The perfectoid open unit disk

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These are expanded notes of a talk given at the workshop "The Galois Group of $\mathbb{Q}_p$ as Geometric Fundamental Group". Its purpose was to explain §3.2 and some initial steps in §4 of the paper [Wei].

1 Perfectoid generic fibers

Let K be a perfectoid field with ring of integers o_K of residue characteristic p. We fix an element $0 \neq \varpi \in o_K$ such that $|p| \leq |\varpi| < 1$.

We consider an o_K-algebra S which is complete and Hausdorff with respect to the I-adic topology for some ideal $I \subseteq S$. For simplicity we assume that $\varpi \in I$ (cf. Remark 1.6). We equip the factor ring $S/\varpi S$ with the $I/\varpi S$-adic topology. The projective limit of $o_K/\varpi o_K$-algebras

$$S^\flat := \varprojlim_{(.)^p} S/\varpi S = \varprojlim(\ldots \xrightarrow{(.)^p} S/\varpi S \xrightarrow{(.)^p} \ldots \xrightarrow{(.)^p} S/\varpi S)$$

equipped with the projective limit topology is a topological $o_{K^\flat}$-algebra, which is perfect. We denote by $\Phi_{S^\flat}$ the p-Frobenius automorphism of $S^\flat$.

Remark 1.1. *The rings $S/\varpi S$ and $S^\flat$ are complete.*

Proof. The vertical maps in the commutative diagrams

$$\begin{tikzcd}
S/I^{n+1} \arrow[r, "\varpi"] \arrow[d, two heads] & S/I^{n+1} \arrow[r, "\mathrm{pr}"] \arrow[d, two heads] & (S/\varpi S)/(I/\varpi S)^{n+1} \arrow[r] \arrow[d, two heads] & 0 \\
S/I^n \arrow[r, "\varpi"] & S/I^n \arrow[r, "\mathrm{pr}"] & (S/\varpi S)/(I/\varpi S)^n \arrow[r] & 0
\end{tikzcd}$$

are surjective. Hence the lower horizontal map in the commutative diagram

$$\begin{tikzcd}
S \arrow[r, two heads, "\mathrm{pr}"] \arrow[d, "\cong"'] & S/\varpi S \arrow[d] \\
\varprojlim_n S/I^n \arrow[r, two heads] & \varprojlim_n (S/\varpi S)/(I/\varpi S)^n
\end{tikzcd}$$

is surjective. Since the left vertical map is bijective by assumption the right vertical map at least is surjective. This amounts to the completeness of $S/\varpi S$. The completeness of $S^\flat$ then is a formal consequence. $\square$

A fundamental system of open neighbourhoods of zero in $S^\flat$ is given by the ideals

$$J_{n,i} := \{(\dots,\alpha_1,\alpha_0) \in S^\flat : \alpha_i \in (I/\varpi S)^n\} \qquad \text{for } n \geq 1 \text{ and } i \geq 0.$$

One checks that

$$\Phi^i_{S^\flat}(J_{n,0}) = J_{n,i}, \Phi^i_{S^\flat}(J_{1,0}) \subseteq J_{1,0}^{p^i} \subseteq J_{p^i,0}, \text{ and hence } \Phi^{i+j}_{S^\flat}(J_{1,0}) \subseteq J_{p^j,i}.$$

This shows the following.

Remark 1.2. *The descending sequence of ideals $\Phi^n_{S^\flat}(J_{1,0})$, for $n \geq 0$, forms a fundamental systems of open neighbourhoods of zero in $S^\flat$. In particular, $\Phi_{S^\flat}$ is a topological automorphism.*

Let $\alpha = (\dots,\alpha_i,\dots,\alpha_0) \in S^\flat$ be any element. We choose elements $a_i, a'_i \in S$ such that $a_i \bmod \varpi S = \alpha_i = a_i \bmod \varpi S$. The usual standard computation shows that the sequences $(a_i^{p^i})_i$ and $({a'}_i^{p^i})_i$ are ϖ-adic Cauchy sequences and that their difference $(a_i^{p^i} - {a'}_i^{p^i})_i$ is a ϖ-adic zero sequence. Since $\varpi \in I$ the same then holds true I-adically. But S is I-adically complete and Hausdorff. It follows that

$$\begin{aligned} S^\flat &\longrightarrow S \\ \alpha &\longmapsto \alpha^\sharp := \lim_{i\to\infty} a_i^{p^i} \end{aligned}$$

is a well defined multiplicative map such that $\alpha^\sharp \bmod \varpi S = \alpha_0$. In fact, we also could have used the following observation (which I learned from L. Ramero).

Remark 1.3. *The ring S is ϖ-adically complete and Hausdorff.*

Proof. The Hausdorff property is clear since $\bigcap_n \varpi^n S \subseteq \bigcap_n I^n = 0$. Let $(a_i)_i$ in S be a Cauchy sequence for the ϖ-adic and hence for the I-adic topology. It has an I-adic limit a. We have to show that a also is the ϖ-adic limit of the sequence. For this we may replace the original sequence by any convenient subsequence. Hence we may assume that $a_{i+1} - a_i \in \varpi^i S$ for any $i \geq 1$. Let $a_{i+1} - a_i = \varpi^i z_i$ with $z_i \in S$. For $j \geq 1$ we then have

$$a_{i+j} - a_i = (a_{i+j} - a_{i+j-1}) + \ldots + (a_{i+1} - a_i) = \varpi^i(\varpi^{j-1} z_{i+j-1} + \ldots + \varpi z_{i+1} + z_i) \ .$$

We put $y_{i,j} := \varpi^{j-1} z_{i+j-1} + \ldots + \varpi z_{i+1} + z_i$ and obtain

$$y_{i,j} - y_{i,m} = \varpi^{j-1} z_{i+j-1} + \ldots + \varpi^m z_{i+m} \in \varpi^m S$$

for $j > m$. It follows that the limit $y_i := \lim_{j\to\infty} y_{i,j}$ with respect to the I-adic topology exists. We finally compute

$$a - a_i = \lim_{j\to\infty} (a_{i+j} - a_i) = \varpi^i \lim_{j\to\infty} y_{i,j} = \varpi^i y_i \in \varpi^i S \ .$$

This shows that the sequence $(a_i)_i$ converges to a also in the ϖ-adic topology. $\square$

Lemma 1.4. *The map $^\sharp : S^\flat \longrightarrow S$ is continuous.*

Proof. We show that $(\alpha + J_{n+1,n})^\sharp \subseteq \alpha^\sharp + I^{n+1}$. If $\beta = (\dots,\beta_0) \in \alpha + J_{n+1,n}$ then $\beta_n - \alpha_n = c \bmod \varpi S$ for some $c \in I^{n+1}$. For $i \geq n$ we choose $a_i, b_i \in S$ such that $a_i \bmod \varpi S = \alpha_i$, $b_i \bmod \varpi S = \beta_i$, and $b_n - a_n \in I^{n+1}$. We have $a_i^{p^i} \equiv a_n^{p^n} \bmod \varpi^{n+1} S$ and $b_i^{p^i} \equiv b_n^{p^n} \bmod \varpi^{n+1} S$. It follows that $b_i^{p^i} - a_i^{p^i} \in I^{n+1}$ for any $i \geq n$ and hence that $\beta^\sharp - \alpha^\sharp \in I^{n+1}$. $\square$

Lemma 1.5. *The map*

$$\varprojlim_{(.)^p} S \xrightarrow{\simeq} S^\flat$$
$$(\ldots, a_i, \ldots, a_0) \longmapsto (\ldots, a_i \bmod \varpi S, \ldots, a_0 \bmod \varpi S)$$

is a multiplicative homeomorphism with inverse $\alpha \longmapsto (\ldots, (\alpha^{1/p^i})^\sharp, \ldots, \alpha^\sharp)$. *In particular,* $S^\flat$ *is Hausdorff.*[1]

Proof. Of course, the asserted projection map is multiplicative and continuous. Let $\alpha = (\ldots, \alpha_i, \ldots, \alpha_0) \in S^\flat$ and choose $a_i \in S$ such that $a_i \bmod \varpi S = \alpha_i$. Then $\alpha^{1/p^i} = (\ldots, \alpha_i)$ and hence $(\alpha^{1/p^i})^\sharp = \lim_{j\to\infty} a_{i+j}^{p^j}$, and we compute

$$((\alpha^{1/p^{i+1}})^\sharp)^p = \lim_{j\to\infty} a_{i+1+j}^{p^{j+1}} = \lim_{j\to\infty} a_{i+j}^{p^j} = (\alpha^{1/p^i})^\sharp \ .$$

Since $(\alpha^{1/p^i})^\sharp \bmod \varpi S = \alpha_i$, we see that the asserted candidate for the inverse is, at least, a left inverse of the projection map. It is continuous by Remark 1.2 and Lemma 1.4. It remains to check that the projection map is injective. Let $(\ldots, a_i, \ldots, a_0)$ and $(\ldots, b_i, \ldots, b_0)$ elements in $\varprojlim S$ such that $a_i \equiv b_i \bmod \varpi S$ for any $i \geq 0$. We deduce that $a_i = a_{i+j}^{p^j} \equiv b_{i+j}^{p^j} = b_i \bmod \varpi^{j+1} S$ for any $j \geq 0$. It follows that $a_i - b_i \in \bigcap_j I^j = 0$. □

Remark 1.6. *If the topology of the topological* o_K*-algebra* S *is the* I'*-adic one for some ideal* $I' \subseteq S$ *then it also is the* I*-adic one for* $I := \varpi S + I'$.

Proof. Being a topological o_K-algebra we must have $\varpi^m \in I'$ for some $m \geq 1$. It follows that $I^m \subseteq I'$. □

Lemma 1.5 and Remark 1.6 show that the topological $o_{K^\flat}$-algebra $S^\flat$ does not depend on the choice of the element ϖ. It will be technically convenient to assume in the following that $|p| \leq |\varpi|^2 < |\varpi| < 1$ and $\varpi \in I$. Then $pS \subseteq \varpi^2 S \subseteq I^2$.

Following [GR] §13.1 we now will make additional assumptions on S. The first one is

$$(S/pS)^p = S/pS \ . \tag{1}$$

Remark 1.7. *Suppose that* K *has characteristic zero; we then have* $(S/pS)^p = S/pS$ *if and only if* $(S/\varpi S)^p = S/\varpi S$.

Proof. The other direction being trivial we assume that $(S/\varpi S)^p = S/\varpi S$. Because of the density of the value group we find an element $\varpi_1 \in K$ such that $|\varpi|^{1/p} \leq |\varpi_1| < 1$. It follows that $\varpi S \subseteq \varpi_1^p S$ and hence that $(S/\varpi_1^p S)^p = S/\varpi_1^p S$. Now let $a \in S$ be any element. Inductively we find elements $(b_n)_{n\geq 0}$ and $(a_n)_{n\geq 1}$ in S such that

$$\begin{aligned} a &= b_0^p + \varpi_1^p a_1 \\ a_1 &= b_1^p + \varpi_1^p a_2 \\ &\ \vdots \\ a_n &= b_n^p + \varpi_1^p a_{n+1} \\ &\ \vdots \end{aligned}$$

[1] It seems that $S/\varpi S$ need not to be Hausdorff.

It follows that there exist elements $(c_n)_{n\geq 0}$ in S such that

$$a \equiv c_n^p + \varpi_1^{p(n+1)} a_{n+1} \mod pS \qquad \text{for any } n \geq 0.$$

But $|\varpi_1^{p(n+1)}| \leq |p|$ and hence $\varpi_1^{p(n+1)} S \subseteq pS$ for sufficiently large n (this is where the characteristic zero assumption is needed). This shows that $(S/pS)^p = S/pS$. □

Our further assumptions are:

$$I \text{ is finitely generated.} \tag{2}$$

$$\ker(S/I \xrightarrow{(.)^p} S/I) \text{ is a finitely generated ideal in } S/I. \tag{3}$$

We denote by $I \subseteq I_1 \subseteq S$ the finitely generated ideal such that $I_1/I = \ker(S/I \xrightarrow{(.)^p} S/I)$. Then, of course, the map

$$S/I_1 \xrightarrow[\cong]{(.)^p} S/I \tag{4}$$

is a well defined isomorphism.

Lemma 1.8. *Assuming* (1), (2), *and* (3) *we have:*

i. There are generators $I_1 = (s_1, \ldots, s_r)$ such that $I = (s_1^p, \ldots, s_r^p)$.

ii. The I_1-adic topology on S coincides with the I-adic one.

Proof. We begin by picking generators $I = (a_1, \ldots, a_m)$. As a consequence of assumption (1) we have $(S/I^2)^p = S/I^2$. Hence we find elements $s_j \in S$ such that $a_j - s_j^p \in I^2$. It follows that $s_1^p, \ldots, s_m^p$ generate the S-module I/I^2. But, since S is I-adically complete and Hausdorff, the ideal I is contained in the Jacobson radical of S (for any $a \in I$ and $b \in S$ the series $\sum_{i\geq 0}(ab)^i$ converges to an inverse of $1 - ab$). Therefore the Nakayama lemma applies and shows that $I = (s_1^p, \ldots, s_m^p)$. By construction we have $\{s_1, \ldots, s_m\} \subseteq I_1$. Since I_1 is finitely generated we may enlarge this set to a finite set of generators $s_1 = b_1, \ldots, s_m = b_m, s_{m+1}, \ldots, s_r$ of I_1 which satisfies the assertion i. Since $J^{pr} \subseteq I \subseteq I_1$ the I_1-adic topology coincides with the I-adic one. □

We now consider the ideal

$$I_1^\flat := \{(\ldots, \alpha_1, \alpha_0) \in S^\flat : \alpha_0 \in I_1/\varpi S\}$$

in $S^\flat$.

Proposition 1.9. *Assuming* (1), (2), *and* (3) *we have:*

i. The ideal $I_1^\flat$ is finitely generated.

ii. The topology of $S^\flat$ is the $I_1^\flat$-adic one.

Proof. We fix generators $I_1 = (s_1, \dots, s_r)$ as in Lemma 1.8.i. By the assumption (1) we find elements $\sigma_j = (\dots, \sigma_{j,1}, \sigma_{j,0}) \in S^\flat$, for $1 \leq j \leq r$, such that $\sigma_{j,0} = s_j \bmod \varpi S$. Obviously, the ideal $J := (\sigma_1, \dots, \sigma_r)$ is contained in $I_1^\flat$. Hence

$$\Phi^i_{S^\flat}(J) \subseteq \Phi^i_{S^\flat}(I_1^\flat) = \{(\dots, \alpha_1, \alpha_0) \in S^\flat : \alpha_i \in I_1/\varpi S\} \quad \text{for } i \geq 0.$$

Claim: The natural map $J \longrightarrow I_1^\flat/\Phi_{S^\flat}(I_1^\flat)$ is surjective.

Let $\alpha = (\dots, \alpha_0) \in I_1^\flat$. Since $\alpha_0 \in I_1/\varpi S$ there are $\gamma_{j,0} \in S/\varpi S$ such that

$$\alpha_0 = \gamma_{1,0}\sigma_{1,0} + \ldots + \gamma_{r,0}\sigma_{r,0} \ .$$

Again by (1) we have elements $\gamma_j \in S^\flat$ of the form $\gamma = (\dots, \gamma_{j,0})$. Then

$$\alpha - \sum_{j=1}^r \gamma_j \sigma_j = (\dots, \beta_1, 0) \ .$$

Using the bijectivity of (4) the vanishing $\beta_1^p = 0$ in $S/\varpi S$ implies that $\beta_1 \in I_1/\varpi S$. We conclude that

$$\alpha \in \sum_{j=1}^r \gamma_j \sigma_j + \Phi_{S^\flat}(I_1^\flat) \subseteq J + \Phi_{S^\flat}(I_1^\flat) \ ,$$

which establishes the claim.

We see that all the natural maps $\Phi^i_{S^\flat}(J)/\Phi^{i+1}_{S^\flat}(J) \longrightarrow \Phi^i_{S^\flat}(I_1^\flat)/\Phi^{i+1}_{S^\flat}(I_1^\flat)$, for $i \geq 0$, are surjective. Remark 1.2 and Lemma 1.5 together imply that $\bigcap_i \Phi^i_{S^\flat}(J) = \bigcap_i \Phi^i_{S^\flat}(I_1^\flat) = 0$. On the other hand it follows from Lemma 1.8.ii and Remark 1.2 that the topology of $S^\flat$ is defined by the system of open zero neighbourhoods $\Phi^i_{S^\flat}(I_1^\flat)$. Furthermore, since J is finitely generated, say with r generators, we have

$$\Phi^{p^i r}_{S^\flat}(J) \subseteq J^{p^i r} \subseteq \Phi^i_{S^\flat}(J) \subseteq \Phi^i_{S^\flat}(I_1^\flat) \tag{5}$$

for any $i \geq 0$. First of all this shows that the J-adic topology on $S^\flat$ is finer than the given topology. From Remark 1.1 and Lemma 1.5 we know that $S^\flat$ is complete and Hausdorff. In this situation it follows, by a generalization of the argument in the proof of Remark 1.3 (cf. [GR] Lemma 5.3.10), that $S^\flat$ also is complete and Hausdorff for the J-adic topology. Secondly (5) shows that the J-adic topology on $S^\flat$ coincides with the topology for which the $\Phi^i_{S^\flat}(J)$ form a fundamental system of open zero neighbourhoods. Hence $S^\flat$ is complete and Hausdorff with respect to this latter filtration as well. At this point we have verified all assumptions of [B-CA] III§2.8 Cor. 2 for the inclusion $J \subseteq I_1^\flat$ viewed as a map of filtered abelian groups with respect to the filtrations $(\Phi^i_{S^\flat}(J))_i$ and $(\Phi^i_{S^\flat}(I_1^\flat))_i$, respectively. The above surjectivity therefore implies that $J = I_1^\flat$. In particular, $I_1^\flat$ is finitely generated. Going back once more to (5) we also deduce that the topology of $S^\flat$ is the $I_1^\flat$-adic one. □

We recall ([Sch] Remark 3.5) that in a perfectoid field we can choose the element ϖ, without changing its absolute value, in such a way that there is an element $\varpi^\flat \in o_K^\flat$ such that

$$(((\varpi^\flat)^{1/p^i})^\sharp)^{p^i} = \varpi \qquad \text{for any } i \geq 0.$$

In the following we work with such a pair $(\varpi, \varpi^\flat)$.

The generic fiber $\mathrm{Spf}(S)^{ad}$ of $\mathrm{Spf}(S)$ (only assuming (2)) as a (not necessarily honest) adic space over K is constructed as follows. Choose the finitely generated ideal I as well as generators $I = (\varpi, s_1, \ldots, s_r)$. For any $n \geq 1$ let

$$\hat{S}_n := \varpi\text{-adic completion of } S[Y_1, \ldots, Y_r]/(s_1^n - \varpi Y_1, \ldots, s_r^n - \varpi Y_r) \text{ (cf. Remark 1.3)},$$
$$\hat{S}_n^+ := \text{integral closure of } \hat{S}_n \text{ in } K \otimes_{o_K} \hat{S}_n.$$

There are natural open immersions of adic spaces

$$\mathrm{Spa}(K \otimes_{o_K} \hat{S}_n, \hat{S}_n^+) \hookrightarrow \mathrm{Spa}(K \otimes_{o_K} \hat{S}_{n+1}, \hat{S}_{n+1}^+)$$

over K. Gluing them along these immersions defines the adic space $\mathrm{Spf}(S)^{ad}$. One checks that the resulting space is independent of the choices $I = (\varpi, s_1, \ldots, s_r)$.

If we assume (1), (2), and (3) then, because of Prop. 1.9, we similarly have the adic generic fiber $\mathrm{Spf}(S^\flat)^{ad}$ of $\mathrm{Spf}(S^\flat)$ over $K^\flat$.

Proposition 1.10. *Suppose that S satisfies (1), (2), and (3). Then $\mathrm{Spf}(S)^{ad}$ and $\mathrm{Spf}(S^\flat)^{ad}$ are perfectoid spaces over K and $K^\flat$, respectively, and $\mathrm{Spf}(S^\flat)^{ad} \cong (\mathrm{Spf}(S)^{ad})^\flat$.*

Proof. (Sketch) The proof basically consists in checking that the arguments in the proof of [Sch] Lemma 6.4 generalize to the present situation. We first consider the space $\mathrm{Spf}(S^\flat)^{ad}$. Let $\varpi^\flat, \sigma_1, \ldots, \sigma_r$ be generators of a finitely generated ideal of definition of $S^\flat$. Since $S^\flat$ is perfect it makes sense to introduce

$$(S^\flat)\widehat{\ }_{n,j} := \varpi^\flat\text{-adic completion of}$$
$$S^\flat[Y_1, \ldots, Y_r]/((\sigma_1^n)^{1/p^j} - (\varpi^\flat)^{1/p^j} Y_1, \ldots, (\sigma_r^n)^{1/p^j} - (\varpi^\flat)^{1/p^j} Y_r)$$

for any $n \geq 1$ and $j \geq 0$. We then have the sequence of homomorphisms

$$(S^\flat)\widehat{\ }_n = (S^\flat)\widehat{\ }_{n,0} \longrightarrow \ldots \longrightarrow (S^\flat)\widehat{\ }_{n,j} \longrightarrow S^\flat)\widehat{\ }_{n,j+1} \longrightarrow \ldots$$

with respect to the transition maps sending Y_i to Y_i^p. One checks that after inverting $\varpi^\flat$ these transition maps become isomorphisms. Hence inside $K^\flat \otimes_{o_{K^\flat}} (S^\flat)\widehat{\ }_n$ we have the $o_{K^\flat}$-subalgebra

$$(S^\flat)\widehat{\ }_n^\circ := \varpi^\flat\text{-adic closure of the union of the images of the } (S^\flat)\widehat{\ }_{n,j}.$$

By construction $(S^\flat)\widehat{\ }_n^\circ$ is $\varpi^\flat$-adically complete and Hausdorff and is a flat (since ϖ-torsion free) $o_{K^\flat}$-algebra. It satisfies $K^\flat \otimes_{o_{K^\flat}} (S^\flat)\widehat{\ }_n^\circ = K^\flat \otimes_{o_{K^\flat}} (S^\flat)\widehat{\ }_n$. At this point the arguments in the proof of [Sch] Lemma 6.4, part (i)(K of characteristic p) show that $(S^\flat)\widehat{\ }_n^{\circ a}$ is a perfectoid $o_{K^\flat}^a$-algebra. It then follows from [Sch] Thm. 5.2 that $K^\flat \otimes_{o_{K^\flat}} (S^\flat)\widehat{\ }_n$ is a perfectoid $K^\flat$-algebra. Therefore $\mathrm{Spa}(K^\flat \otimes_{o_{K^\flat}} (S^\flat)\widehat{\ }_n, (S^\flat)\widehat{\ }_n)$ is an affinoid perfectoid (and hence honest by [Sch] Thm. 6.3) adic space over $K^\flat$. By a gluing argument we finally conclude that $\mathrm{Spf}(S^\flat)^{ad}$ is a perfectoid (and hence honest) adic space over $K^\flat$.

In order to treat $\mathrm{Spf}(S)^{ad}$ we first of all make, in the discussion above, a more specific choice for the defining ideal of $S^\flat$. Let I be a defining ideal of S satisfying (1), (2), and (3) and pick generators $I_1 = (s_1, \ldots, s_r) = (\varpi, s_1, \ldots, s_r)$ as in Lemma 1.8. Choose elements $\sigma_j = (\ldots, s_j \bmod \varpi S) \in S^\flat$. In Prop. 1.9 we have seen that $I_1^\flat = (\varpi^\flat, \sigma_1, \ldots, \sigma_r)$ is a defining ideal of $S^\flat$. Since $\sigma_j^\sharp \equiv s_j \bmod \varpi S$ we have $I_1 = (\varpi, \sigma_1^\sharp, \ldots, \sigma_r^\sharp)$, which is a defining ideal of S

by Lemma 1.8.ii. Now one shows as in the proof of [Sch] Lemma 6.4, part (i),(ii)(general K) that the adic spaces $\mathrm{Spa}(K \otimes_{o_K} \hat{S}_n, \hat{S}_n^+)$ are affinoid perfectoid over K with tilts $\mathrm{Spa}(K^\flat \otimes_{o_{K^\flat}} (S^\flat)\widehat{\ }_n, (S^\flat)\widehat{\ }_n)$ (this uses the result in the characteristic p case). Again by gluing $\mathrm{Spf}(S)^{ad}$ is perfectoid. Since the tilting construction commutes with gluing, by [Sch] Prop. 6.17, the isomorphism in the assertion follows as well. $\square$

2 The perfectoid open unit disk

Let $L/\mathbb{Q}_p$ be a finite extension with ring of integers o and residue field k of cardinality $q = p^f$. We also fix, once and for all, a prime element π of o. Moreover, we choose a Frobenius power series ϕ for π, i.e., a formal power series $\phi(X) \in o[[X]]$ which satisfies

$$\phi(X) = \pi X + \text{higher terms} \qquad \text{and} \qquad \phi(X) \equiv X^q \mod \pi o[[X]] \ .$$

The Lubin-Tate formal group law F_ϕ is the unique commutative one dimensional formal group law $F_\phi \in o[[X, Y]]$ such that $\phi \in \mathrm{End}_o(F_\phi)$ is a formal endomorphism of ϕ. There is a unique homomorphism of rings

$$\begin{aligned} o &\longrightarrow \mathrm{End}_o(F_\phi) \\ a &\longmapsto [a]_\phi(X) = aX + \text{higher terms} \end{aligned}$$

such that $[\pi]_\phi = \phi$. In this way the formal o-scheme $H_L := \mathrm{Spf}(o[[X]])$ (for the maximal ideal as an ideal of definition) becomes a formal o-module. Its generic fiber over L is the open unit disk $\mathbf{D}_L$ viewed a a rigid or adic space.

We form the projective limit

$$\tilde{H}_L := \varprojlim_{\phi} H_L := \varprojlim(\ldots \xrightarrow{\mathrm{Spf}(\phi)} H_L \xrightarrow{\mathrm{Spf}(\phi)} H_L \xrightarrow{\mathrm{Spf}(\phi)} \ldots \xrightarrow{\mathrm{Spf}(\phi)} H_L)$$

in the category of formal o-schemes. Equivalently $\widetilde{H}_L = \mathrm{Spf}(\hat{R}_L)$ where $\hat{R}_L$ is the (π, X)-adic completion of the o-algebra

$$R_L := \varinjlim_{\phi} o[[X]] = \varinjlim(o[[X]] \xrightarrow{\phi} \ldots \xrightarrow{\phi} o[[X]] \xrightarrow{\phi} \ldots) \ .$$

This is a formal L-vector space.

Remark 2.1. *$\hat{R}_L$ is flat over o, and*

$$k \otimes_o \hat{R}_L = \hat{R}_k := X\text{-adic completion of } \varinjlim_{(.)^q} k[[X]].$$

Proof. In the ring $o[[X]]$ we have the obvious implication:

$$\text{If } Xf_1 = \pi f_2, \text{ then } f_1 \in \pi o[[X]] \text{ (and } f_2 \in Xo[[X]]).$$

Using the defining properties of a Frobenius power series one checks that this still holds true in R_L:

$$\text{If } f_1, f_2 \in R_L \text{ such that } Xf_1 = \pi f_2, \text{ then } f_1 \in \pi R_L \text{ (and } f_2 \in XR_L).$$

We now consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_{n+1} \arrow[r] \arrow[d, "0"'] & R_L/(\pi^{n+1}, X^{n+1}) \arrow[r, "\pi"] \arrow[d, "\mathrm{pr}"'] & R_L/(\pi^{n+1}, X^{n+1}) \arrow[r, "\mathrm{pr}"] \arrow[d, "\mathrm{pr}"'] & R_L/(\pi, X^{n+1}) \arrow[r] \arrow[d, "\mathrm{pr}"'] & 0 \\
0 \arrow[r] & K_n \arrow[r] & R_L/(\pi^n, X^n) \arrow[r, "\pi"] & R_L/(\pi^n, X^n) \arrow[r, "\mathrm{pr}"] & R_L/(\pi, X^n) \arrow[r] & 0,
\end{tikzcd}$$

where the terms in the left column are defined to be the kernels of multiplication by π. The above observation implies that the vertical map in the first column, indeed, is the zero map. Since countable projective systems with zero transition maps have zero projective limit as well as zero $\varprojlim^1$-term, passing to the projective limit with respect to n in the above exact sequence gives the short exact sequence

$$0 \longrightarrow \varprojlim R_L/(\pi^n, X^n) \xrightarrow{\pi} \varprojlim R_L/(\pi^n, X^n) \longrightarrow \varprojlim R_L/(\pi, X^n) \longrightarrow 0$$

On the one hand, since the ideal sequences (π^n, X^n) and $(\pi, X)^n$ are cofinal, we have $\hat{R}_L = \varprojlim R_L/(\pi^n, X^n)$. On the other hand, using that $\phi(X) \equiv X^q \bmod \pi o[[X]]$ we compute

$$\varprojlim R_L/(\pi, X^n) = \varprojlim (R_L/\pi R_L)/(X^n) = \varprojlim [(\varinjlim_{(.)^q} k[[X]])/(X^n)] \ .$$

Hence we obtain the short exact sequence

$$0 \longrightarrow \hat{R}_L \xrightarrow{\pi} \hat{R}_L \longrightarrow \varprojlim [(\varinjlim_{(.)^q} k[[X]])/(X^n)] \longrightarrow 0 \ .$$

In particular, $\hat{R}_L$ is π-torsion free and therefore flat over o. $\square$

We let $\tilde{\mathbf{D}}_L := \tilde{H}_L^{ad}$ denote the generic fiber of $\tilde{H}_L$ as an adic space over L. We briefly recall the construction: For any $n \geq 1$ let $\hat{R}_{L,n}$ denote the π-adic completion of $\hat{R}_L[Y]/(X^n - \pi Y)$. There are obvious open immersions of adic spaces $\mathrm{Spa}(L \otimes_o \hat{R}_{L,n}, \hat{R}_{L,n}) \hookrightarrow \mathrm{Spa}(L \otimes_o \hat{R}_{L,n+1}, \hat{R}_{L,n+1})$ [2]. Gluing the $\mathrm{Spa}(L \otimes_o \hat{R}_{L,n}, \hat{R}_{L,n})$ along these immersions gives the space $\tilde{\mathbf{D}}_L$.

Let K be a perfectoid field K which contains L. In the following a subscript K (instead of L) indicates the base extension from L to K (or from o to the ring of integers o_K) of any of the above objects. We choose the element $\varpi \in o_K$ such that $|\pi| \leq |\varpi|^2 < 1$. Then $I := (\varpi, X)$ is a defining ideal of $\hat{R}_K$. We will verify that $I \subseteq S := \hat{R}_K$ also satisfies the conditions (1) and (3) in section 1. As in the proof of Remark 2.1 we obtain

$$\begin{aligned}
\hat{R}_K/\varpi\hat{R}_K &= X\text{-adic completion of } \varinjlim_{(.)^q} o_K/\varpi o_K[[X]] \\
&= X\text{-adic completion of } o_K/\varpi o_K \otimes_k \hat{R}_k \\
&= o_K/\varpi o_K \hat{\otimes}_k \hat{R}_k \ .
\end{aligned} \tag{6}$$

To describe this ring completely explicitly let $\mathbb{N}_0[q^{-1}]$ denote the set of all rational numbers of the form $\frac{\ell}{q^m}$ with $\ell, m \geq 0$. Then

$$\hat{R}_K/\varpi\hat{R}_K = \text{ring of all series } \sum_{i \in \mathbb{N}_0[q^{-1}]} \bar{a}_i X^i \text{ with coefficients } \bar{a}_i \in o_K/\varpi o_K \text{ such that}$$
$$\text{for any } n \geq 0 \text{ there are only finitely many } 0 \leq i < n \text{ with } \bar{a}_i \neq 0.$$

[2]Is $\hat{R}_{L,n}$ integrally closed in $L \otimes_o \hat{R}_{L,n}$?

This description immediately shows that the map $(.)^q$ is surjective on $\hat{R}_K/\varpi\hat{R}_K$. Because of Remark 1.7 this settles the condition (1). We also see that the cosets in $\hat{R}_K/I$ are represented by the elements in the set

$$\{\sum_{0\leq\ell<q^m} \bar{a}_\ell X^{\ell/q^m} : m \geq 1\}.$$

Hence the kernel of $(.)^q$ on $\hat{R}_K/I$ is represented by the cosets in

$$\{\sum_{q^{m-1}\leq\ell<q^m} \bar{a}_\ell X^{\ell/q^m} : m \geq 1, \bar{a}_\ell^q = 0\} = \{X^{1/q} \cdot \sum_{0\leq\ell<q^m-q^{m-1}} \bar{b}_\ell X^{\ell/q^m} : m \geq 1, \bar{b}_\ell^q = 0\}.$$

If $\varpi_1 \in o_K$ is such that $|\varpi_1|^q = |\varpi|$ then it follows that $I_1 = (\varpi_1, X^{1/q})$, which settles the condition (3).

Lemma 2.2. $\tilde{\mathbf{D}}_K$ *is a perfectoid space over* K *(called the perfectoid open unit disk), and* $\tilde{\mathbf{D}}_K^\flat \cong \mathrm{Spf}(o_{K^\flat}\hat{\otimes}_k\hat{R}_k)^{ad}$.

Proof. By the above discussion Prop. 1.10 applies and gives that $\tilde{\mathbf{D}}_K$ is perfectoid with $\tilde{\mathbf{D}}_K^\flat \cong \mathrm{Spf}(\hat{R}_K^\flat)^{ad}$. By (6) and since $\hat{R}_k$ is perfect there is an obvious homomorphism

$$o_{K^\flat} \otimes_k \hat{R}_k = \varprojlim_{(.)^q}(o_K/\varpi o_K) \otimes_k \hat{R}_k \longrightarrow \varprojlim_{(.)^q}(o_K/\varpi o_K\hat{\otimes}_k\hat{R}_k) = \hat{R}_K^\flat \ .$$

This extends to an isomorphism $o_{K^\flat}\hat{\otimes}_k\hat{R}_k \cong \hat{R}_K^\flat$ where the left hand side is completed with respect to the $(\varpi^\flat, X)$-adic topology. The reason is that one has

$$(\varpi^\flat)^{q^m} o_{K^\flat} = \{(\ldots, \alpha_1, \alpha_0) \in \varprojlim_{(.)^q} o_K/\varpi o_K : \alpha_m = \ldots = \alpha_0 = 0\} \qquad \text{for any } m \geq 0.$$

□

Of course, the tilt $\tilde{\mathbf{D}}_K^\flat$ is a perfectoid space over the tilt $K^\flat$.

The final purpose of these notes is to explain a second structure as a perfectoid space over the field $\hat{L}_\infty^\flat$ once we remove the origin from $\tilde{\mathbf{D}}_K^\flat$. We first recall the definition of the field L_∞.

Let $\overline{L}$ be an algebraic closure of L with maximal ideal $\mathfrak{M}$ in its ring of integers. The formal o-module structure of H_L induces an actual o-module structure on $\mathfrak{M}$. For any $n \geq 1$ let $\mathcal{F}_n := \{z \in \mathfrak{M} : [\pi^n]_\phi(z) = 0\}$ be the π^n-torsion submodule. By adjoining these subsets to L we obtain the tower of algebraic extensions

$$L \subseteq L_1 := L(\mathcal{F}_1) \subseteq \ldots \subseteq L_n := L(\mathcal{F}_n) \subseteq \ldots \subseteq L_\infty := \bigcup_n L_n \subseteq \overline{L} \ .$$

From Lubin-Tate theory (cf. [LT]) we know:

a. The extension L_n/L does not depend on the choice of ϕ.

b. $\mathcal{F}_n$ is a free $o/\pi^n o$-module of rank one.

c. L_n/L is a totally ramified Galois extension of degree $(q-1)q^{n-1}$.

d. The map

$$\chi_{L,n} : \mathrm{Gal}(L_n/L) \xrightarrow{\cong} (o/\pi^n o)^\times$$
$$\sigma \longmapsto a \text{ such that } \sigma(z) = [a]_\phi(z) \text{ for any } z \in \mathcal{F}_n$$

is an isomorphism.

e. Any generator z_n of $\mathcal{F}_n$ as an $o/\pi^n o$-module generates the ring of integers o_{L_n} in L_n as an o-algebra and is a prime element of o_{L_n}.

Lemma 2.3. *The completion $\hat{L}_\infty$ of the field L_∞ is perfectoid.*

Proof. The property c. above implies that the value group $|\hat{L}_\infty^\times|$ is dense in $\mathbb{R}_{>0}^\times$. Furthermore the generators in e. can be chosen in such a way that $[\pi]_\phi(z_{n+1}) = z_n$ for any $n \geq 1$. Since $\phi(X) \equiv X^q \bmod \pi o[[X]]$, it follows that $z_{n+1}^q \equiv [\pi]_\phi(z_{n+1}) = z_n \bmod \pi o_{L_\infty}$. Again by e. the z_n generate the o-algebra o_{L_∞}. Hence their cosets generate the k-algebra $o_{L_\infty}/\pi o_{L_\infty}$. We conclude that $(o_{L_\infty}/\pi o_{L_\infty})^q = o_{L_\infty}/\pi o_{L_\infty}$. $\square$

We therefore may form the tilt $\hat{L}_\infty^\flat$ of $\hat{L}_\infty$. As a consequence of b. above the o-module

$$T := \varprojlim(\ldots \xrightarrow{[\pi]_\phi(.)} \mathcal{F}_{n+1} \xrightarrow{[\pi]_\phi(.)} \mathcal{F}_n \xrightarrow{[\pi]_\phi(.)} \ldots \xrightarrow{[\pi]_\phi(.)} \mathcal{F}_1)$$

is free of rank one. Since $\phi(X) \equiv X^q \bmod \pi o[[X]]$ we have

$$y_{m+1}^q \equiv y_m \mod \pi o_{L_\infty} \qquad \text{for any } m \geq 1 \text{ and any } (y_n)_{n\geq 1} \in T.$$

Therefore

$$\iota : \quad T \longrightarrow o_{\hat{L}_\infty^\flat}$$
$$(y_n)_{n\geq 1} \longmapsto (\ldots, y_n \bmod \pi o_{L_\infty}, \ldots, y_1 \bmod \pi o_{L_\infty}, 0)$$

is a well defined map (but not a homomorphism). We fix a generator $t = (z_n)_n$ of the o-module T. It follows from e. above that $\iota(t)$ is neither zero nor a unit in $o_{\hat{L}_\infty^\flat}$. Hence we have the well defined homomorphism of k-algebras

$$k[[Z]] \longrightarrow o_{\hat{L}_\infty^\flat}$$
$$f(Z) \longmapsto f(\iota(t)) \ ,$$

which extends to an embedding of fields

$$k((Z)) \longrightarrow \hat{L}_\infty^\flat \ .$$

The image $\mathbf{E}_L$ of the latter is a complete nonarchimedean discretely valued field of characteristic p with residue field k. It is called the field of norms of L. The image of the former map is its ring of integers $o_{\mathbf{E}_L}$. One can show:

- $\mathbf{E}_L$ is independent of the choice of the generator t.
- $\hat{L}_\infty^\flat$ is the completion of the perfect hull of $\mathbf{E}_L$. This implies that sending X to $\iota(t)$ gives an isomorphism

$$\hat{R}_k \xrightarrow{\cong} o_{\hat{L}_\infty^\flat} \ . \tag{7}$$

On the other hand sending $\iota(t)$ to Z gives a homomorphism $o_{\mathbf{E}_L} \longrightarrow o_{K^\flat}[[Z]]$. Using Lemma 2.2 and (7) we obtain isomorphisms

$$(8) \qquad \tilde{\mathbf{D}}_K^\flat \cong \mathrm{Spf}(o_{K^\flat} \hat{\otimes}_k \hat{R}_k)^{ad} = \mathrm{Spf}(o_{K^\flat} \hat{\otimes}_k o_{\mathbf{E}_L} \hat{\otimes}_{o_{\mathbf{E}_L}} o_{\hat{L}_\infty^\flat})^{ad} \cong \mathrm{Spf}(o_{K^\flat}[[Z]] \hat{\otimes}_{o_{\mathbf{E}_L}} o_{\hat{L}_\infty^\flat})^{ad} \ .$$

Note that $\mathrm{Spf}(o_{K^\flat}[[Z]])$ is the formal open unit disk over $o_{K^\flat}$. To go further we have to remove the origin from the open unit disk, i.e., we have to start with the adic space $\mathbf{D}_K^* := \mathbf{D}_K \setminus \{0\}$. It has a perfectoid version $\tilde{\mathbf{D}}_K^*$. To compute its tilt we observe that now we may view the punctured open unit disk $\mathbf{D}_{K^\flat}^*$ (over $K^\flat$) as an adic space over $\mathbf{E}_L$ via the map

$$\begin{aligned} \mathbf{E}_L &\longrightarrow \mathcal{O}(\mathbf{D}_{K^\flat}^*) \\ Z &\longmapsto \iota(t)_{|\mathbf{D}_{K^\flat}^*} \ . \end{aligned}$$

In this setting the formula (8) becomes the following isomorphism. Technically the additional complication comes from the fact that $\mathbf{D}_{K^\flat}^*$ is no longer the generic fiber of an affine formal scheme. So one has to exhaust $\mathbf{D}_{K^\flat}^*$ by affinoid annuli and has to redo everything above for them.

Proposition 2.4. $(\tilde{\mathbf{D}}_K^*)^\flat \cong \mathbf{D}_{K^\flat}^* \hat{\otimes}_{\mathbf{E}_L} \hat{L}_\infty^\flat$.

The formal o-module structure of H_L induces an action of the multiplicative monoid $o\setminus\{0\}$ on the rings $\hat{R}_L$, $\hat{R}_k$, and $\hat{R}_K$ and consequently on the perfectoid spaces $\tilde{\mathbf{D}}_K$ and $\tilde{\mathbf{D}}_K^*$ and their tilts. Under the isomorphism in Lemma 2.2 this action on the left hand side corresponds to the action on the right hand side through the factor $\hat{R}_k$. The action of π on $\hat{R}_k$ is the q-Frobenius.

On the other hand the Galois group $\mathrm{Gal}(L_\infty/L)$ naturally acts on $\hat{L}_\infty^\flat$. The projective limit of the isomorphisms $\chi_{L,n}$ in d. above is an isomorphism $\chi_L : \mathrm{Gal}(L_\infty/L) \xrightarrow{\cong} o^\times$. Hence we have an $o^\times$-action on $\hat{L}_\infty^\flat$. Letting π act as the q-Frobenius this extends to an action of the monoid $o \setminus \{0\}$ on $\hat{L}_\infty^\flat$. One checks that the isomorphism (7) is equivariant for the $o \setminus \{0\}$-actions on the two sides.

The latter implies that in the isomorphism in Prop. 2.4 the action of π on the left hand side corresponds on the right hand side to the map $(\phi^{-1} \otimes \mathrm{id}) \circ \phi_q$, where ϕ_q is the (absolute) q-Frobenius and ϕ on $\mathbf{D}_{K^\flat}^*$ is given by the q-Frobenius on the coefficients $K^\flat$ and $\phi(Z) = Z$.

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