Remarks on Emerton's functor
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These notes grew out of a seminar talk the first author gave in 2003 on the functor constructed by Emerton in [Eme] and called by him the "Jacquet functor". Their only purpose is to explain the general construction of this functor and Emerton's computation of it for smooth representations in a way which is, as we like to think, somewhat less technical.

We fix a finite extension $L/\mathbb{Q}_p$ as well as a spherically complete extension field K of L. We let G be the group of L-rational points of a connected reductive group over L.

Let V be a locally analytic G-representation on a K-vector space of compact type.

We fix a parabolic subgroup $P = MN$ of G with Levi factor M and unipotent radical N. The modulus character δ of P is given by

$$\delta(mn) = \mathrm{vol}_N(m^{-1}N_0m)/\mathrm{vol}_N(N_0)$$

for $m \in M$ and $n \in N$ where $N_0 \subseteq N$ is any compact open subgroup. If U is any P-representation we write $U(\delta)$ for the twist of U by δ.

Let $V(N)$ denote the vector subspace of V generated by all $nv - v$ for $n \in N$ and $v \in V$. Then

$$V_N := V/\overline{V(N)}$$

is a locally analytic M-representation on a vector space of compact type. Let $\mathfrak{n}$ denote the Lie algebra of N. Then $V^{\mathfrak{n}} := V^{\mathfrak{n}=0}$ is closed in V and hence carries a locally analytic P-representation on a vector space of compact type. Note that the N-action on $V^{\mathfrak{n}}$ is smooth. In an obvious way we may form $V^{\mathfrak{n}}(N)$ and

$$(V^{\mathfrak{n}})_N := V^{\mathfrak{n}}/\overline{V^{\mathfrak{n}}(N)}\ .$$

The latter is a locally analytic M-representation on a vector space of compact type. The inclusion $V^{\mathfrak{n}} \subseteq V$ passes to an M-equivariant continuous linear map

$$(V^{\mathfrak{n}})_N \longrightarrow V_N\ .$$

First we look at the left hand term. We fix a strictly positive element z in the center Z_M of M. This means that there is a compact open subgroup $N_0 \subseteq N$ such that the compact open subgroups $N_i := z^iN_0z^{-i}$ in N satisfy

$$\ldots \supset N_{-1} \supset N_0 \supset N_1 \supset N_2 \supset \ldots\ ,\quad \bigcup_{i\in\mathbf{Z}} N_i = N\ ,\ \text{and}\quad \bigcap_{i\in\mathbf{Z}} N_i = \{1\}\ .$$

The subspace V^{N_0} of N_0-fixed vectors is closed in $V^{\mathfrak{n}}$, and we may consider the continuous operator

$$\psi_z : V^{N_0} \xrightarrow{z} V^{\mathfrak{n}} \xrightarrow{\epsilon_{N_0}} V^{N_0} \ .$$

Here we define, quite generally for any compact open subgroup $N_c \subseteq N$, the operator

$$\begin{array}{rcl} \epsilon_{N_c} : V^{\mathfrak{n}} & \longrightarrow & V^{N_c} \\ v & \longmapsto & \frac{1}{[N_c:N_v]} \sum_{n \in N_c/N_v} nv \end{array}$$

where $N_v \subseteq N_c$ is a compact open subgroup which fixes v. Below we will repeatedly use the formula

$$z^{-1} \circ \epsilon_{N_i} \circ z = \epsilon_{N_{i-1}}$$

for any $i \in \mathbb{Z}$. We set

$$V^{N_0}_{z-tor} := \{v \in V^{N_0} : \psi_z^\ell(v) = 0 \text{ for some } \ell \in \mathbb{N}\}$$

Lemma 1: $V^{N_0}_{z-tor} = \ker(V^{N_0} \xrightarrow{\text{pr}} V^{\mathfrak{n}}/V^{\mathfrak{n}}(N))$

Proof: We first recall (cf. [BZ] 2.33) that

$$V^{\mathfrak{n}}(N) = \bigcup V^{\mathfrak{n}}(N_c)$$

where N_c runs over all compact open subgroups of N and where

$$V^{\mathfrak{n}}(N_c) := \{v \in V^{\mathfrak{n}} : \epsilon_{N_c}(v) = 0\} \ .$$

We further observe that due to our assumption that $z^{-1}N_0z \supseteq N_0$ we have

$$\psi_z^\ell(v) = z^\ell(\epsilon_{z^{-\ell}N_0z^\ell}(v)) \ . \tag{1}$$

If $\psi_z^\ell(v) = 0$ we therefore obtain $v \in V^{\mathfrak{n}}(N_{-\ell}) \subseteq V^{\mathfrak{n}}(N)$. Vice versa, if $v \in V^{N_0} \cap V^{\mathfrak{n}}(N)$ then $v \in V^{\mathfrak{n}}(N_c)$, i.e., $\epsilon_{N_c}(v) = 0$, for some compact open subgroup $N_0 \subseteq N_c \subseteq N$. Since z is strictly positive we find an $\ell \in \mathbb{N}$ such that $N_c \subseteq N_{-\ell}$. Then $\epsilon_{z^{-\ell}N_0z^\ell}(v) = 0$ and consequently $\psi_z^\ell(v) = 0$.

Since the N-action on $V^{\mathfrak{n}}$ is smooth the projection map $V^{N_0} \twoheadrightarrow V^{\mathfrak{n}}/V^{\mathfrak{n}}(N)$ is surjective. From the lemma we therefore obtain the continuous bijection

$$V^{N_0}/V^{N_0}_{z-tor} \longrightarrow V^{\mathfrak{n}}/V^{\mathfrak{n}}(N) \ .$$

In the opposite direction we have the map $V^{\mathfrak{n}} \xrightarrow{\epsilon_{N_0}} V^{N_0}$. Because of $\epsilon_{N_0}(V^{\mathfrak{n}}(N)) \subseteq V^{N_0} \cap V^{\mathfrak{n}}(N) = V^{N_0}_{z-tor}$ it passes to a map $V^{\mathfrak{n}}/V^{\mathfrak{n}}(N) \longrightarrow V^{N_0}/V^{N_0}_{z-tor}$ which obviously is inverse to the above bijection. We also note that the operator induced by ψ_z on the left hand side of this bijection corresponds to the action of z on the right hand side.

Proposition 2: *If V is admissible then the maps*

$$V^{N_0}/V^{N_0}_{z-tor} \xrightarrow{\cong} V^{\mathfrak{n}}/V^{\mathfrak{n}}(N)$$

and

$$V^{N_0}/\overline{V^{N_0}_{z-tor}} \xrightarrow{\cong} (V^{\mathfrak{n}})_N$$

are topological isomorphisms; in particular, the operator induced by ψ_z on the left hand terms is a topological automorphism.

Proof: It suffices to consider the first map. By the preceding discussion it furthermore suffices to show that the map $V^{\mathfrak{n}} \xrightarrow{\epsilon_{N_0}} V^{N_0}$ is continuous. Fix a compact open subgroup $G_0 \subseteq G$ such that $N_0 \subseteq G_0$. According to [ST1] Prop. 6.5 we can write V, viewed as an admissible G_0-representation, as a compact inductive limit of a sequence of locally analytic G_0-representations on Banach spaces V_j (with injective transition maps). Hence $V^{\mathfrak{n}}$, resp. V^{N_0}, is the compact inductive limit of the $V_j^{\mathfrak{n}}$, resp. $V_j^{N_0}$ (compare [GKPS] 3.1.16). It therefore suffices to show that each map $V_j^{\mathfrak{n}} \xrightarrow{\epsilon_{N_0}} V_j^{N_0}$ is continuous. But we do find a defining norm on $V_j^{\mathfrak{n}}$ with respect to which the compact group G_0 acts by isometries. A straightforward computation shows that in this norm the map ϵ_{N_0} is norm decreasing.

To avoid the choice of N_0 and to proceed in a more canonical way we observe that, since the N-action on $V^{\mathfrak{n}}$ is smooth, the usual Hecke algebra $\mathcal{H}(N)$ of K-valued locally constant functions with compact support on N acts on $V^{\mathfrak{n}}$. We define

$$E'_P(V) := \mathrm{Hom}_{\mathcal{H}(N)}(\mathcal{H}(N), V^{\mathfrak{n}})$$

equipped with the topology of pointwise convergence. Next we observe that the group P acts on $E'_P(V)$ via

$${}^{p}A(\phi) := p(A(\phi(p\,.\,p^{-1})))$$

through continuous automorphisms. It is well defined since

$$\begin{aligned}
{}^{p}A(\phi_1 * \phi_2) &= p(A((\phi_1 * \phi_2)(p\,.\,p^{-1}))) \\
&= \delta^{-1}(p) \cdot p(A(\phi_1(p\,.\,p^{-1}) * \phi_2(p\,.\,p^{-1}))) \\
&= \delta^{-1}(p) \cdot p(\phi_1(p\,.\,p^{-1})(A(\phi_2(p\,.\,p^{-1})))) \\
&= \phi_1(p(A(\phi_2(p\,.\,p^{-1})))) \\
&= \phi_1({}^{p}A(\phi_2)) \ .
\end{aligned}$$

Moreover, the restriction to N of this action simply is given by

$${}^{n}A(\phi) = A(\phi^{n})$$

where $\phi^n(n') := \phi(n'n^{-1})$.

To see the connection with the previous discussion one checks that the map

$$\begin{array}{rcl}
\mathrm{Hom}_{\mathcal{H}(N)}(\mathcal{H}(N), V^{\mathfrak{n}}) & \xrightarrow{\cong} & \varprojlim_i V^{N_i} \\
A & \longmapsto & (A(\epsilon_{N_i}))_i
\end{array}$$

is a topological isomorphism where the projective limit on the right hand side is formed with respect to the ϵ_{N_i} as transition maps and is equipped with the projective limit topology. The continuous operators

$$\psi_{z,i} : V^{N_i} \xrightarrow{z} V^{N_{i+1}} \xrightarrow{\epsilon_{N_i}} V^{N_i}$$

form a projective system. Because of

$$\begin{aligned}
{}^{z}A(\epsilon_{N_i}) &= \delta(z) \cdot z(A(\epsilon_{z^{-1}N_i z})) = \delta(z) \cdot z(A(\epsilon_{N_{i-1}})) \\
&= \delta(z) \cdot z(\epsilon_{N_{i-1}}(A(\epsilon_{N_i}))) = \delta(z) \cdot \epsilon_{N_i}(z(A(\epsilon_{N_i}))) \\
&= \delta(z) \cdot \psi_{z,i}(A(\epsilon_{N_i}))
\end{aligned}$$

its limit $\varprojlim_i \psi_{z,i}$ corresponds to the action of z twisted by $\delta^{-1}(z)$ on the left hand side and in particular is invertible.

We have the continuous linear map

$$\begin{array}{rcl}
\mathrm{Hom}_{\mathcal{H}(N)}(\mathcal{H}(N), V^{\mathfrak{n}})(\delta^{-1}) & \longrightarrow & V^{\mathfrak{n}}/V^{\mathfrak{n}}(N) \\
A & \longmapsto & A(\epsilon_{N_0}) + V^{\mathfrak{n}}(N) \ .
\end{array}$$

Using the two formulas

$$A(\epsilon_{N_0}) + V^{\mathfrak{n}}(N) = A(\epsilon_{N_c}) + V^{\mathfrak{n}}(N)$$

for any compact open subgroup $N_c \subseteq N$ and

$$\epsilon_{N_0}(p.p^{-1}) = \delta(p) \cdot \epsilon_{pN_0p^{-1}}$$

for any $p \in P$ one easily sees that this map is P-equivariant. It also is surjective since it extends, via the injective linear map

$$\begin{array}{ccl} V^{\mathfrak{n}} & \longrightarrow & \mathrm{Hom}_{\mathcal{H}(N)}(\mathcal{H}(N), V^{\mathfrak{n}})(\delta^{-1}) \\ v & \longmapsto & [\phi \mapsto \phi(v)] \ , \end{array}$$

the surjective projection map $V^{\mathfrak{n}} \twoheadrightarrow V^{\mathfrak{n}}/V^{\mathfrak{n}}(N)$. Its kernel, by Lemma 1, is equal to the subspace of all A in $\mathrm{Hom}_{\mathcal{H}(N)}(\mathcal{H}(N), V^{\mathfrak{n}})$ for which there is, for any $i \in \mathbb{Z}$, an $\ell_i \in \mathbb{N}$ such that $\psi_{z,i}^{\ell_i}(A(\epsilon_{N_i})) = 0$. The latter, by formula (1), is equivalent to $\epsilon_{N_{i-\ell_i}}(A(\epsilon_{N_i})) = A(\epsilon_{N_{i-\ell_i}}) = 0$. This proves the following result for which we introduce the P-invariant subspace

$$\begin{array}{rcl} E'_P(V)_0 & := & \{A \in \mathrm{Hom}_{\mathcal{H}(N)}(\mathcal{H}(N), V^{\mathfrak{n}}) : A(\epsilon_{N_c}) = 0 \text{ for some} \\ & & \qquad\qquad \text{compact open subgroup } N_c \subseteq N\} \end{array}$$

of $E'_P(V)$.

Lemma 3: *The natural map*

$$E'_P(V)/E'_P(V)_0 \xrightarrow{\cong} (V^{\mathfrak{n}}/V^{\mathfrak{n}}(N))(\delta)$$

is a continuous linear M-equivariant isomorphism.

(Is the map in Lemma 3 a topological isomorphism for admissible V?)

At this point we first look in more detail at the smooth case. As a piece of general notation, whenever U is a linear representation of some group H a vector $u \in U$ is called U-finite if it is contained in a finite dimensional H-invariant subspace of U. Then $U_{H-fin} := \{u \in U : u \text{ is } H\text{-finite}\}$ is an H-invariant subspace.

We now suppose that V is admissible smooth. For any $i \in \mathbb{Z}$, the commutative Z_M-equivariant diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{H}(N)}(\mathcal{H}(N), V)(\delta^{-1}) & \xrightarrow{A \mapsto A(\epsilon_{N_i})} & V/V(N_i) & \xrightarrow{pr} & V_N \\ & \nwarrow & \uparrow{\scriptstyle \cong,\ pr} & & \\ & & V^{N_i} & & \end{array}$$

restricts to the commutative diagram

$$\begin{array}{ccc} E'_P(V)_{Z_M-fin}(\delta^{-1}) & \longrightarrow\!\!\!\!\rightarrow & (V/V(N_i))_{Z_M-fin} \\ & \nwarrow \qquad \nearrow \cong & \\ & (V^{N_i})_{Z_M-fin} & . \end{array}$$

In order to establish the surjectivity of the natural map

$$E'_P(V)_{Z_M-fin}(\delta^{-1}) \longrightarrow V_N$$

it therefore suffices to show that any element of V_N can, for some $i \in \mathbb{Z}$, be lifted to $(V/V(N_i))_{Z_M-fin}$. The fact that V_N is admissible smooth as an M-representation implies

$$(V_N)_{Z_M-fin} = V_N \ .$$

Consider now an arbitrary vector $v \in V$. It is fixed by some compact open subgroup $Z_c \subseteq Z_M$. Since the image of v in V_N is Z_M-finite the kernel of the map

$$\begin{array}{rcl} K[Z_M/Z_c] & \longrightarrow & V_N \\ Q & \longmapsto & Q(v) + V(N) \end{array}$$

is an ideal I of finite codimension in the group ring $K[Z_M/Z_c]$. But Z_M/Z_c being a finitely generated abelian group this group ring is noetherian. Hence the ideal I has finitely many generators $Q_1, \ldots, Q_r$. We now choose an $i \in \mathbb{Z}$ such that $Q_1(v), \ldots, Q_r(v) \in V(N_i)$. Then the above map lifts to a map $K[Z_M/Z_c]/I \longrightarrow V/V(N_i)$ which shows that the image of v in $V/V(N_i)$ is Z_M-finite. It follows that the map $E'_P(V)_{Z_M-fin} \longrightarrow\!\!\!\!\rightarrow V_N$ is surjective.

On the other hand we claim that

$$E'_P(V)_{Z_M-fin} \cap E'_P(V)_0 = 0 \ .$$

Let A be an element in the intersection on the left hand side. By the definition of $E'_P(V)_0$ we then find a $j \in \mathbb{Z}$ such that $A(\epsilon_{N_i}) = 0$ for any $i \leq j$. Hence $({}^{z^\ell}A)(\epsilon_{N_j}) = 0$ for any $\ell \geq 0$. Since A is Z_M-finite this forces $({}^{z^{-1}}A)(\epsilon_{N_j}) = 0$ or, equivalently, $A(\epsilon_{N_{j+1}}) = 0$. Inductively we obtain in this way that $A(\epsilon_{N_i}) = 0$ for any $i \in \mathbb{Z}$ which means that $A = 0$.

Together with Lemma 3 this establishes the following result in the smooth case.

Proposition 4: *If V is admissible smooth then the natural map*

$$E'_P(V)_{Z_M-fin} \xrightarrow{\cong} V_N(\delta)$$

is an M-equivariant isomorphism.

Going back to a general V one might ask what properties the M-action on $E'_P(V)$ has. We will give an answer for the center Z_M.

Lemma 5: *The action of Z_M on $E'_P(V)$ is continuous and extends (uniquely) to a separately continuous action of the distribution algebra $D(Z_M, K)$.*

Proof: Obviously this needs to be checked only for the action of a sufficiently small compact open subgroup $Z_c \subseteq Z_M$. We choose Z_c in such a way that it normalizes N_0 and then also each N_i so that the subspaces V^{N_i} are Z_c-invariant. One easily checks that our above topological isomorphism $\mathrm{Hom}_{\mathcal{H}(N)}(\mathcal{H}(N), V^{\mathfrak{n}}) \cong \varprojlim_i V^{N_i}$ is Z_c-equivariant. Since the G-action on V is locally analytic so, too, is the Z_c-action on each closed subspace V^{N_i}. From this we obtain a unique separately continuous $D(Z_c, K)$-action on each V^{N_i} and hence one on their projective limit.

Let Z_M^0 denote the maximal compact subgroup of Z_M. We may write $Z_M = Z_M^0 \times Z$ with a finitely generated free abelian group Z. Then

$$D(Z_M, K) = D(Z_M^0, K)[Z] = D(Z_M^0, K) \otimes_K K[Z] \ .$$

We now view $K[Z]$ as the ring of rational functions on the split K-torus $\mathcal{T}$ with character group Z and we introduce the ring $\mathcal{O}$ of holomorphic functions on $\mathcal{T}$. Since $\mathcal{T}$ is a Stein space the ring $\mathcal{O}$ is a commutative nuclear Fréchet-Stein algebra. We introduce the completed tensor product

$$D^{hol}(Z_M, K) := D(Z_M^0, K) \widehat{\otimes}_K \mathcal{O} \ .$$

One checks that the Fréchet algebra $D^{hol}(Z_M, K)$ is (up to natural isomorphism) independent of the choice of Z. It contains $D(Z_M, K)$ as a dense subalgebra.

We have seen that $E'_P(V)$ is a $D(Z_M, K)$-module. The following definition singles out a submodule on which the $D(Z_M, K)$-action extends (uniquely) to a separately continuous $D^{hol}(Z_M, K)$-action.

Definition: $E_P(V) := \mathrm{Hom}^{cont}_{D(Z_M,K)}(D^{hol}(Z_M, K), E'_P(V)(\delta^{-1}))$ *equipped with the strong topology.*

Remarks: 1. The M-action on $E'_P(V)$ induces by functoriality an action of M by topological automorphisms on $E_P(V)$. It commutes with the obvious $D^{hol}(Z_M, K)$-action.

2. Since $D(Z_M, K)$ is dense in $D^{hol}(Z_M, K)$ evaluation at $1 \in D^{hol}(Z_M, K)$ is a continuous M-equivariant injective linear map

$$E_P(V) \longrightarrow E'_P(V)(\delta^{-1}) = \mathrm{Hom}_{\mathcal{H}(N)}(\mathcal{H}(N), V^{\mathfrak{n}})(\delta^{-1}) \ .$$

3. The idea to consider the M-analytic vectors in $E'_P(V)(\delta^{-1})$ is pointless since, if V is smooth, they contain all of V.

We note that

$$E_P(V) = \mathrm{Hom}^{cont}_{K[Z]}(\mathcal{O}, E'_P(V)(\delta^{-1})) \ .$$

To explore the projective limit structure of $E'_P(V)$ we introduce the submonoid Z^+ of all positive elements in Z, i.e., all elements which satisfy $z^{-1}N_0 z \supseteq N_0$. For any $z \in Z^+$ the continuous operators

$$\psi_{z,i} : V^{N_i} \xrightarrow{z} V^{zN_i z^{-1}} \xrightarrow{\epsilon_{N_i}} V^{N_i}$$

still form a projective system and, because of

$$\begin{aligned} {}^z A(\epsilon_{N_i}) &= \delta(z) \cdot z(A(\epsilon_{z^{-1}N_i z})) = \delta(z) \cdot z(A(\epsilon_{z^{-1}N_i z} * \epsilon_{N_i})) \\ &= \delta(z) \cdot z(\epsilon_{z^{-1}N_i z}(A(\epsilon_{N_i}))) = \delta(z) \cdot \epsilon_{N_i}(z(A(\epsilon_{N_i}))) \\ &= \delta(z) \cdot \psi_{z,i}(A(\epsilon_{N_i})) \ , \end{aligned}$$

its limit $\varprojlim_i \psi_{z,i}$ still corresponds to the action of z twisted by $\delta^{-1}(z)$ on the left hand side of the isomorphism

$$E'_P(V) = \mathrm{Hom}_{\mathcal{H}(N)}(\mathcal{H}(N), V^{\mathfrak{n}}) \cong \varprojlim_i V^{N_i} \ .$$

Observing that, for any two $z_1, z_2 \in Z^+$, we have

$$\begin{aligned} \psi_{z_1,i} \circ \psi_{z_2,i} &= \epsilon_{N_i} \circ z_1 \circ \epsilon_{N_i} \circ z_2 = \epsilon_{N_i} \circ z_1 \circ \epsilon_{N_i} \circ z_1^{-1} \circ (z_1 z_2) \\ &= \epsilon_{N_i} \circ \epsilon_{z_1 N_i z_1^{-1}} \circ (z_1 z_2) = \epsilon_{N_i} \circ (z_1 z_2) \\ &= \psi_{z_1 z_2, i} \end{aligned}$$

we may also introduce the $\mathcal{O}$-modules

$$\mathrm{Hom}^{cont}_{K[Z^+]}(\mathcal{O}, V^{N_i}) := \{\alpha \in \mathrm{Hom}^{cont}_K(\mathcal{O}, V^{N_i}) : \alpha \circ z = \psi_{z,i} \circ \alpha \text{ for any } z \in Z^+\}$$

equipped with the strong topology.

Lemma 6: *The natural map $E_P(V) \xrightarrow{\cong} \mathrm{Hom}^{cont}_{K[Z^+]}(\mathcal{O}, V^{N_0})$ is an $\mathcal{O}$-linear topological isomorphism.*

Proof: Using the universal property of the projective limit we have

$$\begin{aligned} E_P(V) &= \mathrm{Hom}^{cont}_{K[Z]}(\mathcal{O}, E'_P(V)(\delta^{-1})) \\ &= \mathrm{Hom}^{cont}_{K[Z]}(\mathcal{O}, \varprojlim_i V^{N_i}) \\ &= \mathrm{Hom}^{cont}_{K[Z^+]}(\mathcal{O}, \varprojlim_i V^{N_i}) \\ &= \varprojlim_i \mathrm{Hom}^{cont}_{K[Z^+]}(\mathcal{O}, V^{N_i}) \ . \end{aligned}$$

Here the third equality is due to the fact that any element in Z is a quotient of two elements in Z^+. The transition map in the latter projective system is

$$\tau_i := \mathrm{Hom}(\mathcal{O}, \epsilon_{N_{i-1}}) : \mathrm{Hom}^{cont}_{K[Z^+]}(\mathcal{O}, V^{N_i}) \longrightarrow \mathrm{Hom}^{cont}_{K[Z^+]}(\mathcal{O}, V^{N_{i-1}}) \ .$$

On the other hand, choosing our original strictly positive element z in Z^+, we have the continuous linear maps

$$\zeta_i := \mathrm{Hom}(\mathcal{O}, z) : \mathrm{Hom}^{cont}_{K[Z^+]}(\mathcal{O}, V^{N_{i-1}}) \longrightarrow \mathrm{Hom}^{cont}_{K[Z^+]}(\mathcal{O}, V^{N_i}) \ .$$

For the two composites we obtain

$$\tau_i \circ \zeta_i = \mathrm{Hom}(\mathcal{O}, \psi_{z,i-1}) = \mathrm{Hom}(z, V^{N_{i-1}})$$

and

$$\zeta_i \circ \tau_i = \mathrm{Hom}(\mathcal{O}, z \circ \epsilon_{N_{i-1}}) = \mathrm{Hom}(\mathcal{O}, \psi_{z,i}) = \mathrm{Hom}(z, V^{N_i})$$

which both are topological isomorphisms. It follows that the transition maps τ_i are topological isomorphisms.

By the way, the same reasoning as in the above proof shows that

$$E'_P(V)(\delta^{-1}) \xrightarrow{\cong} \mathrm{Hom}_{K[Z^+]}(K[Z], V^{N_0}) \ .$$

Proposition 7: *The locally convex vector space $E_P(V)$ is of compact type and the M-action on it is locally analytic; moreover, the $\mathcal{O}$-action on $E_P(V)$ is separately continuous.*

Proof: We fix a compact open subgroup $M_0 \subseteq M$ which normalizes N_0 so that V^{N_0} is M_0-invariant. Since M acts on $E_P(V)$ by topological automorphisms the local analyticity only needs to be checked for the M_0-action on $E_P(V)$. But, by Lemma 6, $E_P(V)$ is a closed M_0-invariant subspace of $\mathcal{L}_b(\mathcal{O}, V^{N_0})$. It therefore suffices to check that the latter is of compact type and that the M_0-action on it is locally analytic. According to [ST0] Cor. 3.3 this is equivalent to checking that the strong dual $\mathcal{L}_b(\mathcal{O}, V^{N_0})'_b$ is a nuclear Fréchet space and that the M_0-action on it extends to a separately continuous $D(M_0, K)$-action. Certainly the strong dual $(V^{N_0})'_b$ has both these properties. It follows from [NFA] p.134 that

$$\mathcal{L}_b(\mathcal{O}, V^{N_0}) = (\mathcal{O} \,\widehat{\otimes}_{K,\pi}\, (V^{N_0})'_b)'_b \ .$$

On the other hand [NFA] 19.11, 20.4, and 20.14 imply that with $\mathcal{O}$ and $(V^{N_0})'_b$ also $\mathcal{O} \,\widehat{\otimes}_{K,\pi}\, (V^{N_0})'_b$ is a nuclear Fréchet space and, in particular, is reflexive so that

$$\mathcal{L}_b(\mathcal{O}, V^{N_0})'_b = \mathcal{O} \,\widehat{\otimes}_{K,\pi}\, (V^{N_0})'_b \ .$$

The projective tensor product $\mathcal{O} \otimes_{K,\pi} (V^{N_0})'_b$ carries obvious separately continuous $\mathcal{O}$- and $D(M_0, K)$-actions through the first and second factor, respectively. By the universal property of the completion and the Banach-Steinhaus theorem these actions extend to separately continuous action on the completed tensor product $\mathcal{O} \widehat{\otimes}_{K,\pi} (V^{N_0})'_b$.

Proposition 8: *If V is admissible smooth then the image of the injective map $E_P(V) \hookrightarrow E'_P(V)(\delta^{-1})$ given by evaluation in $1 \in \mathcal{O}$ is equal to $E'_P(V)_{Z_M-fin}$; in particular, we have a natural M-equivariant isomorphism*

$$E_P(V) \xrightarrow{\cong} V_N \ .$$

Proof: Suppose first that $A \in E'_P(V)$ is Z_M-finite. Then the map

$$\begin{array}{ccc} K[Z] & \longrightarrow & E'_P(V) \\ z' & \longmapsto & {}^{z'}A \end{array}$$

has finite dimensional image. Its kernel I therefore is an ideal of finite codimension. Since I is finitely generated the ideal $I\mathcal{O}$ it generates in the Fréchet-Stein algebra $\mathcal{O}$ is closed. It follows that the dense inclusion $K[Z] \subseteq \mathcal{O}$ induces the surjection $K[Z]/I \twoheadrightarrow \mathcal{O}/I\mathcal{O}$ of finite dimensional Hausdorff vector spaces. Since $\mathcal{O}$ is faithfully flat over $K[Z]$ the latter map also is injective. This shows that A extends continuously to $\mathcal{O}$ which means that $A = \alpha(1)$ for some $\alpha \in E_P(V)$.

Because of Prop. 4 it remains to show that the composed map

$$E_P(V) \longrightarrow E'_P(V) \longrightarrow V_N$$

is injective. But we have the commutative diagram

$$\begin{array}{ccccc} E_P(V) & \longrightarrow & E'_P(V) & \longrightarrow & V_N \\ \cong \downarrow & & & & \uparrow \cong \\ \mathrm{Hom}^{cont}_{K[Z^+]}(\mathcal{O}, V^{N_0}) & & \xrightarrow{\beta \mapsto \beta(1)} & & V^{N_0}/V^{N_0}_{z-tor} \end{array}$$

where the perpendicular arrows are isomorphisms by Lemma 6 and Prop. 2, respectively. So we are reduced to showing that the lower horizontal arrow is injective. Let $\beta \in \mathrm{Hom}^{cont}_{K[Z^+]}(\mathcal{O}, V^{N_0})$. Then β is a continuous map from the Fréchet space $\mathcal{O}$ into the vector space V^{N_0} with its finest locally convex topology. The latter, in particular, is a countable locally convex inductive limit of finite dimensional Hausdorff vector spaces. Therefore the map β has a finite dimensional image ([NFA] 8.9. Since multiplication by z is invertible on $\mathcal{O}$ the map ψ_z must restrict to a surjective and hence bijective endomorphism of $\mathrm{im}(\beta)$. It follows that for nonzero β the value $\beta(1)$ cannot lie in $V^{N_0}_{z-tor}$.

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