\alpha : x \mapsto x - \alpha(x)\check{\alpha} \mod X_*(\mathrm{Z}) \otimes_{\mathbb{Z}} \mathbb{R} .$$

The subgroup of the transformations of $\mathscr{A}$ generated by these reflections identifies with the finite Weyl group W_0. The finite Weyl group W_0 acts by conjugation on T and this induces a faithful linear action on $\mathscr{A}$. Thus W_0 identifes with a subgroup of the transformations of $\mathscr{A}$ and this subgroup is the one generated by the reflections s_α for all $\alpha \in \Phi$. To an element $g \in T$ corresponds a vector $\nu(g) \in \mathbb{R} \otimes_{\mathbb{Z}} X_*(T)$ defined by

$$\langle \nu(g), \chi \rangle = -\operatorname{val}_{\mathfrak{F}}(\chi(g)) \qquad \text{for any } \chi \in X^*(T).$$

The quotient of T by $\ker(\nu) = T^0$ is a free abelian group Λ with rank equal to $\dim(T)$, and ν induces an isomorphism $\Lambda \cong X_*(T)$. The group Λ acts by translation on $\mathscr{A}$ via ν. The actions of W_0 and Λ combine into an action of W on $\mathscr{A}$. The extended affine Weyl group W is the semi-direct product $W_0 \ltimes \Lambda$ if we identify W_0 with the subgroup of W that fixes any lift of x_0 in the extended building of G.

2.1.2. *Affine roots and root subgroups.* We now recall the definition of the affine roots and the properties of the associated root subgroups. To a root α is attached a unipotent subgroup $\mathcal{U}_\alpha$ of G such that for any $u \in \mathcal{U}_\alpha \setminus \{1\}$, the intersection $\mathcal{U}_{-\alpha} u \mathcal{U}_{-\alpha} \cap N(T)$ consists in only one element called $m_\alpha(u)$. The image in W of this element $m_\alpha(u)$ is the reflection at the affine hyperplane $\{x \in \mathscr{A},\ \alpha(x) = -\mathfrak{h}_\alpha(u)\}$ for a certain $\mathfrak{h}_\alpha(u) \in \mathbb{R}$. Denote by Γ_α the discrete unbounded subset of $\mathbb{R}$ given by $\{\mathfrak{h}_\alpha(u),\ u \in \mathcal{U}_\alpha \setminus \{1\}\}$. Since our group $\mathbf{G}$ is $\mathfrak{F}$-split we have $\{\mathfrak{h}_\alpha(u),\ u \in \mathcal{U}_\alpha \setminus \{1\}\} = \mathbb{Z}$. The affine functions

$$(\alpha, \mathfrak{h}) := \alpha(\,.\,) + \mathfrak{h} \qquad \text{for } \alpha \in \Phi \text{ and } \mathfrak{h} \in \mathbb{Z}$$

are called the affine roots.

We identify an element α of Φ with the affine root $(\alpha, 0)$ so that the set of affine roots Φ_{aff} contains Φ. The action of W_0 on Φ extends to an action of W on Φ_{aff}. Explicitly, if $w = w_0 t_\lambda \in W$ is the composition of the translation by $\lambda \in \Lambda$ with $w_0 \in W_0$, then the action of w on the affine root $(\alpha, \mathfrak{h})$

$$(w_0(\alpha), \mathfrak{h} + (\operatorname{val}_{\mathfrak{F}} \circ \alpha)(\lambda)) = (w_0(\alpha), \mathfrak{h} - \langle \nu(\lambda), \alpha \rangle) .$$

Define a filtration of $\mathcal{U}_\alpha$, $\alpha \in \Phi$ by

$$\mathcal{U}_{\alpha,r} := \{u \in \mathcal{U}_\alpha \setminus \{1\},\ \mathfrak{h}_\alpha(u) \geq r\} \cup \{1\} \text{ for } r \in \mathbb{R} .$$

For $(\alpha, \mathfrak{h}) \in \Phi_{aff}$, we put $\mathcal{U}_{(\alpha,\mathfrak{h})} := \mathcal{U}_{\alpha,\mathfrak{h}}$. Obviously, for $r \in \mathbb{R}$ a real number, $\mathfrak{h} \geq r$ is equivalent to $\mathcal{U}_{(\alpha,\mathfrak{h})} \subseteq \mathcal{U}_{\alpha,r}$.

By abuse of notation we write throughout the paper wMw^{-1}, for some $w \in W$ and some subgroup $M \subseteq G$, whenever the result of this conjugation is independent of the choice of a representative of w in $N(T)$. For example, for $(\alpha, \mathfrak{h}) \in \Phi_{aff}$ and $w \in W$, we have

$$w\mathcal{U}_{(\alpha,\mathfrak{h})}w^{-1} = \mathcal{U}_{w(\alpha,\mathfrak{h})} . \tag{1}$$

For any non empty subset $\mathbf{Y} \subset \mathscr{A}$, define

$$\begin{aligned} f_{\mathbf{Y}} : \Phi &\longrightarrow \mathbb{R} \cup \{\infty\} \\ \alpha &\longmapsto -\inf_{x \in \mathbf{Y}} \alpha(x) . \end{aligned}$$

and the subgroup of G

$$\mathcal{U}_{\mathbf{Y}} = < \mathcal{U}_{\alpha, f_{\mathbf{Y}}(\alpha)},\ \alpha \in \Phi > \tag{2}$$

generated by all $\mathcal{U}_{\alpha,f_{\mathbf{Y}}(\alpha)}$ for $\alpha\in\Phi$. We have ([SchSt] Page 103, point 3.)

$$\mathcal{U}_{\mathbf{Y}}\cap\mathcal{U}_\alpha=\ \mathcal{U}_{\alpha,f_{\mathbf{Y}}(\alpha)}\ \text{ for any } \alpha\in\Phi\ . \tag{3}$$

2.1.3. *Positive roots and length .* The choice of the chamber C determines the subset Φ^+ of the positive roots, namely the set of $\alpha\in\Phi$ taking nonnegative values on C. Denote by Π a basis for Φ^+. Likewise, the set of positive affine roots Φ^+_{aff} is defined to be the set of affine roots taking nonnegative values on C. The set of negative affine roots is $\Phi^-_{aff}:=-\Phi^+_{aff}$.

Lemma 2.1. i. *We have $\mathcal{U}_{\alpha,f_C(\alpha)}=\mathcal{U}_{\alpha,0}$ for $\alpha\in\Phi^+$ and $\mathcal{U}_{\alpha,f_C(\alpha)}=\mathcal{U}_{\alpha,1}$ for $\alpha\in\Phi^-$.*
ii. *The pro-p Iwahori subgroup I is generated by T^1 and $\mathcal{U}_C$, namely by T^1 and all root subgroups $\mathcal{U}_A$ for $A\in\Phi^+_{aff}$.*
iii. *For $\alpha\in\Phi$, we have $I\cap\mathcal{U}_\alpha=\mathcal{U}_{\alpha,f_{\mathbf{C}}(\alpha)}$.*

Proof. i., ii. This is given by [SchSt] Prop. I.2.2 (recalled in [OS1] Proof of Lemma 4.8) and [OS1] Proof of Lemma 4.2. iii. This also follows from [SchSt] Prop. I.2.2. □

The finite Weyl group W_0 is a Coxeter system generated by the set $S:=\{s_\alpha:\alpha\in\Pi\}$ of reflections associated to the simple roots Π. It is endowed with a length function denoted by ℓ. This length extends to W in such a way that the length of an element $w\in W$ is the cardinality of $\{A\in\Phi^+_{aff},\, w(A)\in\Phi^-_{aff}\}$. For any affine root $(\alpha,\mathfrak{h})$, we have in W the reflection $s_{(\alpha,\mathfrak{h})}$ at the affine hyperplane $\alpha(\,.\,)=-\mathfrak{h}$. The affine Weyl group is defined as the subgroup W_{aff} of W generated by all s_A for all $A\in\Phi_{aff}$.

There is a partial order on Φ given by $\alpha\leq\beta$ if and only if $\beta-\alpha$ is a linear combination with (integral) nonnegative coefficients of elements in Π. Let $\Phi^{min}:=\{\alpha\in\Phi:\alpha \text{ is minimal for } \leq\}$ and $\Pi_{aff}:=\Pi\cup\{(\alpha,1):\alpha\in\Phi^{min}\}\subseteq\Phi^+_{aff}$. Let $S_{aff}:=\{s_A:A\in\Pi_{aff}\}$, then the pair (W_{aff},S_{aff}) is a Coxeter system and the length function ℓ restricted to W_{aff} coincides with the length function of this Coxeter system. For any $s\in S_{aff}$ there is a unique positive affine root $A_s\in\Phi^+_{aff}$ such that $sA_s\in\Phi^-_{aff}$. In fact A_s lies in Π_{aff}.

We have the following formula, for every $A\in\Pi_{aff}$ and $w\in W$ ([Lu] §1):

$$\ell(ws_A)=\begin{cases}\ell(w)+1 & \text{if } w(A)\in\Phi^+_{aff},\\ \ell(w)-1 & \text{if } w(A)\in\Phi^-_{aff}.\end{cases} \tag{4}$$

The Bruhat-Tits decomposition of G says that G is the disjoint union of the double cosets JwJ for $w\in W$. As in [OS1] §4.3 (see the references there) we will denote by Ω the abelian subgroup of W of all elements with length zero and recall that Ω normalizes S_{aff}. Furthermore, W is the semi-direct product $W=\Omega\ltimes W_{aff}$. The length function is constant on the double cosets $\Omega w\Omega$ for $w\in W$. The stabilizer $\mathcal{P}^\dagger_C$ of C in G is the disjoint union of the double cosets $J\omega J$ for $\omega\in\Omega$ ([OS1] Lemma 4.9). We have

$$I\subseteq J\subseteq\mathcal{P}^\dagger_C$$

and I, the pro-unipotent radical of the parahoric subgroup J, is normal in $\mathcal{P}^\dagger_C$ (see [OS1] §4.5). In fact, J is also normal in $\mathcal{P}^\dagger_C$ since J is generated by T^0 and I, and since the action by conjugation of $\omega\in\Omega$ on T normalizes T^0.

2.1.4. Recall that we denote by $\widetilde{W}$ the quotient of $N(T)$ by T^1 and obtain the exact sequence

$$0 \to T^0/T^1 \to \widetilde{W} \to W \to 0 \ .$$

The length function ℓ on W pulls back to a length function ℓ on $\widetilde{W}$ ([Vig05] Prop. 1). The Bruhat-Tits decomposition of G says that G is the disjoint union of the double cosets IwI for $w \in \widetilde{W}$. We will denote by $\widetilde{\Omega}$ the preimage of Ω in $\widetilde{W}$. It contains T^0/T^1.

Obviously $\widetilde{W}$ acts by conjugation on its normal subgroup T^0/T^1, and we denote this action simply by $(w,t) \mapsto w(t)$. But, since T/T^1 is abelian the action in fact factorizes through $W_0 = \widetilde{W}/T$. On the other hand W_0, by definition, acts on T and this action stabilizes the maximal compact subgroup T^0 and its pro-p Sylow T^1. Therefore we again have an action of W_0 on T^0/T^1. These two actions, of course, coincide.

2.1.5. *On certain open compact subgroups of the pro-p Iwahori subgroup.* Let $g \in G$. We let

$$I_g := I \cap gIg^{-1} \ . \tag{5}$$

Since this definition depends only on gJ, we may consider $I_w := I \cap wIw^{-1}$ for any $w \in W$ or $w \in \widetilde{W}$. Since I is normal in $\mathcal{P}_C^\dagger$, we have $I_{w\omega} = I_w$ for any $\omega \in \Omega$ and any $w \in W$ (but in general not $I_{\omega w} = I_w$).

Note that if $\tilde{w} \in \widetilde{W}$ lifts $w \in W$, then $I_{\tilde{w}} = I_w$.

Lemma 2.2. *Let $v, w \in W$ such that $\ell(vw) = \ell(v) + \ell(w)$. We have*

$$I_{vw} \subseteq I_v \tag{6}$$

and

$$I \subseteq I_{v^{-1}} wIw^{-1} \ . \tag{7}$$

Proof. (6): The claim is clear when w has length 0 since we then have $I_{vw} = I_v$. By induction, it suffices to treat the case when $w = s \in S_{aff}$. Note first that adjoining vsC to a given minimal gallery of $\mathscr{X}$ between C and vC gives a minimal gallery between C and vsC.

Let $y \in I_{vs}$. The apartment $y\mathscr{A}$ contains vsC and C and therefore it contains any minimal gallery from C to vsC by [BT1] 2.3.6. In particular, it contains vC. Let F be the facet with codimension 1 which is contained in both the closures of the chambers vC and vsC. It is fixed by y and therefore the image C' of the chamber vC under the action of y also contains F in its closure. By [BT1] 1.3.6, only two chambers of $y\mathscr{A}$ contain F in their closure, so $C' = vC$. Therefore, $y \in v\mathcal{P}_C^\dagger v^{-1} \cap I = \mathcal{P}_{vC}^\dagger \cap I$ where $\mathcal{P}_{vC}^\dagger$ denotes the stabilizer in G of the chamber vC. By [OS1] Lemma 4.10, the intersection $\mathcal{P}_{vC}^\dagger \cap I$ is contained in the parahoric subgroup vJv^{-1} of $\mathcal{P}_{vC}^\dagger$, and since it is a pro-p group, it is contained in its pro-p Sylow subgroup vIv^{-1}. We have proved that y lies in I_v.

(7): Let $w \in W$. We prove the following statement by induction on $\ell(v)$: let $v \in W$ such that $\ell(vw) = \ell(v) + \ell(w)$; then any $A \in \Phi_{aff}^+$ such that $vA \in \Phi_{aff}^-$ satisfies $w^{-1}A \in \Phi_{aff}^+$. Using (1) and Lemma 2.1.ii, this means that for $v \in W$ such that $\ell(vw) = \ell(v) + \ell(w)$ and $A \in \Phi_{aff}^+$ we have $vA \in \Phi_{aff}^+$ and $\mathcal{U}_A \subseteq v^{-1}\mathcal{U}_C v^{-1} \cap I \subseteq I_{v^{-1}}$, or $vA \in \Phi_{aff}^-$ and $w^{-1}A \in \Phi_{aff}^+$ so $\mathcal{U}_A \subseteq w\mathcal{U}_C w^{-1} \subseteq wIw^{-1}$. This implies that for $v \in W$ such that $\ell(vw) = \ell(v) + \ell(w)$ we have $\mathcal{U}_C \subseteq I_{v^{-1}} wIw^{-1}$ and again using Lemma 2.1.ii, that $I \subseteq I_{v^{-1}} wIw^{-1}$. We now proceed to the proof of the claim by induction.

When v has length zero the claim is clear. Now let v such that $\ell(vw) = \ell(v) + \ell(w)$ and $s \in S_{aff}$ such that $\ell(sv) = \ell(v) - 1$. This implies that $\ell(svw) \leq \ell(sv) + \ell(w) = \ell(vw) - 1$ and therefore $\ell(svw) = \ell(vw) - 1 = \ell(sv) + \ell(w)$. In particular we have $(vw)^{-1}A_s \in \Phi^-_{aff}$ with the notation introduced in §2.1.3 (see [Lu] Section 1, recalled in [OS1] (4.2)). By induction hypothesis, any $A \in \Phi^+_{aff}$ such that $svA \in \Phi^-_{aff}$ satisfies $w^{-1}A \in \Phi^+_{aff}$. Now let $A \in \Phi^+_{aff}$ such that $vA \in \Phi^-_{aff}$. We need to show that $w^{-1}A \in \Phi^+_{aff}$. If $svA \in \Phi^-_{aff}$ then it follows from the induction hypothesis. Otherwise, it means that $vA = -A_s$ and therefore $w^{-1}A = -(vw)^{-1}A_s \in \Phi^+_{aff}$. □

Lemma 2.3. *Let $w \in W$. The product map induces a bijection*

$$\prod_{\alpha\in\Phi^-} \mathcal{U}_{\alpha,f_{C\cup wC}(\alpha)} \times T^1 \times \prod_{\alpha\in\Phi^+} \mathcal{U}_{\alpha,f_{C\cup wC}(\alpha)} \xrightarrow{\sim} I_w \tag{8}$$

where the products on the left hand side are ordered in some arbitrarily chosen way.

Proof. The multiplication in G induces an injective map

$$\prod_{\alpha\in\Phi^-} \mathcal{U}_{\alpha} \times T \times \prod_{\alpha\in\Phi^+} \mathcal{U}_{\alpha} \hookrightarrow G\ .$$

In the notation of [SchSt] §I.2 we have $I = R_C$ and $wIw^{-1} = R_{wC}$. Therefore [SchSt] Prop. I.2.2 says that the above map restricts to bijections

$$\prod_{\alpha\in\Phi^-} \mathcal{U}_{\alpha,f_C(\alpha)} \times T^1 \times \prod_{\alpha\in\Phi^+} \mathcal{U}_{\alpha,f_C(\alpha)} \xrightarrow{\sim} I \tag{9}$$

and

$$\prod_{\alpha\in\Phi^-} \mathcal{U}_{\alpha,f_{wC}(\alpha)} \times T^1 \times \prod_{\alpha\in\Phi^+} \mathcal{U}_{\alpha,f_{wC}(\alpha)} \xrightarrow{\sim} wIw^{-1}\ ,$$

and hence to the bijection

$$\prod_{\alpha\in\Phi^-} \mathcal{U}_{\alpha,f_C(\alpha)} \cap \mathcal{U}_{\alpha,f_{wC}(\alpha)} \times T^1 \times \prod_{\alpha\in\Phi^+} \mathcal{U}_{\alpha,f_C(\alpha)} \cap \mathcal{U}_{\alpha,f_{wC}(\alpha)} \xrightarrow{\sim} I_w\ .$$

Since, obviously, $f_{C\cup wC}(\alpha) = \max(f_C(\alpha), f_{wC}(\alpha))$ we have $\mathcal{U}_{\alpha,f_C(\alpha)} \cap \mathcal{U}_{\alpha,f_{wC}(\alpha)} = \mathcal{U}_{\alpha,f_{C\cup wC}(\alpha)}$. □

Remark 2.4. Let $\alpha \in \Phi$ and $w \in W$. Define $g_w(\alpha) := \min\{m \in \mathbb{Z},\ (\alpha, m) \in \Phi^+_{aff} \cap w\Phi^+_{aff}\}$. We have $\mathcal{U}_{\alpha,g_w(\alpha)} = \mathcal{U}_{\alpha,f_{C\cup wC}(\alpha)}$. This is Lemma 2.1.i when $w = 1$.

Proof. First note that $\Phi^+_{aff} \cap w\Phi^+_{aff}$ is the set of affine roots which are positive on $C \cup wC$. Let $\alpha \in \Phi$. Since $\alpha + g_w(\alpha) \geq 0$ on $C \cup wC$ we have $f_{C\cup wC}(\alpha) \leq g_w(\alpha)$ and $\mathcal{U}_{\alpha,g_w(\alpha)} \subseteq \mathcal{U}_{\alpha,f_{C\cup wC}(\alpha)}$. Now let $u \in \mathcal{U}_{\alpha,f_{C\cup wC}(\alpha)} \setminus \{1\}$. It implies $\mathfrak{h}_\alpha(u) \geq f_{C\cup wC}(\alpha)$ so $\alpha + \mathfrak{h}_\alpha(u) \geq 0$ on $C \cup wC$, therefore $\mathfrak{h}_\alpha(u) \geq g_w(\alpha)$ so $u \in \mathcal{U}_{\alpha,g_w(\alpha)}$. □

Corollary 2.5. *Let $v, w \in W$ and $s \in S_{aff}$ with respective lifts $\tilde{v}, \tilde{w}$ and $\tilde{s}$ in $\widetilde{W}$. We have:*

i. $|I/I_w| = q^{\ell(w)}$;
ii. *if* $\ell(vw) = \ell(v) + \ell(w)$ *then* $I\tilde{v}I \cdot I\tilde{w}I = I\tilde{v}\tilde{w}I$;
iii. *if* $\ell(ws) = \ell(w) + 1$ *then* I_{ws} *is a normal subgroup of* I_w *of index* q.

Proof. Points i. and ii. are well known. Compare i. with [IM] Prop. 3.2 and §I.5 and ii. with [IM] Prop. 2.8(i). For the convenience of the reader we add the arguments.

i. We obtain the result by induction on $\ell(w)$. Suppose that $\ell(ws) = \ell(w) + 1$. Again by [Lu] Section 1 (recalled in [OS1] (4.2)) we have $wA_s \in \Phi^+_{aff}$ and $\Phi^+_{aff} \cap ws\Phi^+_{aff} = (\Phi^+_{aff} \cap w\Phi^+_{aff}) \setminus \{wA_s\}$. So if we let $(\beta, m) := wA_s$, then using Remark 2.4 we have

$$\mathfrak{U}_{\alpha, f_{C\cup wC}(\alpha)} = \mathfrak{U}_{\alpha, f_{C\cup wsC}(\alpha)} \quad \text{for any } \alpha \in \Phi,\ \alpha \neq \beta \tag{10}$$

and

$$\mathfrak{U}_{\beta, f_{C\cup wC}(\beta)} = \mathfrak{U}_{wA_s} = \mathfrak{U}_{(\beta,m)} \quad \text{and} \quad \mathfrak{U}_{\beta, f_{C\cup wsC}(\beta)} = \mathfrak{U}_{(\beta,1+m)}. \tag{11}$$

Hence using Lemma 2.3 we deduce that $I_{ws} \subseteq I_w$ and

$$I_w/I_{ws} \simeq \mathfrak{U}_{\beta, f_{C\cup wC}(\beta)}/\mathfrak{U}_{\beta, f_{C\cup wsC}(\beta)} = \mathfrak{U}_{\beta,m}/\mathfrak{U}_{\beta,m+1}\ , \tag{12}$$

which has cardinality q by [Tits] 1.1.

ii. It suffices to treat the case $v = s \in S_{aff}$. The claim then follows by induction on $\ell(v)$. Using Lemma 2.1 we have $I\tilde{s}I = I\tilde{s}\mathfrak{U}_{A_s}$ since $sA \in \Phi^+_{aff}$ for any $A \in \Phi^+_{aff} \setminus \{A_s\}$. Now $\ell(sw) = \ell(w) + 1$ means that $w^{-1}A_s \in \Phi^+_{aff}$ therefore (again by Lemma 2.1.ii) $I\tilde{s}I\tilde{w}I = I\tilde{s}\mathfrak{U}_{A_s}\tilde{w}I = I\tilde{s}\tilde{w}\mathfrak{U}_{w^{-1}A_s}I = I\tilde{s}\tilde{w}I$.

iii. We first treat the case that $w = 1$. By i. we only need to show that I_s is normal in I. Let F be the 1-codimensional facet common to C and sC. The pro-unipotent radical I_F of the parahoric subgroup $\mathbf{G}^\circ_F(\mathfrak{O})$ attached to F is a normal subgroup of $\mathbf{G}^\circ_F(\mathfrak{O})$ ([SchSt] I.2). It follows from [SchSt] Prop. I.2.11 and its proof that I_F is a normal subgroup of $I = I_C$. Obviously $I_s \subseteq I_F$. Hence it suffices to show equality. By [SchSt] Prop. I.2.2 (see also [OS1] Proof of Lemma 4.8), the product map induces a bijection

$$\prod_{\alpha\in\Phi^-} \mathfrak{U}_{\alpha, f^*_F(\alpha)} \times T^1 \times \prod_{\alpha\in\Phi^+} \mathfrak{U}_{\alpha, f^*_F(\alpha)} \xrightarrow{\sim} I_F$$

where $f^*_F(\alpha) = f_F(\alpha)$ if $\alpha \neq \alpha_0$ and $f^*_F(\alpha_0) = f_F(\alpha_0) + 1 = \epsilon + 1$ with $(\alpha_0, \epsilon) = A_s$. In view of Lemma 2.3 it remains to show that $\mathfrak{U}_{\alpha, f^*_F(\alpha)} = \mathfrak{U}_{\alpha, f_{C\cup sC}(\alpha)}$ for any $\alpha \in \Phi$. But this is immediate from (10) and (11) applied with $w = 1$.

Coming back to the general case we see that $wIw^{-1} \cap wsI(ws)^{-1}$ is normal in wIw^{-1}. Hence $I_w \cap I_{ws} = I \cap wIw^{-1} \cap wsI(ws)^{-1}$ is normal in I_w. But in the proof of i. we have seen that $I_{ws} \subseteq I_w$. The assertion about the index also follows from i. $\square$

2.1.6. *Chevalley basis and double cosets decompositions.* Let $\mathbf{G}_{x_0}$ denote the Bruhat-Tits group scheme over $\mathfrak{O}$ corresponding to the hyperspecial vertex x_0 (cf. [Tits]). As part of a Chevalley basis we have (cf. [BT2] 3.2), for any root $\alpha \in \Phi$, a homomorphism $\varphi_\alpha : \mathrm{SL}_2 \longrightarrow \mathbf{G}_{x_0}$ of $\mathfrak{O}$-group schemes which restricts to isomorphisms

$$\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\} \xrightarrow{\cong} \mathfrak{U}_\alpha \qquad \text{and} \qquad \{\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\} \xrightarrow{\cong} \mathfrak{U}_{-\alpha}\ .$$

Moreover, one has $\check{\alpha}(x) = \varphi_\alpha(\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right))$. We let the subtorus $\mathbf{T}_{s_\alpha} \subseteq \mathbf{T}$ denote the image (in the sense of algebraic groups) of the cocharacter $\check{\alpha}$. We always view these as being defined over $\mathfrak{O}$ as subtori of $\mathbf{G}_{x_0}$. The group of $\mathbb{F}_q$-rational points $\mathbf{T}_{s_\alpha}(\mathbb{F}_q)$ can be viewed as a subgroup of $T^0/T^1 \xrightarrow{\cong} \mathbf{T}(\mathbb{F}_q)$ (and is abstractly isomorphic to $\mathbb{F}_q^\times$). Given $z \in \mathbb{F}_q^\times$, we consider $[z] \in \mathfrak{O}^\times$ the Teichmüller representative ([Se2] II.4 Prop. 8) and denote by $\check{\alpha}([\_])$ the composite morphism of groups

$$\check{\alpha}([\_])\ :\ \mathbb{F}_q^\times \xrightarrow{[\_]} \mathfrak{O}^\times \xrightarrow{\check{\alpha}} \mathbf{T}(\mathfrak{O}) = T^0\ . \tag{13}$$

We will denote its kernel by $\mu_{\check{\alpha}}$. Looking at the commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{O}^\times & \xrightarrow{\check{\alpha}} & \mathbf{T}_{s_\alpha}(\mathfrak{O}) & \xrightarrow{\subseteq} & \mathbf{T}(\mathfrak{O}) & \xrightarrow{\mathrm{pr}} & T^0/T^1 \\
\downarrow{\scriptstyle red} & & \downarrow{\scriptstyle red} & & \downarrow{\scriptstyle red} & \swarrow{\scriptstyle \cong,\ red} & \\
\mathbb{F}_q^\times & \xrightarrow{\check{\alpha}} & \mathbf{T}_{s_\alpha}(\mathbb{F}_q) & \xrightarrow{\subseteq} & \mathbf{T}(\mathbb{F}_q) & &
\end{array}$$

of reduction maps and using that the Teichmüller map is a section of the reduction map $\mathfrak{O}^\times \twoheadrightarrow \mathbb{F}_q^\times$ we deduce that the reduction map induces isomorphisms

$$\check{\alpha}([\mathbb{F}_q^\times]) \xrightarrow{\cong} \check{\alpha}(\mathbb{F}_q^\times) \qquad \text{and} \qquad \mu_{\check{\alpha}} \xrightarrow{\cong} \ker(\check{\alpha}_{|\mathbb{F}_q^\times}) \ .$$

Remark 2.6. In Propositions 5.6 and 8.2 we will need to differentiate between an element $t \in \check{\alpha}([\mathbb{F}_q^\times]) \subset T^0$ and its image in $T^0/T^1 \subset \widetilde{W}$ which we will denote by $\bar{t}$.

Remark 2.7. If φ_α is not injective then its kernel is $\{(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}) : a = \pm 1\}$ (cf. [Jan] II.1.3(7)). It follows that $\mu_{\check{\alpha}}$ has cardinality 1 or 2.

For the following two lemmas below, recall that the notation for the action of W on T^0/T^1 was introduced in §2.1.4.

Lemma 2.8. *Suppose that* $\mathbf{G}$ *is semisimple simply connected, then:*

i. $\check{\alpha}(\mathbb{F}_q^\times) = \mathbf{T}_{s_\alpha}(\mathbb{F}_q)$ *for any* $\alpha \in \Phi$*;*

ii. $T^0/T^1 = \widetilde{\Omega}$ *is generated by the union of its subgroups* $\mathbf{T}_{s_\alpha}(\mathbb{F}_q)$ *for* $\alpha \in \Pi$*;*

Proof. By our assumption that $\mathbf{G}$ is semisimple simply connected the set $\{\check{\alpha} : \alpha \in \Pi\}$ is a basis of the cocharacter group $X_*(T)$. This means that

$$\prod_{\alpha \in \Pi} \mathbb{G}_m \xrightarrow{\prod_\alpha \check{\alpha}} \mathbf{T}$$

is an isomorphism of algebraic tori. It follows that multiplication induces an isomorphism

$$\prod_{\alpha \in \Pi} \mathbf{T}_{s_\alpha}(\mathbb{F}_q) \xrightarrow{\cong} \mathbf{T}(\mathbb{F}_q) \ .$$

This implies ii. But, since any root is part of some basis of the root system, we also obtain that $\Phi \cap 2X_*(T) = \emptyset$. Hence φ_α and $\check{\alpha}$ are injective for any $\alpha \in \Phi$ (cf. [Jan] II.1.3(7)). For ii. it therefore remains to notice that $\check{\alpha} : \mathbb{F}_q^\times \to \mathbf{T}_{s_\alpha}(\mathbb{F}_q)$ is a map between finite sets of the same cardinality. □

Lemma 2.9. *For any* $t \in T^0/T^1$ *and any* $A = (\alpha, \mathfrak{h}) \in \Phi_{aff}$*, we have* $s_A(t)t^{-1} \in \check{\alpha}([\mathbb{F}_q^\times])$*.*

Proof. Recall from §2.1.4 that the action of s_A on t is inflated from the action of its image $s_\alpha \in W_0$. The action of W_0 on $X_*(T)$ is induced by its action on T, i.e., we have

$$(w(\xi))(x) = w(\xi(x)) \qquad \text{for any } w \in W_0,\ \xi \in X_*(T), \text{ and } x \in T.$$

On the other hand the action of s_α on $\xi \in X_*(T)$ is given by

$$s_\alpha(\xi) = \xi - \langle \xi, \alpha \rangle \check{\alpha} \ .$$

So for any $\xi \in X_*(T)$ and any $y \in \mathbb{F}_q^\times$ we have

$$s_\alpha(\xi([y])) = (s_\alpha(\xi))([y]) = \xi([y])\check{\alpha}([y])^{-\langle \xi, \alpha \rangle} \in \xi([y])\check{\alpha}([\mathbb{F}_q^\times]) \ .$$

It remains to notice that, if $\xi_1, \ldots, \xi_m$ is a basis of $X_*(T)$, then $\prod_i \xi([\mathbb{F}_q^\times]) = T^0/T^1$. □

For any $\alpha \in \Phi$ we have the additive isomorphism $x_\alpha : \mathfrak{F} \xrightarrow{\cong} \mathfrak{U}_\alpha$ defined by

$$x_\alpha(u) := \varphi_\alpha(\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)) \ . \tag{14}$$

Let $(\alpha, \mathfrak{h}) \in \Pi_{aff}$ and $s = s_{(\alpha,\mathfrak{h})}$. We put

$$n_s := \varphi_\alpha(\begin{pmatrix} 0 & \pi^{\mathfrak{h}} \\ -\pi^{-\mathfrak{h}} & 0 \end{pmatrix}) \in N(T) \ .$$

We have $n_s^2 = \check{\alpha}(-1) \in T^0$ and $n_s T^0 = s \in W$. We set:

$$\tilde{s} = n_s T^1 \ \in \widetilde{W} \ . \tag{15}$$

From (12), Lemma 2.3 and [Tits] 1.1 we deduce that $\{x_\alpha(\pi^{\mathfrak{h}} u)\}$ is a system of representatives of I/I_s when u ranges over a system of representatives of $\mathfrak{O}/\pi\mathfrak{O}$. We have the decomposition

$$I n_s I = n_s I \,\dot{\cup} \dot{\bigcup}_{z \in \mathbb{F}_q^\times} x_\alpha(\pi^{\mathfrak{h}}[z]) n_s I \ . \tag{16}$$

Note that since $[-1] = -1 \in \mathfrak{O}$, we have $x_\alpha(\pi^{\mathfrak{h}}[z])^{-1} = x_\alpha(-\pi^{\mathfrak{h}}[z]) = x_\alpha(\pi^{\mathfrak{h}}[-z])$ and for $z \in \mathbb{F}_q^\times$, we compute, using φ_α, that

$$x_\alpha(\pi^{\mathfrak{h}}[z]) \check{\alpha}([z]) n_s = n_s x_\alpha(\pi^{\mathfrak{h}}[-z^{-1}]) n_s x_\alpha(\pi^{\mathfrak{h}}[-z]) \in n_s I n_s I \tag{17}$$

because the Teichmüller map $[\_] : \mathbb{F}_q^\times \to \mathfrak{O}^\times$ is a morphism of groups. Since $I n_s I = n_s I \,\dot{\cup} \dot{\bigcup}_{z \in \mathbb{F}_q^\times} x_\alpha(\pi^{\mathfrak{h}}[-z^{-1}]) n_s I$, it follows that

$$n_s I n_s I = I n_s^2 \,\dot{\cup} \dot{\bigcup}_{z \in \mathbb{F}_q^\times} x_\alpha(\pi^{\mathfrak{h}}[z]) \check{\alpha}([z]) n_s I \subset I n_s^2 \,\dot{\cup} \bigcup_{z \in \mathbb{F}_q^\times} I \check{\alpha}([z]) n_s I \tag{18}$$

and hence

$$I n_s I \cdot I n_s I = I n_s^2 \,\dot{\cup} \dot{\bigcup}_{t \in \check{\alpha}([\mathbb{F}_q^\times])} I t n_s I \ . \tag{19}$$

Remark 2.10. Let $s \in S_{aff}$ with lift $\tilde{s} \in \widetilde{W}$. Let $w \in \widetilde{W}$ such that $\ell(\tilde{s}w) = \ell(w) - 1$. From (19) and Cor. 2.5.ii, we deduce that

$$I\tilde{s}I \cdot IwI = I\tilde{s}I \cdot I\tilde{s}I \cdot I(\tilde{s}^{-1}w)I = I\tilde{s}wI \,\dot{\cup} \dot{\bigcup}_{t \in \check{\alpha}([\mathbb{F}_q^\times])} ItwI \ . \tag{20}$$

Lemma 2.11. *For $u, v, w \in \widetilde{W}$ satisfying $IuI \subseteq IvI \cdot IwI$, we have:*

$$|\ell(w) - \ell(v)| \leq \ell(u) \leq \ell(w) + \ell(v) \ .$$

Proof. It is enough the prove that for $u, v, w \in W$ satisfying $JuJ \subseteq JvJ \cdot JwJ$, we have:

$$|\ell(w) - \ell(v)| \leq \ell(u) \leq \ell(w) + \ell(v) \ .$$

Let $w \in W$. We prove by induction with respect to $\ell(v)$ that for $u \in W$ such that $JuJ \subseteq JvJ \cdot JwJ$ we have

$$\ell(w) - \ell(v) \leq \ell(u) \leq \ell(w) + \ell(v) \ . \tag{21}$$

This will prove the lemma because $Ju^{-1}J \subseteq Jw^{-1}J \cdot Jv^{-1}J$ so we have $\ell(u) = \ell(u^{-1}) \geq \ell(v^{-1}) - \ell(w^{-1}) = \ell(v) - \ell(w)$. If v has length 0 then $JvJwJ = JvwJ$ since v normalizes J.

Therefore $u = vw$ and $\ell(u) = \ell(w)$ so the claim is true. Now suppose v has length 1 meaning $v = \omega s$ for some $s \in S_{aff}$ and $\omega \in \Omega$. Recall that $s^2 = 1$. From (19) we deduce

$$JsJ \cdot JsJ = J \dot\cup JsJ \ , \tag{22}$$

since $J = T^0 I$ contains n_s^2 and $\check\alpha([\mathbb{F}_q^\times])$, where $(\alpha, \mathfrak{h}) \in \Pi_{aff}$ is such that $s = s_{(\alpha,\mathfrak{h})}$.

If $\ell(sw) = \ell(w) + 1$, then $\ell(vw) = \ell(v) + \ell(w)$ and $JvJ \cdot JwJ = JvwJ$ using Cor. 2.5.ii. Therefore (21) is obviously satisfied when $u = vw$. Otherwise, $\ell(sw) = \ell(w) - 1$ and $\ell(vw) = \ell(w) - 1$. By (22) we get

$$JsJ \cdot JwJ = JsJ \cdot JsJ \cdot JswJ = JswJ \dot\cup JwJ. \tag{23}$$

Therefore $JvJ \cdot JwJ = J\omega swJ \dot\cup J\omega wJ$ and we see that (21) is satisfied for all $u \in \{vw, \omega w\}$.

Now let $v \in W$ with length > 1 and $s \in S_{aff}$ such that $\ell(sv) = \ell(v) - 1$. By induction hypothesis, $JsvJ \cdot JwJ$ is the disjoint union of double cosets of the form $Ju'J$ with

$$\ell(w) - \ell(v) + 1 \leq \ell(u') \leq \ell(w) + \ell(v) - 1 \tag{24}$$

and, using the previous case, $JvJ \cdot JwJ = JsJ \cdot JsvJ \cdot JwJ$ is a union of double cosets of the form JuJ with

$$\ell(u') - 1 \leq \ell(u) \leq \ell(u') + 1 \ . \tag{25}$$

Combining (24) and (25), we see that u satisfies $\ell(w) - \ell(v) \leq \ell(u) \leq \ell(w) + \ell(v)$. □

2.2. **The pro-p Iwahori-Hecke algebra.** We start from the compact induction $\mathbf{X} := \mathrm{ind}_I^G(1)$ of the trivial I-representation. It can be seen as the space of compactly supported functions $G \to k$ which are constant on the left cosets mod I. It lies in $\mathrm{Mod}(G)$. For Y a compact subset of G which is right invariant under I, we denote by char_Y the characteristic function of Y. It is an element of $\mathbf{X}$.

The pro-p Iwahori-Hecke algebra is defined to be the k-algebra

$$H := \mathrm{End}_{k[G]}(\mathbf{X})^{\mathrm{op}} \ .$$

We often will identify H, as a right H-module, via the map

$$H \xrightarrow{\cong} \mathbf{X}^I$$
$$h \longmapsto (\mathrm{char}_I)h$$

with the submodule $\mathbf{X}^I$ of I-fixed vectors in $\mathbf{X}$. The Bruhat-Tits decomposition of G says that G is the disjoint union of the double cosets IwI for $w \in \widetilde{W}$. Hence we have the I-equivariant decomposition

$$\mathbf{X} = \oplus_{w \in \widetilde{W}} \mathbf{X}(w) \quad \text{with} \quad \mathbf{X}(w) := \mathrm{ind}_I^{IwI}(1) \ ,$$

where the latter denotes the subspace of those functions in $\mathbf{X}$ which are supported on the double coset IwI. In particular, we have $\mathbf{X}(w)^I = k\tau_w$ where $\tau_w := \mathrm{char}_{IwI}$ and hence

$$H = \oplus_{w \in \widetilde{W}} k\tau_w$$

as a k-vector space. If $g \in IwI$ we sometimes also write $\tau_g := \tau_w$. The defining relations of H are the braid relations (see [Vig05] Thm. 1)

$$\tau_w \tau_{w'} = \tau_{ww'} \qquad \text{for } w, w' \in \tilde{W} \text{ such that } \ \ell(ww') = \ell(w) + \ell(w') \tag{26}$$

together with the quadratic relations which we describe now (compare with [OS1] §4.8, and see more references therein). We refer to the notation introduced in §2.1.6, see (13) in particular. To any $s = s_{(\alpha,\mathfrak{h})} \in S_{aff}$, we attach the following idempotent element:

$$\theta_s := -|\mu_{\check\alpha}| \sum_{t\in\check\alpha([\mathbb{F}_q^\times])} \tau_t \ \in H \ . \tag{27}$$

The quadratic relations in H are:

$$\tau_{n_s}^2 = -\tau_{n_s}\,\theta_s = -\theta_s\,\tau_{n_s} \qquad \text{for any } s \in S_{aff}, \tag{28}$$

Since in the existing literature the definition of θ_s is not correct we will give a proof further below. Recall that we defined $\tilde s = n_s T^1 \in \widetilde W$ in (15). The quadratic relation says that $\tau_{\tilde s}^2 = -\theta_s\tau_{\tilde s} = -\tau_{\tilde s}\theta_s$. A general element $w \in \widetilde W$ can be decomposed into $w = \omega\tilde{s_1}\dots\tilde{s_\ell}$ with $\omega \in \widetilde\Omega$, $s_i \in S_{aff}$, and $\ell = \ell(w)$. The braid relations imply $\tau_w = \tau_\omega\tau_{\tilde{s_1}}\dots\tau_{\tilde{s_\ell}}$.

The subgroup of G generated by all parahoric subgroups is denoted by G_{aff} as in [OS1] §4.5. It is a normal subgroup of G and we have $G/G_{aff} = \Omega$. By Bruhat decomposition, G_{aff} is the disjoint union of all IwI for w ranging over the preimage $\widetilde W_{aff}$ of W_{aff} in $\widetilde W$. The subalgebra of H of the functions with support in G_{aff} is denoted by H_{aff} and has basis the set of all τ_w, $w \in \widetilde W_{aff}$. When $\mathbf{G}$ is simply connected semisimple, we have $G = G_{aff}$ and $H = H_{aff}$.

Recall that there is a unique involutive automorphism of H satisfying

$$\iota(\tau_{n_s}) = -\tau_{n_s} - \theta_s \quad \text{and } \iota(\tau_\omega) = \tau_\omega \text{ for all } s \in S_{aff} \text{ and } \omega \in \widetilde\Omega \tag{29}$$

(see for example [OS1] §4.8). It restricts to an involutive automorphism of H_{aff}.

Proof of (28). First we notice that θ_s is indeed an idempotent because

$$\begin{aligned}\theta_s^2 = |\mu_{\check\alpha}|^2 \sum_{u,t\in\check\alpha([\mathbb{F}_q^\times])} \tau_{ut} &= |\mu_{\check\alpha}|^2|\alpha([\mathbb{F}_q^\times])| \sum_{t\in\check\alpha([\mathbb{F}_q^\times])}\tau_t \\ &= (q-1)|\mu_{\check\alpha}| \sum_{t\in\check\alpha([\mathbb{F}_q^\times])}\tau_t = -|\mu_{\check\alpha}|\sum_{t\in\check\alpha([\mathbb{F}_q^\times])}\tau_t = \theta_s \ .\end{aligned}$$

The support of $\tau_{n_s}^2$ is contained in In_sIn_sI. Its value at $h \in G$ is equal to $|In_sI \cap hIn_s^{-1}I/I|\cdot 1_k$, and by (19) we need to consider the cases of $h = n_s^2$ and of $h = tn_s$ for $t \in \check\alpha([\mathbb{F}_q^\times])$. For $h = n_s^2$ this value is equal to $|In_sI/I|\cdot 1_k = |I/I_s|\cdot 1_k = q\cdot 1_k = 0$. From (18), we deduce that for $t = \check\alpha([\zeta])$ where $\zeta \in \mathbb{F}_q^\times$, we have

$$tn_sIn_s^{-1}I = It \,\dot\cup \dot{\bigcup}_{z\in\mathbb{F}_q^\times} tx_\alpha(\pi^{\mathfrak{h}}[z])\check\alpha([z])n_s^{-1}I = I\check\alpha(\zeta) \,\dot\cup \dot{\bigcup}_{z\in\mathbb{F}_q^\times} \check\alpha(\zeta)x_\alpha(\pi^{\mathfrak{h}}[z])\check\alpha([z])n_s^{-1}I \ .$$

For $z \in \mathbb{F}_q^\times$ we compute

$$\check\alpha([\zeta])x_\alpha(\pi^{\mathfrak{h}}[z])\check\alpha([z])n_s^{-1} = x_\alpha(\pi^{\mathfrak{h}}[\zeta^2 z])\check\alpha([-\zeta z])n_s \in I\check\alpha([-\zeta z])n_sI \ .$$

So

$$tn_sIn_s^{-1}I \cap In_sI = \dot{\bigcup}_{z\in\mathbb{F}_q^\times,\, \check\alpha([-\zeta z])=1} \check\alpha([\zeta])x_\alpha(\pi^{\mathfrak{h}}[z])\check\alpha([z])n_s^{-1}I$$

and $|tn_sIn_s^{-1}I \cap In_sI/I| = |\mu_{\check\alpha}|$. We have proved that

$$\tau_{n_s}^2 = |\mu_{\check\alpha}| \sum_{t\in\check\alpha([\mathbb{F}_q^\times])} \tau_{tn_s} = -\theta_s n_s \ .$$

Noticing that $\check{\alpha}([\mathbb{F}_q^\times])n_s = n_s\check{\alpha}([\mathbb{F}_q^\times])$, we then obtain $\tau_{n_s}^2 = -\tau_{n_s}\theta_s$. □

2.2.1. *Idempotents in H.* We now introduce idempotents in H (compare with [OS2] §3.2.4). To any k-character $\lambda : T^0/T^1 \to k^\times$ of T^0/T^1, we associate the following idempotent in H:

$$e_\lambda := (-1)^{\dim \mathbf{T}} \sum_{t\in T^0/T^1} \lambda(t^{-1})\tau_t \ . \tag{30}$$

It satisfies

$$e_\lambda\, \tau_t = \tau_t\, e_\lambda = \lambda(t)\, e_\lambda \tag{31}$$

for any $t \in T^0/T^1$. In particular, when λ is the trivial character we obtain the idempotent element denoted by e_1. Let $s = s_{(\alpha,\mathfrak{h})} \in S_{aff}$ with the corresponding idempotent θ_s as in (27). Using (31), we easily see that $e_\lambda\theta_s = -\sum_{z\in\mathbb{F}_q^\times} \lambda(\check{\alpha}([z]))e_\lambda$ therefore

$$e_\lambda\theta_s = \begin{cases} e_\lambda & \text{if the restriction of } \lambda \text{ to } \check{\alpha}([\mathbb{F}_q^\times]) \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

The action of the finite Weyl group W_0 on T^0/T^1 gives an action on the characters $T^0/T^1 \to k$. This actions inflates to an action of $\widetilde{W}$ denoted by $(w, \lambda) \mapsto {}^w\lambda$. From the braid relations (26), one sees that for $w \in \widetilde{W}$, we have

$$\tau_w\, e_\lambda = e_{{}^w\lambda}\, \tau_w \ . \tag{33}$$

Suppose for a moment that $\mathbb{F}_q \subseteq k$. We then denote by $\widehat{T^0/T^1}$ the set of k-characters of T^0/T^1. The family $\{e_\lambda\}_\lambda \in \widehat{T^0/T^1}$ is a family of orthogonal idempotents with sum equal to 1. It gives the following ring decomposition

$$k[T^0/T^1] = \prod_{\lambda\in\widehat{T^0/T^1}} ke_\lambda \ . \tag{34}$$

Let Γ denote the set of W_0-orbits in $\widehat{T^0/T^1}$. To $\gamma \in \Gamma$ we attach the element $e_\gamma := \sum_{\lambda\in\gamma} e_\lambda$. It is a central idempotent in H (see (33)), and we have the obvious ring decomposition

$$H = \prod_{\gamma\in\Gamma} He_\gamma \ . \tag{35}$$

2.2.2. *Characters of H and H_{aff}.* Since for $s \in S_{aff}$ we have $\tau_{n_s}(\tau_{n_s} + \theta_s) = 0$ where θ_s is an idempotent element, we see that a character $H \to k$ (resp. $H_{aff} \to k$) takes value 0 or -1 at τ_{n_s}. In fact, the following morphisms of k-algebras $H \to k$ are well defined (compare with [OS1] Definition after Remark 6.13):

$$\chi_{triv} : \tau_{\tilde{s}} \longmapsto 0,\ \tau_\omega \longmapsto 1,\ \text{for any } s \in S_{aff} \text{ and } \omega \in \widetilde{\Omega}\ , \tag{36}$$

$$\chi_{sign} : \tau_{\tilde{s}} \longmapsto -1,\ \tau_\omega \longmapsto 1,\ \text{for any } s \in S_{aff} \text{ and } \omega \in \widetilde{\Omega}\ . \tag{37}$$

They satisfy in particular $\chi_{triv}(e_1) = \chi_{sign}(e_1) = 1$. They are called the trivial and the sign character of H, respectively. Notice that $\chi_{sign} = \chi_{triv} \circ \iota$ (see (29)). The restriction to H_{aff} of χ_{triv} (resp. χ_{sign}) is called the trivial (resp. sign) character of H_{aff}.

We call a twisted sign character of H_{aff} a character $\chi : H_{aff} \to k$ such that $\chi(\tau_{n_s}) = -1$ for all $s \in S_{aff}$. The precomposition by ι of a twisted sign character of H_{aff} is called a twisted trivial character. This definition given in [Vig15] coincides with the definition given in [Oll2] §5.4.2 but it is simpler and more concise.

Remark 2.12.
i. A twisted sign character χ of H_{aff} satisfies $\chi(\theta_s) = 1$ for all $s = s_{(\alpha,\mathfrak{h})} \in S_{aff}$ which is equivalent to $\chi(\tau_t) = 1$ for all $t \in \check{\alpha}([\mathbb{F}_q^\times])$. This is also true for a twisted trivial character since the involution ι fixes τ_t for $t \in T^0/T^1$. Therefore, the twisted sign (resp. trivial) characters of H_{aff} are in bijection with the characters $\lambda : T^0/T^1 \to k$ which are equal to 1 on the subgroup $(T^0/T^1)'$ of T^0/T^1 generated by all $\check{\alpha}([\mathbb{F}_q^\times])$ for $\alpha \in \Phi$, equivalently by all $\check{\alpha}([\mathbb{F}_q^\times])$ for $\alpha \in \Pi$.

ii. The twisted trivial characters of H_{aff} are characterized by the fact that they send τ_w to 0 for all $w \in \widetilde{W}_{aff}$ with length > 0 and τ_t to 1 for all $t \in (T^0/T^1)'$.

iii. By Lemma 2.9, the action of $\widetilde{W}$ on T^0/T^1, which is inflated from the action of W_0, induces the trivial action on the quotient $(T^0/T^1)/(T^0/T^1)'$. This has the following consequences:
   - $(T^0/T^1)'$ is a normal subgroup of $\widetilde{W}$.
   - given $\lambda : T^0/T^1 \to k^\times$ a character which is equal to 1 on $(T^0/T^1)'$, the corresponding idempotent e_λ is central in H (use (33))
   - the natural action $(\omega, \chi) \mapsto \chi(\tau_\omega^{-1}\_\tau_\omega)$ of $\widetilde{\Omega}$ on the characters of H_{aff} fixes the twisted trivial characters. Since ι fixes the elements τ_ω for $\omega \in \widetilde{\Omega}$, this action also fixes the twisted sign characters.

iv. When $\mathbf{G}$ is semisimple simply connected, then $H = H_{aff}$ and $(T^0/T^1)' = T^0/T^1$ (Lemma 2.8.i) so the trivial (resp. sign) character of $H = H_{aff}$ is the only twisted trivial (resp. sign) character.

As in [Vig15] §1.4, we notice that the Coxeter system (W_{aff}, S_{aff}) is the direct product of the irreducible affine Coxeter systems $(W^i_{aff}, S^i_{aff})_{1\le i\le r}$ corresponding to the irreducible components $(\Phi^i, \Pi^i)_{1\le i\le r}$ of the based root system (Φ, Π). For $i \in \{1, \ldots, r\}$, the k-module of basis $(\tau_w)_{w\in\widetilde{W}^i_{aff}}$ is a subalgebra of H_{aff}. Remark that it contains $\{\tau_t, t \in T^0/T^1\}$. Remark also that the anti-involution ι restricts to an anti-involution of H^i_{aff}. We call a twisted sign character of H^i_{aff} a character $\chi : H^i_{aff} \to k$ such that $\chi(\tau_{n_s}) = -1$ for all $s \in S^i_{aff}$. The precomposition by ι of a twisted sign character of H^i_{aff} is called a twisted trivial character. The algebras $(H^i_{aff})_{1\le i\le r}$ will be called the irreducible components of H_{aff}.

A character of $H_{aff} \to k$ will be called **supersingular** if, for every $i \in \{1, \ldots, r\}$, it does not restrict to a twisted trivial or sign character of H^i_{aff}. This terminology is justified in the following subsection.

2.3. **Supersingularity.** We refer here to definitions and results of [Oll2] §2 and §5. Note that there the field k was algebraically closed, but is easy to see that the claims that we are going to use are valid when k is not necessarily algebraically closed. In fact, in [Vig15], these definitions and results are generalized to the case where k is an arbitrary field of characteristic p and $\mathbf{G}$ is a general connected reductive $\mathfrak{F}$-group.

In [Oll2] §2.3.1. a central subalgebra $\mathcal{Z}^0(H)$ of H is defined (it is denoted by $\mathcal{Z}_T$ in [Vig15]). This algebra is isomorphic to the affine semigroup algebra $k[X_*^{dom}(T)]$, where $X_*^{dom}(T)$ denotes the semigroup of all dominant cocharacters of T ([Oll2] Prop. 2.10). The cocharacters $\lambda \in X_*^{dom}(T) \setminus (-X_*^{dom}(T))$ generate a proper ideal of $k[X_*^{dom}(T)]$, the image of which in $\mathcal{Z}^0(H)$ is denoted by $\mathfrak{J}$ as in [Oll2] §5.2 (it coincides with the ideal $\mathcal{Z}_{T,\ell>0}$ of [Vig15]).

Generalizing [Oll2] Prop.-Def. 5.10 and [Vig15] Def. 6.10 we call an H-module M supersingular if any element in M is annihilated by a power of $\mathfrak{J}$. Recall that a finite length H-module is always finite dimensional (see for example [OS1] Lemma 6.9). Hence, if M has finite length, then it is supersingular if and only if it is annihilated by a power of $\mathfrak{J}$. Also note that supersingularity can be tested after an arbitrary extension of the coefficient field k.

The supersingular characters of H_{aff} were defined at the end of §2.2.2. The two notions of supersingularity are related by the following fact.

Lemma 2.13. - *Let $\chi : H_{aff} \to k$ be a supersingular character of H_{aff}. The left (resp. right) H-module $H \otimes_{H_{aff}} \chi$ (resp. $\chi \otimes_{H_{aff}} H$) is annihilated by $\mathfrak{J}$. In particular, it is a supersingular H-module.*
- *Let $\chi : H_{aff} \to k$ be a twisted trivial or sign character of H_{aff}. The left (resp. right) H-module $H \otimes_{H_{aff}} \chi$ (resp. $\chi \otimes_{H_{aff}} H$) does not have any nonzero supersingular subquotient.*

Proof. Since supersingularity can be tested after an arbitrary extension of the coefficient field k, we may assume in this proof that k is algebraically closed. We only need to show that the generator $1 \otimes 1$ of $H \otimes_{H_{aff}} \chi$ is annihilated by $\mathfrak{J}$. In that case, Theorem 5.14 in [Oll2] (when the root system is irreducible) and Corollary 6.13 in [Vig15] state that, if a simple (left) H-module M contains a supersingular character χ of H_{aff}, then it is a supersingular H-module (in fact they state that this is an equivalence). The proof of the statement consists in picking an element m in M supporting the character χ and proving that $\mathfrak{J}$ acts by zero on it, the simplicity of M being used only to ensure that $\mathfrak{J}$ acts by zero on the whole M. Therefore the arguments apply to the left H-module $H \otimes_{H_{aff}} \chi$ when χ is a supersingular character of H_{aff} (although this module may not even be of finite length).

Now let $\chi : H_{aff} \to k$ be a twisted trivial (resp. sign) character of H_{aff}. As an H_{aff}-module, $H \otimes_{H_{aff}} \chi$ is isomorphic to a direct sum of copies of χ (Remark 2.12.iii). A nonzero H-module which is a subquotient of $H \otimes_{H_{aff}} \chi$ is therefore also a direct sum of copies of χ as an H_{aff}-module. Suppose that $H \otimes_{H_{aff}} \chi$ has a nonzero supersingular subquotient. Then it has a nonzero supersingular finitely generated subquotient. Since the latter admits a nonzero simple quotient, the H-module $H \otimes_{H_{aff}} \chi$ has a nonzero simple supersingular subquotient M. This is not compatible with M being a direct sum of copies of χ as an H_{aff}-module, as proved in [Oll2] Lemma 5.12 when the root system is irreducible or in [Vig15] Corollary 6.13.

The proof is the same for right H-modules. □

Define the decreasing filtration

$$F^n H := \oplus_{\ell(w) \geq n} k\tau_w \qquad \text{for } n \geq 0 \tag{38}$$

of H as a bimodule over itself.

Lemma 2.14. *Under the hypothesis that $\mathbf{G}$ is semisimple simply connected with irreducible root system, we have:*

i. *As an H-module on the left and on the right, $F^m H / F^{m+1} H$, for any $m \geq 1$, is annihilated by $\mathfrak{J}$; in particular:*
$$\mathfrak{J}^{m-1} \cdot F^1 H = F^1 H \cdot \mathfrak{J}^{m-1} \subset F^m H \ .$$
ii. *As an H-module on the left and on the right, $(1 - e_1) \cdot (F^0 H / F^1 H)$ is annihilated by $\mathfrak{J}$; in particular:*
$$\mathfrak{J}^m \cdot [(1 - e_1) \cdot F^0 H + F^1 H] \subset F^m H \ .$$

Proof. In this proof we consider left H-modules. The arguments are the same for the structures of right modules. Recall that under the hypothesis of the lemma, we have $W = W_{aff}$ and $H = H_{aff}$. Furthermore, we may assume that $\mathbb{F}_q \subseteq k$ and that $\mathbf{G} \neq 1$.

i. We will show that $F^m H/F^{m+1}H$, in fact, is a direct sum of supersingular characters. Since $\mathbb{F}_q \subseteq k$, a basis for $F^m H/F^{m+1}H$ is given by all $e_\lambda \tau_{\tilde{w}}$ for $w \in W$ with length m and lift $\tilde{w} \in \widetilde{W}$ and all $\lambda \in \widehat{T^0/T^1}$ (notation in §2.2.1). For $s = s_{(\alpha,\mathfrak{h})} \in S_{aff}$ we pick the lift $\tilde{s}$ as in (15). Using (33), (26) and (28):

$$\tau_{\tilde{s}} \cdot e_\lambda \tau_{\tilde{w}} = \begin{cases} e_{^s\lambda}\, \tau_{\tilde{s}}^2\, \tau_{\tilde{s}^{-1}\tilde{w}} = -\theta_s\, e_{^s\lambda}\, \tau_{\tilde{w}} & \text{if } \ell(sw) = \ell(w) - 1, \\ e_{^s\lambda} \tau_{\tilde{s}\tilde{w}} & \text{if } \ell(sw) = \ell(w) + 1. \end{cases}$$

So in $F^m H/F^{m+1}H$, we have, using (32):

$$\tau_{\tilde{s}} \cdot e_\lambda \tau_{\tilde{w}} = \begin{cases} 0 & \text{if } \ell(sw) = \ell(w) - 1 \text{ and } \lambda|_{\check{\alpha}([\mathbb{F}_q^\times])} \neq 1, \\ -e_{^s\lambda} \tau_{\tilde{w}} & \text{if } \ell(sw) = \ell(w) - 1 \text{ and } \lambda|_{\check{\alpha}([\mathbb{F}_q^\times])} = 1, \\ 0 & \text{if } \ell(sw) = \ell(w) + 1. \end{cases}$$

Using Lemma 2.9, notice that if λ is trivial on $\check{\alpha}([\mathbb{F}_q^\times])$, then ${}^s\lambda = \lambda$. This proves that $e_\lambda \tau_{\tilde{w}}$ supports the character $\chi : H_{aff} \to k$ defined by $\chi(\tau_t) = \lambda(t)$ and

$$\chi(\tau_{\tilde{s}}) = \begin{cases} 0 & \text{if } \ell(sw) = \ell(w) - 1 \text{ and } \lambda|_{\check{\alpha}([\mathbb{F}_q^\times])} \neq 1 \text{ or if } \ell(sw) = \ell(w) + 1, \\ -1 & \text{if } \ell(sw) = \ell(w) - 1 \text{ and } \lambda|_{\check{\alpha}([\mathbb{F}_q^\times])} = 1 \end{cases}$$

for $s \in S_{aff}$. If there is $s_{(\alpha,\mathfrak{h})} \in S_{aff}$ such that $\lambda|_{\check{\alpha}([\mathbb{F}_q^\times])} \neq 1$ then χ is supersingular (Remark 2.12). Otherwise λ is trivial on T^0/T^1 and we want to check that χ is not a twisted sign or a twisted trivial character. This is because $m \geq 1$ and the root system of $\mathbf{G}$ is irreducible of rank > 0. Therefore there are $s, s' \in S_{aff}$ such that $\ell(sw) = \ell(w) - 1$ and $\ell(s'w) = \ell(w) + 1$. Since H_{aff} has only one irreducible component we see that χ is a supersingular character. By Lemma 2.13, the left H-module $F^m H/F^{m+1}H$ then is annihilated by $\mathfrak{J}$, which concludes the proof of i.

ii. Again we will show that $(1 - e_1) \cdot (F^0 H/F^1 H)$, in fact, is a direct sum of supersingular characters. A basis for $(1 - e_1) \cdot (F^0 H/F^1 H)$ is given by all e_λ for all $\lambda \in \widehat{T^0/T^1} \setminus \{1\}$. But $\tau_{\tilde{s}} e_\lambda \in F^1 H$ by (26). This proves that e_λ supports the character $\chi : H_{aff} \to k$ defined by $\chi(\tau_t) = \lambda(t)$ (see (31)) and $\chi(\tau_{\tilde{s}}) = 0$ for $s \in S_{aff}$. It is not a twisted sign or a twisted trivial character since λ is nontrivial on $(T^0/T^1)' = T^0/T^1$. As the root system is irreducible, it is a supersingular character. Conclude using point i. and Lemma 2.13. ☐

3. The Ext-algebra

3.1. **The definition.** In order to introduce the algebra in the title we again start from the compact induction $\mathbf{X} = \mathrm{ind}_I^G(1)$ of the trivial I-representation, which lies in $\mathrm{Mod}(G)$. We form the graded Ext-algebra

$$E^* := \mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})^{\mathrm{op}}$$

over k with the multiplication being the (opposite of the) Yoneda product. Obviously

$$H := E^0 = \mathrm{End}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})^{\mathrm{op}}$$

is the usual pro-p Iwahori-Hecke algebra over k. By using Frobenius reciprocity for compact induction and the fact that the restriction functor from $\mathrm{Mod}(G)$ to $\mathrm{Mod}(I)$ preserves injective objects we obtain the identification

$$E^* = \mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})^{\mathrm{op}} = H^*(I, \mathbf{X}) \ . \tag{39}$$

The only part of the multiplicative structure on E^* which is still directly visible on the cohomology $H^*(I, \mathbf{X})$ is the right multiplication by elements in $E^0 = H$, which is functorially induced by the right action of H on $\mathbf{X}$. It is one of the main technical issues of this paper to make the full multiplicative structure visible on $H^*(I, \mathbf{X})$. We recall that for $* = 0$ the above identification is given by

$$\begin{aligned} H &\xrightarrow{\cong} \mathbf{X}^I \\ \tau &\longmapsto (\mathrm{char}_I)\tau \ . \end{aligned}$$

3.2. **The technique.** The technical tool for studying the algebra E^* is the I-equivariant decomposition

$$\mathbf{X} = \oplus_{w \in \widetilde{W}} \mathbf{X}(w)$$

introduced in section 2.2. Noting that the cohomology of profinite groups commutes with arbitrary sums, we obtain

$$H^*(I, \mathbf{X}) = \oplus_{w \in \widetilde{W}} H^*(I, \mathbf{X}(w)) \ .$$

Similarly as we write IwI, since this double coset only depends on the coset $w \in N(T)/T^1$, we will silently abuse notation in the following whenever something only depends on the coset w. We have the isomorphism of I-representations

$$\begin{aligned} \mathrm{ind}_I^{IwI}(1) &\xrightarrow{\cong} \mathrm{ind}_{I_w}^I(1) \\ f &\longmapsto \phi_f(a) := f(aw) \ . \end{aligned}$$

This gives rise to the left hand cohomological isomorphism

$$H^*(I, \mathbf{X}(w)) \xrightarrow{\cong} H^*(I, \mathrm{ind}_{I_w}^I(1)) \xrightarrow{\cong} H^*(I_w, k)$$

which we may combine with the right hand Shapiro isomorphism. For simplicity we will call in the following the above composite isomorphism the Shapiro isomorphism and denote it by Sh_w. Equivalently, it can be described as the composite map

$$\mathrm{Sh}_w : H^*(I, \mathbf{X}(w)) \xrightarrow{\mathrm{res}} H^*(I_w, \mathbf{X}(w)) \xrightarrow{H^*(I_w, \mathrm{ev}_w)} H^*(I_w, k) \tag{40}$$

where

$$\begin{aligned} \mathrm{ev}_w : \mathbf{X}(w) &\longrightarrow k \\ f &\longrightarrow f(w) \ . \end{aligned}$$

We leave it as an exercise to the reader to check that the map

$$\mathrm{Sh}_w^{-1} : H^*(I_w, k) \xrightarrow{\mathrm{i}_w} H^*(I_w, \mathbf{X}(w)) \xrightarrow{\mathrm{cores}} H^*(I, \mathbf{X}(w)) \ , \tag{41}$$

where

$$\begin{aligned} \mathrm{i}_w : k &\longrightarrow \mathbf{X}(w) \\ a &\longrightarrow a\, \mathrm{char}_{wI} \ , \end{aligned}$$

is the inverse of the Shapiro isomorphism Sh_w.

3.3. **The cup product.** There is a naive product structure on the cohomology $H^*(I, \mathbf{X})$. By multiplying maps we obtain the G-equivariant map

$$\begin{aligned} \mathbf{X} \otimes_k \mathbf{X} &\longrightarrow \mathbf{X} \\ f \otimes f' &\longmapsto f f' \ . \end{aligned}$$

It gives rise to the cup product

$$H^i(I, \mathbf{X}) \otimes_k H^j(I, \mathbf{X}) \xrightarrow{\cup} H^{i+j}(I, \mathbf{X}) \tag{42}$$

which, quite obviously, has the property that

$$H^i(I, \mathbf{X}(v)) \cup H^j(I, \mathbf{X}(w)) = 0 \quad \text{whenever } v \neq w. \tag{43}$$

On the other hand, since $\mathrm{ev}_w(f f') = \mathrm{ev}_w(f)\, \mathrm{ev}_w(f')$ and since the cup product is functorial and commutes with cohomological restriction maps, we have the commutative diagrams

$$\begin{array}{ccc} H^i(I, \mathbf{X}(w)) \otimes_k H^j(I, \mathbf{X}(w)) & \xrightarrow{\ \cup\ } & H^{i+j}(I, \mathbf{X}(w)) \\ \Big\downarrow {\scriptstyle \mathrm{Sh}_w \otimes \mathrm{Sh}_w} & & \Big\downarrow {\scriptstyle \mathrm{Sh}_w} \\ H^i(I_w, k) \otimes_k H^j(I_w, k) & \xrightarrow{\ \cup\ } & H^{i+j}(I_w, k) \end{array} \tag{44}$$

for any $w \in \widetilde{W}$, where the bottom row is the usual cup product on the cohomology algebra $H^*(I_w, k)$. In particular, we see that the cup product (42) is anticommutative.

4. Representing cohomological operations on resolutions

4.1. **The Shapiro isomorphism.** The Shapiro isomorphism (40) also holds for nontrivial coefficients provided we choose once and for all, as we will do in the following, a representative $\dot{w} \in N(T)$ for each $w \in \widetilde{W}$. Compact induction is an exact functor

$$\begin{aligned} \mathrm{ind}_I^G : \mathrm{Mod}(I) &\longrightarrow \mathrm{Mod}(G) \\ Y &\longmapsto \mathrm{ind}_I^G(Y) \ . \end{aligned}$$

Moreover, as before we have the decomposition $\mathrm{ind}_I^G(Y) = \oplus_{w \in \widetilde{W}} \mathrm{ind}_I^{IwI}(Y)$ and the isomorphism

$$\begin{aligned} \mathrm{ind}_I^{IwI}(Y) &\xrightarrow{\cong} \mathrm{ind}_{I_{\dot{w}}}^I(\dot{w}_* \operatorname{res}^I_{I_{\dot{w}^{-1}}}(Y)) \\ f &\longmapsto \phi_f(a) := f(a\dot{w}) \end{aligned}$$

as I-representations. On cohomology we obtain the commutative diagram

$$\begin{array}{ccccc} H^*(I, \mathrm{ind}_I^{IwI}(Y)) & \xrightarrow{\cong} & H^*(I, \mathrm{ind}_{I_w}^I(\dot{w}_* \operatorname{res}^I_{I_{w^{-1}}}(Y)) & \xrightarrow{\cong} & H^*(I_w, \dot{w}_* Y) \\ & {\scriptstyle \mathrm{res}} \searrow & & \nearrow {\scriptstyle H^*(I_w, \mathrm{ev}_{\dot{w}})} & \\ & & H^*(I_w, \mathrm{ind}_I^{IwI}(Y)) & & \end{array} \tag{45}$$

in which $\mathrm{ev}_{\dot{w}}$ now denotes the evaluation map in $\dot{w}$ and in which the composite map in the top row is an isomorphism denoted by $\mathrm{Sh}_{\dot{w}}$.

To lift this to the level of complexes we first make the following observation.

Lemma 4.1. *If $\mathcal{J}$ is an injective object in* $\mathrm{Mod}(I)$ *then* $\mathrm{ind}_I^{IwI}(\mathcal{J})$*, for any $w \in \widetilde{W}$, is an injective object in* $\mathrm{Mod}(I)$ *as well.*

Proof. We use the isomorphism $\mathrm{ind}_I^{IwI}(\mathcal{J}) \cong \mathrm{ind}_{I_w}^{I}(\dot{w}_* \operatorname{res}^I_{I_{w^{-1}}}(\mathcal{J}))$. As recalled before, the restriction functor to open subgroups preserves injective objects. The functor $\dot{w}_* : \mathrm{Mod}(I_{w^{-1}}) \xrightarrow{\simeq} \mathrm{Mod}(I_w)$ is an equivalence of categories and hence preserves injective objects. Finally, the induction functor from open subgroups of finite index also preserves injective objects (cf. [Vig96] I.5.9.b)). □

Let now $k \xrightarrow{\sim} \mathcal{I}^\bullet$ and $k \xrightarrow{\sim} \mathcal{J}^\bullet$ be any two injective resolutions in $\mathrm{Mod}(I)$ of the trivial representation. By Lemma 4.1 then $\mathbf{X}(w) \xrightarrow{\sim} \mathrm{ind}_I^{IwI}(\mathcal{J}^\bullet)$ is an injective resolution in $\mathrm{Mod}(I)$ as well. Hence

$$H^*(I, \mathbf{X}(w)) = \mathrm{Hom}_{K(I)}(\mathcal{I}^\bullet, \mathrm{ind}_I^{IwI}(\mathcal{J}^\bullet)[*]) \ ,$$

i.e., any cohomology class in $H^*(I, \mathbf{X}(w))$ is of the form $[\alpha^\bullet]$ for some homomorphism of complexes $\alpha^\bullet : \mathcal{I}^\bullet \longrightarrow \mathrm{ind}_I^{IwI}(\mathcal{J}^\bullet)[*]$ in $\mathrm{Mod}(I)$ (which is unique up to homotopy). Composition with the evaluation map gives rise to the homomorphism of injective complexes

$$\mathrm{Sh}_{\dot{w}}(\alpha^\bullet) := \mathrm{ev}_{\dot{w}} \circ \alpha^\bullet : \mathcal{I}^\bullet \longrightarrow \dot{w}_* \mathcal{J}^\bullet[*]$$

in $\mathrm{Mod}(I_w)$ whose cohomology class is $\mathrm{Sh}_w([\alpha^\bullet]) \in H^*(I_w, k) = \mathrm{Hom}_{K(I_w)}(\mathcal{I}^\bullet, \dot{w}_*\mathcal{J}^\bullet[*])$.

In fact it will be more convenient later on to use the following modified version of the Shapiro isomorphism. For this we assume that $k \xrightarrow{\sim} \mathcal{J}^\bullet$ is actually an injective resolution in $\mathrm{Mod}(G)$ (and hence in $\mathrm{Mod}(I)$). Then the composite map

$$\mathrm{Sh}'_{\dot{w}}(\alpha^\bullet) : \mathcal{I}^\bullet \xrightarrow{\mathrm{Sh}_{\dot{w}}(\alpha^\bullet)} \dot{w}_*\mathcal{J}^\bullet[*] \xrightarrow{y \mapsto \dot{w}y} \mathcal{J}^\bullet[*] \tag{46}$$

is defined and is also a homomorphism of injective complexes in $\mathrm{Mod}(I_w)$ representing the same cohomology class as $\mathrm{Sh}_{\dot{w}}(\alpha^\bullet)$ but viewed in $H^*(I_w, k) = \mathrm{Hom}_{K(I_w)}(\mathcal{I}^\bullet, \mathcal{J}^\bullet[*])$, i.e., we have

$$[\mathrm{Sh}'_{\dot{w}}(\alpha^\bullet)] = [\mathrm{Sh}_{\dot{w}}(\alpha^\bullet)] = \mathrm{Sh}_w([\alpha^\bullet]) \ . \tag{47}$$

The homomorphism $\alpha^\bullet$ can be reconstructed from $\mathrm{Sh}'_{\dot{w}}(\alpha^\bullet)$ by the formula

$$\alpha^\bullet(x)(a\dot{w}b) = b^{-1}(({}^{a^{-1}}(\alpha^\bullet(x)))(\dot{w})) = b^{-1}(\alpha^\bullet(a^{-1}x)(\dot{w})) = (a\dot{w}b)^{-1}a(\mathrm{Sh}'_{\dot{w}}(\alpha^\bullet)(a^{-1}x)) \tag{48}$$

for any $x \in \mathcal{I}^\bullet$ and any $a, b \in I$.

4.2. **The Yoneda product.** Here we consider an injective resolution $\mathbf{X} \xrightarrow{\sim} \mathcal{I}^\bullet$ of our G-representation $\mathbf{X}$ in $\mathrm{Mod}(G)$. Then

$$E^* = \mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X}) = \mathrm{Hom}_{D(G)}(\mathbf{X}, \mathbf{X}[*]) = \mathrm{Hom}_{K(G)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet[*]) \ ,$$

and the Yoneda product is the obvious composition of homomorphisms of complexes (cf. [Har] Cor. I.6.5). We recall, though, that our convention is to consider the opposite of this composition. For our purposes it is crucial to replace $\mathcal{I}^\bullet$ by a quasi-isomorphic complex constructed as follows.

We begin with an injective resolution $k \xrightarrow{\sim} \mathcal{J}^\bullet$ in $\mathrm{Mod}(I)$ of the trivial representation. Then $\mathbf{X} \xrightarrow{\sim} \mathrm{ind}_I^G(\mathcal{J}^\bullet) = \oplus_{w \in \widetilde{W}} \mathrm{ind}_I^{IwI}(\mathcal{J}^\bullet)$ is a resolution in $\mathrm{Mod}(G)$. By Lemma 4.1 each term $\mathrm{ind}_I^{IwI}(\mathcal{J}^\bullet)$ is injective in $\mathrm{Mod}(I)$. Since the cohomology functor $H^*(I, -)$ commutes with arbitrary sums in $\mathrm{Mod}(I)$ it follows that each term $\mathrm{ind}_I^G(\mathcal{J}^\bullet)$ is an $H^0(I, -)$-acyclic object in $\mathrm{Mod}(I)$. But by Frobenius reciprocity we have the isomorphism $\mathrm{Hom}_{\mathrm{Mod}(G)}(\mathrm{ind}_I^G(1), -) \cong$

$H^0(I, -)$ of left exact functors on $\mathrm{Mod}(G)$. We conclude that $\mathbf{X} \xrightarrow{\sim} \mathrm{ind}_I^G(\mathcal{J}^\bullet)$ is a resolution of $\mathbf{X}$ in $\mathrm{Mod}(G)$ by $\mathrm{Hom}_{\mathrm{Mod}(G)}(\mathbf{X}, -)$-acyclic objects. It follows that

$$
\begin{aligned}
\mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X}) &= h^*(\mathrm{Hom}_{\mathrm{Mod}(G)}(\mathbf{X}, \mathrm{ind}_I^G(\mathcal{J}^\bullet))) \\
&= h^*(\mathrm{ind}_I^G(\mathcal{J}^\bullet)^I) \\
&= \oplus_{w \in \widetilde{W}}\, h^*(\mathrm{ind}_I^{IwI}(\mathcal{J}^\bullet)^I) \cong \oplus_{w \in \widetilde{W}}\, h^*(\mathrm{ind}_{I_w}^I(\dot{w}_* \,\mathrm{res}^I_{I_{w^{-1}}}(\mathcal{J}^\bullet))^I) \\
&\cong \oplus_{w \in \widetilde{W}}\, h^*((\mathcal{J}^\bullet)^{I_{w^{-1}}}) \ .
\end{aligned} \tag{49}
$$

In order to lift these equalities to the level of complexes we consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X}) & & \\
\| & & \\
\mathrm{Hom}_{D(G)}(\mathrm{ind}_I^G(\mathcal{J}^\bullet), \mathrm{ind}_I^G(\mathcal{J}^\bullet)[*]) & = & h^*(\mathrm{Hom}_{\mathrm{Mod}(G)}(\mathbf{X}, \mathrm{ind}_I^G(\mathcal{J}^\bullet))) \\
\uparrow & & \\
\mathrm{Hom}_{K(G)}(\mathrm{ind}_I^G(\mathcal{J}^\bullet), \mathrm{ind}_I^G(\mathcal{J}^\bullet)[*]) & & \cong \uparrow \text{Frobenius reciprocity} \\
\text{Frobenius reciprocity} \uparrow \cong & & \\
\mathrm{Hom}_{K(I)}(\mathcal{J}^\bullet, \mathrm{ind}_I^G(\mathcal{J}^\bullet)[*]) & \longrightarrow & H^*(I, \mathbf{X})) \\
\uparrow & & \| \\
\oplus_{w \in \widetilde{W}} \mathrm{Hom}_{K(I)}(\mathcal{J}^\bullet, \mathrm{ind}_I^{IwI}(\mathcal{J}^\bullet)[*]) & \xrightarrow{\cong} & \oplus_{w \in \widetilde{W}} H^*(I, \mathbf{X}(w)).
\end{array}
$$

The isomorphism in the bottom row is a consequence of Lemma 4.1. The computation (49) shows that the composite map in the first column is an isomorphism.

We point out that these two ways of representing E^* by homomorphisms of complexes, through $\mathcal{I}^\bullet$ and through $\mathrm{ind}_I^G(\mathcal{J}^\bullet)$, are related by the unique (up to homotopy) homomorphism of complexes in $\mathrm{Mod}(G)$ which makes the diagram

$$
\begin{array}{ccc}
 & & \mathrm{ind}_I^G(\mathcal{J}^\bullet) \\
 & \overset{\sim}{\nearrow} & \downarrow \\
\mathbf{X} & & \\
 & \underset{\sim}{\searrow} & \downarrow \\
 & & \mathcal{I}^\bullet
\end{array}
$$

commutative.

Consider any classes $[\alpha^\bullet] \in H^i(I, \mathbf{X}(v))$ and $[\beta^\bullet] \in H^j(I, \mathbf{X}(w))$ represented by homomorphisms of complexes

$$
\alpha^\bullet : \mathcal{J}^\bullet \longrightarrow \mathrm{ind}_I^{IvI}(\mathcal{J}^\bullet)[i] \subseteq \mathrm{ind}_I^G(\mathcal{J}^\bullet)[i] \quad \text{and} \quad \beta^\bullet : \mathcal{J}^\bullet \longrightarrow \mathrm{ind}_I^{IwI}(\mathcal{J}^\bullet)[j] \subseteq \mathrm{ind}_I^G(\mathcal{J}^\bullet)[j] \ ,
$$

respectively. According to the above diagram these induce, by Frobenius reciprocity, homomorphisms of complexes $\tilde{\alpha}^\bullet : \mathrm{ind}_I^G(\mathcal{J}^\bullet) \longrightarrow \mathrm{ind}_I^G(\mathcal{J}^\bullet)[i]$ and $\tilde{\beta}^\bullet : \mathrm{ind}_I^G(\mathcal{J}^\bullet) \longrightarrow \mathrm{ind}_I^G(\mathcal{J}^\bullet)[j]$ which represent our original classes when viewed in $\mathrm{Ext}^i_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})$ and $\mathrm{Ext}^j_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})$, respectively. The Yoneda product of the latter is represented by the composite $\tilde{\beta}^\bullet[i] \circ \tilde{\alpha}^\bullet$, which we may write as $\tilde{\beta}^\bullet[i] \circ \tilde{\alpha}^\bullet = \tilde{\gamma}^\bullet$ for a homomorphism of complexes $\gamma^\bullet : \mathcal{J}^\bullet \longrightarrow \mathrm{ind}_I^G(\mathcal{J}^\bullet)[i+j]$. By writing out the Frobenius reciprocity isomorphism we see that

$$\tilde{\beta}^\bullet : \mathrm{ind}_I^G(\mathcal{J}^\bullet) \longrightarrow \mathrm{ind}_I^G(\mathcal{J}^\bullet)[j] \qquad (50)$$
$$f \longmapsto \sum_{g \in G/I} g\beta^\bullet(f(g)) \ .$$

We deduce that, if f has support in the subset $S \subseteq G/I$, then $\tilde{\beta}^\bullet(f) = \sum_{g\in S} g\beta^\bullet(f(g))$ has support in $S \cdot IwI$. Applying this to functions in the image of $\alpha^\bullet$, which are supported in IvI, we obtain that $\gamma^\bullet$, in fact, is a homomorphism of complexes

$$\gamma^\bullet : \mathcal{J}^\bullet \longrightarrow \mathrm{ind}_I^{IvI\cdot IwI}(\mathcal{J}^\bullet)[i+j] \ .$$

This shows that, if $\cdot$ denotes the multiplication on $H^*(I, \mathbf{X})$ induced by the opposite of the Yoneda product, then we have $[\alpha^\bullet] \cdot [\beta^\bullet] = (-1)^{ij}[\gamma^\bullet]$ and hence

$$H^i(I, \mathbf{X}(v)) \cdot H^j(I, \mathbf{X}(w)) \subseteq H^{i+j}(I, \mathrm{ind}_I^{IvI\cdot IwI}(1)) \ . \qquad (51)$$

4.3. **The cup product.** Let U be any profinite group. It is well known that, under the identification $H^*(U,k) = \mathrm{Ext}^*_{\mathrm{Mod}(U)}(k,k)$, the cup product pairing

$$H^i(U,k) \times H^j(U,k) \xrightarrow{\cup} H^{i+j}(U,k)$$

coincides with the Yoneda composition product

$$\mathrm{Ext}^i_{\mathrm{Mod}(U)}(k,k) \times \mathrm{Ext}^j_{\mathrm{Mod}(U)}(k,k) \xrightarrow{\circ} \mathrm{Ext}^{i+j}_{\mathrm{Mod}(U)}(k,k) \ .$$

For discrete groups this is, for example, explained in [Bro] V§4. The argument there uses projective resolutions and therefore cannot be generalized directly to profinite groups. Instead one may use the axiomatic approach in [Lan] Chap. IV (see also p. 136).

We will use this in the following way. Let $k \xrightarrow{\sim} \mathcal{I}^\bullet$, $k \xrightarrow{\sim} \mathcal{J}^\bullet$, and $k \xrightarrow{\sim} \mathcal{K}^\bullet$ be three injective resolutions in $\mathrm{Mod}(U)$. Any two cohomology classes $\alpha \in H^i(U,k)$ and $\beta \in H^j(U,k)$ can be represented by homomorphisms of complexes $\alpha^\bullet : \mathcal{J}^\bullet \to \mathcal{K}^\bullet[i]$ and $\beta : \mathcal{I}^\bullet \to \mathcal{J}^\bullet[j]$, respectively. Then $\alpha \cup \beta \in H^{i+j}(U,k)$ is represented by the composite $\alpha^\bullet[j] \circ \beta^\bullet : \mathcal{I}^\bullet \to \mathcal{K}^\bullet[i+j]$.

4.4. **Conjugation.** The cohomology of profinite groups is functorial in pairs (ξ, f) where $\xi : V' \to V$ is a continuous homomorphism of profinite groups and $f : M \to M'$ is a k-linear map between an M in $\mathrm{Mod}(V)$ and an M' in $\mathrm{Mod}(V')$ such that

$$f(\xi(g)m) = gf(m) \qquad \text{for any } g \in V \text{ and } m \in M.$$

One method to construct the corresponding map on cohomology $(\xi, f)^* : H^i(V, M) \to H^i(V', M')$ proceeds as follows. We pick injective resolutions $M \xrightarrow{\sim} \mathcal{I}^\bullet_M$ in $\mathrm{Mod}(V)$ and $M' \xrightarrow{\sim} \mathcal{I}^\bullet_{M'}$ in $\mathrm{Mod}(V')$. By viewing, via ξ, $M \xrightarrow{\sim} \mathcal{I}^\bullet_M$ as a resolution in $\mathrm{Mod}(V')$ we see that f extends to a homomorphism of complexes $f^\bullet : \mathcal{I}^\bullet_M \to \mathcal{I}^\bullet_{M'}$ such that

$$f^i(\xi(g)x) = gf^i(x) \qquad \text{for any } i \geq 0,\ g \in V, \text{ and } x \in \mathcal{I}^i_M. \qquad (52)$$

Then

$$(\xi, f)^* : H^i(V, M) = \mathrm{Hom}_{K(V)}(k, \mathcal{I}^\bullet_M[i]) \longrightarrow \mathrm{Hom}_{K(V')}(k, \mathcal{I}^\bullet_{M'}[i]) = H^i(V', M')$$
$$\alpha^\bullet \longmapsto f^\bullet[i] \circ \alpha^\bullet \ .$$

We are primarily interested in the case where $M = k$ and $M' = k$ are the trivial representations, $f = \mathrm{id}_k$, and ξ is an isomorphism. We simply write $\xi^* := (\xi, \mathrm{id}_k)^*$ in this case.

We may then take $\mathcal{I}^\bullet_{M'} := \xi^*\mathcal{I}^\bullet_k$ to be $\mathcal{I}^\bullet_k$ but with V' acting through ξ and $f^\bullet := \mathrm{id}_{\mathcal{I}^\bullet_k}$. Let $k \xrightarrow{\sim} \mathcal{I}^\bullet$ be another injective resolution in $\mathrm{Mod}(V)$. We have the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{K(V)}(k, \mathcal{I}^\bullet_k[i]) & \xrightarrow{\alpha^\bullet \mapsto \alpha^\bullet} & \mathrm{Hom}_{K(V')}(k, \xi^*\mathcal{I}^\bullet_k[i]) \\
\cong \uparrow & & \uparrow \cong \\
\mathrm{Hom}_{K(V)}(\mathcal{I}^\bullet, \mathcal{I}^\bullet_k[i]) & \xrightarrow{\alpha^\bullet \mapsto \alpha^\bullet} & \mathrm{Hom}_{K(V')}(\xi^*\mathcal{I}^\bullet, \xi^*\mathcal{I}^\bullet_k[i]).
\end{array}$$

In other words, if the cohomology class $\alpha \in H^i(V, k)$ is represented by the homomorphism of complexes $\alpha^\bullet : \mathcal{I}^\bullet \to \mathcal{I}^\bullet_k[i]$, then its image $\xi^*\alpha \in H^i(V', k)$ is represented by

(53) $\qquad \xi^*\alpha^\bullet := \alpha^\bullet : \xi^*\mathcal{I}^\bullet \to \xi^*\mathcal{I}^\bullet_k[i]$ (viewed as a V'-equivariant homomorphism via ξ).

A specific instance of this situation is the following. Assume that the profinite group V is a subgroup of some topological group H, let $h \in H$ be a fixed element, $V' := hVh^{-1}$, and $\xi : V' \to V$ be the isomorphism given by conjugation by h^{-1}. We then write

$$h_* = (h^{-1})^* := \xi^* : H^i(V, k) \longrightarrow H^i(hVh^{-1}, k)$$

for the map on cohomology and $h_*\alpha^\bullet$ for the representing homomorphisms. We suppose now that V is open in H, in which case there is the following alternative description. We choose injective resolutions $k \xrightarrow{\sim} \mathcal{I}^\bullet$ and $k \xrightarrow{\sim} \mathcal{J}^\bullet$ in $\mathrm{Mod}(H)$. Then they are also injective resolutions in $\mathrm{Mod}(V)$ and $\mathrm{Mod}(V')$, so that we may take $\mathcal{I}^\bullet_k := \mathcal{J}^\bullet$. The map $f^\bullet : \mathcal{J}^\bullet \xrightarrow{h\cdot} \mathcal{J}^\bullet$ satisfies the condition (52), and we obtain

$$\begin{aligned} h_* : H^i(V, k) = \mathrm{Hom}_{K(V)}(k, \mathcal{J}^\bullet[i]) &\longrightarrow \mathrm{Hom}_{K(hVh^{-1})}(k, \mathcal{J}^\bullet[i]) = H^i(hVh^{-1}, k) \\ \alpha^\bullet &\longmapsto h\alpha^\bullet \ . \end{aligned}$$

This time one checks that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{K(V)}(k, \mathcal{J}^\bullet[i]) & \xrightarrow{\alpha^\bullet \mapsto h\alpha^\bullet} & \mathrm{Hom}_{K(hVh^{-1})}(k, \mathcal{J}^\bullet[i]) \\
\cong \uparrow & & \uparrow \cong \\
\mathrm{Hom}_{K(V)}(\mathcal{I}^\bullet, \mathcal{J}^\bullet[i]) & \xrightarrow{\alpha^\bullet \mapsto h\alpha^\bullet h^{-1}} & \mathrm{Hom}_{K(hVh^{-1})}(\mathcal{I}^\bullet, \mathcal{J}^\bullet[i])
\end{array}$$

is commutative. We conclude that, if the cohomology class $\alpha \in H^i(V, k)$ is represented by the homomorphism of complexes $\alpha^\bullet : \mathcal{I}^\bullet \to \mathcal{J}^\bullet[i]$, then its image $h_*\alpha \in H^i(hVh^{-1}, k)$ is represented by

(54) $$^{h_*}\alpha^\bullet := h\alpha^\bullet h^{-1} : \mathcal{I}^\bullet \to \mathcal{J}^\bullet[i] \ .$$

4.5. **The corestriction.** Let U be a profinite group with open subgroup $V \subseteq U$ and let M be in $\mathrm{Mod}(U)$. In this situation we have the corestriction map $\mathrm{cores}^V_U : H^*(V, M) \to H^*(U, M)$. It can be constructed as follows (cf. [NSW] I§5.4). Let $M \xrightarrow{\sim} \mathcal{I}^\bullet_M$ be an injective resolution in $\mathrm{Mod}(U)$. Then

$$\begin{aligned} \mathrm{cores}^V_U : H^i(V, M) = \mathrm{Hom}_{K(V)}(k, \mathcal{I}^\bullet_M[i]) &\longrightarrow \mathrm{Hom}_{K(U)}(k, \mathcal{I}^\bullet_M[i]) = H(U, M) \\ \alpha^\bullet &\longmapsto \sum_{g \in U/V} g\alpha^\bullet \ . \end{aligned}$$

For a variant of this, which we will need, let $k \xrightarrow{\sim} \mathcal{I}^\bullet$ be an injective resolution in $\mathrm{Mod}(U)$. One easily checks that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{K(V)}(k, \mathcal{I}_M^\bullet[i]) & \xrightarrow{\alpha^\bullet \mapsto \sum_{g \in U/V} g\alpha^\bullet} & \mathrm{Hom}_{K(U)}(k, \mathcal{I}_M^\bullet[i]) \\ \cong \uparrow & & \uparrow \cong \\ \mathrm{Hom}_{K(V)}(\mathcal{I}^\bullet, \mathcal{I}_M^\bullet[i]) & \xrightarrow{\alpha^\bullet \mapsto \sum_{g \in U/V} g\alpha^\bullet g^{-1}} & \mathrm{Hom}_{K(U)}(\mathcal{I}^\bullet, \mathcal{I}_M^\bullet[i]). \end{array}$$

is commutative. This means that, if the cohomology class $\alpha \in H^i(V, M)$ is represented by the homomorphism of complexes $\alpha^\bullet : \mathcal{I}^\bullet \to \mathcal{I}_M^\bullet[i]$, then its image $\mathrm{cores}_U^V(\alpha) \in H^i(U, M)$ is represented by

$$\sum_{g \in U/V} g\alpha^\bullet g^{-1} : \mathcal{I}^\bullet \to \mathcal{I}_M^\bullet[i] \ . \tag{55}$$

4.6. **Basic properties.** For later reference we record from [NSW] Prop. 1.5.4 that on cohomology restriction as well as corestriction commute with conjugation and from [NSW] Prop. 1.5.3(iv) that the projection formulas

$$\mathrm{cores}_U^V(\alpha \cup \mathrm{res}_V^U(\beta)) = \mathrm{cores}_U^V(\alpha) \cup \beta \quad \text{and} \quad \mathrm{cores}_U^V(\mathrm{res}_V^U(\beta) \cup \alpha) = \beta \cup \mathrm{cores}_U^V(\alpha)$$

hold when V is an open subgroup of the profinite group U and $\alpha \in H^*(V, M)$, $\beta \in H^*(U, M)$.

5. The product in E^*

5.1. **A technical formula relating the Yoneda and cup products.** We fix classes $[\alpha^\bullet] \in H^i(I, \mathbf{X}(v))$ and $[\beta^\bullet] \in H^j(I, \mathbf{X}(w))$ represented by homomorphisms of complexes

$$\alpha^\bullet : \mathcal{J}^\bullet \longrightarrow \mathrm{ind}_I^{IvI}(\mathcal{J}^\bullet)[i] \quad \text{and} \quad \beta^\bullet : \mathcal{J}^\bullet \longrightarrow \mathrm{ind}_I^{IwI}(\mathcal{J}^\bullet)[j] \ ,$$

respectively. Here we always take $k \xrightarrow{\sim} \mathcal{I}^\bullet$ and $k \xrightarrow{\sim} \mathcal{J}^\bullet$ to be injective resolutions in $\mathrm{Mod}(G)$ (and hence in $\mathrm{Mod}(I)$). By (50) and (51) their Yoneda product $[\gamma^\bullet] := (-1)^{ij}[\alpha^\bullet] \cdot [\beta^\bullet]$ is represented by the homomorphism

$$\begin{aligned} \gamma^\bullet : \mathcal{J}^\bullet &\longrightarrow \mathrm{ind}_I^{IvI \cdot IwI}(\mathcal{J}^\bullet)[i+j] \\ x &\longmapsto \sum_{g \in IvI/I} g\beta^\bullet[i](\alpha^\bullet(x)(g)) \ . \end{aligned}$$

In fact, we introduce, for any $u \in \widetilde{W}$ such that $IuI \subseteq IvI \cdot IwI$, the homomorphism

$$\gamma_u^\bullet(-) := \gamma^\bullet(-)_{|IuI} : \mathcal{J}^\bullet \longrightarrow \mathrm{ind}_I^{IuI}(\mathcal{J}^\bullet)[i+j] \ .$$

Then

$$(-1)^{ij}[\alpha^\bullet] \cdot [\beta^\bullet] = \sum_{IuI \subseteq IvI \cdot IwI} [\gamma_u^\bullet] \ . \tag{56}$$

Our goal here is to give a formula for the class $[\mathrm{Sh}'_{\dot{u}}(\gamma_u^\bullet)] = \mathrm{Sh}_u([\gamma_u^\bullet]) \in H^{*+i+j}(I_u, k)$ (cf. (47)) in terms of group cohomological operations. We fix throughout a $u \in \widetilde{W}$ such that $IuI \subseteq IvI \cdot IwI$.

Remark 5.1. The map

$$\begin{aligned} \{a \in I/I_v : v^{-1}au \in IwI\} &\xrightarrow{\simeq} I_{v^{-1}}\backslash(v^{-1}Iu \cap IwI) \\ a &\longmapsto v^{-1}a^{-1}\dot{u} \end{aligned}$$

is a well defined bijection.

Proof. The map is well defined since $v^{-1}I_v = I_{v^{-1}}v^{-1}$. It is obviously surjective. For injectivity suppose that $I_{v^{-1}}v^{-1}a\dot{u} = I_{v^{-1}}v^{-1}b\dot{u}$ for some $a, b \in I$. Then $v^{-1}I_v a^{-1} = v^{-1}I_v b^{-1}$ and hence $aI_v = bI_v$. ☐

Using the above formula for $\gamma^\bullet$ and Remark 5.1 we compute

$$\begin{aligned} \mathrm{Sh}'_{\dot{u}}(\gamma^\bullet_u)(x) &= \dot{u}(\gamma^\bullet_u(x)(\dot{u})) = \dot{u}(\gamma^\bullet(x)(\dot{u})) = \dot{u}\big(\big(\sum_{a \in I/I_v} av\beta^\bullet[i](\alpha^\bullet(x)(av))\big)(\dot{u})\big) \\ &= \sum_{a \in I/I_v} \dot{u}(\beta^\bullet[i](\alpha^\bullet(x)(av))(v^{-1}a^{-1}\dot{u})) \\ &= \sum_{a \in I/I_v, v^{-1}a^{-1}u \in IwI} \dot{u}(\beta^\bullet[i](\alpha^\bullet(x)(av))(v^{-1}a^{-1}\dot{u})) \\ &= \sum_{h \in I_{v^{-1}}\backslash(v^{-1}Iu \cap IwI)} \dot{u}(\beta^\bullet[i](\alpha^\bullet(x)(\dot{u}h^{-1}))(h)) \ . \end{aligned} \tag{57}$$

We fix, for the moment, an element $h \in v^{-1}Iu \cap IwI$ written as $h = c\dot{w}d = \dot{v}^{-1}a^{-1}\dot{u}$ with $a, c, d \in I$ and put

$$\begin{aligned} \Gamma^\bullet_{\dot{u},h}(x) := \dot{u}(\beta^\bullet[i](\alpha^\bullet(x)(\dot{u}h^{-1}))(h)) &= \dot{u}(\beta^\bullet[i](\alpha^\bullet(x)(a\dot{v}))(c\dot{w}d)) \\ &= \dot{u}h^{-1}({}^{c_*}\,\mathrm{Sh}'_{\dot{w}}(\beta^\bullet[i])(\alpha^\bullet(x)(a\dot{v}))) \\ &= \dot{u}h^{-1}({}^{c_*}\,\mathrm{Sh}'_{\dot{w}}(\beta^\bullet[i])(\dot{v}^{-1}a^{-1}({}^{a_*}\,\mathrm{Sh}'_{\dot{v}}(\alpha^\bullet)(x)))) \\ &= \dot{u}h^{-1}\dot{v}^{-1}a^{-1}\big({}^{(a\dot{v})_*c_*}\,\mathrm{Sh}'_{\dot{w}}(\beta^\bullet[i])({}^{a_*}\,\mathrm{Sh}'_{\dot{v}}(\alpha^\bullet)(x))\big) \\ &= {}^{(a\dot{v}c)_*}\,\mathrm{Sh}'_{\dot{w}}(\beta^\bullet[i])({}^{a_*}\,\mathrm{Sh}'_{\dot{v}}(\alpha^\bullet)(x)) \ . \end{aligned}$$

In the second and third line we have used the reconstruction formula (48) for $\beta^\bullet$ and $\alpha^\bullet$, respectively, as well as (54).

Lemma 5.2. i. *In the commutative diagram of surjective projection maps*

$$\begin{array}{ccc} I_{v^{-1}}\backslash(v^{-1}Iu \cap IwI) & \longrightarrow & I_{v^{-1}}\backslash(v^{-1}Iu \cap IwI)/I_{u^{-1}} \\ \downarrow & \swarrow & \\ I_{v^{-1}}\backslash(v^{-1}IuI \cap IwI)/I & & \end{array}$$

the lower oblique arrow is bijective.

ii. *For $h \in v^{-1}Iu \cap IwI$ the map $b \mapsto I_{v^{-1}}hb$ from the set $(I_{u^{-1}} \cap h^{-1}Ih)\backslash I_{u^{-1}}$ to the fiber of the projection map $I_{v^{-1}}\backslash(v^{-1}Iu \cap IwI) \twoheadrightarrow I_{v^{-1}}\backslash(v^{-1}Iu \cap IwI)/I_{u^{-1}}$ in the point $I_{v^{-1}}hI_{u^{-1}}$ is a bijection.*

Proof. i. Let $I_{v^{-1}}hI = I_{v^{-1}}h'I$ with $h = \dot{v}^{-1}a^{-1}\dot{u}$, $h' = \dot{v}^{-1}a'^{-1}\dot{u}$, and $a, a' \in I$. Then $I_v a^{-1}\dot{u}I = I_v a'^{-1}\dot{u}I$ and hence $a'^{-1} = Aa^{-1}\dot{u}B\dot{u}^{-1}$ for some $A \in I_v$ and $B \in I$. It follows that $B \in I_{u^{-1}}$ and $h' = \dot{v}^{-1}Aa^{-1}\dot{u}B\dot{u}^{-1}\dot{u} = (\dot{v}^{-1}A\dot{v})hB \in I_{v^{-1}}hI_{u^{-1}}$.

ii. The equality $I_{v^{-1}}hb = I_{v^{-1}}h$ for some $b \in I_{u^{-1}}$ is equivalent to

$$b \in h^{-1}I_{v^{-1}}h = h^{-1}Ih \cap h^{-1}v^{-1}Ivh = h^{-1}Ih \cap u^{-1}Iu \ .$$

But the latter is equivalent to $b \in I \cap u^{-1}Iu \cap h^{-1}Ih = I_{u^{-1}} \cap h^{-1}Ih$. □

Coming back to $\Gamma^\bullet_{\dot{u},h}$ we note that, for $b \in I_{u^{-1}}$, we have $hb = cw(db) = \dot{v}^{-1}(\dot{u}b^{-1}\dot{u}^{-1}a)^{-1}\dot{u}$, where $c, db, \dot{u}b^{-1}\dot{u}^{-1}a \in I$. It follows that

$$\Gamma^\bullet_{\dot{u},hb}(x) = \dot{u}b^{-1}\dot{u}^{-1}\Gamma^\bullet_{\dot{u},h}(\dot{u}b\dot{u}^{-1}x) \ . \tag{58}$$

By inserting (58) into (57) and by using Lemma 5.2 we obtain

$$\begin{aligned}\mathrm{Sh}'_{\dot{u}}(\gamma^\bullet_u)(x) &= \sum_{h \in I_{v^{-1}} \backslash (v^{-1}Iu \cap IwI)} \Gamma^\bullet_{\dot{u},h}(x) \\ &= \sum_{h \in I_{v^{-1}} \backslash (v^{-1}Iu \cap IwI)/I_{u^{-1}}} \ \sum_{b \in (I_{u^{-1}} \cap h^{-1}Ih) \backslash I_{u^{-1}}} \dot{u}b^{-1}\dot{u}^{-1}\Gamma^\bullet_{\dot{u},h}(\dot{u}b\dot{u}^{-1}x) \\ &= \sum_{h \in I_{v^{-1}} \backslash (v^{-1}Iu \cap IwI)/I_{u^{-1}}} \ \sum_{b \in (I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}) \backslash I_u} b^{-1}\Gamma^\bullet_{\dot{u},h}(bx) \ .\end{aligned} \tag{59}$$

Above and in the following every summation over h is understood to be over a chosen set of representatives in G of the respective double cosets. It also follows from (58) that

$$b\Gamma^\bullet_{\dot{u},h}(-) = \Gamma^\bullet_{\dot{u},h}(b-) \qquad \text{for any } b \in I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}.$$

This says that $\Gamma^\bullet_{\dot{u},h} : \mathcal{J}^\bullet \longrightarrow \mathcal{J}^\bullet[*+i+j]$ is a homomorphism of injective complexes in $\mathrm{Mod}(I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1})$ and therefore defines a cohomology class $[\Gamma^\bullet_{\dot{u},h}] \in H^{*+i+j}(I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}, k)$. By (55) the equality (59) then gives rise on cohomology to the equality

$$\mathrm{Sh}_u([\gamma^\bullet_u]) = [\mathrm{Sh}'_{\dot{u}}(\gamma^\bullet_u)] = \sum_{h \in I_{v^{-1}} \backslash (v^{-1}Iu \cap IwI)/I_{u^{-1}}} \mathrm{cores}^{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u}([\Gamma^\bullet_{\dot{u},h}]) \tag{60}$$

in $H^{*+i+j}(I_u, k)$.

We recall that $h = c\dot{w}d = \dot{v}^{-1}a^{-1}\dot{u}$ with $a, c, d \in I$ and

$$\Gamma^\bullet_{\dot{u},h}(x) = {}^{(a\dot{v}c)_*}\,\mathrm{Sh}'_{\dot{w}}(\beta^\bullet[i])({}^{a_*}\,\mathrm{Sh}'_{\dot{v}}(\alpha^\bullet)(x)) \ .$$

Note that both groups $aI_va^{-1} = I \cap \dot{u}h^{-1}Ih\dot{u}^{-1}$ and $(a\dot{v}c)I_w(a\dot{v}c)^{-1} = uIu^{-1} \cap \dot{u}h^{-1}Ih\dot{u}^{-1}$ contain $I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}$; in fact the latter is the intersection of the former two. Therefore the above identity should, more precisely, be written as

$$\Gamma^\bullet_{\dot{u},h} = \mathrm{res}^{uIu^{-1} \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}\left({}^{(a\dot{v}c)_*}\,\mathrm{Sh}'_{\dot{w}}(\beta^\bullet[i])\right) \circ \mathrm{res}^{I \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}\left({}^{a_*}\,\mathrm{Sh}'_{\dot{v}}(\alpha^\bullet)\right) \ .$$

Using Subsection 4.3 as well as (47) we deduce that on cohomology classes we have the equality

$$[\Gamma^\bullet_{\dot{u},h}] = \mathrm{res}^{uIu^{-1} \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}\left((a\dot{v}c)_*\,\mathrm{Sh}_w([\beta^\bullet])\right) \cup \mathrm{res}^{I \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}\left(a_*\,\mathrm{Sh}_v([\alpha^\bullet])\right) \tag{61}$$

in $H^{*+i+j}(I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}, k)$.

Proposition 5.3. *For any cohomology classes* $\alpha \in H^i(I, \mathbf{X}(v))$ *and* $\beta \in H^j(I, \mathbf{X}(w))$ *we have* $\alpha \cdot \beta = \sum_{u \in \widetilde{W}, IuI \subseteq IvI \cdot IwI} \gamma_u$ *with* $\gamma_u \in H^{i+j}(I, \mathbf{X}(u))$ *and*

$$\mathrm{Sh}_u(\gamma_u) = \sum_{h \in I_{v^{-1}} \backslash (v^{-1}Iu \cap IwI)/I_{u^{-1}}} \mathrm{cores}_{I_u}^{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}} \big(\tilde{\Gamma}_{u,h}\big)$$

with

$$\tilde{\Gamma}_{u,h} := \mathrm{res}^{I \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}} \big(a_* \, \mathrm{Sh}_v(\alpha)\big) \cup \mathrm{res}^{uIu^{-1} \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}} \big((a\dot{v}c)_* \, \mathrm{Sh}_w(\beta)\big) \ ,$$

where $h = c\dot{w}d = \dot{v}^{-1}a^{-1}\dot{u}$ *with* $a, c, d \in I$.

Proof. Insert (61) into (60) and use the anticommutativity of the cup product together with (56). □

Remark 5.4. Let $u, v, w \in \widetilde{W}$ such that $IuI \subseteq IvI \cdot IwI$. Suppose that $I_{v^{-1}} \backslash (v^{-1}Iu \cap IwI)/I_{u^{-1}}$ contains a single element $I_{v^{-1}}hI_{u^{-1}}$ and that $I_u \subset \dot{u}h^{-1}Ih\dot{u}^{-1}$. Then for any cohomology classes $\alpha \in H^i(I, \mathbf{X}(v))$ and $\beta \in H^j(I, \mathbf{X}(w))$ the component γ_u of $\alpha \cdot \beta$ in $H^{i+j}(I, \mathbf{X}(u))$ is such that

$$\mathrm{Sh}_u(\gamma_u) = \mathrm{res}^{I \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u} \big(a_* \, \mathrm{Sh}_v(\alpha)\big) \cup \mathrm{res}^{uIu^{-1} \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u} \big((a\dot{v}c)_* \, \mathrm{Sh}_w(\beta)\big)$$

with a and c as in the proposition. If $i = 0$ and $\alpha = \tau_v$ (resp. $j = 0$ and $\beta = \tau_w$) it is simply equal to $\mathrm{res}^{uIu^{-1} \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u} \big((a\dot{v}c)_* \, \mathrm{Sh}_w(\beta)\big)$ (resp. $\mathrm{res}^{I \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u} \big(a_* \, \mathrm{Sh}_v(\alpha)\big)$) because τ_v (resp. τ_w) corresponds to the constant function equal to 1 in $H^0(I_v, k)$ (resp. $H^0(I_w, k)$). Therefore for general α and β in the context of this remark, the components of $\alpha \cdot \beta$ and of $\alpha \cdot \tau_w \cup \tau_v \cdot \beta$ in $H^{i+j}(I, \mathbf{X}(u))$ coincide.

Corollary 5.5. *Let* $v, w \in \widetilde{W}$ *such that* $\ell(vw) = \ell(v) + \ell(w)$. *For any cohomology classes* $\alpha \in H^i(I, \mathbf{X}(v))$ *and* $\beta \in H^j(I, \mathbf{X}(w))$ *we have* $\alpha \cdot \beta \in H^{i+j}(I, \mathbf{X}(vw))$, *and*

$$\alpha \cdot \beta = (\alpha \cdot \tau_w) \cup (\tau_v \cdot \beta) \ , \tag{62}$$

where we use the cup product in the sense of subsection 3.3; moreover

$$\mathrm{Sh}_{vw}(\alpha \cdot \tau_w) = \mathrm{res}^{I_v}_{I_{vw}} \big(\mathrm{Sh}_v(\alpha)\big) \quad \text{and} \quad \mathrm{Sh}_{vw}(\tau_v \cdot \beta) = \mathrm{res}^{vI_wv^{-1}}_{I_{vw}} \big(v_* \, \mathrm{Sh}_w(\beta)\big) \ . \tag{63}$$

Proof. Note before starting the proof that, for $x \in \widetilde{W}$, we have $\mathrm{Sh}_x(\tau_x) = 1 \in H^0(I_x, k) = k$. If $\ell(vw) = \ell(v) + \ell(w)$, then $IvI \cdot IwI = IvwI$ (cf. Cor. 2.5.ii) and there is only $u = vw$ to consider in Prop. 5.3. We have $\alpha \cdot \tau_w \in H^i(I, \mathbf{X}(vw))$ and $\tau_v \cdot \beta \in H^j(I, \mathbf{X}(vw))$ and $\alpha \cdot \beta \in H^{i+j}(I, \mathbf{X}(vw))$ by Prop. 5.3.

From Lemma 5.2.i we know that the projection map

$$I_{v^{-1}} \backslash (v^{-1}Ivw \cap IwI)/I_{(vw)^{-1}} \xrightarrow{\sim} I_{v^{-1}} \backslash (v^{-1}IvwI \cap IwI)/I$$

is a bijection. On the other hand we have the inclusions

$$IwI \subseteq I_{v^{-1}}wI \subseteq v^{-1}IvwI \cap IwI \ ,$$

the left one coming from (7) and the right one being trivial. It follows that, in fact,

$$I_{v^{-1}}wI = v^{-1}IvwI \cap IwI \ .$$

Furthermore, it is straightforward to check

$$I_{v^{-1}}wI_{(vw)^{-1}} \subseteq v^{-1}Ivw \cap IwI \subseteq I_{v^{-1}}wI \ .$$

We claim that the left inclusion actually is an equality. Let $g \in v^{-1}Ivw \cap IwI$ be an arbitrary element. Using the second inclusion above we may find a $g_0 \in I_{v^{-1}}w \subseteq v^{-1}Ivw \cap IwI$ and a $g_1 \in I$ such that $g = g_0 g_1$. The above bijectivity then implies that necessarily $g_1 \in I_{(vw)^{-1}}$. We conclude that $g \in I_{v^{-1}}wI_{(vw)^{-1}}$. This proves that indeed

$$v^{-1}Ivw \cap IwI = I_{v^{-1}}wI_{(vw)^{-1}} \ .$$

Hence it suffices to consider $h = \dot{w}$. We have $I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1} = I_u \cap I_v = I_{vw}$ by (6). Using Remark 5.4, we have proved (62). Moreover, we have $I \cap \dot{u}h^{-1}Ih\dot{u}^{-1} = I_v$ and $uIu^{-1} \cap \dot{u}h^{-1}Ih\dot{u}^{-1} = vI_wv^{-1}$ for obvious reasons. Furthermore, we may take $c = 1$ and $a \in T^1$ such that $\dot{u} = a\dot{v}\dot{w}$. Note that a is contained in both vI_wv^{-1} and I_v so its acts trivially on the cohomology spaces $H^i(I_v, k)$ and $H^j(vI_wv^{-1}, k)$. Therefore in this case the formula of Proposition 5.3 gives:

$$\mathrm{Sh}_{vw}(\alpha \cdot \tau_w) = \mathrm{res}^{I_v}_{I_{vw}}\big(\mathrm{Sh}_v(\alpha)\big) \quad \text{and} \quad \mathrm{Sh}_{vw}(\tau_v \cdot \beta) = \mathrm{res}^{vI_wv^{-1}}_{I_{vw}}\big(\dot{v}_* \,\mathrm{Sh}_w(\beta)\big)$$

and

$$\mathrm{Sh}_{vw}(\alpha \cdot \beta) = \mathrm{Sh}_{vw}(\alpha \cdot \tau_w) \cup \mathrm{Sh}_{vw}(\tau_v \cdot \beta) \ .$$

□

5.2. **Explicit left action of H on the Ext-algebra.** Here we draw from Prop. 5.3 the formula for the explicit left action of H on E^*. The proposition and its proof use notation introduced in §2.1.6. See in particular Remark 2.6.

Proposition 5.6. *Let $\beta \in H^j(I, \mathbf{X}(w))$ with $w \in \widetilde{W}$ and $j \geq 0$. For $\omega \in \widetilde{\Omega}$, we have $\tau_\omega \cdot \beta \in H^j(I, \mathbf{X}(\omega w))$ and*

$$\mathrm{Sh}_{\omega w}(\tau_\omega \cdot \beta) = \omega_* \,\mathrm{Sh}_w(\beta) \ . \tag{64}$$

For $s = s_{(\alpha,\mathfrak{h})} \in S_{aff}$ we have either $\ell(\tilde{s}w) = \ell(w) + 1$ and $\tau_{\tilde{s}} \cdot \beta \in H^j(I, \mathbf{X}(\tilde{s}w))$ with

$$\mathrm{Sh}_{\tilde{s}w}(\tau_{\tilde{s}} \cdot \beta) = \mathrm{res}^{\tilde{s}I_w\tilde{s}^{-1}}_{I_{sw}}\big(\tilde{s}_* \,\mathrm{Sh}_w(\beta)\big) \ , \tag{65}$$

or $\ell(\tilde{s}w) = \ell(w) - 1$ and

$$\tau_{\tilde{s}} \cdot \beta = \gamma_{\tilde{s}w} + \sum_{t \in \check{\alpha}([\mathbb{F}_q^\times])} \gamma_{\bar{t}w} \in H^j(I, \mathbf{X}(\tilde{s}w)) \oplus \bigoplus_{t \in \check{\alpha}([\mathbb{F}_q^\times])} H^j(I, \mathbf{X}(\bar{t}w)) \tag{66}$$

with

$$\mathrm{Sh}_{\tilde{s}w}(\gamma_{\tilde{s}w}) = \mathrm{cores}^{\tilde{s}I_w\tilde{s}^{-1}}_{I_{\tilde{s}w}}\big(\tilde{s}_* \,\mathrm{Sh}_w(\beta)\big) \ \text{and} \tag{67}$$

$$\mathrm{Sh}_{\bar{t}w}(\gamma_{\bar{t}w}) = \sum_{z \in \mathbb{F}_q^\times,\, \check{\alpha}([z]) = t} (n_s t^{-1} x_\alpha(\pi^{\mathfrak{h}}[z]) n_s^{-1})_* \,\mathrm{Sh}_w(\beta) \ . \tag{68}$$

(Note that the statements above in the case $\ell(\tilde{s}w) = \ell(w) + 1$ are true for any lift $\tilde{s}$ for s in $\widetilde{W}$ whereas (66) and the subsequent formulas are valid only for the specific choice of $\tilde{s}$ made in (15).)

Proof. Except for the case of $\ell(\tilde{s}w) = \ell(w) - 1$, the statements follow easily from Corollary 5.5, see in particular formula (63). So we consider the remaining case $\ell(\tilde{s}w) = \ell(w) - 1$ and recall that there is $(\alpha,\mathfrak{h}) \in \Pi_{aff}$ such that $s = s_{(\alpha,\mathfrak{h})}$ and that $\tilde{s} = n_s T^1$ was defined in (15). Using the notation from §2.1.6 (see in particular (18)), we have

$$n_s I n_s^{-1} I = I \,\dot\cup \dot\bigcup_{z\in\mathbb{F}_q^\times} x_\alpha(\pi^\mathfrak{h}[z])\check\alpha([z]) n_s^{-1} I \subset I \,\dot\cup \dot\bigcup_{t\in\check\alpha([\mathbb{F}_q^\times])} I t n_s^{-1} I$$

and hence, using Cor. 2.5.ii,

$$\begin{aligned}(69)\qquad n_s I w I = n_s I n_s^{-1} I n_s \dot w I &= I n_s \dot w I \,\dot\cup \dot\bigcup_{z\in\mathbb{F}_q^\times} x_\alpha(\pi^\mathfrak{h}[z])\check\alpha([z]) n_s^{-1} I n_s \dot w I \\ &\subset I n_s \dot w I \,\dot\cup \dot\bigcup_{t\in\check\alpha([\mathbb{F}_q^\times])} I t \dot w I \ .\end{aligned}$$

This proves (66) using Proposition 5.3. It remains to compute $\gamma_{\tilde s w}$ and $\gamma_{\bar t w}$ for $t \in \check\alpha([\mathbb{F}_q^\times])$.

Let $u := \tilde s w$. From (7) we deduce that $\tilde s^{-1} I \tilde s w \subset I_{s^{-1}} w I$, therefore $\tilde s^{-1} I \tilde s w I \cap I w I = I_{s^{-1}} w I$, and using Lemma 5.2.i we see that $I_{s^{-1}} \backslash (\tilde s^{-1} I u \cap I w I)/I_{u^{-1}}$ is made of the single double coset $I_{s^{-1}} w I_{u^{-1}}$. We have $I_u = I_{\tilde s w}$ and

$$I_u \cap u w^{-1} I w u^{-1} = I_{\tilde s w} \cap \tilde s I \tilde s^{-1} = \tilde s (w I w^{-1} \cap \tilde s^{-1} I \tilde s \cap I)\tilde s^{-1} = \tilde s (I_w \cap \tilde s^{-1} I \tilde s)\tilde s^{-1} = \tilde s I_w \tilde s^{-1} \ ,$$

where the last equality is justified by (6). Furthermore

$$u I u^{-1} \cap u w^{-1} I w u^{-1} = \tilde s (w I w^{-1} \cap I)\tilde s^{-1} = \tilde s I_w \tilde s^{-1} \ .$$

So Proposition 5.3 says that the component $\gamma_{\tilde s w}$ in $H^{i+j}(I, \mathbf{X}(\tilde s w))$ of $\tau_{\tilde s} \cdot \beta$ is given by

$$\mathrm{Sh}_{\tilde s w}(\gamma_{\tilde s w}) = \mathrm{cores}^{\tilde s I_w \tilde s^{-1}}_{I_{\tilde s w}} \big(\mathrm{res}^{\tilde s I_w \tilde s^{-1}}_{\tilde s I_w \tilde s^{-1}} (\tilde s_* \mathrm{Sh}_w(\beta)\big) = \mathrm{cores}^{\tilde s I_w \tilde s^{-1}}_{I_{\tilde s w}} \big(\tilde s_* \mathrm{Sh}_w(\beta)\big) \ .$$

Let $t \in \check\alpha([\mathbb{F}_q^\times])$ and $u_t := \bar t w$. We pick $\dot u_t := t \dot w \in N(T)$. We have $n_s^{-1} I u_t I \cap I w I = n_s^{-1}(I t w I \cap n_s I w I)$. From (69) we obtain that

$$I \bar t w I \cap n_s I w I = \dot\bigcup_{z\in\mathbb{F}_q^\times,\, \check\alpha([z])=t} x_\alpha(\pi^\mathfrak{h}[z]) t n_s^{-1} I n_s \dot w I = \dot\bigcup_{z\in\mathbb{F}_q^\times,\, \check\alpha([z])=t} I_s x_\alpha(\pi^\mathfrak{h}[z]) t \dot w I \ .$$

The second equality comes from the fact that t and $x_\alpha(\pi^\mathfrak{h}[z])$ normalize I_s (see Cor. 2.5.iii and Lemma 2.1 for the latter) and from (7). Therefore

$$n_s^{-1} I u_t I \cap I w I = \dot\bigcup_{z\in\mathbb{F}_q^\times,\, \check\alpha([z])=t} I_{s^{-1}} n_s^{-1} x_\alpha(\pi^\mathfrak{h}[z]) t \dot w I \ .$$

Let $z \in \mathbb{F}_q^\times$ such that $\check\alpha([z]) = t$ and $h_{t,z} := n_s^{-1} x_\alpha(\pi^\mathfrak{h}[z]) \dot u_t$. It lies in $n_s^{-1} I \dot u_t \cap I w I$. Using Lemma 5.2 and the above equalities, we obtain that $I_{s^{-1}} \backslash (n_s^{-1} I \dot u_t \cap I w I)/I_{u_t^{-1}}$ is made of the (distinct) double cosets $I_{s^{-1}} h_{t,z} I_{u_t^{-1}}$ where $z \in \mathbb{F}_q^\times$ is such that $\check\alpha([z]) = t$. (By Remark 2.7, there is one or two such double cosets.) Furthermore, we have $I_{u_t} = I_w$ and $\dot u_t h_{t,z}^{-1} I h_{t,z} \dot u_t^{-1} = x_\alpha(\pi^\mathfrak{h}[z])^{-1} n_s I n_s^{-1} x_\alpha(\pi^\mathfrak{h}[z])$. Therefore

$$\begin{aligned} I_{u_t} \cap \dot u_t h_{t,z}^{-1} I h_{t,z} \dot u^{-1} &= I \cap w I w^{-1} \cap x_\alpha(\pi^\mathfrak{h}[z])^{-1} n_s I n_s^{-1} x_\alpha(\pi^\mathfrak{h}[z]) \\ &= x_\alpha(\pi^\mathfrak{h}[z])^{-1} I_s x_\alpha(\pi^\mathfrak{h}[z]) \cap w I w^{-1} = I_s \cap w I w^{-1} = I_s \cap I_w = I_w = I_{u_t} \ ,\end{aligned}$$

where the third equality uses Cor. 2.5.iii and the fifth equality uses (6). Now to apply the formula of Prop. 5.3, we need to find $a_{t,z}$ and $c_{t,z}$ in I such that $h_{t,z} = n_s^{-1} a_{t,z}^{-1} \dot u_t \in c_{t,z} \dot w I$ where $z \in \mathbb{F}_q^\times$ is such that $\check\alpha([z]) = t$. Before giving them explicitly, first notice that $a_{t,z} n_s c_{t,z}$ lies in $\bar t w I w^{-1}$ thus it normalizes $w I w^{-1}$ and it also lies in $I n_s I$ thus normalizes I_s by Corollary 2.5.iii. By (6) we have $I_w = w I w^{-1} \cap I_s$ hence $a_{t,z} n_s c_{t,z}$ normalizes I_w and it follows that

$(a_{t,z}n_sc_{t,z})I_w(a_{t,z}n_sc_{t,z})^{-1} = u_tIu_t^{-1} \cap u_th_{t,z}^{-1}Ih_{t,z}u_t{}^{-1}$ coincides with $I_w = I_{u_t}$. Therefore, we have

$$\mathrm{Sh}_{u_t}(\gamma_{u_t}) = \sum_{z\in\mathbb{F}_q^\times,\, \check{\alpha}([z])=t} (a_{t,z}n_sc_{t,z})_* \,\mathrm{Sh}_w(\beta)\ .$$

By the above definitions we have $a_{t,z} = x_\alpha(\pi^{\mathfrak{h}}[z])^{-1}$. To find a suitable element $c_{t,z}$, notice that

$$n_sx_\alpha(\pi^{\mathfrak{h}}[z^{-1}])h_{t,z}\dot{w}^{-1}n_s^{-1} = n_sx_\alpha(\pi^{\mathfrak{h}}[z^{-1}])n_s^{-1}x_\alpha(\pi^{\mathfrak{h}}[z])tn_s^{-1} = x_\alpha(\pi^{\mathfrak{h}}[-z]) \in \mathcal{U}_{(\alpha,\mathfrak{h})}$$

by (17). By (1), we have $\dot{w}^{-1}n_s^{-1}\mathcal{U}_{(\alpha,\mathfrak{h})}n_s\dot{w} = \mathcal{U}_{(s\hat{w})^{-1}(\alpha,\mathfrak{h})}$ where $\hat{w}$ denotes the image of w in W. By (4), $(s\hat{w})^{-1}(\alpha,\mathfrak{h})$ lies in Φ_{aff}^+ and from Lemma 2.1.ii we deduce that $\mathcal{U}_{(s\hat{w})^{-1}(\alpha,\mathfrak{h})}$ is contained in I. Therefore $\dot{w}^{-1}x_\alpha(\pi^{\mathfrak{h}}[z^{-1}])h_{t,z} \in I$ and we may pick $c_{t,z} := x_\alpha(\pi^{\mathfrak{h}}[z^{-1}])^{-1}$. Lastly using (17), we see that with this choice we have $a_{t,z}n_sc_{t,z} = n_st^{-1}x_\alpha(\pi^{\mathfrak{h}}[z])n_s^{-1}$, which concludes the proof of (68). ☐

5.3. **Appendix.** In [Ron] §4.1 a groupoid cohomology class is a G-equivariant function

$$f : (G\times G)/(I\times I) \longrightarrow \oplus_{(g_1,g_2)\in G^2/I^2} H^*(g_1Ig_1^{-1}\cap g_2Ig_2^{-1}, k)$$

such that

- f is supported on finitely many G-orbits, and
- $f(g_1I, g_2I) \in H^*(g_1Ig_1^{-1}\cap g_2Ig_2^{-1}, k)$ for any $(g_1,g_2)\in G^2$.

The product of two such functions f and $\tilde{f}$ is defined as follows. We use the maps

$$\begin{aligned} \iota_{\mu,\nu} : G\times G\times G &\longrightarrow G\times G \\ (g_1,g_2,g_3) &\longmapsto (g_\mu, g_\nu)\ , \end{aligned}$$

for $1\le\mu<\nu\le 3$, in order to first introduce the pulled back functions

$$(\iota_{1,2}^*f)(g_1,g_2,g_3) := \mathrm{res}^{g_1Ig_1^{-1}\cap g_2Ig_2^{-1}}_{g_1Ig_1^{-1}\cap g_2Ig_2^{-1}\cap g_3Ig_3^{-1}} f(g_1,g_2)$$

and

$$(\iota_{2,3}^*f)(g_1,g_2,g_3) := \mathrm{res}^{g_2Ig_2^{-1}\cap g_3Ig_3^{-1}}_{g_1Ig_1^{-1}\cap g_2Ig_2^{-1}\cap g_3Ig_3^{-1}} f(g_2,g_3)$$

(with value in $H^*(g_1Ig_1^{-1}\cap g_2Ig_2^{-1}\cap g_3Ig_3^{-1}, k)$) on G^3/I^3. These functions are no longer supported on finitely many G-orbits. Nevertheless we consider their cup product

$$F(g_1,g_2,g_3) := (\iota_{1,2}^*f)(g_1,g_2,g_3)\cup(\iota_{2,3}^*f)(g_1,g_2,g_3)\ .$$

One can check (in fact, it follows from the subsequent computations) that its push forward

$$(\iota_{1,3*}F)(g_1,g_3) := \sum_{g\in(g_1Ig_1^{-1}\cap g_3Ig_3^{-1})\backslash G/I} \mathrm{cores}^{g_1Ig_1^{-1}\cap gIg^{-1}\cap g_3Ig_3^{-1}}_{g_1Ig_1^{-1}\cap g_3Ig_3^{-1}} F(g_1,g,g_3)$$

is well defined and again is a groupoid cohomology class, which is defined in [Ron] Prop. 7 to be the product $f\cdot\tilde{f}$.

We now fix two "ordinary" cohomology classes $\alpha\in H^i(I,\mathbf{X}(v))$ and $\beta\in H^j(I,\mathbf{X}(w))$ and introduce the corresponding groupoid cohomology classes f_α and f_β supported on $G(1,v)I^2$ and $G(1,w)I^2$ by

$$f_\alpha(1,v) := \mathrm{Sh}_v(\alpha)\in H^i(I_v,k) \quad\text{and}\quad f_\beta(1,w) := \mathrm{Sh}_w(\beta)\in H^j(I_w,k)\ ,$$

respectively. In the following we compute the product $f_\alpha \cdot f_\beta$. By the G-equivariance it suffices to compute the classes

$$(f_\alpha \cdot f_\beta)(1,u) \in H^{i+j}(I_u, k) \qquad \text{for } u \in \widetilde{W}.$$

We have the three injective maps between double coset spaces

$$\begin{aligned} D_v : I_v\backslash G/I &\hookrightarrow G\backslash G^3/I^3 \\ h &\mapsto (1, v, h), \\ D_w : I_w\backslash G/I &\hookrightarrow G\backslash G^3/I^3 \\ h &\mapsto (h, 1, w), \text{ and} \\ D_u : I_u\backslash G/I &\hookrightarrow G\backslash G^3/I^3 \\ h &\mapsto (1, h, u). \end{aligned}$$

The functions $\iota_{1,2}^* f_\alpha$ and $\iota_{2,3}^* f_\beta$ are supported on the G-orbits in $\mathrm{im}(D_v)$ and $\mathrm{im}(D_w)$, respectively. Hence $F := \iota_{1,2}^* f_\alpha \cup \iota_{2,3}^* f_\beta$ is supported on the G-orbits in $\mathrm{im}(D_v) \cap \mathrm{im}(D_w)$. Moreover, we have

$$(f_\alpha \cdot f_\beta)(1,u) = (\iota_{1,3*}F)(1,u) = \sum_{h \in I_u\backslash G/I} \mathrm{cores}_{I_u}^{I_u \cap hIh^{-1}} F(1,h,u)\ . \tag{70}$$

Of course, on the right hand side only those h can occur for which we have $D_u(h) \in \mathrm{im}(D_v) \cap \mathrm{im}(D_w)$.

Lemma 5.7. $D_u^{-1}(\mathrm{im}(D_v) \cap \mathrm{im}(D_w)) = I_u\backslash(uIw^{-1}I \cap IvI)/I$.

Proof. Let $h = \dot{u}x_0\dot{w}^{-1}x_1 = x_2\dot{v}x_3$ with $x_j \in I$. Then

$$D_u(h) = G(1, x_2\dot{v}x_3, \dot{u})I^3 = Gx_2(1, \dot{v}, x_2^{-1}\dot{u})(x_2^{-1}, x_3, 1)I^3 = G(1, \dot{v}, x_2^{-1}\dot{u})I^3 \in \mathrm{im}(D_v)$$

and

$$\begin{aligned} D_u(h) &= G(1, \dot{u}x_0\dot{w}^{-1}x_1, \dot{u})I^3 = G\dot{u}x_0\dot{w}^{-1}(\dot{w}x_0^{-1}\dot{u}^{-1}, 1, \dot{w})(1, x_1, x_0^{-1})I^3 \\ &= G(\dot{w}x_0^{-1}\dot{u}^{-1}, 1, \dot{w})I^3 \in \mathrm{im}(D_w)\ . \end{aligned}$$

On the other hand, let $h \in G$ be such that $D_u(h) = G(1,h,u)I^3 \in \mathrm{im}(D_v) \cap \mathrm{im}(D_w)$. Then $(1, h, \dot{u}) \in G(1, v, \tilde{h})I^3 \cap G(\bar{h}, 1, w)I^3$ for some $\tilde{h}, \bar{h} \in G$. Write

$$(1, h, \dot{u}) = (gx_1, g\dot{v}x_2, g\tilde{h}x_3) = (g'\bar{h}y_1, g'y_2, g'\dot{w}y_3) \qquad \text{with } g, g' \in G \text{ and } x_j, y_j \in I.$$

We see that $1 = gx_1$ and $h = g\dot{v}x_2 = x_1^{-1}\dot{v}x_2 \in IvI$ as well as $\dot{u} = g'\dot{w}y_3$ and $h = g'y_2 = \dot{u}y_3^{-1}\dot{w}^{-1}y_2 \in uIw^{-1}I$. □

We deduce that (70) simplifies to

$$(f_\alpha \cdot f_\beta)(1,u) = \sum_{h \in I_u\backslash(uIw^{-1}I \cap IvI)/I} \mathrm{cores}_{I_u}^{I_u \cap hIh^{-1}} F(1,h,u)\ .$$

Recall that

$$\begin{aligned} F(1,h,u) &= (\iota_{1,2}^* f_\alpha)(1,h,u) \cup (\iota_{2,3}^* f_\beta)(1,h,u) \\ &= \mathrm{res}^{I \cap hIh^{-1}}_{I_u \cap hIh^{-1}} f_\alpha(1,h) \cup \mathrm{res}^{uIu^{-1} \cap hIh^{-1}}_{I_u \cap hIh^{-1}} f_\beta(h,u)\ . \end{aligned}$$

We write

$$h = \dot{u}A\dot{w}^{-1}B = C_h\dot{v}D \qquad \text{with } A, B, C_h, D \in I$$

and put $x_h := \dot{u}A\dot{w}^{-1}$. Then

$$(1, h) = C_h(1, \dot{v})(C_h^{-1}, D) \quad \text{and} \quad (h, \dot{u}) = x_h(1, \dot{w})(B, A^{-1})$$

and hence, by G-equivariance,

$$f_\alpha(1, h) = C_{h*}\,\mathrm{Sh}_v(\alpha) \quad \text{and} \quad f_\beta(h, u) = x_{h*}\,\mathrm{Sh}_w(\beta)\ .$$

Inserting this into the above formulas we arrive at

$$(71)\quad (f_\alpha \cdot f_\beta)(1, u) = \sum_{h \in I_u\backslash(uIw^{-1}I \cap IvI)/I} \mathrm{cores}_{I_u}^{I_u \cap hIh^{-1}} \left(\mathrm{res}^{I \cap hIh^{-1}}_{I_u \cap hIh^{-1}} C_{h*}\,\mathrm{Sh}_v(\alpha) \cup \mathrm{res}^{uIu^{-1} \cap hIh^{-1}}_{I_u \cap hIh^{-1}} x_{h*}\,\mathrm{Sh}_w(\beta)\right)\ .$$

Remark 5.8. 1. $hIh^{-1} \cap uIu^{-1} = C_hvIv^{-1}C_h^{-1} \cap uIu^{-1}$.
2. $C_hI_vC_h^{-1} = I \cap hIh^{-1} = I \cap x_hIx_h^{-1}$.
3. $uIu^{-1} \cap hIh^{-1} = x_hI_wx_h^{-1} = x_hIx_h^{-1} \cap x_hwIw^{-1}x_h^{-1} = x_hIx_h^{-1} \cap uIu^{-1}$.
4. The coset C_hI_v only depends on the coset hI (for $c, c' \in I$, we have $cvI = c'vI \iff c^{-1}c' \in I_v$).
5. The coset x_hI_w only depends on the coset hI.
6. $x_hIx_h^{-1}uI = x_hIwI$.

Lemma 5.9. i. *The projection map*

$$I_u\backslash(uIw^{-1} \cap IvI)/I_w \xrightarrow{\simeq} I_u\backslash(uIw^{-1}I \cap IvI)/I$$

is bijective.

ii. *The map*

$$I_{v^{-1}}\backslash(v^{-1}Iu \cap IwI)/I_{u^{-1}} \xrightarrow{\simeq} I_u\backslash(uIw^{-1} \cap IvI)/I_w$$
$$h = cwd \longmapsto \dot{u}h^{-1}c$$

is bijective.

Proof. i. Replace in Lemma 5.2.i the elements v^{-1}, u, w with u, w^{-1}, v.

ii. First of all we note that the coset cI_w only depends on w (and h). We therefore wrote $cwd = h$ by a slight abuse of notation. The map is well defined since $I_uuh^{-1}c = I_uud^{-1}\dot{w}^{-1} \subseteq uIw^{-1}$ and, if $h = \dot{v}^{-1}a\dot{u}$, then $\dot{u}h^{-1}c = a^{-1}\dot{v}c \in IvI$. The bijectivity follows by checking that the map $h' = CvD \mapsto Dh'^{-1}u$ is a well defined inverse. □

We note that if $h = c\dot{w}d = \dot{v}^{-1}a^{-1}\dot{u}$ then $h' := \dot{u}h^{-1}c = \dot{u}d^{-1}\dot{w}^{-1} = a\dot{v}c$ and $x_{h'} = a\dot{v}c$ and $C_{h'} = a$. Hence, rewriting the right hand sum in (71) by using the composite bijection in Lemma 5.9, we obtain

$$(f_\alpha \cdot f_\beta)(1, u) = \sum_{h \in I_{v^{-1}}\backslash(v^{-1}Iu \cap IwI)/I_{u^{-1}}} \mathrm{cores}_{I_u}^{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}} \left(\mathrm{res}^{I \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}} a_*\,\mathrm{Sh}_v(\alpha)\right.$$
$$\left.\cup\, \mathrm{res}^{uIu^{-1} \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}} (a\dot{v}c)_*\,\mathrm{Sh}_w(\beta)\right)\ ,$$

where $h = c\dot{w}d = \dot{v}^{-1}a^{-1}\dot{u}$ with $a, c, d \in I$. By comparing this equality with the equality in Prop. 5.3 we deduce the following result.

Proposition 5.10. *Let $\alpha \in H^i(I, \mathbf{X}(v))$, $\beta \in H^j(I, \mathbf{X}(w))$, and $\alpha \cdot \beta = \sum_u \gamma_u$ with $\gamma_u \in H^{i+j}(I, \mathbf{X}(u))$ as in 5.3; then $f_\alpha \cdot f_\beta = \sum_u f_{\gamma_u}$.*

6. An involutive anti-automorphism of the algebra E^*

For $w \in \widetilde{W}$, we have $I_{w^{-1}} = w^{-1} I_w w$ and a linear isomorphism

$$(w^{-1})_* : H^i(I_w, k) \xrightarrow{\cong} H^i(I_{w^{-1}}, k) \ ,$$

for all $i \geq 0$. Recall that conjugation by an element in $T^1 \subset I_w$ is a trivial operator on $H^i(I_w, k)$ and therefore the conjugation above is well defined and does not depend on the chosen lift for w^{-1} in $N(T)$. Via the Shapiro isomorphism (40), this induces the linear isomorphism $\mathcal{J}_w$:

$$\begin{CD} H^i(I, \mathbf{X}(w)) @>{\mathcal{J}_w}>{\cong}> H^i(I, \mathbf{X}(w^{-1})) \\ @V{\mathrm{Sh}_w}VV @V{\cong}V{\mathrm{Sh}_{w^{-1}}}V \\ H^i(I_w, k) @>{(w^{-1})_*}>{\cong}> H^i(I_{w^{-1}}, k) \end{CD} \tag{72}$$

Summing over all $w \in \widetilde{W}$, the maps $(\mathcal{J}_w)_{w \in \widetilde{W}}$ induce a linear isomorphism

$$\mathcal{J} : H^i(I, \mathbf{X}) \xrightarrow{\cong} H^i(I, \mathbf{X}) \ .$$

Proposition 6.1. *The map $\mathcal{J}$ defines an involutive anti-automorphism of the graded* Ext-*algebra E^*, namely*

$$\mathcal{J}(\alpha \cdot \beta) = (-1)^{ij} \mathcal{J}(\beta) \cdot \mathcal{J}(\alpha)$$

where $\alpha \in H^i(I, \mathbf{X})$ and $\beta \in H^j(I, \mathbf{X})$ for all $i, j \geq 0$. Restricted to $H^0(I, \mathbf{X})$ it yields the anti-involution

$$\tau_g \mapsto \tau_{g^{-1}} \text{ for any } g \in G$$

of the algebra H.

Proof. First note that, for $w \in \widetilde{W}$, the element $\tau_w \in H^0(I, \mathbf{X}(w)) = \mathbf{X}(w)^I$ corresponds to $1 \in H^0(I_w, k) = k$. Therefore, $\mathcal{J}(\tau_w) = \tau_{w^{-1}}$. Now we turn to the proof of the first statement of the proposition. Let $\alpha \in H^i(I, \mathbf{X}(v))$ and $\beta \in H^j(I, \mathbf{X}(w))$. On the one hand, recall that we have

$$\alpha \cdot \beta = \sum_{u \in \widetilde{W}, IuI \subseteq IvI \cdot IwI} \gamma_u$$

with $\gamma_u \in H^{i+j}(I, \mathbf{X}(u))$ as in Proposition 5.3 given by

$$\mathrm{Sh}_u(\gamma_u) = (-1)^{ij} \sum_{h \in I_{v^{-1}} \backslash (v^{-1} I u \cap I w I)/I_{u^{-1}}} \mathrm{cores}_{I_u}^{I_u \cap \dot{u} h^{-1} I h \dot{u}^{-1}} \big(\Gamma_{u,h}\big)$$

where

$$\Gamma_{u,h} := \mathrm{res}^{uIu^{-1} \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}} \big((a\dot{v}c)_*(\mathrm{Sh}_w(\beta))\big) \cup \mathrm{res}^{I \cap \dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u \cap \dot{u}h^{-1}Ih\dot{u}^{-1}} \big(a_*(\mathrm{Sh}_v(\alpha))\big)$$

and $h = c\dot{w}d = \dot{v}^{-1}a^{-1}\dot{u}$ with $a, c, d \in I$.

We compute $\mathcal{J}(\beta) \cdot \mathcal{J}(\alpha)$. Recall that $\mathrm{Sh}_{v^{-1}}(\mathcal{J}(\alpha)) = (v^{-1})_* \, \mathrm{Sh}_v(\alpha)$ and $\mathrm{Sh}_{w^{-1}}(\mathcal{J}(\beta)) = (w^{-1})_* \, \mathrm{Sh}_w(\beta)$. Note that the map $u \mapsto u^{-1}$ yields a bijection between the set of $u \in \widetilde{W}$ such that $IuI \subseteq IvI \cdot IwI$ and the set of $u' \in \widetilde{W}$ such that $Iu'I \subseteq Iw^{-1}I \cdot Iv^{-1}I$. Therefore, by

Proposition 5.3, we have $\mathcal{J}(\beta)\cdot\mathcal{J}(\alpha) = \sum_{u\in\widetilde{W}, IuI\subseteq IvI\cdot IwI}\delta_{u^{-1}}$ with $\delta_{u^{-1}}\in H^{i+j}(I,\mathbf{X}(u^{-1}))$ given by

$$\mathrm{Sh}_{u^{-1}}(\delta_{u^{-1}}) = (-1)^{ij}\sum_{h'\in I_w\backslash(wIu^{-1}\cap Iv^{-1}I)/I_u}\mathrm{cores}^{I_{u^{-1}}\cap\dot{u}^{-1}h'^{-1}Ih'\dot{u}}_{I_{u^{-1}}}\big(\Delta_{u^{-1},h'}\big)$$

where

$$\begin{aligned}&\Delta_{u^{-1},h'}\\ &=\mathrm{res}^{u^{-1}Iu\cap\dot{u}^{-1}h'^{-1}Ih'\dot{u}}_{I_{u^{-1}}\cap\dot{u}^{-1}h'^{-1}Ih'\dot{u}}\big((a'\dot{w^{-1}}c')_*(\mathrm{Sh}_{v^{-1}}(\mathcal{J}(\alpha)))\big)\cup\mathrm{res}^{I\cap\dot{u}^{-1}h'^{-1}Ih'\dot{u}}_{I_{u^{-1}}\cap\dot{u}^{-1}h'^{-1}Ih'\dot{u}}\big({a'}_*(\mathrm{Sh}_{w^{-1}}(\mathcal{J}(\beta)))\big)\\ &=\mathrm{res}^{u^{-1}Iu\cap\dot{u}^{-1}h'^{-1}Ih'\dot{u}}_{I_{u^{-1}}\cap\dot{u}^{-1}h'^{-1}Ih'\dot{u}}\big((a'\dot{w^{-1}}c'\dot{v^{-1}})_*(\mathrm{Sh}_{v}(\alpha))\big)\cup\mathrm{res}^{I\cap\dot{u}^{-1}h'^{-1}Ih'\dot{u}}_{I_{u^{-1}}\cap\dot{u}^{-1}h'^{-1}Ih'\dot{u}}\big((a'\dot{w^{-1}})_*(\mathrm{Sh}_{w}(\beta))\big)\end{aligned}$$

with $h' = c'\dot{v^{-1}}d' = (\dot{w^{-1}})^{-1}{a'}^{-1}\dot{u^{-1}}$ and $a', c', d'\in I$.

Now we compute $\mathcal{J}(\mathcal{J}(\beta)\cdot\mathcal{J}(\alpha))$. Recall that corestriction commutes with conjugation (cf. §4.6). We have:

$$\mathrm{Sh}_u(\mathcal{J}(\delta_{u^{-1}})) = u_*\,\mathrm{Sh}_{u^{-1}}(\delta_{u^{-1}}) = (-1)^{ij}\sum_{h'\in I_w\backslash(wIu^{-1}\cap Iv^{-1}I)/I_u}\mathrm{cores}^{I_u\cap h'^{-1}Ih'}_{I_u}u_*\big(\Delta_{u^{-1},h'}\big)$$

To an element $h'\in wIu^{-1}\cap Iv^{-1}I$ written in the form $h' = c'\dot{v^{-1}}d' = (\dot{w^{-1}})^{-1}{a'}^{-1}\dot{u^{-1}}$ as above, we attach the double coset $I_{v^{-1}}hI_{u^{-1}}$ where $h := {c'}^{-1}h'\dot{u} = \dot{v^{-1}}d'\dot{u}\in v^{-1}Iu\cap IwI$. This is well defined because d' is defined up to multiplication on the left by an element in I_v. It is easy to see that this yields a map $I_w\backslash(wIu^{-1}\cap Iv^{-1}I)/I_u\to I_{v^{-1}}\backslash(v^{-1}Iu\cap IwI)/I_{u^{-1}}$. One can check that the map in the opposite direction induced by attaching to $h = c\dot{w}d\in v^{-1}Iu\cap IwI$ the double coset $I_wh'I_u$ where $h' = c^{-1}h\dot{u}^{-1} = \dot{w}d\dot{u}^{-1}$ is well defined. Therefore, these maps are bijective. Note that for h' and h corresponding to each other as above, we have $I_u\cap h'^{-1}Ih' = I_u\cap\dot{u}h^{-1}Ih\dot{u}^{-1}$ and $h = c\dot{w}d = \dot{v}^{-1}a^{-1}\dot{u}$ with $a^{-1}\in T^1d'$, $c^{-1}\in T^1c'$ and $d^{-1}\in a'T^1$, therefore we compute

$$\begin{aligned}&u_*(\Delta_{u^{-1},h'})\\ &=\dot{u}_*\,\mathrm{res}^{u^{-1}Iu\cap h^{-1}Ih}_{I_{u^{-1}}\cap h^{-1}Ih}\big((a'\dot{w^{-1}}c'\dot{v^{-1}})_*(\mathrm{Sh}_v(\alpha))\big)\cup\mathrm{res}^{I\cap h^{-1}Ih}_{I_{u^{-1}}\cap h^{-1}Ih}\big((a'\dot{w^{-1}})_*(\mathrm{Sh}_w(\beta))\big)\\ &=\mathrm{res}^{I\cap\dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_{u^{-1}}\cap h^{-1}Ih}\big((\dot{u}a'\dot{w^{-1}}c'\dot{v^{-1}})_*(\mathrm{Sh}_v(\alpha))\big)\cup\mathrm{res}^{\dot{u}I\dot{u}^{-1}\cap\dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u\cap\dot{u}h^{-1}Ih\dot{u}^{-1}}\big((\dot{u}a'\dot{w^{-1}})_*(\mathrm{Sh}_w(\beta))\big)\\ &=\mathrm{res}^{I\cap\dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_{u^{-1}}\cap h^{-1}Ih}\big((d'^{-1})_*(\mathrm{Sh}_v(\alpha))\big)\cup\mathrm{res}^{\dot{u}I\dot{u}^{-1}\cap\dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u\cap\dot{u}h^{-1}Ih\dot{u}^{-1}}\big((\dot{u}a'\dot{w^{-1}})_*(\mathrm{Sh}_w(\beta))\big)\\ &=\mathrm{res}^{I\cap\dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_{u^{-1}}\cap h^{-1}Ih}\big(a_*(\mathrm{Sh}_v(\alpha))\big)\cup\mathrm{res}^{\dot{u}I\dot{u}^{-1}\cap\dot{u}h^{-1}Ih\dot{u}^{-1}}_{I_u\cap\dot{u}h^{-1}Ih\dot{u}^{-1}}\big((a\dot{v}c)_*(\mathrm{Sh}_w(\beta))\big)\\ &=(-1)^{ij}\Gamma_{u,h}\end{aligned}$$

and $\mathrm{Sh}_u(\mathcal{J}(\delta_{u^{-1}})) = (-1)^{ij}\,\mathrm{Sh}_u(\gamma_u)$. We proved $\mathcal{J}(\mathcal{J}(\beta)\cdot\mathcal{J}(\alpha)) = (-1)^{ij}\alpha\cdot\beta$. □

Remark 6.2. By (44) our cup product commutes with the Shapiro isomorphism. It also commutes with conjugation of the group ([NSW] Prop. 1.5.3(i)). Therefore the anti-automorphism $\mathcal{J}$ respects the cup product.

Remark 6.3. We want to show that the anti-involution of H induced by $\mathcal{J}$ preserves the ideal $\mathfrak{J}$ of §2.3. Consider the space $\mathbb{Z}[G/I]$ of finitely supported functions $G/I\to\mathbb{Z}$. The ring of its G-equivariant $\mathbb{Z}$-endomorphisms is isomorphic to the convolution ring $\mathbb{Z}[I\backslash G/I]$ with product given by $f\star f'(\_) = \sum_{x\in G/I}f(x^{-1}\_)f'(x)$. The opposite ring is denoted by $H_{\mathbb{Z}}$. One easily checks that the map $f\mapsto[g\mapsto f(g^{-1})]$ defines an anti-involution $j_{\mathbb{Z}}$ of the convolution ring

$\mathbb{Z}[I\backslash G/I]$. It induces an anti-involution of the k-algebra $H = H_{\mathbb{Z}} \otimes k$ which coincides with $\mathcal{J}$. A basis $(z_{\{\lambda\}})_{\{\lambda\}\in\widetilde{\Lambda}/W_0}$ of the center of the ring $H_{\mathbb{Z}}$ is described in [Vig05]. It is indexed by the set of W_0-orbits in the preimage $\widetilde{\Lambda} = T/T^1$ of $\Lambda = T/T^0$ in $\widetilde{W}$ (see §2.1.4). From [Oll2] Lemma 3.4 we deduce that $j_{\mathbb{Z}}(z_{\{\lambda\}}) = z_{\{\lambda^{-1}\}}$ for any $\lambda \in \widetilde{\Lambda}$ where $\{\lambda^{-1}\}$ denotes the W_0-orbit of λ^{-1}.

Recall that the map ν defined in §2.1.1 induces an isomorphism $\Lambda \cong X_*(T)$. As in [Oll2] 1.2.6, notice that the map $X_*(T) \to T/T^1$, $\xi \mapsto \xi(\pi^{-1}) \bmod T^1$ composed with ν splits the exact sequence

$$0 \longrightarrow T^0/T^1 \longrightarrow \widetilde{\Lambda} \longrightarrow \Lambda \longrightarrow 1$$

and we may see Λ as a subgroup of $\widetilde{\Lambda}$ which is preserved by the action of W_0. The set Λ/W_0 of all W_0-orbits of in Λ contains the set $(\Lambda/W_0)'$ of orbits of elements with nonzero length (when seen in W). Note that via the map ν, it is indexed by the set $X_*^{dom}(T) \setminus (-X_*^{dom}(T))$. By the above remarks, the $\mathbb{Z}$-linear subspace of $H_{\mathbb{Z}}$ with basis $(z_{\{\lambda\}})_{\{\lambda\}\in(\Lambda/W_0)'}$ is preserved by $j_{\mathbb{Z}}$. Therefore, its image $\mathfrak{J}$ in H (as defined in [Oll2] 5.2) is preserved by $\mathcal{J}$. Note in passing that the algebra $\mathcal{Z}^0(H)$ (as introduced in §2.3) has basis $(z_{\{\lambda\}})_{\{\lambda\}\in\Lambda/W_0}$.

We deduce from this that if M is a left (resp. right) supersingular module in the sense of §2.3, namely if any element of M is annihilated by a power of $\mathfrak{J}$, then $M^{\mathcal{J}}$ (resp. ${}^{\mathcal{J}}M$) is a right (resp. left) supersingular module.

Alternatively, if $\mathbf{G}$ is semisimple, one can argue that $\mathcal{J}$ preserves supersingular modules as follows. First of all it suffices to show this for cyclic supersingular modules and such modules have finite length. (This is because H is finitely generated over $\mathcal{Z}^0(H)$ by [Oll2] Prop. 2.5ii and since $\mathbf{G}$ is semisimple, $\mathfrak{J}$ has finite codimension in $\mathcal{Z}^0(H)$ and therefore in H.) It then further suffices to do this after a suitable extension of the coefficient field. By the equivalence in the proof of Lemma 2.13 this finally reduces us to quotients of $H \otimes_{H_{aff}} \chi$ (or $\chi \otimes_{H_{aff}} H$) where χ is a supersingular character of H_{aff}. It is easy to see that the composite $\chi \circ \mathcal{J} : H_{aff} \to k$ is also a supersingular character. The right (resp. left) H-module $(H \otimes_{H_{aff}} \chi)^{\mathcal{J}}$ (resp. ${}^{\mathcal{J}}(\chi \otimes_{H_{aff}} H)$) is generated as an H-module by $1 \otimes 1$ which supports the character $\chi \circ \mathcal{J}$ of H_{aff}. Therefore by Lemma 2.13 it is annihilated by $\mathfrak{J}$ and supersingular as an H-module.

7. Dualities

7.1. **Finite and twisted duals.** Given a vector space Y, we denote by $Y^\vee$ the dual space $Y^\vee := \mathrm{Hom}_k(Y, k)$ of Y. If Y is a left, resp. right, module over H, then $Y^\vee$ is naturally a right, resp. left, module over H. Recall that H is endowed with an anti-involution respecting the product and given by the map $\mathcal{J}$ (see Proposition 6.1). We may twist the action of H on a left, resp. right, module Y by $\mathcal{J}$ and thus obtain the right, resp. left module $Y^{\mathcal{J}}$, resp. ${}^{\mathcal{J}}Y$, with the twisted action of H given by $(y, h) \mapsto \mathcal{J}(h)y$, resp. $(h, y) \mapsto y\mathcal{J}(h)$. If Y is an H-bimodule, then we may define the twisted H-bimodule ${}^{\mathcal{J}}Y^{\mathcal{J}}$ the obvious way.

Remark 7.1. For a left, resp right, resp. bi-, H-module, the identity map yields an isomorphism of right, resp. left, resp. bi-, H-modules

$$({}^{\mathcal{J}}Y)^\vee = (Y^\vee)^{\mathcal{J}}, \quad \text{resp. } (Y^{\mathcal{J}})^\vee = {}^{\mathcal{J}}(Y^\vee), \quad \text{resp. } ({}^{\mathcal{J}}Y^{\mathcal{J}})^\vee = {}^{\mathcal{J}}(Y^\vee)^{\mathcal{J}}.$$

Since $\mathbf{X}$ is a right H-module, the space $\mathbf{X}^\vee$ is naturally a left H-module via $(h, \varphi) \mapsto \varphi(\_h)$. It is also endowed with a left action of G which commutes with the action of H via $(g, \varphi) \mapsto \varphi(g^{-1}\_)$. It is however not a smooth representation of G. Since $\mathbf{X}$ decomposes into $\oplus_{w\in\widetilde{W}}\mathbf{X}(w)$ as a vector space, $\mathbf{X}^\vee$ identifies with $\prod_{w\in\widetilde{W}}\mathbf{X}(\omega)^\vee$ which contains $\oplus_{w\in\widetilde{W}}\mathbf{X}(w)^\vee$.

We denote by $\mathbf{X}^{\vee,f}$ the image of the latter in $\mathbf{X}^\vee$. It is stable under the action of G on $\mathbf{X}^\vee$, and $\mathbf{X}^{\vee,f}$ is a smooth representation of G. Moreover it follows from Cor. 2.5.ii and (20) that $\mathbf{X}^{\vee,f}$ is an H-submodule of $\mathbf{X}^\vee$.

More generally, for Y a vector space which decomposes into a direct sum $Y = \oplus_{w\in\widetilde{W}} Y_w$, we denote by $Y^{\vee,f}$ the so-called finite dual of Y which is defined to be the image in $Y^\vee = \prod_{w\in\widetilde{W}} Y_w^\vee$ of $\oplus_{w\in\widetilde{W}} Y_w^\vee$.

For $g\in G$ denote by ev_g the evaluation map $\mathbf{X}\to k$, $f\mapsto f(g)$. This is an element in $\mathbf{X}^{\vee,f}$. For $g_0, g\in G$ and $f\in\mathbf{X}$ we have $({}^{g_0}\mathrm{ev}_g)(f) = \mathrm{ev}_g({}^{g_0^{-1}}f) = ({}^{g_0^{-1}}f)(g) = f(g_0g) = \mathrm{ev}_{g_0g}(f)$. In particular, $\mathrm{ev}_1\in\mathbf{X}^{\vee,f}$ is fixed under the action of I and there is a well defined morphism of smooth representations of G:

$$\tag{73} \begin{aligned} \mathbf{ev} : \mathbf{X} &\longrightarrow \mathbf{X}^{\vee,f} \\ \mathrm{char}_{gI} = {}^g\mathrm{char}_I &\longmapsto \mathrm{ev}_g = {}^g\mathrm{ev}_1 \quad \text{for any } g\in G. \end{aligned}$$

It is clearly a bijection. The basis $(\mathrm{ev}_g)_{g\in G/I}$ of $\mathbf{X}^{\vee,f}$ is dual to the basis $(\mathrm{char}_{gI})_{g\in G/I}$ of $\mathbf{X}$. For $w\in\widetilde{W}$, the space $\mathbf{X}(w)$ corresponds to $\mathbf{X}(w)^\vee$ under the isomorphism $\mathbf{ev}$.

Lemma 7.2. *The map* $\mathbf{ev}$ *induces an isomorphism of right* H*-modules* $\mathbf{X}\xrightarrow{\cong}(\mathbf{X}^{\vee,f})^{\mathcal{J}}$.

Proof. We only need to show that the composite map $\mathbf{X}\xrightarrow{\mathbf{ev}}(\mathbf{X}^{\vee,f})^{\mathcal{J}}\xrightarrow{\subseteq}(\mathbf{X}^\vee)^{\mathcal{J}}$ is right H-equivariant. Since $\mathbf{X}$ and $(\mathbf{X}^\vee)^{\mathcal{J}}$ are (G,H)-bimodules and since $\mathbf{X}$ is generated by char_I under the action of G, it is enough to prove that $\mathbf{ev}((\mathrm{char}_I)\tau) = \mathcal{J}(\tau)\mathbf{ev}(\mathrm{char}_I)$ for any $\tau\in H$, or equivalently, that

$$\mathbf{ev}(\mathrm{char}_{IwI}) = \tau_{w^{-1}}\mathbf{ev}(\mathrm{char}_I)$$

for any $w\in\widetilde{W}$. Decompose IwI into simple cosets $IwI = \sqcup_x xwI$ for $x\in I\cap wIw^{-1}\backslash I$. Then on the one hand, $\mathbf{ev}(\mathrm{char}_{IwI}) = \sum_x \mathrm{ev}_{xw}$. For $g\in G$, it sends the function char_{gI} onto 1 if and only if $g\in IwI$ and to 0 otherwise. On the other hand, we have $(\tau_{w^{-1}}\mathrm{ev}_1)(\mathrm{char}_{gI}) = \mathrm{ev}_1(\mathrm{char}_{gI}\mathrm{char}_{Iw^{-1}I}) = \mathrm{ev}_1(\mathrm{char}_{gIw^{-1}I})$. It is equal to 1 if and only if $g^{-1}\in Iw^{-1}I$ and to 0 otherwise. This proves the lemma. □

7.2. **Duality between E^i and E^{d-i} when I is a Poincaré group of dimension d.** In this section we always **assume** that the pro-p Iwahori group I is torsion free. This forces the field $\mathfrak{F}$ to be a finite extension of $\mathbb{Q}_p$ with $p\geq 5$. Then I is a Poincaré group of dimension d where d is the dimension of G as a p-adic Lie group: According to [Laz] Thm. V.2.2.8 and [Ser1] the group I has finite cohomological dimension; then [Laz] Thm. V.2.5.8 implies that I is a Poincaré group of dimension d. Any open subgroup of a Poincaré group is a Poincaré group of the same dimension (cf. [S-CG] Cor. I.4.5). This applies to our groups I_w for any $w\in\widetilde{W}$. It follows that

$$E^* = H^*(I,\mathbf{X}) = 0 \qquad \text{for } * > d$$

and that

$$\tag{74} H^d(I,\mathbf{X}(w))\cong H^d(I_w,k) \text{ is one dimensional for any } w\in\widetilde{W}.$$

Remark 7.3. Let L be a proper open subgroup of I. By [S-CG] Chap. 1 Prop. 30(4) and Exercise 5) respectively, $\mathrm{cores}_I^L : H^d(L,k)\to H^d(I,k)$ is a linear isomorphism while $\mathrm{res}_L^I : H^d(I,k)\to H^d(L,k)$ is the zero map.

Let $\mathcal{S} \in \mathbf{X}^\vee$ be the linear map given by

$$\mathcal{S} := \sum_{g \in G/I} \mathrm{ev}_g \ . \tag{75}$$

It is easy to check that $\mathcal{S} : \mathbf{X} \to k$ is G-equivariant when k is endowed with the trivial action of G. We denote by $\mathcal{S}^i := H^i(I, \mathcal{S})$ the maps induced on cohomology.

Remark 7.4. We may decompose $\mathcal{S} = \sum_{w \in \widetilde{W}} \mathcal{S}_w$ where $\mathcal{S}_w = \sum_{g \in IwI/I} \mathrm{ev}_g$. Each summand $\mathcal{S}_w : \mathbf{X} \to k$ is I-equivariant and $\mathcal{S}_w|_{\mathbf{X}(v)} = 0$ if $v \neq w \in \widetilde{W}$ and the following diagram is commutative:

$$\begin{array}{ccc} & & H^i(I, \mathbf{X}(w)) \\ & \swarrow^{\mathrm{Sh}_w} & \Big\downarrow {\scriptstyle H^i(I, \mathcal{S}_w)} \\ H^i(I_w, k) & & \\ & \searrow_{\mathrm{cores}_I^{I_w}} & \\ & & H^i(I, k) \end{array}$$

Proof. We contemplate the larger diagram

$$\begin{array}{ccccc} H^i(I, \mathbf{X}(w)) & & \xrightarrow{\quad H^i(I, \mathcal{S}_w) \quad} & & H^i(I, k) \\ & \nwarrow^{\mathrm{cores}_I^{I_w}} & & \nwarrow^{\mathrm{cores}_I^{I_w}} & \\ \Big\downarrow {\scriptstyle \mathrm{Sh}_w} & & H^i(I_w, \mathbf{X}(w)) \xrightarrow{\quad H^i(I_w, \mathcal{S}_w) \quad} & & H^i(I_w, k) \\ & \nearrow^{?} & & \nearrow_{=} & \\ H^i(I_w, k). & & & & \end{array}$$

Here the map ? is induced by the map between coefficients which sends $a \in k$ to $a\,\mathrm{char}_{wI}$. The parallelogram is commutative since the corestriction is functorial in the coefficients. The right lower triangle is commutative since $\mathcal{S}_w(a\,\mathrm{char}_{wI}) = a$. The left triangle is commutative since the composite of the upwards pointing arrows is the inverse of the Shapiro isomorphism by (41). ☐

Lemma 7.5. *For $0 \leq i \leq d$, the bilinear map defined by the composite*

$$H^i(I, \mathbf{X}) \otimes_k H^{d-i}(I, \mathbf{X}) \xrightarrow{\cup} H^d(I, \mathbf{X}) \xrightarrow{\mathcal{S}^d} H^d(I, k) \cong k$$

is nondegenerate.

Proof. Let $w \in \widetilde{W}$. We consider the diagram

$$\begin{array}{ccccc} H^i(I, \mathbf{X}(w)) \otimes_k H^{d-i}(I, \mathbf{X}(w)) & \xrightarrow{\ \cup\ } & H^d(I, \mathbf{X}(w)) & \xrightarrow{H^d(I, \mathcal{S}_w)} & H^d(I, k) \\ \cong \Big\downarrow {\scriptstyle \mathrm{Sh}_w \otimes \mathrm{Sh}_w} & & \cong \Big\downarrow {\scriptstyle \mathrm{Sh}_w} & & \Big\| \\ H^i(I_w, k) \otimes_k H^{d-i}(I_w, k) & \xrightarrow{\ \cup\ } & H^d(I_w, k) & \xrightarrow[\cong]{\mathrm{cores}_I^{I_w}} & H^d(I, k), \end{array}$$

where the lower right corestriction map is an isomorphism by Remark 7.3. The left square is commutative by (44) and the right one by Remark 7.4. The lower pairing is nondegenerate since I_w is a Poincaré group of dimension d. Therefore, the top horizontal composite induces a perfect pairing. Using (43), this proves the lemma. ☐

7.2.1. *Congruence subgroups.* We consider a smooth affine group scheme $\mathcal{G} = \mathrm{Spec}(A)$ over $\mathfrak{O}$ of dimension δ. In particular, A is an $\mathfrak{O}$-algebra via a homomorphism $\alpha : \mathfrak{O} \to A$. A point $s \in \mathcal{G}(\mathfrak{O})$ is an $\mathfrak{O}$-algebra homomorphism $s : A \to \mathfrak{O}$; it necessarily satisfies $s \circ \alpha = \mathrm{id}$. The reduction map is

$$\begin{aligned} \mathcal{G}(\mathfrak{O}) &\longrightarrow \mathcal{G}(\mathfrak{O}/\pi\mathfrak{O}) \\ s &\longmapsto \bar{s} := [A \xrightarrow{s} \mathfrak{O} \xrightarrow{\mathrm{pr}} \mathfrak{O}/\pi\mathfrak{O}] \ . \end{aligned}$$

Let $\epsilon : A \to \mathfrak{O}$ denote the unit element in $\mathcal{G}(\mathfrak{O})$; then $\bar{\epsilon}$ is the unit element in $\mathcal{G}(\mathfrak{O}/\pi\mathfrak{O}$.

Let $\mathfrak{p} := \ker(\epsilon)$. The formal completion $\widehat{\mathcal{G}}$ of $\mathcal{G}$ in the unit section is the formal group scheme $\widehat{\mathcal{G}} := \mathrm{Spf}(\widehat{A}^{\mathfrak{p}})$ where $\widehat{A}^{\mathfrak{p}}$ is the $\mathfrak{p}$-adic completion of A. By our smoothness assumption $\widehat{A}^{\mathfrak{p}} = \mathfrak{O}[[X_1, \ldots, X_\delta]]$ is a formal power series ring in δ many variables $X_1, \ldots, X_\delta$. A point in $\widehat{\mathcal{G}}(\mathfrak{O})$ is a point $s : A \to \mathfrak{O}$ in $\mathcal{G}(\mathfrak{O})$ which extends to a continuous homomorphism $s : \widehat{A}^{\mathfrak{p}} \to \mathfrak{O}$, i.e., which satisfies $s(\mathfrak{p}) \subseteq \mathfrak{M}$, or equivalently, $\bar{s}(\mathfrak{p}) = 0$. One checks that $\bar{s}(\mathfrak{p}) = 0$ if and only if $\bar{s} = \bar{\epsilon}$. This shows that

$$\widehat{\mathcal{G}}(\mathfrak{O}) = \ker\Big(\mathcal{G}(\mathfrak{O}) \xrightarrow{\text{reduction}} \mathcal{G}(\mathfrak{O}/\pi\mathfrak{O})\Big) \ .$$

On the other hand we have the bijection

$$\begin{aligned} \xi : \widehat{\mathcal{G}}(\mathfrak{O}) &\xrightarrow{\simeq} \mathfrak{M}^\delta \\ s &\longmapsto (s(X_1), \ldots s(X_\delta)) \ . \end{aligned}$$

We see that $\widehat{\mathcal{G}}(\mathfrak{O})$ is a standard formal group in the sense of [S-LL] II Chap. IV §8. We then have in $\widehat{\mathcal{G}}(\mathfrak{O})$ the descending sequence of normal subgroups

$$\widehat{\mathcal{G}}_m(\mathfrak{O}) := \xi^{-1}((\pi^m\mathfrak{O})^\delta) \qquad \text{for } m \geq 1$$

(loc. cit. II Chap. IV §9). It is clear that

$$\widehat{\mathcal{G}}_m(\mathfrak{O}) = \ker\Big(\mathcal{G}(\mathfrak{O}) \xrightarrow{\text{reduction}} \mathcal{G}(\mathfrak{O}/\pi^m\mathfrak{O})\Big) \ .$$

Proposition 7.6. *Suppose that $\mathfrak{O} = \mathbb{Z}_p$; then $\widehat{\mathcal{G}}_m(\mathbb{Z}_p)$, for any $m \geq 1$ if $p \neq 2$, resp. $m \geq 2$ if $p = 2$, is a uniform pro-p group.*

Proof. By [DDMS] §13.2 (the discussion before Lemma 13.21) and Exercise 5 the group $\widehat{\mathcal{G}}_m(\mathfrak{O})$ is standard in the sense of loc. cit. Def. 8.22. Hence it is uniform by loc. cit. Thm. 8.31. □

To treat the general case we observe that the Weil restriction $\mathcal{G}_0 := \mathrm{Res}_{\mathfrak{O}/\mathbb{Z}_p}(\mathcal{G})$ is a smooth affine group scheme over $\mathbb{Z}_p$ (cf. [BLR] §7.6 Thm. 4 and Prop. 5). Let $e(\mathfrak{F}/\mathbb{Q}_p)$ denote the ramification index of the extension $\mathfrak{F}/\mathbb{Q}_p$. By the definition of the Weil restriction we have

$$\mathcal{G}_0(\mathbb{Z}_p) = \mathcal{G}(\mathfrak{O}) \quad \text{and} \quad \mathcal{G}_0(\mathbb{Z}_p/p^m\mathbb{Z}_p) = \mathcal{G}(\mathfrak{O}/p^m\mathfrak{O}) = \mathcal{G}(\mathfrak{O}/\pi^{me(\mathfrak{F}/\mathbb{Q}_p)}\mathfrak{O}) \quad \text{for } m \geq 1.$$

We therefore obtain the following consequence of the above proposition.

Corollary 7.7. *Let $m = je(\mathfrak{F}/\mathbb{Q}_p)$ with $j \geq 1$ if $p \neq 2$, resp. $j \geq 2$ if $p = 2$. Then $\widehat{\mathcal{G}}_m(\mathfrak{O})$ is a uniform pro-p group.*

7.2.2. *Bruhat-Tits group schemes.* We fix a facet F in the standard apartment $\mathscr{A}$. Let $\mathbf{G}_F$ denote the Bruhat-Tits group scheme over $\mathfrak{O}$ corresponding to F (cf. [Tits]). It is affine smooth with general fiber $\mathbf{G}$, and $\mathbf{G}_F(\mathfrak{O})$ is the pointwise stabilizer in G of of the preimage of F in the extended building (denoted by $\mathscr{G}_{\mathrm{pr}^{-1}(F)}$ in [Tits] 3.4.1). Its neutral component is denoted by $\mathbf{G}_F^\circ$. The group of points $K_F := \mathbf{G}_F^\circ(\mathfrak{O})$ is the parahoric subgroup associated with the facet F. We introduce the descending sequence of normal congruence subgroups

$$K_{F,m} := \ker\Big(\mathbf{G}_F^\circ(\mathfrak{O}) \xrightarrow{\text{reduction}} \mathbf{G}_F^\circ(\mathfrak{O}/\pi^m\mathfrak{O})\Big) \quad \text{for } m \geq 1$$

in K_F. Let $\mathcal{P}_F^\dagger$ denote the stabilizer of F in G. It follows from [Tits] 3.4.3 (or [BT2] 4.6.17) that each $K_{F,m}$, in fact, is a normal subgroup of $\mathcal{P}_F^\dagger$. Note that, given F, any open subgroup of G contains some $K_{F,m}$.

Corollary 7.8. *For any $m = je(\mathfrak{F}/\mathbb{Q}_p)$ with $j \geq 2$ the group $K_{F,m}$ is a uniform pro-p group.*

Proof. Apply Cor. 7.7 with $\mathcal{G} := \mathbf{G}_F^\circ$. □

In the following we will determine the groups $K_{F,m}$ in terms of the root subgroups $\mathcal{U}_\alpha$ and the torus $\mathbf{T}$. Let $\mathscr{T}$ over $\mathfrak{O}$ denote the neutral component of the Neron model of $\mathbf{T}$. We have $\mathscr{T}(\mathfrak{O}) = T^0$, and we put

$$T^m := \ker\Big(\mathscr{T}(\mathfrak{O}) \xrightarrow{\text{reduction}} \mathscr{T}(\mathfrak{O}/\pi^m\mathfrak{O})\Big) \quad \text{for } m \geq 1.$$

By [BT2] 5.2.2-4 the group scheme $\mathbf{G}_F^\circ$ possesses, for each root $\alpha \in \Phi$, a smooth closed $\mathfrak{O}$-subgroup scheme $\mathscr{U}_{\alpha,F}$ such that

$$\mathscr{U}_{\alpha,F}(\mathfrak{O}) = \mathcal{U}_{\alpha,f_F(\alpha)} \ . \tag{76}$$

Moreover the product map induces an open immersion of $\mathfrak{O}$-schemes

$$\prod_{\alpha\in\Phi^-} \mathscr{U}_{\alpha,F} \times \mathscr{T} \times \prod_{\alpha\in\Phi^+} \mathscr{U}_{\alpha,F} \hookrightarrow \mathbf{G}_F^\circ \ . \tag{77}$$

Proposition 7.9. *For any $m \geq 1$ the map (77) induces the equality*

$$\prod_{\alpha\in\Phi^-} \mathcal{U}_{\alpha,f_F(\alpha)+m} \times T^m \times \prod_{\alpha\in\Phi^+} \mathcal{U}_{\alpha,f_F(\alpha)+m} = K_{F,m} \ .$$

Proof. Let $\mathscr{Y}$ denote the left hand side of the open immersion (77). Because of (76) the left hand side of our assertion is equal to the subset of all points in $\mathscr{Y}(\mathfrak{O})$ which reduce to the unit element modulo π^m and hence is contained in $K_{F,m}$. On the other hand it follows from [SchSt] Prop. I.2.2 and (76) that any point in $\mathbf{G}_F^\circ(\mathfrak{O})$, which reduces to a point of the unipotent radical of its special fiber, already lies in $\mathscr{Y}(\mathfrak{O})$. It follows that $K_{F,m}$ corresponds to points in $\mathscr{Y}(\mathfrak{O})$ which reduce to the unit element modulo π^m and hence is contained in the left hand side of the assertion. □

Remark 7.10. In Chap. I of [SchSt] certain pro-p subgroups $U_F^{(e)} \subseteq G$ for $e \geq 0$ were introduced and studied. If $F = x$ is a hyperspecial vertex then $K_{x,m} = U_x^{(m-1)}$. On the other hand, if $F = D$ is a chamber then $K_{D,m} = U_D^{(m)} T^m$.

Corollary 7.11. *Suppose that m is large enough so that $K_{F,m}$ is uniform. Then the Frattini quotient $(K_{F,m})_\Phi$ of $K_{F,m}$ satisfies*

$$\prod_{\alpha\in\Phi^-} \frac{\mathcal{U}_{\alpha,f_F(\alpha)+m}}{\mathcal{U}^p_{\alpha,f_F(\alpha)+m}} \times \frac{T^m}{(T^m)^p} \times \prod_{\alpha\in\Phi^+} \frac{\mathcal{U}_{\alpha,f_F(\alpha)+m}}{\mathcal{U}^p_{\alpha,f_F(\alpha)+m}} \xrightarrow{\sim} (K_{F,m})_\Phi \ .$$

Proof. As a consequence of Prop. 7.9 the map in the assertion, which is given by multiplication, exists and is a surjection of $\mathbb{F}_p$-vector spaces. But both sides have the same dimension d. Hence the map is an isomorphism. □

In the case where the facet is a vertex x in the closure of our fixed chamber C we also introduce the notation

$$\Phi_x := \{(\alpha,\mathfrak{h}) \in \Phi_{aff} : (\alpha,\mathfrak{h})(x) = 0\},\ \Phi_x^{\pm} := \Phi_x \cap \Phi_{aff}^{\pm},$$
$$\Pi_x := \Phi_x \cap \Pi_{aff},\ S_x := \{s \in S_{aff} : s(x) = x\},$$
$$W_x := \text{subgroup of } W_{aff} \text{ generated by all } s_{(\alpha;\mathfrak{h})} \text{ such that } (\alpha,\mathfrak{h}) \in \Phi_x\}.$$

The pair (W_x, S_x) is a Coxeter system with finite group W_x (cf. [OS1] §4.3 and the references therein).

For any such vertex we have the inclusions $K_{x,1} \subseteq I \subseteq J \subseteq K_x$.

Lemma 7.12. *The parahoric subgroup K_x is the disjoint union of the double cosets JwJ for all $w \in W_x$.*

Proof. See [OS1] Lemma 4.9. □

7.2.3. *Triviality of actions on the top cohomology.* We recall from section 2.1.3 that I and J are normal subgroups of $\mathcal{P}_C^{\dagger}$ and that $\mathcal{P}_C^{\dagger} = \bigcup_{\omega\in\Omega} \omega J$. In the case where the root system is irreducible the following result was shown in [Koz2] Thm. 7.1. The first part of our proof is essentially a repetition of his arguments.

Lemma 7.13. *For $g \in \mathcal{P}_C^{\dagger}$, the endomorphism g_* on the one dimensional k-vector space $H^d(I,k)$ is the identity.*

Proof. As noted above, g normalizes each subgroup $K_{C,m}$. Moreover, by Lemma 2.1.ii and Prop. 7.9, $K_{C,m}$ is contained in I. Hence the same argument as at the beginning of the proof of Lemma 7.15 reduces us to showing that the endomorphism g_* on the one dimensional k-vector space $H^d(K_{C,m},k)$ is the identity. Using Cor. 7.8 we may, by choosing m large enough, assume that $K_{C,m}$ is a uniform pro-p group.

Then, by [Laz] V.2.2.6.3 and V.2.2.7.2, the one dimensional k-vector space $H^d(K_{C,m},k)$ is the maximal exterior power (via the cup product) of the d-dimensional k-vector space $H^1(K_{C,m},k)$. Conjugation commuting with the cup product, we see that the endomorphism g_* on $H^d(K_{C,m},k)$ is the determinant of g_* on $H^1(K_{C,m},k)$. We have

$$H^1(K_{C,m},k) = \mathrm{Hom}_{\mathbb{F}_p}((K_{C,m})_\Phi, k)$$

where $(K_{C,m})_\Phi$ is the Frattini quotient of the group $K_{C,m}$. This further reduces us to showing that the conjugation by g on $(K_{C,m})_\Phi$ has trivial determinant. In Cor. 7.11 we computed this Frattini quotient to be

$$(K_{C,m})_\Phi = \prod_{\alpha\in\Phi^-} \frac{\mathcal{U}_{\alpha,f_C(\alpha)+m}}{\mathcal{U}^p_{\alpha,f_C(\alpha)+m}} \times \frac{T^m}{(T^m)^p} \times \prod_{\alpha\in\Phi^+} \frac{\mathcal{U}_{\alpha,f_C(\alpha)+m}}{\mathcal{U}^p_{\alpha,f_C(\alpha)+m}}\ .$$

using Lemma 2.1.i this simplifies to

$$(K_{C,m})_\Phi = \prod_{\alpha\in\Phi^-} \frac{\mathcal{U}_{\alpha,m+1}}{\mathcal{U}^p_{\alpha,m+1}} \times \frac{T^m}{(T^m)^p} \times \prod_{\alpha\in\Phi^+} \frac{\mathcal{U}_{\alpha,m}}{\mathcal{U}^p_{\alpha,m}}\ . \tag{78}$$

Recall that, for any $\alpha \in \Phi$, we have the additive isomorphism $x_\alpha : \mathfrak{F} \xrightarrow{\cong} \mathfrak{U}_\alpha$ defined in (14) by $x_\alpha(u) := \varphi_\alpha((\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}))$. Put $s_0 := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathfrak{F})$ and $n_\alpha := \varphi_\alpha(s_0)$. We observe that

- $n_\alpha = n_{s_{(\alpha,0)}}$, and
- $x_{-\alpha}(u) = n_\alpha x_\alpha(u) n_\alpha^{-1} = \varphi_\alpha(\left(\begin{smallmatrix} 1 & 0 \\ -u & 1 \end{smallmatrix}\right))$ for any $u \in \mathfrak{F}$.

By [Tits] 1.1 and 1.4 the map x_α restricts, for any $r \in \mathbb{Z}$, to an isomorphism $\pi^r\mathfrak{O} \xrightarrow{\cong} \mathfrak{U}_{\alpha,r}$. This implies that all the $\mathbb{F}_p$-vector spaces $\frac{\mathfrak{U}_{\alpha,m}}{\mathfrak{U}_{\alpha,m}^p}$ and $\frac{\mathfrak{U}_{-\alpha,m+1}}{\mathfrak{U}_{-\alpha,m+1}^p}$, for $\alpha \in \Phi^+$, have the same dimension equal to $[\mathfrak{F} : \mathbb{Q}_p]$.

First let $g \in T^0$. Obviously g centralizes T^m. It acts on $\mathfrak{U}_\alpha$, resp. $\mathfrak{U}_{-\alpha}$, via α, resp. $-\alpha$. Therefore on the right hand side of (78) the conjugation by g visibly has trivial determinant. Since the conjugation action of I on $H^d(I,k)$ is trivial we obtain our assertion for any $g \in J$.

For the rest of the proof we fix an $\omega \in \Omega$. It remains to establish our assertion for the elements $g \in \omega J$. In fact, by the above observation, it suffices to do this for one specific $\ddot{\omega} \in \omega J$, which we choose as follows. We write the image of ω in W as a reduced product $s_{\alpha_1} \cdots s_{\alpha_\ell}$ of simple reflections and put

$$w_\omega := n_{\alpha_1} \cdots n_{\alpha_\ell} \in K_{x_0} \ .$$

Then $t := \omega w_\omega^{-1} \in T$, and we now define

$$\ddot{\omega} := t w_\omega \ .$$

Let $\Phi = \Phi_1 \dot\cup \ldots \dot\cup \Phi_r$ be the decomposition into orbits of (the image in W of) ω and put

$$\Theta_i := \Big(\prod_{\alpha \in \Phi^- \cap \Phi_i} \frac{\mathfrak{U}_{\alpha,m+1}}{\mathfrak{U}_{\alpha,m+1}^p} \Big) \times \Big(\prod_{\alpha \in \Phi^+ \cap \Phi_i} \frac{\mathfrak{U}_{\alpha,m}}{\mathfrak{U}_{\alpha,m}^p} \Big) \ .$$

The Chevalley basis $(x_\alpha)_{\alpha\in\Phi}$ has the following property (cf. [BT2] 3.2):

(79) For any $\alpha \in \Phi$ there exists $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ such that
$$x_{s_\beta(\alpha)}(u) = n_\beta x_\alpha(\epsilon_{\alpha,\beta} u) n_\beta^{-1} \quad \text{for any } u \in \mathfrak{F}.$$

This implies that the conjugation by w_ω preserves each Θ_i. The conjugation t_* preserves each root subgroup. Since $\ddot{\omega}_*$ preserves $(K_{C,m})_\Phi$ it follows that $\ddot{\omega}_*$ preserves each Θ_i. Setting $\Theta_0 := \frac{T^m}{(T^m)^p}$ we conclude that

$$(K_{C,m})_\Phi = \Theta_0 \times \Theta_1 \times \cdots \times \Theta_r$$

is an $\ddot{\omega}_*$-invariant decomposition. We will determine the determinant of $\ddot{\omega}_*$ on each factor.

The $\ddot{\omega}_*$-action on

$$\Theta_0 = \frac{X_*(T)}{pX_*(T)} \otimes_{\mathbb{F}_p} \frac{1 + \pi^m \mathfrak{O}}{(1 + \pi^m \mathfrak{O})^p}$$

is through the product $s_{\alpha_1} \cdots s_{\alpha_\ell} \in W$ acting on the left factor. Note that the dimension of the $\mathbb{F}_p$-vector space $\frac{1+\pi^m\mathfrak{O}}{(1+\pi^m\mathfrak{O})^p}$ is equal to $[\mathfrak{F} : \mathbb{Q}_p]$. Let $Q^\vee \subseteq X_*(T)$ denote the coroot lattice. By [B-LL] VI.1.9 Prop. 27 the action of a simple reflection s_β on the quotient $X_*(T)/Q^\vee$ is trivial. Therefore in the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(X_*(T)/Q^\vee, \mathbb{F}_p) \longrightarrow Q^\vee \otimes_{\mathbb{Z}} \mathbb{F}_p \longrightarrow X_*(T) \otimes_{\mathbb{Z}} \mathbb{F}_p \longrightarrow (X_*(T)/Q^\vee) \otimes_{\mathbb{Z}} \mathbb{F}_p \longrightarrow 0$$

the action of s_β on the two outer terms is trivial. Hence the determinant of s_β on $X_*(T)/pX_*(T)$ is equal to its determinant on $Q^\vee/pQ^\vee$. If $p \neq 2$ then s_β is a reflection on the $\mathbb{F}_p$-vector space

$Q^\vee/pQ^\vee$ and therefore has determinant -1. For $p=2$, as $s_\beta^2 = \mathrm{id}$, the determinant is $-1=1$ as well. We deduce that

(80) $$\text{the determinant of } \ddot{\omega}_* \text{ on } \Theta_0 \text{ is equal to } (-1)^{\ell[\mathfrak{F}:\mathbb{Q}_p]}.$$

Next we consider Θ_i for some $1 \le i \le r$. Let c_i denote its cardinality, and fix some root $\beta \in \Phi_i$. We distinguish two cases.

Case 1: $-\Phi_i \cap \Phi_i = \emptyset$. Note that $\Phi_{i'} := -\Phi_i$ is the orbit of $-\beta$. By interchanging i and i' we may assume that $\beta \in \Phi^+$. We then have for completely formal reasons that

$$\text{determinant of } \ddot{\omega}_* \text{ on } \Theta_i = (-1)^{(c_i-1)[\mathfrak{F}:\mathbb{Q}_p]} \cdot \text{determinant of } \ddot{\omega}_*^{c_i} \text{ on } \frac{\mathfrak{U}_{\beta,m}}{\mathfrak{U}_{\beta,m}^p}$$

and correspondingly that

$$\text{determinant of } \ddot{\omega}_* \text{ on } \Theta_{i'} = (-1)^{(c_i-1)[\mathfrak{F}:\mathbb{Q}_p]} \cdot \text{determinant of } \ddot{\omega}_*^{c_i} \text{ on } \frac{\mathfrak{U}_{-\beta,m+1}}{\mathfrak{U}_{-\beta,m+1}^p}.$$

We have $\ddot{\omega}^{c_i} = w_\omega^{c_i} t_i$ for some $t_i \in T$. The determinants of t_{i*} on $\frac{\mathfrak{U}_{\beta,m}}{\mathfrak{U}_{\beta,m}^p}$ and $\frac{\mathfrak{U}_{-\beta,m+1}}{\mathfrak{U}_{-\beta,m+1}^p}$ (we must have $\beta(t_i) \in \mathfrak{O}^\times$) are inverse to each other. On the other hand let w_i denote the image of $w_\omega^{c_i}$ in W, which fixes β. As a consequence of the property (79) of the Chevalley basis, which we have recalled above, there are signs $\epsilon_{\pm\beta,w_i} \in \{\pm 1\}$ such that

$$x_{\pm\beta}(u) = w_\omega^{c_i} x_{\pm\beta}(\epsilon_{\pm\beta,w_i} u) w_\omega^{-c_i} \qquad \text{for any } u \in \mathfrak{F}.$$

Altogether we obtain that

$$\text{determinant of } \ddot{\omega}_* \text{ on } \Theta_i \times \Theta_{i'} = (\epsilon_{\beta,w_i}\epsilon_{-\beta,w_i})^{[\mathfrak{F}:\mathbb{Q}_p]} .$$

But we have $\epsilon_{\beta,w_i} = \epsilon_{-\beta,w_i}$. This follows by a straightforward induction from the fact that $\epsilon_{-\beta,\alpha} = \epsilon_{\beta,\alpha}$ (cf. [Spr] Lemma 9.2.2(ii)). Hence in the present case we deduce that

(81) $$\text{the determinant of } \ddot{\omega}_* \text{ on } \Theta_i \times \Theta_{i'} \text{ is equal to } 1.$$

Case 2: $-\Phi_i = \Phi_i$. Again we may assume that $\beta \in \Phi^+$. Then c_i is even, and $\ddot{\omega}_*^{c_i/2}$ preserves the product $\frac{\mathfrak{U}_{\beta,m}}{\mathfrak{U}_{\beta,m}^p} \times \frac{\mathfrak{U}_{-\beta,m+1}}{\mathfrak{U}_{-\beta,m+1}^p}$. The formal argument now says that

$$\text{determinant of } \ddot{\omega}_* \text{ on } \Theta_i = \text{determinant of } \ddot{\omega}_*^{c_i/2} \text{ on } \frac{\mathfrak{U}_{\beta,m}}{\mathfrak{U}_{\beta,m}^p} \times \frac{\mathfrak{U}_{-\beta,m+1}}{\mathfrak{U}_{-\beta,m+1}^p}.$$

This time we write $\ddot{\omega}^{c_i/2} = w_\omega^{c_i/2} t_i$ for some $t_i \in T$ and we let w_i denote the image of $w_\omega^{c_i/2}$ in W, which maps β to $-\beta$. We have $\beta(t_i) \in \pi\mathfrak{O}^\times$. Using the Chevalley basis we compute the above right hand determinant as being equal to

$$(-1)^{[\mathfrak{F}:\mathbb{Q}_p]} (\beta(t_i)\epsilon_{\beta,w_i})^{[\mathfrak{F}:\mathbb{Q}_p]} (\beta(t_i)^{-1}\epsilon_{-\beta,w_i})^{[\mathfrak{F}:\mathbb{Q}_p]} = (-1)^{[\mathfrak{F}:\mathbb{Q}_p]} .$$

Hence in this case we deduce that

(82) $$\text{the determinant of } \ddot{\omega}_* \text{ on } \Theta_i \text{ is equal to } (-1)^{[\mathfrak{F}:\mathbb{Q}_p]}.$$

Combining (80), (81), and (82) we have established at this point that

(83) $$\text{the determinant of } \ddot{\omega}_* \text{ on } (K_{C,m})_\Phi \text{ is equal to } (-1)^{(N+\ell)[\mathfrak{F}:\mathbb{Q}_p]}$$

where N is the number of ω-orbits $\Phi_i = -\Phi_i$ and, we repeat, ℓ is the length of w_ω. It remains to show that the sum $N+\ell$ always is even. This will be done in the subsequent lemma in a more general situation. □

Lemma 7.14. *Let $w \in W$ be any element and let $N(w)$ be the number of $w^{\mathbb{Z}}$-orbits $\Psi \subseteq \Phi$ with the property that $-\Psi = \Psi$; then the number $N(w) + \ell(w)$ is even.*

Proof. We have to show that $(-1)^{N(w)} = (-1)^{\ell(w)}$ holds true for any $w \in W$.

Step 1: The map $w \mapsto (-1)^{\ell(w)}$ is a homomorphism. This is immediate from the fact that $(-1)^{\ell(w)}$ is equal to the determinant of w in the reflection representation of W (cf. [Hum]).

Step 2: The map $w \mapsto (-1)^{N(w)}$ is a homomorphism. For this let $\mathbb{Z}[S]$ denote the free abelian group on a set S. We consider the obvious action of W on $\mathbb{Z}[\Phi]$. If $\Phi = \Phi_1 \dot{\cup} \ldots \dot{\cup} \Phi_r$ is the decomposition into $w^{\mathbb{Z}}$-orbits, then

$$\mathbb{Z}[\Phi] = \mathbb{Z}[\Phi_1] \oplus \ldots \oplus \mathbb{Z}[\Phi_r]$$

is a w-invariant decomposition. Obviously $\det(w|\mathbb{Z}[\Phi_i]) = (-1)^{|\Phi_i|-1}$. If $-\Phi_i \neq \Phi_i$ then $-\Phi_i = \Phi_j$ for some $j \neq i$. If $-\Phi_i = \Phi_i$ then $|\Phi|$ is even. It follows that $\det(w|\mathbb{Z}[\Phi]) = (-1)^{N(w)}$.

Step 3: If $w = s$ is a reflection at some $\alpha \in \Phi$ then $N(s) = 1$ and $(-1)^{\ell(s)} = -1$. □

Lemma 7.15. *Let L be an open subgroup of I. For $g \in K_x$ normalizing L, the endomorphism g_* on the one dimensional k-vector space $H^d(L,k)$ is the identity.*

Proof. We choose $m \geq 1$ large enough so that $K_{x,m}$ is contained in L. Recall that $\mathrm{cores}_L^{K_{x,m}} : H^d(K_{x,m},k) \xrightarrow{\cong} H^d(L,k)$ is an isomorphism by Remark 7.3. Since corestriction commutes with conjugation (§4.6), the following diagram commutes:

$$\begin{array}{ccc} H^d(L,k) & \xrightarrow{g_*} & H^d(L,k) \\ \cong \uparrow \mathrm{cores}_L^{K_{x,m}} & & \mathrm{cores}_L^{K_{x,m}} \uparrow \cong \\ H^d(K_{x,m},k) & \xrightarrow{g_*} & H^d(K_{x,m},k) \ . \end{array}$$

Therefore it is enough to prove the assertion for $L = K_{x,m}$. In this case the group K_x acts by conjugation on the one dimensional space $H^d(K_{x,m},k)$. This action is given by a character $\xi : K_x \to k^\times$. Its kernel $\Xi := \ker(\xi)$ is a normal subgroup of K_x.

First of all we recall again that the corestriction map commutes with conjugation, that $\mathrm{cores}_I^{K_{x,m}} : H^d(K_{x,m},k) \xrightarrow{\cong} H^d(I,k)$ is an isomorphism, and that conjugation by g_*, for $g \in I$, induces the identity on the cohomology $H^*(I,k)$. Therefore we have the commutative diagram

$$\begin{array}{ccc} & & H^d(K_{x,m},k) \\ & \swarrow^{\mathrm{cores}_I^{K_{x,m}}}_{\cong} & \\ H^d(I,k) & & \uparrow g_* \\ & \nwarrow^{\cong}_{\mathrm{cores}_I^{K_{x,m}}} & \\ & & H^d(K_{x,m},k) \ . \end{array}$$

This shows that $I \subseteq \Xi$. Since $J = IT^0$ we deduce from Lemma 7.13 that even $J \subseteq \Xi$. Since (W_x, S_x) is a Coxeter system with finite group W_x, we may consider its (unique) longest element w_x. The normal subgroup Ξ of K_x then must contain $Jw_xJw_x^{-1}J$. For any $s \in S_x$, we have $\ell(sw_x) = \ell(w) - 1$. By (20) we have

$$JsJ \cdot Jw_xJ = Jsw_xJ \,\dot{\cup}\, Jw_xJ$$

and hence $\dot{s}J\dot{w}_x \cap Jw_xJ \neq \emptyset$. It follows that

$$\dot{s} \in Jw_xJw_x^{-1}J \subseteq \Xi$$

and therefore $\Xi = K_x$ by Lemma 7.12. $\square$

Proposition 7.16. *For $g \in G$, and $v, w \in \widetilde{W}$ such that $\ell(vw) = \ell(v) + \ell(w)$, the following diagrams of one dimensional k-vector spaces are commutative:*

$$\begin{array}{ccc} & & H^d(I_g,k) \\ & \swarrow^{\mathrm{cores}_I^{I_g}}_{\cong} & \\ H^d(I,k) & & \uparrow g_* \\ & \nwarrow^{\cong}_{\mathrm{cores}_I^{I_{g^{-1}}}} & \\ & & H^d(I_{g^{-1}},k) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & H^d(I_{vw},k)\ . \\ & \swarrow^{\mathrm{cores}_I^{I_{vw}}}_{\cong} & \\ H^d(I,k) & & \uparrow v_* \\ & \nwarrow^{\cong}_{\mathrm{cores}_I^{v^{-1}I_{vw}v}} & \\ & & H^d(v^{-1}I_{vw}v,k). \end{array}$$

Proof. We prove the commutativity of the left diagram. We will see along the way that the commutativity of the right one follows.

Step 1: We claim that it suffices to establish the commutativity of the left diagram for elements $w \in \widetilde{W}$.

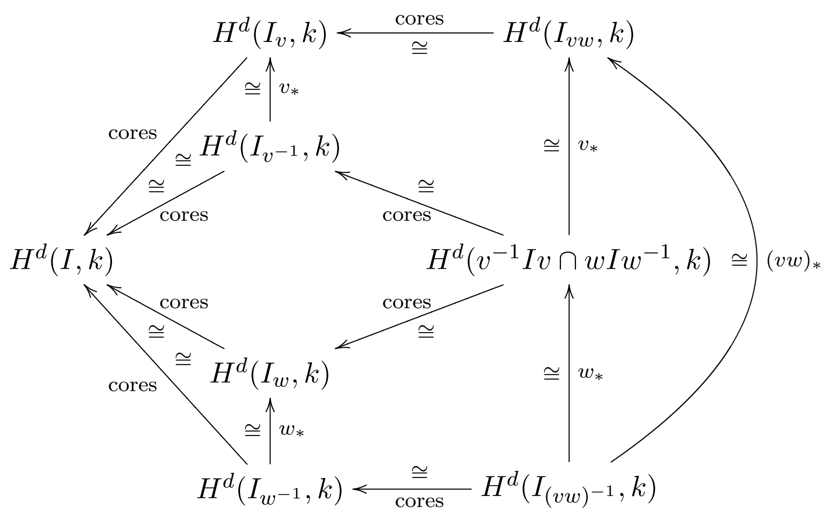
Let $g \in G$ and $h_1, h_2 \in I$. We have the commutative diagram

$$\begin{array}{ccccccc} H^d(I_{(h_1gh_2)^{-1}},k) & \xrightarrow{h_{2*}} & H^d(I_{g^{-1}},k) & \xrightarrow{g_*} & H^d(I_g,k) & \xrightarrow{h_{1*}} & H^d(I_{h_1gh_2},k) \\ {\scriptstyle \mathrm{cores}_I^{I_{(h_1gh_2)^{-1}}}}\downarrow\cong & & {\scriptstyle \mathrm{cores}_I^{I_{g^{-1}}}}\downarrow\cong & & {\scriptstyle \mathrm{cores}_I^{I_g}}\downarrow\cong & & {\scriptstyle \mathrm{cores}_I^{I_{h_1gh_2}}}\downarrow\cong \\ H^d(I,k) & \xrightarrow{=} & H^d(I,k) & \xrightarrow{?} & H^d(I,k) & \xrightarrow{=} & H^d(I,k), \end{array}$$

where the equality signs in the lower row use the facts that corestriction commutes with conjugation (cf. §4.6) and that the conjugation by an element in I is trivial on $H^d(I,k)$. This shows that the commutativity of the left diagram in the assertion only depends on the double coset IgI.

Step 2: Let $v, w \in \widetilde{W}$ be two elements for which the left diagram commutes and such that $\ell(vw) = \ell(v) + \ell(w)$; we claim that then the left diagram for vw as well as the right diagram commute.

This is straightforward from the following commutative diagram (cf. Lemma 2.2):

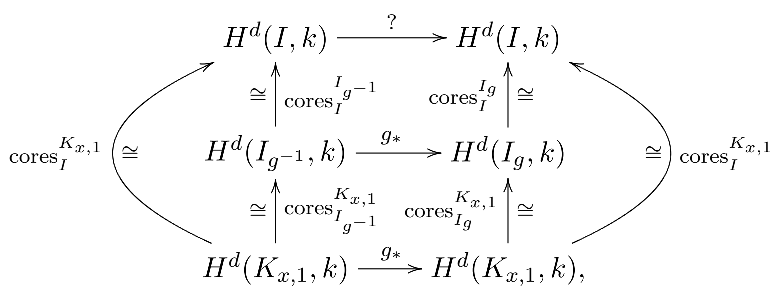


At this point we are reduced to establishing the commutativity of the left diagram in our assertion in the following two cases:

(A) $g = \dot{\omega}$ for $\omega \in \widetilde{\Omega}$. In that case, the claim is given by Lemma 7.13.

(B) $g = n_s$ for $s = s_{(\beta,\mathfrak{h})}$ where $(\beta, \mathfrak{h}) \in \Pi_{aff}$ is any simple affine root.

Step 3: It remains to treat case (B). We pick a vertex x in the closure of the chamber C such that $s(x) = x$. Then g lies in K_x and therefore normalizes $K_{x,1}$. Hence $K_{x,1} \subseteq I_g \subseteq I$, and we have the commutative diagram



using again that corestriction commutes with conjugation. This reduces us to showing that the endomorphism g_* on $H^d(K_{x,1}, k)$ is the identity. This claim is given by Lemma 7.15. $\square$

Corollary 7.17. *We have* $\mathcal{S}^d \circ \mathcal{J} = \mathcal{S}^d$ *on* $E^d = H^d(I, \mathbf{X})$.

Proof. Let $w \in \widetilde{W}$ and $\alpha \in H^d(I, \mathbf{X}(w))$. Recalling that $\mathcal{J}(\alpha) \in H^d(I, \mathbf{X}(w^{-1}))$ satisfies $\mathrm{Sh}_{w^{-1}}(\mathcal{J}(\alpha)) = (w^{-1})_* \mathrm{Sh}_w(\alpha)$, we have

$$\begin{aligned}
\mathcal{S}^d \circ \mathcal{J}(\alpha) &= H^d(I, \mathcal{S}_{w^{-1}}) \circ \mathcal{J}(\alpha) = H^d(I, \mathcal{S}_{w^{-1}}) \circ \mathrm{Sh}_{w^{-1}}^{-1} \circ (w^{-1})_*(\mathrm{Sh}_w(\alpha)) \\
&= \mathrm{cores}_I^{I_{w^{-1}}} \circ (w^{-1})_*(\mathrm{Sh}_w(\alpha)) && \text{by Remark 7.4} \\
&= \mathrm{cores}_I^{I_w}(\mathrm{Sh}_w(\alpha)) && \text{by Prop. 7.16} \\
&= H^d(I, \mathcal{S}_w)(\alpha) = \mathcal{S}^d(\alpha) && \text{by Remark 7.4.}
\end{aligned}$$

$\square$

7.2.4. *The duality theorem.* We fix an isomorphism $\eta : H^d(I,k) \longrightarrow k$. By Lemma 7.5 the map

$$\Delta^i : E^i = H^i(I,\mathbf{X}) \longrightarrow H^{d-i}(I,\mathbf{X})^\vee = (E^{d-i})^\vee \tag{84}$$
$$\alpha \longmapsto l_\alpha(\beta) := \eta \circ \mathcal{S}^d(\alpha \cup \beta)$$

is a linear injectionwith image $(E^{d-i})^{\vee,f}$. The space E^{d-i} is naturally a bimodule under H. As in §7.1, we consider the twisted H-bimodule ${}^{\mathcal{J}}(E^{d-i})^{\mathcal{J}}$ namely the space E^{d-i} with the action of H on $\beta \in E^{d-i}$ given by

$$(\tau,\beta,\tau') \mapsto \mathcal{J}(\tau')\cdot\beta\cdot\mathcal{J}(\tau) \quad \text{for } \tau,\tau' \in H.$$

Proposition 7.18. *The map (84) induces an injective morphism of H-bimodules*

$$E^i \longrightarrow ({}^{\mathcal{J}}(E^{d-i})^{\mathcal{J}})^\vee$$

with image $({}^{\mathcal{J}}(E^{d-i})^{\mathcal{J}})^{\vee,f}$.

Proof. The fact that the map is injective with image $(E^{d-i})^{\vee,f}$ comes directly from Lemma 7.5 and its proof.

We prove that for all $\alpha \in H^i(I,\mathbf{X})$ and $\beta \in H^{d-i}(I,\mathbf{X})$ and all $\tau,\tau' \in H$, we have $l_{\tau\cdot\alpha\cdot\tau'}(\beta) = l_\alpha(\mathcal{J}(\tau)\cdot\beta\cdot\mathcal{J}(\tau'))$ namely

$$\mathcal{S}^d(\tau\cdot\alpha\cdot\tau' \cup \beta) = \mathcal{S}^d(\alpha \cup \mathcal{J}(\tau)\cdot\beta\cdot\mathcal{J}(\tau'))\ . \tag{85}$$

We first show that

$$\mathcal{S}^d(\alpha\cdot\tau' \cup \beta) = \mathcal{S}^d(\alpha \cup \beta\cdot\mathcal{J}(\tau'))\ . \tag{86}$$

The right action of H on $H^*(I,\mathbf{X})$ being through the coefficients it is enough to prove that for any $a,b \in \mathbf{X}$ and $\tau' \in H$ we have

$$\mathcal{S}((a\cdot\tau')b - a(b\cdot\mathcal{J}(\tau'))) = 0\ .$$

We check this equality for $a = \mathrm{char}_{xI}$ and $b = \mathrm{char}_{yI}$ with $x,y \in G$ and $\tau' = \tau_g$ with $g \in G$. We then have

$$(a\cdot\tau')b - a(b\cdot\mathcal{J}(\tau')) = \mathrm{char}_{xIgI\cap yI} - \mathrm{char}_{xI\cap yIg^{-1}I}\ .$$

It lies in the kernel of $\mathcal{S}$ if and only if $xIgI\cap yI/I$ and $xI\cap yIg^{-1}I/I$ have the same cardinality. But observe that $xIgI\cap yI$ is equal to yI if $x^{-1}y \in IgI$ and is empty otherwise, while $xI\cap yIg^{-1}I$ is equal to xI if $y^{-1}x \in Ig^{-1}I$ and is empty otherwise. This proves the equality and (86) follows.

That

$$\mathcal{S}^d(\tau\cdot\alpha \cup \beta) = \mathcal{S}^d(\alpha \cup \mathcal{J}(\tau)\cdot\beta)\ . \tag{87}$$

holds true as well follows now from the following computation: We may assume that $\alpha \in E^i$, $\beta \in E^j$, and $\gamma \in E^m$. We then compute

$$\begin{aligned}
\mathcal{S}^d(\tau \cdot \alpha \cup \beta) &= \mathcal{S}^d(\mathcal{J}(\tau \cdot \alpha \cup \beta)) && \text{by Cor. 7.17} \\
&= \mathcal{S}^d(\mathcal{J}(\tau \cdot \alpha) \cup \mathcal{J}(\beta)) && \text{by Remark 6.2} \\
&= \mathcal{S}^d(\mathcal{J}(\alpha) \cdot \mathcal{J}(\tau) \cup \mathcal{J}(\beta)) && \text{by Prop. 6.1} \\
&= \mathcal{S}^d(\mathcal{J}(\alpha) \cup \mathcal{J}(\beta) \cdot \tau) && \text{by (86)} \\
&= \mathcal{S}^d(\mathcal{J}(\alpha) \cup \mathcal{J}(\mathcal{J}(\tau) \cdot \beta)) && \text{by Prop. 6.1} \\
&= \mathcal{S}^d(\mathcal{J}(\alpha \cup \mathcal{J}(\tau) \cdot \beta)) && \text{by Remark 6.2} \\
&= \mathcal{S}^d(\alpha \cup \mathcal{J}(\tau) \cdot \beta) && \text{by Cor. 7.17.}
\end{aligned}$$

□

Corollary 7.19. *For any $i \in \{0, \cdots, d\}$ the space $(E^i)^{\vee,f}$ is a sub-H-bimodule of $(E^i)^\vee$.*

8. The structure of E^d

In this section we still **assume**, as in §7.2, that the pro-p Iwahori group I is torsion free. In this section we describe the top cohomology space $E^d = H^d(I, \mathbf{X})$ as an H-bimodule.

Remark 8.1. The space E^d as a right H-module had already been computed in [SDGA] §5.1.

Recall that the pairing $E^d \times E^0 \to k$, $(\alpha, \beta) \mapsto \eta \circ \mathcal{S}^d(\alpha \cup \beta)$ induces the isomorphism of H-bimodules

$$\Delta^d : \quad E^d \xrightarrow{\cong} ({}^{\mathcal{J}}E^{0\,\mathcal{J}})^{\vee,f} \tag{88}$$

of Proposition 7.18 when $i = d$. We denote by $(\phi_w)_{w\in\widetilde{W}}$ the basis of E^d obtained by dualizing the basis of $(\tau_w)_{w\in\widetilde{W}}$ of E^0. For each $w \in \widetilde{W}$, the element ϕ_w is the only one in $H^d(I, \mathbf{X}(w))$ satisfying $\eta(\mathcal{S}^d(\phi_w)) = 1$ (see (43)). Now $\mathcal{J}(\phi_w) \in H^d(I, \mathbf{X}(w^{-1}))$ and $\eta(\mathcal{S}^d(\mathcal{J}(\phi_w)))$ and $\eta(\mathcal{S}^d(\phi_{w^{-1}}))$ are both equal to 1 by Corollary 7.17. Therefore,

$$\mathcal{J}(\phi_w) = \phi_{w^{-1}} \ . \tag{89}$$

We now describe the explicit action of H on the elements $(\phi_w)_{w\in\widetilde{W}}$ in E^d. For any $s \in S_{aff}$ recall that we introduced the idempotent element θ_s in (27):

$$\theta_s := -|\mu_{\check{\alpha}}| \sum_{t\in\check{\alpha}([\mathbb{F}_q^\times])} \tau_t \ \in H \ .$$

Proposition 8.2. *Let $w \in \widetilde{W}$, $\omega \in \widetilde{\Omega}$ and $s \in S_{aff}$. We have the formulas:*

$$\phi_w \cdot \tau_\omega = \phi_{w\omega} \ , \ \tau_\omega \cdot \phi_w = \phi_{\omega w}, \tag{90}$$

$$\phi_w \cdot \tau_{n_s} = \begin{cases} \phi_{w\tilde{s}} + |\mu_{\check{\alpha}}| \sum_{t\in\check{\alpha}([\mathbb{F}_q^\times])} \phi_{w\bar{t}} = \phi_{w\tilde{s}} - \phi_w \cdot \theta_s & \text{if } \ell(w\tilde{s}) = \ell(w) - 1, \\ 0 & \text{if } \ell(w\tilde{s}) = \ell(w) + 1, \end{cases} \tag{91}$$

$$\tau_{n_s} \cdot \phi_w = \begin{cases} \phi_{\tilde{s}w} + |\mu_{\check{\alpha}}| \sum_{t\in\check{\alpha}([\mathbb{F}_q^\times])} \phi_{\bar{t}w} = \phi_{\tilde{s}w} - \theta_s \cdot \phi_w & \text{if } \ell(\tilde{s}w) = \ell(w) - 1, \\ 0 & \text{if } \ell(\tilde{s}w) = \ell(w) + 1. \end{cases} \tag{92}$$

Proof. Before proving the proposition, we recall that the quadratic relations in H are given by $\tau_{n_s}^2 = -\theta_s\tau_{n_s} = \tau_{n_s}\theta_s$ for any $s \in S_{aff}$. Let $w \in \widetilde{W}$, $s \in S_{aff}$, and $\omega \in \Omega$. We study the right action of τ_ω and of τ_{n_s} on $\phi_w \in E^d$. Recall that $\phi_w \cdot \tau_\omega(\_) = \phi_w(\_\tau_{\omega^{-1}})$ and $\phi_w \cdot \tau_{n_s}(\_) = \phi_w(\_\tau_{n_s^{-1}})$ (see (85)). Let $w' \in \widetilde{W}$. Below we use the braid relation (26) repeatedly.

- We have $\phi_w \cdot \tau_\omega\,(\tau_{w'}) = \phi_w(\tau_{w'\omega^{-1}})$. It is nonzero if and only if $w' = w\omega$ in which case $\phi_w \cdot \tau_\omega(\tau_{w'}) = 1$.
- If $\ell(w'\tilde{s}) = \ell(w') + 1$ then $\phi_w \cdot \tau_{n_s}\,(\tau_{w'}) = \phi_w(\tau_{w'}\tau_{n_s^{-1}}) = \phi_w(\tau_{w'\tilde{s}^{-1}})$ and it is nonzero if and only if IwI is contained in $Iw'\tilde{s}^{-1}I$ which is equivalent to $w\tilde{s} = w'$ and in which case $\phi_w.\tau_{n_s}\,(\tau_{w'}) = 1$.
- If $\ell(w'\tilde{s}) = \ell(w') - 1$ then $\tau_{w'} = \tau_{w'\tilde{s}}\tau_{\tilde{s}^{-1}}$ and $\phi_w \cdot \tau_{n_s}(\tau_{w'}) = \phi_w(\tau_{w'\tilde{s}}\tau_{n_s^{-1}}\tau_{n_s^{-1}}) = \phi_w(\tau_{w'\tilde{s}}\tau_{n_s}^2\tau_{n_s^2}) = -\phi_w(\tau_{w'}\theta_s)$ which is nonzero if and only if there is $t \in \check{\alpha}([\mathbb{F}_q^\times])$ such that $w'\bar{t} = w$ and in which case $\phi_w \cdot \tau_{n_s}(\tau_{w'}) = |\mu_{\check{\alpha}}|$.

From the discussion above, we immediately deduce that $\phi_w \cdot \tau_\omega = \phi_{w\omega}$. Now we compute $\phi_w \cdot \tau_{n_s}$. Suppose that $\ell(w\tilde{s}) = \ell(w) + 1$, then for any $w' \in \widetilde{W}$ we have $\phi_w \cdot \tau_{n_s}(\tau_{w'}) = 0$, so $\phi_w \cdot \tau_{n_s} = 0$. Suppose that $\ell(w\tilde{s}) = \ell(w) - 1$, then for any $w' \in \widetilde{W}$ we have $\phi_w \cdot \tau_{n_s}(\tau_{w'}) = 0$ except if $w' = w\tilde{s}$ in which cases $\phi_w \cdot \tau_{n_s}(\tau_{w'}) = 1$, or if there is $t \in \check{\alpha}([\mathbb{F}_q^\times])$ such that $w' = w\bar{t}^{-1}$ in which case $\phi_w \cdot \tau_{\tilde{s}}(\tau_{w'}) = |\mu_{\check{\alpha}}|$, so $\phi_w \cdot \tau_{n_s} = \phi_{w\tilde{s}} + |\mu_{\check{\alpha}}| \sum_{t\in\check{\alpha}([\mathbb{F}_q^\times])} \phi_{w\bar{t}}$.

Since (88) is an isomorphism of right H-modules, the calculation above gives the right action of H on the ϕ's. The formulas for the left action follow using (89), Proposition 6.1 and

$$\begin{aligned}\tau_x \cdot \phi_w &= \mathcal{J}(\mathcal{J}(\tau_x \cdot \phi_w)) = \mathcal{J}(\mathcal{J}(\phi_w) \cdot \mathcal{J}(\tau_x)) \\ &= \mathcal{J}(\phi_{w^{-1}} \cdot \tau_{x^{-1}}).\end{aligned}$$

□

Remark 8.3. i. The formula for the right action coincides with the one given in [SDGA] §5.1 p.9.

ii. Recall that $\iota(\tau_{n_s}) = -\tau_{n_s} - \theta_s$ where ι is the involutive automorphism of H defined in (29). Formulas (91) and (92) can be given in the form:

$$\phi_w \cdot \iota(\tau_{n_s}) = \begin{cases} -\phi_{w\tilde{s}} & \text{if } \ell(w\tilde{s}) = \ell(w) - 1 \\ -\phi_w \cdot \theta_s & \text{if } \ell(w\tilde{s}) = \ell(w) + 1 \end{cases}, \quad \iota(\tau_{n_s}) \cdot \phi_w = \begin{cases} -\phi_{\tilde{s}w} & \text{if } \ell(\tilde{s}w) = \ell(w) - 1 \\ -\theta_s \cdot \phi_w & \text{if } \ell(\tilde{s}w) = \ell(w) + 1. \end{cases}$$

Recall that we defined in (75) the G-equivariant map $\mathcal{S} = \sum_{g\in G/I} \mathrm{ev}_g \ : \mathbf{X} \longrightarrow k$, where k is endowed with the trivial action of G. Note that the restriction of $\mathcal{S}$ to $H = \mathbf{X}^I$ coincides with the trivial character $\chi_{triv} : H \to k$ as defined in §2.2.2. This is because for $w \in \widetilde{W}$ we have $\mathcal{S}(\tau_w) = \mathcal{S}(\mathrm{char}_{IwI}) = |IwI/I| = |I/I_w| = q^{\ell(w)}.1_k$ by Corollary 2.5.i. Since $\mathbf{X}$ is generated by char_I as a representation of G, it follows that $\mathcal{S}$ is a morphism of (G,H) modules $\mathbf{X} \to k_{triv}$ where k_{triv} denotes the (G,H)-bimodule k with the trivial action of G and the action of H via χ_{triv}.

The cohomology group $H^d(I, k_{triv}) = \mathrm{Ext}^d_{\mathrm{Mod}(G)}(\mathbf{X}, k_{triv})$ is naturally an H-bimodule where the left action comes from the right action on $\mathbf{X}$ and the right action comes from the trivial action on k_{triv}.

Proposition 8.4. i. *The left as well as the right H-action on $H^d(I, k_{triv})$ are trivial, i.e., are through χ_{triv}.*

ii. *The map $\mathcal{S}^d$ induced on cohomology by $\mathcal{S}$ yields an exact sequence of H-bimodules*

$$0 \longrightarrow \ker(\mathcal{S}^d) \longrightarrow E^d \longrightarrow H^d(I, k_{triv}) \longrightarrow 0 \ . \tag{93}$$

Proof. i. The triviality of the right action is obvious. For the left action we first consider the following diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, V) & & \xrightarrow{\quad\tau_g\quad} & & \mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, V) \\
\downarrow{=} & & & & \downarrow{=} \\
H^*(I, V) & \xrightarrow{\mathrm{res}} & H^*(I \cap g^{-1}Ig, V) & & H^*(I, V) \\
\searrow{g_*} & & \searrow{g_*} & & \nearrow{\mathrm{cores}} \\
& H^*(gIg^{-1}, V) & \xrightarrow{\mathrm{res}} & H^*(I \cap gIg^{-1}, V), &
\end{array}$$

where V is an arbitrary object in $\mathrm{Mod}(G)$ and where the upper horizontal arrow is the action of $\tau_g \in H$ induced by its right action on $\mathbf{X}$. For its commutativity it suffices, by using an injective resolution of V, to consider the case $* = 0$, i.e., the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{k[G]}(\mathbf{X}, V) & & \xrightarrow{f \mapsto f(_\tau_g)} & & \mathrm{Hom}_{k[G]}(\mathbf{X}, V) \\
{=}\downarrow{f \mapsto f(\mathrm{char}_I)} & & & & {f \mapsto f(\mathrm{char}_I)}\downarrow{=} \\
V^I & & & & V^I \\
\searrow{g} & & & & \nearrow{\sum_{h \in I/I \cap gIg^{-1}} h} \\
& V^{gIg^{-1}} & \xrightarrow{\subseteq} & V^{I \cap gIg^{-1}}. &
\end{array}$$

Its commutativity is verified easily by direct inspection. So our assertion reduces to the claim that the composed map

$$H^d(I, k) \xrightarrow{\mathrm{res}} H^d(I_{g^{-1}}, k) \xrightarrow{g_*} H^d(I_g, k) \xrightarrow{\mathrm{cores}} H^d(I, k)$$

coincides with the multiplication by $\chi_{triv}(\tau_g)$. Suppose that $g \in IwI$ with $w \in \widetilde{W}$. By Cor. 2.5.i we have $I_{g^{-1}} \subsetneqq I$ if and only if $\ell(w) > 0$. In this case the left restriction map above is the zero map by Remark 7.3 and $\chi_{triv}(\tau_w) = 0$. If $\ell(w) = 0$ then $\chi_{triv}(\tau_w) = 1$ and the above composed map simply is the map $H^d(I, k) \xrightarrow{w_*} H^d(I, k)$, which is the identity by Lemma 7.13.

ii. The map $\mathcal{S}^d : \mathrm{Ext}^d_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X}) \longrightarrow \mathrm{Ext}^d_{\mathrm{Mod}(G)}(\mathbf{X}, k_{triv})$ is surjective, for example, by Remark 7.4. It is right H-equivariant because $\mathcal{S}$ is right H-equivariant, and because the right action of H on the cohomology spaces is through the coefficients. Finally, it is left H-equivariant since the left actions are functorially induced by the right H-action on $\mathbf{X}$. □

Remark 8.5. Using directly the Hecke operators $\mathrm{cores} \circ g_* \circ \mathrm{res}$ in order to define the left H-action on $H^d(I, k_{triv})$ the Prop. 8.4.i is proved independently in [Koz2] Thm. 7.1 in case the root system is irreducible.

The kernel $\ker(\chi_{triv})$ of χ_{triv} is a sub-H-bimodule of $E^0 = H$. Passing to duals, we have the restriction map

$$(E^0)^\vee \longrightarrow \ker(\chi_{triv})^\vee \ . \tag{94}$$

Its kernel is a one dimensional vector space isomorphic to χ_{triv} as an H-bimodule. It is generated by the linear map

$$\phi : \tau_\omega \mapsto 1, \text{ for } \omega \in \widetilde{\Omega} \text{ and } \tau_w \mapsto 0, \text{ for } w \in \widetilde{W} \text{ with length} > 0. \tag{95}$$

Assume that Ω is finite. Then ϕ lies in $(E^0)^{\vee,f}$ and we have a short exact sequence of H-bimodules

$$0 \longrightarrow k\phi \longrightarrow (E^0)^{\vee,f} \longrightarrow \ker(\chi_{triv})^{\vee,f} \longrightarrow 0 \tag{96}$$

where $\ker(\chi_{triv})^{\vee,f}$ denotes the image of $(E^0)^{\vee,f} \subset (E^0)^\vee$ by (94).

Proposition 8.6. *If Ω is finite and $|\Omega|$ is invertible in k, then we have a decomposition of $(E^0)^{\vee,f}$ into a direct sum of H-bimodules*

$$(E^0)^{\vee,f} \cong \chi_{triv} \oplus \ker(\chi_{triv})^{\vee,f}$$

where χ_{triv} is supported by the element ϕ defined in (95) and $\ker(\chi_{triv})^{\vee,f}$ is the image of $(E^0)^{\vee,f}$ by the restriction map (94). Via the isomorphism (88), this corresponds to the decomposition

$$E^d \cong H^d(I, k_{triv}) \oplus \ker(\mathcal{S}^d). \tag{97}$$

Proof. Note that the image of ϕ in E^d is $\sum_{\omega\in\widetilde{\Omega}} \phi_\omega$. Combining (93), (96) and (88), we obtain a diagram of morphisms of H-bimodules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}^{\partial}(k\phi)^{\partial} & \longrightarrow & ({}^{\partial}E^0\,{}^{\partial})^{\vee,f} & \longrightarrow & ({}^{\partial}\ker(\chi_{triv})^{\partial})^{\vee,f} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \partial} & & {\scriptstyle \Delta^d}\Big\uparrow{\scriptstyle \cong} & & \Big\uparrow{\scriptstyle \varepsilon} & & \\
0 & \longleftarrow & H^d(I, k_{triv}) & \longleftarrow & E^d & \longleftarrow & \ker(\mathcal{S}^d) & \longleftarrow & 0
\end{array}$$

where ∂ and ε are such that the diagram commutes. The map ∂ is a map between one dimensional vector spaces, and it is not trivial since

$$\partial(\phi) = \mathcal{S}^d(\sum_{\omega\in\widetilde{\Omega}} \phi_\omega) = |\widetilde{\Omega}|\eta^{-1}(1) = -|\Omega|\eta^{-1}(1) \neq 0\ .$$

This implies that the second short exact sequence splits. By a standard argument, ε is also an isomorphism and the first exact sequence also splits. $\square$

Recall that we defined in (38) a decreasing filtration of H as an H-bimodule. For the sake of homogeneity of notations, we denote it here by $(F^nE^0)_{n\geq 0}$ and recall that

$$F^nE^0 = \oplus_{\ell(w)\geq n} k\tau_w\ . \tag{98}$$

For $n \geq 1$, define $(F^nE^0)^{\vee,f}$ to be the H-bimodule image of $(E^0)^{\vee,f}$ by the surjective restriction map $(E^0)^\vee \to (F^nE^0)^\vee$. Since Ω is assumed to be finite, the kernel of $(E^0)^{\vee,f} \twoheadrightarrow (F^nE^0)^{\vee,f}$ coincides with the dual space $(E^0/F^nE^0)^\vee$. Furthermore $\ker(\chi_{triv})^{\vee,f}$ is the increasing union of the subspaces $(\ker(\chi_{triv})/F^nE^0)^\vee$ of all linear maps φ in $\ker(\chi_{triv})^\vee$ which are trivial on F^nE^0 for some $n \geq 1$. The supersingular H-modules were defined in §2.3.

Corollary 8.7. *If* **G** *is semisimple simply connected with irreducible root system, then the H-module $\ker(\mathcal{S}^d)$ is a union of H-bimodules which are (finite length) supersingular on the right and on the left.*

Proof. It suffices to prove, for $n \geq 1$ that the finite dimensional space ${}^{\mathcal{J}}((\ker(\chi_{triv})/F^nE^0)^\vee)^{\mathcal{J}}$ is a supersingular H-module on the left and on the right. Since $\Omega = \{1\}$, we have

$$\ker(\chi_{triv}) = (1 - e_1)F^0E^0 + F^1E^0 \ ,$$

and $\ker(\chi_{triv})/F^nE^0$, by Lemma 2.14.ii, is annihilated by the action of $\mathfrak{J}^n$ on the left (resp. right). Therefore, $(\ker(\chi_{triv})/F^nE^0)^\vee$ is annihilated by the action of $\mathfrak{J}^n$ on the right (resp. left) and therefore supersingular. Due to Remark 6.3 this remains the case after twisting by $\mathcal{J}$. □

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