Nonarchimedean Functional Analysis

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This book grew out of a course which I gave during the winter term 1997/98 at the Universität Münster. The course covered the material which here is presented in the first three chapters. The fourth more advanced chapter was added to give the reader a rather complete tour through all the important aspects of the theory of locally convex vector spaces over nonarchimedean fields. There is one serious restriction, though, which seemed inevitable to me in the interest of a clear presentation. In its deeper aspects the theory depends very much on the field being spherically complete or not. To give a drastic example, if the field is not spherically complete then there exist nonzero locally convex vector spaces which do not have a single nonzero continuous linear form. Although much progress has been made to overcome this problem a really nice and complete theory which to a large extent is analogous to classical functional analysis can only exist over spherically complete fields. I therefore allowed myself to restrict to this case whenever a conceptual clarity resulted.

Although I hope that this text will also be useful to the experts as a reference my own motivation for giving that course and writing this book was different. I had the reader in mind who wants to use locally convex vector spaces in the applications and needs a text to quickly grasp this theory. There are several areas, mostly in number theory like p-adic modular forms and deformations of Galois representations and in representation theory of p-adic reductive groups, in which one can observe an increasing interest in methods from nonarchimedean functional analysis. By the way, discretely valued fields like p-adic number fields as they occur in these applications are spherically complete.

This is a textbook which is self-contained in the sense that it requires only some basic knowledge in linear algebra and point set topology. Everything presented is well known, nothing is new or original. Some of the material in the last chapter appears in print for the first time, though. In the references I have listed all the sources I have drawn upon. At the same time this list shows to the reader who the protagonists are in this area of mathematics. I certainly do not belong to this group.

Münster, May 2001 Peter Schneider

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Chap. I: Foundations

In this chapter we introduce the basic notions and constructions of nonarchimedean functional analysis. We begin in §1 with a brief but self-contained review of nonarchimedean fields. The main objective of functional analysis is the investigation of a certain class of topological vector spaces over a fixed nonarchimedean field K. This is the class of locally convex vector spaces. The more traditional analytic point of view characterizes locally convex topologies as those vector space topologies which can be defined by a family of (nonarchimedean) seminorms. But the presence of the ring of integers o inside the field K allows for an equivalent algebraic point of view. A locally convex topology on a K-vector space V is a vector space topology defined by a class of o-submodules of V which are required to generate V as a vector space. In §§2 and 4 we thoroughly discuss these two concepts and their equivalence. Throughout the book we usually will present the theory from both angles. But sometimes there will be a certain bias towards the algebraic point of view.

The most basic methods to actually construct locally convex vector spaces along with concrete examples are treated in §§3 and 5. In §6 we explain how the notion of a bounded subset leads to a systematic method to equip the vector space of continuous linear maps between two given locally convex vector spaces with a natural class of locally convex topologies. The two most important ones among them are the weak and the strong topology. The important concepts of completeness and quasi-completeness are discussed in §7. The construction of the completion of a locally convex vector space is one of the places where we find an algebraic treatment preferable since conceptually simpler. Banach spaces as already introduced in §3 are complete. They are included in the very important class of Fréchet spaces. These are the complete locally convex vector spaces whose topology is metrizable. Their importance partly derives from the validity of the closed graph and open mapping theorems for linear maps between Fréchet spaces. These basic results are established in §8 using Baire category theory. In the final §9 of this chapter we begin the investigation of the continuous linear dual of a locally convex vector space. Provided the field K is spherically complete we establish the Hahn-Banach theorem about the existence of continuous linear forms. This is then applied to obtain the first properties of the duality maps into the various forms of the linear bidual. In this section we encounter for the first time the phenomenon in nonarchimedean functional analysis that crucial aspects of the theory depend on special properties of the nonarchimedean field K. The ultimate reason for this difficulty is that K need not to be locally compact. A satisfactory substitute for compact subsets in locally convex K-vector spaces only exists if the field K is spherically complete. This will be discussed systematically in §12 of the third chapter.

§1 Nonarchimedean fields

Let K be a field. A nonarchimedean absolute value on K is a function $| \ | : K \longrightarrow \mathbb{R}$ such that, for any $a, b \in K$ we have

(i) $|a| \geq 0$,

(ii) $|a| = 0$ if and only if $a = 0$,

(iii) $|ab| = |a| \cdot |b|$,

(iv) $|a + b| \leq \max(|a|, |b|)$.

The condition (iv) is called the strict triangle inequality. Because of (iii) the map $| \ | : K^\times \longrightarrow \mathbb{R}_+^\times$ is a homomorphism of groups. In particular we have $|1| = |-1| = 1$. We always will assume in addition that $| \ |$ is non-trivial, i.e., that

(v) there is an $a_0 \in K$ such that $|a_0| \neq 0, 1$.

It follows immediately that $|n \cdot 1| \leq 1$ for any $n \in \mathbb{Z}$. Moreover, if $|a| \neq |b|$ for some $a, b \in K$ then the strict triangle inequality actually can be sharpened into the equality

$$|a + b| = \max(|a|, |b|) \ .$$

To see this we may assume that $|a| < |b|$. Then $|a| < |b| = |b + a - a| \leq \max(|b+a|, |a|)$, hence $|a| < |a+b|$ and therefore $|b| \leq |a+b| \leq \max(|a|, |b|) = |b|$.

Via the distance function $d(a, b) := |b - a|$ the set K is a metric and hence topological space. The subsets

$$B_\epsilon(a) := \{b \in K : |b - a| \leq \epsilon\}$$

for any $a \in K$ and any real number $\epsilon > 0$ are called *closed balls* or simply *balls* in K. They form a fundamental system of neighbourhoods of a in the metric space K. Likewise the *open balls*

$$B_\epsilon^-(a) := \{b \in K : |b - a| < \epsilon\}$$

form a fundamental system of neighbourhoods of a in K. As we will see below $B_\epsilon(a)$ and $B_\epsilon^-(a)$ are both open and closed subsets of K. Talking about open and closed balls therefore does not refer to a topological distinction but only to the nature of the inequality sign in the definition.

We point out the following two simple facts.

1) If $| \ |_\infty$ denotes the usual archimedean absolute value on $\mathbb{R}$ then, for any $b \in B_\epsilon^-(a)$, we have $||b| - |a||_\infty = ||(b-a)+a| - |a||_\infty \leq |\max(|b-a|, |a|) - |a||_\infty < \epsilon$. This means that the absolute value $| \ | : K \longrightarrow \mathbb{R}$ is a continuous function.

2) For $b_0 \in B_\epsilon^-(a_0)$ and $b_1 \in B_\epsilon^-(a_1)$ we have $b_0 + b_1 \in B_\epsilon^-(a_0 + a_1)$ and $b_0 b_1 \in B^-_{\epsilon \cdot \max(\epsilon, |a_0|, |a_1|)}(a_0 a_1)$. The latter follows from $b_0 b_1 - a_0 a_1 = (b_0 - a_0)(b_1 - a_1) + (b_0 - a_0)a_1 + a_0(b_1 - a_1)$. This says that addition $+ : K \times K \longrightarrow K$ and multiplication $\cdot : K \times K \longrightarrow K$ are continuous maps.

Lemma 1.1:

i. $B_\epsilon(a)$ *is open and closed in* K*;*

ii. if $B_\epsilon(a) \cap B_\epsilon(a') \neq \emptyset$ *then* $B_\epsilon(a) = B_\epsilon(a')$*;*

iii. If B *and* B' *are any two balls in* K *with* $B \cap B' \neq \emptyset$ *then either* $B \subseteq B'$ *or* $B' \subseteq B$*;*

iv. K *is totally disconnected.*

Proof: The assertions i. and ii. are immediate consequences of the strict triangle inequality. The assertion iii. follows from ii. To see iv. let $M \subseteq K$ be a nonempty connected subset. Pick a point $a \in M$. By i. the intersection $M \cap B_\epsilon(a)$ is open and closed in M. It follows that M is contained in any ball around a and therefore must be equal to $\{a\}$.

Clearly the assertions i.-iii. hold similarly for open balls. The assertion ii. says that any point of a (open) ball can serve as its midpoint. On the other hand the real number ϵ is not uniquely determined by the set $B_\epsilon(a)$ and therefore cannot be considered as the "radius" of this ball.

Another consequence of the strict triangle inequality is the fact that a sequence $(a_n)_{n \in \mathbb{N}}$ in K is a Cauchy sequence if and only if the consecutive distances $|a_{n+1} - a_n|$ converge to zero if n goes to infinity.

Definition:

The field K *is called* <u>*nonarchimedean*</u> *if it is equipped with a nonarchimedean absolute value such that the corresponding metric space* K *is complete (i.e., every Cauchy sequence in* K *converges).*

From now on throughout the book K always denotes a nonarchimedean field with absolute value $|\ |$.

Lemma 1.2:

i. $o := \{a \in K : |a| \leq 1\}$ *is an integral domain with quotient field* K*;*

ii. $\mathfrak{m} := \{a \in K : |a| < 1\}$ *is the unique maximal ideal of* o*;*

iii. $o^\times = o \backslash \mathfrak{m}$*;*

iv. every finitely generated ideal in o *is principal.*

Proof: The assertions i.-iii. again are simple consequences of the strict triangle inequality. For iv. consider an ideal $\mathfrak{a} \subseteq o$ generated by the finitely many elements $a_1, \ldots, a_m$. Among the generators choose one, say a, of maximal absolute value. Then $\mathfrak{a} = oa$.

The ring o is a valuation ring and is called the *ring of integers* of K. The field $o/\mathfrak{m}$ is called the *residue class field* of K.

The reader should convince himself that, for any $a \in o$ and any $\epsilon \leq 1$, the ball $B_\epsilon(a)$ is an additive coset $a + \mathfrak{b}$ for an appropriate ideal $\mathfrak{b} \subseteq o$.

Examples:

1) The completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the p-adic absolute value $|a|_p := p^{-r}$ if $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p. The field $\mathbb{Q}_p$ is locally compact.

2) The p-adic absolute value $|\ |_p$ extends uniquely to any given finite field extension K of $\mathbb{Q}_p$. (Remember that there are plenty of such extensions since the algebraic closure $\overline{\mathbb{Q}}_p$ is not finite over $\mathbb{Q}$.) Any such K again is locally compact.

3) The completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$. This field is not locally compact since the set of absolute values $|\mathbb{C}_p|$ is dense in $\mathbb{R}_+$ (though countable).

4) The field of formal Laurent series $\mathbb{C}\{\{T\}\}$ in one variable over $\mathbb{C}$ with the absolute value $|\sum_{n\in\mathbb{Z}} a_n T^n| := e^{-\min\{n : a_n \neq 0\}}$. The ring of integers of this field is the ring of formal power series $\mathbb{C}[[T]]$ over $\mathbb{C}$. Since $\mathbb{C}[[T]]$ is the infinite disjoint union of the open subsets $a + T \cdot \mathbb{C}[[T]]$ with a running over the complex numbers the field $\mathbb{C}\{\{T\}\}$ is not locally compact.

The above examples show that the topological properties of the field K can be quite different. It therefore may come as no surprise that there is in fact a stronger notion of completeness. To explain this we consider any decreasing sequence $B_1 \supseteq B_2 \supseteq \ldots$ of balls in K. If K happens to be locally compact then the intersection $\bigcap_{n\in\mathbb{N}} B_n$, of course, is nonempty. For a general field K there is the following additional condition which ensures the same. For any nonempty subset $A \subseteq K$ call $d(A) := \sup\{|a-b| : a, b \in A\}$ the diameter of A. If we require in addition that the diameters $d(B_n)$ converge to zero if n goes to infinity then choosing points $a_n \in B_n$ we obtain a Cauchy sequence $(a_n)_n$ which has to converge and whose limit has to lie in the intersection $\bigcap_{n\in\mathbb{N}} B_n$. But in general, without any further condition, this intersection $\bigcap_{n\in\mathbb{N}} B_n$ indeed can be empty as the following construction shows.

Let the field be $K = \mathbb{C}_p$. Fix any sequence $(a_n)_n$ in $\mathbb{C}_p$ which as a subset is dense in $\mathbb{C}_p$ (e.g., one can take the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}_p$ written as a sequence). In addition fix a sequence $(\epsilon_n)_n$ of real numbers such that $1 > \epsilon_1 > \epsilon_2 > \ldots > \frac{1}{2}$. Consider now the equivalence relation on $\mathbb{C}_p$ defined by $a \underset{1}{\sim} b$ if $|b-a| \leq \epsilon_1$. The equivalence classes clearly are balls. Since the value group $|\mathbb{C}_p^\times| = p^{\mathbb{Q}}$ is dense in $\mathbb{R}_+^\times$ their diameter is ϵ_1. Moreover there certainly is more than one equivalence class. In particular, we may fix an equivalence class B_1 such that $a_1 \notin B_1$. Repeating this procedure with the equivalence relation on B_1 defined by $a \underset{2}{\sim} b$ if $|b-a| \leq \epsilon_2$ we find a ball $B_2 \subseteq B_1$ of diameter ϵ_2 such that $a_2 \notin B_2$. Continuing with this construction we inductively obtain a decreasing sequence of balls $B_1 \supseteq B_2 \supseteq \ldots$ in $\mathbb{C}_p$ such that

$$d(B_n) = \epsilon_n \quad \text{and} \quad a_n \notin B_n$$

for every $n \in \mathbb{N}$. We claim that the intersection $\bigcap_{n\in\mathbb{N}} B_n$ is empty. Otherwise let $b \in \bigcap_{n\in\mathbb{N}} B_n$. We then have $B_n = B_{\epsilon_n}(b)$ for any $n \in \mathbb{N}$ and hence $B_{1/2}(b) \subseteq \bigcap_{n\in\mathbb{N}} B_n$. As a consequence none of the a_n can be contained in the nonempty open subset $B_{1/2}(b)$. This contradicts the density of the sequence $(a_n)_n$.

Definition:

The field K is called <u>*spherically complete*</u> *if for any decreasing sequence of balls $B_1 \supseteq B_2 \supseteq \ldots$ in K the intersection $\bigcap_{n\in\mathbb{N}} B_n$ is nonempty.*

Any finite extension K of $\mathbb{Q}_p$ is locally compact and hence spherically complete. On the other hand the field $\mathbb{C}_p$, by the above discussion, is not spherically complete.

Lemma 1.3:

Let K be spherically complete and let $(B_i)_{i\in I}$ be any family of balls in K such that $B_i \cap B_j \neq \emptyset$ for any two $i, j \in I$; then $\bigcap_{i\in I} B_i \neq \emptyset$.

Proof: Choose a sequence $(i_n)_{n\in\mathbb{N}}$ of indices in I such that $d(B_{i_1}) \geq d(B_{i_2}) \geq \ldots$ and such that for every index $i \in I$ there is a natural number n such that $d(B_i) \geq d(B_{i_n})$. It follows from Lemma 1.1.iii that then $B_{i_1} \supseteq B_{i_2} \supseteq \ldots$ and that for any $i \in I$ there is an $n \in \mathbb{N}$ such that $B_i \supseteq B_{i_n}$. Hence $\bigcap_{i\in I} B_i = \bigcap_{n\in\mathbb{N}} B_{i_n}$ is nonempty.

Another important class of nonarchimedean fields is formed by those for which the value group $|K^\times|$ is a discrete subset of $\mathbb{R}_+^\times$. These fields are called *discretely valued*. Examples of discretely valued fields K are finite extensions of $\mathbb{Q}_p$ and the field of Laurent series $\mathbb{C}\{\{T\}\}$. The field $\mathbb{C}_p$ on the other hand is not discretely valued.

Lemma 1.4:

The subgroup $|K^\times| \subseteq \mathbb{R}_+^\times$ either is dense or is discrete; in the latter case there is a real number $0 < r < 1$ such that $|K^\times| = r^{\mathbb{Z}}$.

Proof: Let us assume that $|K^\times|$ is not dense in $\mathbb{R}_+^\times$. Then $\log|K^\times|$ is not dense in $\mathbb{R}$. Set $\rho := \sup(\log|K^\times| \cap (-\infty, 0))$. We claim that ρ actually is the maximum of this set. Otherwise there is a sequence $\rho_1 < \rho_2 < \ldots$ in $\log|K^\times|$

which converges to ρ. But then $\rho_i - \rho_{i+1}$ is a sequence in $\log|K^\times| \cap (-\infty, 0)$ converging to zero which implies that $\rho = 0$. In this case we find for any $\epsilon > 0$ a $\sigma \in \log|K^\times|$ such that $-\epsilon < \sigma < 0$. Consider now an arbitrary $\tau \in \mathbb{R}$ and choose an integer $m \in \mathbb{Z}$ such that $m\sigma \leq \tau < (m-1)\sigma$. It follows that $0 \leq \tau - m\sigma < -\sigma < \epsilon$ and hence that $\log|K^\times|$ is dense in $\mathbb{R}$ which is a contradiction. This establishes the existence of this maximum and consequently also the existence of $r := \max(|K^\times| \cap (0,1))$. Given any $s \in |K^\times|$ there is an $m \in \mathbb{Z}$ such that $r^{m+1} < s \leq r^m$. We then have $r < s/r^m \leq 1$ which, by the maximality of r, implies that $s = r^m$. This shows that $|K^\times| = r^{\mathbb{Z}}$.

Lemma 1.5:

The ring of integers o of a discretely valued field K is a principal ideal domain.

Proof: Let $\mathfrak{a} \subseteq o$ be an ideal. By the discreteness we find an $a \in \mathfrak{a}$ such that $|a| = \max\{|b| : b \in \mathfrak{a}\}$. Then $\mathfrak{a} = ao$.

Lemma 1.6:

Any discretely valued field K is spherically complete.

Proof: Let $B_1 \supseteq B_2 \supseteq \ldots$ be any decreasing sequence of balls in K. Then $d(B_n)$ is a decreasing sequence of numbers in $|K^\times|$ which, by the discreteness, either becomes constant (so that the intersection $\bigcap_{n \in \mathbb{N}} B_n$ even contains a ball) or converges to zero (so that we know from our initial discussion that the intersection $\bigcap_{n \in \mathbb{N}} B_n$ is nonempty).

§2 Seminorms

Let V be a K-vector space throughout this section. A (nonarchimedean) seminorm q on V is a function $q : V \longrightarrow \mathbb{R}$ such that

(i) $q(av) = |a| \cdot q(v)$ for any $a \in K$ and $v \in V$,

(ii) $q(v + w) \leq \max(q(v), q(w))$ for any $v, w \in V$.

Since in the following exclusively nonarchimedean seminorms will appear we simply speak of seminorms. Note that as an immediate consequence of (i) and (ii) one has:

- $q(0) = |0| \cdot q(0) = 0$,

- $q(v) = \max(q(v), q(-v)) \geq q(v - v) = q(0) = 0$ for any $v \in V$,

- $|q(v) - q(w)|_\infty \leq q(v - w)$ for any $v, w \in V$.

Moreover, with the same proof as before, one has

- $q(v + w) = \max(q(v), q(w))$ for any $v, w \in V$ such that $q(v) \neq q(w)$.

The vector space V in particular is an o-module so that we can speak about o-submodules of V.

Definition:

A subset $A \subseteq V$ is called <u>*convex*</u> *if either A is empty or is of the form $A = v + A_0$ for some vector $v \in V$ and some o-submodule $A_0 \subseteq V$.*

Note that in the above definition the submodule A_0 is uniquely determined by the convex subset A. The following properties are immediately clear:

- If the convex subset A contains the zero vector then it is an o-submodule;

- if A is convex then $v + A$ and $b \cdot A$ are convex for any $v \in V$ and any $b \in K$;

- if A and B are convex then so, too, is $A + B = \{v + w : v \in A, w \in B\}$;

- the image as well as the preimage under a K-linear map of a convex subset again is convex.

Lemma 2.1:

Let $(A_i)_{i \in I}$ be a family of convex subsets in V; we then have:

i. the intersection $\bigcap_{i \in I} A_i$ is convex;

ii. if for any two $i, j \in I$ there is a third $k \in I$ such that $A_i \cup A_j \subseteq A_k$ then the union $\bigcup_{i \in I} A_i$ is convex.

Proof: i. We only need to consider the case where the intersection is nonempty. Fix a vector $v \in \bigcap_{i \in I} A_i$. Then $A_i = v + B_i$, for any $i \in I$, with some o-submodule $B_i \subseteq V$. We therefore see that $\bigcap_{i \in I} A_i = v + \bigcap_{i \in I} B_i$ is convex.

ii. Similarly we may assume that there is a vector $v \in \bigcup_{i \in I} A_i$. Put $J := \{i \in I : v \in A_i\}$. By the assumption we are making in the assertion we have $\bigcup_{i \in I} A_i = \bigcup_{i \in J} A_i$. For $i \in J$ we may write $A_i = v + B_i$ with some o-submodule $B_i \subseteq V$. It follows that $\bigcup_{i \in I} A_i = v + \bigcup_{i \in J} B_i$. But again as a consequence of our assumption $\bigcup_{i \in J} B_i$ is, indeed, an o-submodule.

Definition:

A <u>*lattice*</u> *L in V is an o-submodule which satisfies the condition that for any vector $v \in V$ there is a nonzero scalar $a \in K^\times$ such that $av \in L$.*

In fact, for a lattice $L \subseteq V$ the natural map

$$\begin{array}{ccc} K \underset{o}{\otimes} L & \xrightarrow{\cong} & V \\ a \otimes v & \longmapsto & av \end{array}$$

is a bijection. The surjectivity holds by definition. For the injectivity (which holds for any o-submodule $L \subseteq V$) consider any linear equation $\sum_{i=1}^{n} a_i v_i = 0$ with the vectors v_i lying in L. Choose $a \in K^\times$ and $b_i \in o$ such that $a_i = ab_i$ for any $1 \leq i \leq n$. In $K \otimes_o L$ we then obtain $\sum_i a_i \otimes v_i = \sum_i a \otimes b_i v = a \otimes (\sum_i b_i v) = a \otimes 0 = 0$.

On the other hand a lattice in our sense does not need to be free as an o-module. The preimage of a lattice under a K-linear map again is a lattice. As the following argument shows the intersection $L \cap L'$ of two lattices $L, L' \subseteq V$ again is a lattice. Let $v \in V$ and $a, a' \in K^\times$ such that $av \in L$ and $a'v \in L'$. If $a \notin o$ then $a^{-1} \in o$ and hence $v = (a^{-1})av \in a^{-1}L \subseteq L$. We therefore may assume that $a, a' \in o$. Then $aa'v \in L \cap L'$.

For any lattice $L \subseteq V$ we define its *gauge* p_L by

$$\begin{aligned} p_L : V &\longrightarrow \mathbb{R} \\ v &\longmapsto \inf_{v \in aL} |a| \ . \end{aligned}$$

We claim that p_L is a seminorm on V. First of all, for any $b \in K^\times$ and any $v \in V$, we compute

$$p_L(bv) = \inf_{bv \in aL} |a| = \inf_{v \in b^{-1}aL} |a| = \inf_{v \in aL} |ba| = |b| \cdot \inf_{v \in aL} |a| = |b| \cdot p_L(v) \ .$$

Secondly, the inequality $p_L(v + w) \leq \max(p_L(v), p_L(w))$ is an immediate consequence of the following observation: For $a, b \in K$ such that $|b| \leq |a|$ we have $aL + bL = aL$.

On the other hand for any given seminorm q on V we define the o-submodules

$$L(q) := \{v \in V : q(v) \leq 1\} \quad \text{and} \quad L^-(q) := \{v \in V : q(v) < 1\} \ .$$

We claim that $L^-(q) \subseteq L(q)$ are lattices in V. But, since we assumed the absolute value $|\ |$ to be non-trivial, we find an $a \in K^\times$ such that $|a^n|$ converges to zero if $n \in \mathbb{N}$ goes to infinity. This means that for any given vector $v \in V$ we find an $n \in \mathbb{N}$ such that $q(a^n v) = |a^n| \cdot q(v) < 1$.

Lemma 2.2:

i. For any lattice $L \subseteq V$ we have $L^-(p_L) \subseteq L \subseteq L(p_L)$;

ii. for any seminorm q on V we have $c_o \cdot p_{L(q)} \leq q \leq p_{L(q)}$ where $c_o := \sup_{|b|<1} |b|$.

Proof: i. By construction we have $p_L(v) \leq 1$ for $v \in L$. On the other hand, if $p_L(v) < 1$ then $v \in aL$ for some $a \in K$ of absolute value < 1; hence $v \in L$.

ii. Let $a \in K^\times$; then a vector v lies in $aL(q)$ if and only if $q(v) \leq |a|$. We therefore have

$$p_{L(q)}(v) = \inf_{q(v) \leq |a|} |a| \ .$$

This shows that $p_{L(q)} \geq q$. Moreover, if $|b| < 1$ then $|b| \cdot \inf_{q(v) \leq |a|} |a| < p_{L(q)}(v)$. It follows that there must be an $a \in K$ such that $q(v) \leq |a|$ and $|ba| < p_{L(q)}(v)$. The latter inequality means that $v \notin baL(q)$ and hence that $|b| \cdot |a| < q(v)$. We obtain $c_\mathrm{o} p_{L(q)}(v) \leq c_\mathrm{o}|a| \leq q(v)$.

§3 Normed vector spaces

In this section we study a particular class of seminorms on a K-vector space V.

Definition:

A seminorm q on V is called a <u>*norm*</u> *if*

(iii) $q(v) = 0$ *implies that* $v = 0$.

Moreover, a K-vector space equipped with a norm is called a <u>*normed*</u> *K-vector space.*

It is the usual convention to denote norms by $\| \;\|$ (and not by q). A normed vector space $(V, \| \;\|)$ will always be considered as a metric space with respect to the metric $d(v,w) := \|v-w\|$. It is therefore in particular a Hausdorff topological space. Extending the language of the first section we introduce the *closed balls* (or simply *balls*)

$$B_\epsilon(v) := \{w \in V : \|w - v\| \leq \epsilon\}$$

and the *open balls*

$$B_\epsilon^-(v) := \{w \in V : \|w - v\| < \epsilon\}$$

for any $v \in V$ and any $\epsilon > 0$. For a fixed v and varying ϵ each of them form a fundamental system of open neighbourhoods of v in V. It is immediately clear that $B_\epsilon(0)$ and $B_\epsilon^-(0)$ are lattices in V and that $B_\epsilon(v) = v + B_\epsilon(0)$ and $B_\epsilon^-(v) = v + B_\epsilon^-(0)$ are convex subsets. As in the first section one shows:

1) Addition and scalar multiplication in V as well as the norm on V are continuous maps. (The continuity of the former two maps is usually expressed by saying that a normed vector space is a topological vector space.)

2) (Open) balls are open and closed subsets.

3) If the intersection of two balls $B_\epsilon(v)$ and $B_\epsilon(w)$ is nonempty then $B_\epsilon(v) = B_\epsilon(w)$; a corresponding statement holds for open balls.

4) If B and B' are two (open) balls with nonempty intersection then $B \subseteq B'$ or $B' \subseteq B$.

Definition:

A normed K-vector space is called a K-Banach space if the corresponding metric space is complete.

Examples:

1) Any finite dimensional vector space K^n with the norm $\|(a_1, \ldots, a_n)\| := \max_{1 \leq i \leq n} |a_i|$ is a K-Banach space.

2) Let X be any set; then

$$\ell^\infty(X) := \text{all bounded functions } \phi : X \to K$$

with pointwise addition and scalar multiplication and the norm

$$\|\phi\|_\infty := \sup_{x \in X} |\phi(x)|$$

is a K-Banach space. The following vector subspaces are closed and therefore Banach spaces in their own right:

- $c_o(X) := \{\phi \in \ell^\infty(X) : \text{for any } \epsilon > 0 \text{ there are at most finitely many } x \in X \text{ such that } |\phi(x)| \geq \epsilon\}$ (e.g., $c_o(\mathbb{N})$ is the space of all zero sequences in K);

- $BC(X) := \{\phi \in \ell^\infty(X) : \phi \text{ is continuous}\}$ provided X is a topological space.

3) Let X be any locally compact topological space; then

$$C_c(X) := \{\phi \in BC(X) : \phi \text{ has compact support}\}$$

is a vector subspace of $\ell^\infty(X)$ which in general is not closed (recall that the support of a function ϕ is the closure of the subset $\{x \in X : \phi(x) \neq 0\}$). Hence $C_c(X)$ is a normed vector space but in general not a Banach space.

Let $(V, \| \ \|)$ and $(W, \| \ \|)$ be two normed vector spaces. The continuous linear operators form a vector subspace

$$\mathcal{L}(V, W) := \{f \in \mathrm{Hom}_K(V, W) : f \text{ is continuous}\}$$

of $\mathrm{Hom}_K(V, W)$.

Proposition 3.1:

For a K-linear map $f : V \longrightarrow W$ the following assertions are equivalent:

i. f is continuous;

ii. there is a real number $c \geq 0$ such that $\|f(v)\| \leq c \cdot \|v\|$ for any $v \in V$.

Proof: Let us first assume that the second assertion holds true. Consider an arbitrary sequence $(v_n)_{n\in\mathbb{N}}$ in V converging to some vector $v \in V$. Then the sequence $v_n - v$ converges to the zero vector and hence the norms $\|v_n - v\|$ converge to zero. It follows from our assumption that the norms $\|f(v_n) - f(v)\|$ converge to zero as well. This implies that the sequence $f(v_n)$ converges to $f(v)$ and shows that f is continuous.

We now assume vice versa that f is continuous. There is then a $0 < \epsilon < 1$ such that $f^{-1}(B_1(0)) \supseteq B_\epsilon(0)$. Since the absolute value $|\ |$ is non-trivial we may assume ϵ to be of the form $\epsilon = |a|$ for some $a \in K$. This means that $\|f(v)\| \leq 1$ provided $\|v\| \leq |a|$. Let now v be an arbitrary nonzero vector in V and choose an integer $m \in \mathbb{Z}$ such that $|a|^{m+2} < \|v\| \leq |a|^{m+1}$. We compute

$$\|f(v)\| = |a|^m \cdot \|f(a^{-m}v)\| \leq |a|^m < |a|^{-2} \cdot \|v\| \ .$$

Corollary 3.2:

$\mathcal{L}(V,W)$ is a normed K-vector space with respect to the norm

$$\|f\| := \sup\{\frac{\|f(v)\|}{\|v\|} : v \in V\backslash\{0\}\} = \sup\{\frac{\|f(v)\|}{\|v\|} : v \in V \text{ such that } 0 < \|v\| \leq 1\}.$$

The above norm on $\mathcal{L}(V,W)$ is called the *operator norm*. We warn the reader that since the set of values $\|V\|$ may be different from the set of absolute values $|K|$ we in general have

$$\|f(v)\| \neq \sup\{\|f(v)\| : v \in V \text{ such that } \|v\| = 1\}.$$

Proposition 3.3:

If W is a Banach space so, too, is $\mathcal{L}(V,W)$.

Proof: Let $(f_n)_{n\in\mathbb{N}}$ be any Cauchy sequence in $\mathcal{L}(V,W)$. Then, in particular, $\|f_n\|$ is a Cauchy sequence in $\mathbb{R}$ so that the limit $\lim\limits_{n\to\infty} \|f_n\|$ exists. Moreover, because of

$$\|f_{n+1}(v) - f_n(v)\| = \|(f_{n+1} - f_n)(v)\| \leq \|f_{n+1} - f_n\| \cdot \|v\|$$

$f_n(v)$, for any $v \in V$, is a Cauchy sequence in W. By assumption the limit $f(v) := \lim\limits_{n\to\infty} f_n(v)$ exists in W. It is obvious that

$$f(av) = af(v) \quad \text{for any } a \in K \ .$$

For $v, v' \in V$ we compute

$$\begin{aligned} f(v) + f(v') &= \lim_{n\to\infty} f_n(v) + \lim_{n\to\infty} f_n(v') = \lim_{n\to\infty} (f_n(v) + f_n(v')) \\ &= \lim_{n\to\infty} f_n(v+v') \\ &= f(v+v') \ . \end{aligned}$$

This means that $v \longmapsto f(v)$ is a K-linear map which we denote by f. Since

$$\|f(v)\| = \lim_{n\to\infty} \|f_n(v)\| \le (\lim_{n\to\infty} \|f_n\|) \cdot \|v\|$$

it follows from Prop. 3.1 that f is continuous, i.e., that $f \in \mathcal{L}(V, W)$. Finally the inequality

$$\begin{aligned} \|f - f_n\| &= \sup\{\tfrac{\|(f-f_n)(v)\|}{\|v\|}\} = \sup\{\tfrac{\lim\limits_{m\to\infty} \|f_m(v) - f_n(v)\|}{\|v\|}\} \\ &\le \sup_{m\ge n} \|f_{m+1} - f_m\| \end{aligned}$$

shows that f indeed is the limit of the sequence $(f_n)_n$ in $\mathcal{L}(V, W)$.

Corollary 3.4:

$V' := \mathcal{L}(V, K)$ is a Banach space.

Definition:

V' is called the <u>*dual*</u> *Banach space to V.*

We list two further simple properties:

1) The linear map

$$\begin{array}{ccc} \mathcal{L}(V, W) & \longrightarrow & \mathcal{L}(W', V') \\ f & \longmapsto & f'(\ell) := \ell \circ f \end{array}$$

is continuous satisfying $\|f'\| \le \|f\|$ (observe that $\|\ell \circ f\| \le \|\ell\| \cdot \|f\|$).

2) The linear map

$$\begin{array}{ccc} V & \longrightarrow & V'' \\ v & \longmapsto & \delta_v(\ell) := \ell(v) \end{array}$$

is continuous satisfying $\|\delta_v\| \le \|v\|$.

It unfortunately turns out that in the nonarchimedean world there are nonzero normed vector spaces whose dual Banach space is zero, i.e., which do not possess a single nonzero continuous linear form. But we will see later on that this phenomenon does not occur if the field K is spherically complete. Meanwhile we want to compute the dual Banach space in one basic case.

Example:

Let X be an arbitrary set and, for any $x \in X$, let 1_x denote the function on X defined by $1_x(y) = 1$ if $y = x$ and $= 0$ otherwise; then the map

$$\begin{array}{ccc} c_{\mathrm{o}}(X)' & \xrightarrow{\cong} & \ell^\infty(X) \\ \ell & \longmapsto & \phi_\ell(x) := \ell(1_x) \end{array}$$

is an isometric linear isomorphism.

We first have to discuss the notion of *summability* in K before we can establish this isometry. It follows from the strict triangle inequality that a series $\sum_{n=1}^{\infty} a_n$ in K converges if and only if $(a_n)_n$ is a zero sequence; moreover, in this case one has $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}$ for any permutation σ of $\mathbb{N}$. Let now $\phi \in c_{\mathrm{o}}(X)$ be an arbitrary function. We claim that $\phi(x) \neq 0$ for at most countably many $x \in X$. By assumption the set $X_m := \{x \in X : |\phi(x)| \geq \frac{1}{m}\}$ is finite for any $m \in \mathbb{N}$. Hence the union $\bigcup_m X_m = \{x \in X : |\phi(x)| > 0\} = \{x \in X : \phi(x) \neq 0\}$ must be countable. Having established this claim we find an injective map $\iota : \mathbb{N} \longrightarrow X$ such that

- $\phi(x) = 0$ for $x \notin \iota(\mathbb{N})$, and

- $(\phi(\iota(n))_{n\in\mathbb{N}}$ is a zero sequence.

We obtain that

$$\sum_{x\in X} \phi(x) := \sum_{n=1}^{\infty} \phi(\iota(n))$$

is a well defined element in K which does not depend on the choice of the map ι.

Let us now look at the above example. The image map ϕ_ℓ indeed lies in $\ell^\infty(X)$ since

$$\|\phi_\ell\|_\infty = \sup_{x\in X} |\phi_\ell(x)| = \sup_{x\in X} |\ell(1_x)| \leq \|\ell\| \ .$$

It is clear that the map $\ell \longmapsto \phi_\ell$ is linear. To see that it is isometric we need to show the opposite inequality $\|\ell\| \leq \|\phi_\ell\|_\infty$. Fix a nonzero function $\psi \in c_{\mathrm{o}}(X)$ and choose an injective map $\iota : \mathbb{N} \longrightarrow X$ such that $\psi(x) = 0$ for $x \notin \iota(\mathbb{N})$ and $(\psi(\iota(n))_{n\in\mathbb{N}}$ is a zero sequence. We then have the convergent series $\psi = \sum_{n=1}^{\infty} \psi(\iota(n)) \cdot 1_{\iota(n)}$ in $c_o(X)$. Applying the continuous linear form ℓ gives $\ell(\psi) = \sum_{n=1}^{\infty} \psi(\iota(n)) \cdot \ell(1_{\iota(n)})$ and hence

$$\frac{|\ell(\psi)|}{\|\psi\|} \leq \frac{\sup_{x\in X}|\psi(x) \cdot \ell(1_x)|}{\|\psi\|} \leq \sup_{x\in X} |\ell(1_x)| = \sup_{x\in X} |\phi_\ell(x)| = \|\phi_\ell\|_\infty \ .$$

Since ψ was arbitrary it follows that $\|\ell\| \leq \|\phi_\ell\|_\infty$. This shows that our map is isometric and a fortiori injective. For the surjectivity take any $\phi \in \ell^\infty(X)$. We define a linear form $\ell \in c_o(X)'$ through

$$\ell(\psi) := \sum_{x\in X} \psi(x)\phi(x) \ .$$

Its continuity is a consequence of the inequality $|\sum_{x\in X} \psi(x)\phi(x)| \leq \|\psi\|_\infty \cdot \|\phi\|_\infty$. We obviously have $\phi_\ell = \phi$.

§4 Locally convex vector spaces

Let $(L_j)_{j\in J}$ be a nonempty family of lattices in the K-vector space V such that we have

(lc1) for any $j \in J$ and any $a \in K^\times$ there exists a $k \in J$ such that $L_k \subseteq aL_j$, and

(lc2) for any two $i, j \in J$ there exists a $k \in J$ such that $L_k \subseteq L_i \cap L_j$.

The second condition implies that the intersection of two convex subsets $v + L_i$ and $v' + L_j$ either is empty or contains a convex subset of the form $w + L_k$. This means that the convex subsets $v + L_j$ for $v \in V$ and $j \in J$ form the basis of a topology on V which will be called the *locally convex topology on V defined by the family* (L_j). For any vector $v \in V$ the convex subsets $v + L_j$, for $j \in J$, form a fundamental system of open and closed neighbourhoods of v in this topology.

Definition:

A locally convex K-vector space is a K-vector space equipped with a locally convex topology.

Lemma 4.1:

If V is locally convex then addition $V \times V \xrightarrow{+} V$ and scalar multiplication $K \times V \xrightarrow{\cdot} V$ are continuous maps.

Proof: For the continuity of the addition we only need to observe that $(v + L_j) + (w + L_j) \subseteq (v + w) + L_j$. For the continuity of the scalar multiplication consider arbitrary elements $a \in K$, $v \in V$, and $j \in J$. Since L_j is a lattice we find a scalar $b \in K^\times$ such that $bv \in L_j$, and by (lc1) and (lc2) we find a $k \in J$ such that $aL_k + bL_k \subseteq L_j$. We then have $(a + bo) \cdot (v + L_k) \subseteq av + L_j$.

Since on a nonzero K-vector space the scalar multiplication cannot be continuous for the discrete topology we see that the discrete topology is not locally convex. In the following we want to discuss an alternative way to describe locally convex topologies with the help of seminorms.

Let $(q_i)_{i \in I}$ be a family of seminorms on the K-vector space V. The topology on V defined by this family $(q_i)_{i \in I}$, by definition, is the coarsest topology on V such that

- all $q_i : V \longrightarrow \mathbb{R}$, for $i \in I$, are continuous, and

- all translation maps $v + . : V \longrightarrow V$, for $v \in V$, are continuous.

For any finitely many norms $q_{i_1}, \ldots, q_{i_r}$ in the given family and any real number $\epsilon > 0$ we set

$$V(q_{i_1}, \ldots, q_{i_r}; \epsilon) := \{v \in V : q_{i_1}, \ldots, q_{i_r}(v) \leq \epsilon\} \ .$$

Lemma 4.2:

$V(q_{i_1}, \ldots, q_{i_r}; \epsilon)$ is a lattice in V.

Proof: Since $V(q_{i_1}, \ldots, q_{i_r}; \epsilon) = V(q_{i_1}; \epsilon) \cap \ldots \cap V(q_{i_r}; \epsilon)$ and since the intersection of two lattices again is a lattice it suffices to consider a single $V(q_i; \epsilon)$. It is obviously an o-submodule. Choose an $a \in K^\times$ such that $|a| \leq \epsilon$. Then $V(q_i; \epsilon)$ contains the lattice $aL(q_i)$ and therefore must also be a lattice.

Clearly the family of lattices $V(q_{i_1}, \ldots, q_{i_r}; \epsilon)$ in V has the properties (lc1) and (lc2) and hence defines a locally convex topology on V.

Proposition 4.3:

The topology on V defined by the family of seminorms $(q_i)_{i \in I}$ coincides with the locally convex topology defined by the family of lattices $\{V(q_{i_1}, \ldots, q_{i_r}; \epsilon) : i_1, \ldots, i_r \in I, \epsilon > 0\}$.

Proof: Let $\mathcal{T}$, resp. $\mathcal{T}'$, denote the topology defined by the seminorms, resp. by the lattices. By the defining properties for $\mathcal{T}$ all the convex sets $v + V(q_{i_1}, \ldots, q_{i_r}; \epsilon)$ are open for $\mathcal{T}$. This means that $\mathcal{T}$ is finer than $\mathcal{T}'$. To obtain the equality of the two topologies it remains to show that $\mathcal{T}'$ satisfies the two defining properties for $\mathcal{T}$. The translation maps are continuous in $\mathcal{T}'$ by Lemma 4.1. To check the continuity of the seminorm q_i in $\mathcal{T}'$ let $(\alpha, \beta) \subseteq \mathbb{R}$ be an open interval and $v_0 \in q_i^{-1}(\alpha, \beta)$ be a vector. If $q_i(v_0) > 0$ we choose a $0 < \epsilon < q_i(v_0)$. Because of $q_i(v_0 + v) = q_i(v_0)$ for $v \in V(q_i; \epsilon)$ we then have $v_0 + V(q_i; \epsilon) \subseteq q_i^{-1}(\alpha, \beta)$. If, on the other hand, $q_i(v_0) = 0$ then we choose a $0 < \epsilon < \beta$ and obtain $\alpha < 0 \leq q_i(v_0 + v) \leq q_i(v) \leq \epsilon < \beta$ for $v \in V(q_i; \epsilon)$ which again means that $v_0 + V(q_i; \epsilon) \subseteq q_i^{-1}(\alpha, \beta)$.

We see in particular that the normed vector spaces of the previous section are locally convex. The above result has the following converse.

Proposition 4.4:

A locally convex topology on V defined by the family of lattices $(L_j)_{j \in J}$ can also be defined by the family of gauges $(p_{L_j})_{j \in J}$.

Proof: Let $\mathcal{T}'$, resp. $\mathcal{T}$, denote the topology defined by the lattices L_j, resp. by the seminorms $p_j := p_{L_j}$. Given an $\epsilon > 0$ we fix an $a \in K^\times$ such that $|a| \leq \epsilon$. It follows from Lemma 2.2.i that $aL_j \subseteq V(p_j; \epsilon)$. Using the condition (lc2) we deduce that $V(p_j; \epsilon)$ is open for $\mathcal{T}'$. This implies, by Prop. 4.3, that $\mathcal{T} \subseteq \mathcal{T}'$. For the converse we fix a $b \in K$ such that $0 < |b| < 1$. Again from Lemma 2.2.i we obtain that $V(p_j; |b|) \subseteq L_j$ which means that L_j is open for $\mathcal{T}$ and hence that $\mathcal{T}' \subseteq \mathcal{T}$.

These two results together show that the concept of a locally convex topology is the same as the concept of a topology defined by a family of seminorms. We finish this discussion with several useful observations belonging to this context. For the rest of this section we let V be a locally convex K-vector space.

Lemma 4.5:

Let L be a lattice in V and q be a seminorm on V; we then have:

i. The seminorm q is continuous if and only if the lattice $L^-(q)$, or equivalently the lattice $L(q)$, is open in V;

ii. the lattice L is open in V if and only if its gauge p_L is continuous.

Proof: i. Being the preimage under q of an open subset in $\mathbb{R}_{\geq 0}$ the lattice $L^-(q)$ is open if q is continuous. Furthermore, $L(q)$ being a union of additive translates of $L^-(q)$ is open as soon as $L^-(q)$ is open. Assuming finally that $L(q)$ is open let $(\alpha, \beta) \subseteq \mathbb{R}$ be an open interval and $v_0 \in q^{-1}(\alpha, \beta)$ be a vector. At the end of the proof of Prop. 4.3 we have seen that there is then an $a \in K^\times$ such that $q^{-1}(\alpha, \beta)$ contains the open neighbourhood $v_0 + aL(q)$ of v_0. This means that q is continuous.

ii. As we have just seen if p_L is continuous then $L^-(p_L)$ is open. But $L^-(p_L) \subseteq L$ by Lemma 2.2.i so that L is open as well. If on the other hand L is open then, again by Lemma 2.2.i, the lattice $L(p_L)$ also is open. By assertion i., this amounts to the continuity of p_L.

Lemma 4.6:

Assume that the topology of V is defined by the family of lattices $(L_j)_{j\in J}$, resp. by the family of seminorms $(q_i)_{i\in I}$; then the closure of $\{0\}$ in V is the o-module $\bigcap_{j\in J} L_j = \bigcap_{i\in I} q_i^{-1}(0)$; in particular, the following assertions are equivalent:

i. V is Hausdorff;

ii. for any nonzero vector $v \in V$ there is a $j \in J$ such that $v \notin L_j$;

iii. for any nonzero vector $v \in V$ there is an $i \in I$ such that $q_i(v) \neq 0$.

Proof: The lattices L_j are open and therefore closed. Hence we have $\overline{\{0\}} \subseteq \bigcap_j L_j$. If the vector v is not in $\overline{\{0\}}$ we find a lattice L_k such that $0 \notin v + L_k$ and consequently $v \notin L_k$. This gives the equality $\overline{\{0\}} = \bigcap_j L_j$. On the other hand, by Prop. 4.3 the lattices $V(q_{i_1}, \ldots, q_{i_r}; \epsilon)$ form a fundamental system of neighbourhoods of the zero vector. It follows that $\bigcap_i q_i^{-1}(0) = \bigcap_{i,\epsilon} V(q_{i_1}, \ldots, q_{i_r}; \epsilon) = \bigcap_j L_j$. For the equivalence of the assertions i.-iii. one only has to observe in addition that as a consequence of the translation invariance of any locally convex topology (Lemma 4.1) V is Hausdorff if and only if any nonzero vector can be separated from the zero vector.

Remark 4.7:

Assume that the topology of V is defined by the family of seminorms $(q_i)_{i\in I}$. For any finite subset $F \subseteq I$ we may form the continuous seminorm $q_F := \max_{i\in F} q_i$. The family $(q_F)_F$ is a defining family for the topology on V which has the additional property that the convex subsets $v + V(q_F; \epsilon)$ form a basis of the topology.

Next we want to investigate the topological properties of convex subsets in V.

Lemma 4.8:

Let $A \subseteq V$ be a convex subset; we then have:

i. The closure $\overline{A}$ of A is convex;

ii. if A is not open then its interior is empty;

iii. if A is open then it is also closed;

iv. if A is an open neighbourhood of the zero vector then A is a lattice.

Proof: We may assume that A is nonempty. By a translation we are furthermore reduced to the case that A is an o-submodule. As a consequence of Lemma 4.1 the closure $\overline{A}$ also is an o-submodule and hence is convex. If A is open then, by the definition of locally convex topologies, it must contain an open lattice and therefore is a lattice as well; being the complement of a union of additive cosets of A it is closed. Finally, if v is a vector in the interior of A then A must contain $v + L$ for some open lattice $L \subseteq V$; being an o-module it then also contains L and therefore has to be open.

The *convex hull* of a subset $S \subseteq V$ is defined to be

$$\mathrm{Co}(S) := \bigcap \{S \subseteq A \subseteq V : A \text{ is convex}\} \ .$$

By Lemma 2.1.i this is the smallest convex subset of V which contains S. Because of Lemma 4.8.i we have

$$\overline{\mathrm{Co}(S)} = \overline{\mathrm{Co}(\overline{S})} \ .$$

Lemma 4.9:

For any subset $S \subseteq V$ we have:

i. If S is open then its convex hull $\mathrm{Co}(S)$ *is open;*

ii. $\overline{\mathrm{Co}(S)} = \bigcap \{S \subseteq A \subseteq V : A \text{ is convex and closed}\}$.

Proof: i. If S is empty then $\mathrm{Co}(S)$ is empty. Otherwise we may assume, by translation, that S contains the zero vector so that $\mathrm{Co}(S)$ is an o-submodule. Since S and hence $\mathrm{Co}(S)$ then contain an open lattice $\mathrm{Co}(S)$ is open. ii. It follows from Lemma 4.8.i that $\overline{\mathrm{Co}(S)}$ is convex and closed.

In a metric space and hence in a normed vector space it is clear what is meant by a bounded subset. It is of utmost importance that this concept can also be introduced in locally convex vector spaces.

Definition:

A subset $B \subseteq V$ is called <u>bounded</u> *if for any open lattice $L \subseteq V$ there is an $a \in K$ such that $B \subseteq aL$.*

It is almost immediate that any finite set is bounded, and that any finite union of bounded subsets is bounded. We leave it to the reader to check the following: Assume that the topology on V is defined by the family of seminorms $(q_i)_{i \in I}$. Then a subset $B \subseteq V$ is bounded if and only if $\sup\limits_{v \in B} q_i(v) < \infty$ for any $i \in I$.

Lemma 4.10:

Let $B \subseteq V$ be a bounded subset; then the closure of the o-submodule of V generated by B and a fortiori the convex hull $\mathrm{Co}(B)$ *are bounded.*

Proof: Let $L \subseteq V$ be an open lattice and $a \in K$ such that $B \subseteq aL$. Since aL is a closed o-submodule it necessarily contains the closed o-submodule generated by B.

As a first application of this concept of boundedness we will derive an intrinsic characterization of those locally convex vector spaces which underlie a normed vector space.

Proposition 4.11:

The topology of V can be defined by a single seminorm if and only if there exists a bounded open lattice in V.

Proof: If q is a defining seminorm then $L(q)$ is a bounded open lattice. Let, on the other hand, L_o be a bounded open lattice and put $q := p_{L_o}$. According to Lemma 4.5.ii the gauge q is continuous so that the lattices $V(q;\epsilon)$ are open in V. Since L_o is bounded we find for any open lattice $L \subseteq V$ a scalar $a_L \in K^\times$ such that $L_o \subseteq a_L L$. Using Lemma 2.2.i it follows that $V(q; (|a_L|+1)^{-1}) \subseteq L$. This shows that q defines the topology of V.

Corollary 4.12:

Assume V to be Hausdorff; then the topology of V can be defined by a norm if and only if there is a bounded open lattice in V.

Proof: This follows from Lemma 4.6 and Prop. 4.11.

Proposition 4.13:

The only locally convex and Hausdorff topology on a finite dimensional vector space K^n is the one defined by the norm $\|(a_1,\ldots,a_n)\| := \max_{1\le i\le n} |a_i|$.

Proof: We will divide the argument into three steps. Step 1: The topology defined by the norm $\| \;\|$ is finer than any other locally convex topology on K^n. To see this let $e_1,\ldots,e_n$ denote the standard basis of K^n and let q be an arbitrary seminorm on K^n. We then have

$$q(v) \le (\max_{1\le i\le n} q(e_i)) \cdot \|v\|$$

for any $v \in V$ which amounts to our claim. Step 2: Any locally convex and Hausdorff topology on K^n can be weakened to a topology defined by a single norm p. Let $(q_i)_{i\in I}$ be a defining family of seminorms for the given topology. By Lemma 4.6 we have $\{0\} = \bigcap_{i\in I} q_i^{-1}(0)$. Since each $q_i^{-1}(0)$ is a vector subspace it follows from finite dimensionality that $\{0\} = q_{i_1}^{-1}(0)\cap\ldots\cap q_{i_r}^{-1}(0)$ for finitely many appropriate $i_1,\ldots,i_r \in I$. Then $p := \max(q_{i_1},\ldots,q_{i_r})$ is a norm with the required property. Step 3: It remains to show that, given an arbitrary norm p on K^n, the identity map $(K^n, p) \xrightarrow{\mathrm{id}} (K^n, \| \ \|)$ is continuous. According to Prop. 3.1 we have to find a $c > 0$ such that

$$\|v\| \leq c \cdot p(v) \qquad \text{for any } v \in K^n \ .$$

This will be achieved by induction with respect to n. The case $n = 1$ is obvious with $c := p(1)$. Applying the induction hypothesis to the restriction $p|K^{n-1}$ we obtain a constant $c_1 > 0$ such that $\|v\| \leq c_1 \cdot p(v)$ for any $v \in V := Ke_1 \oplus \ldots \oplus Ke_{n-1} \subseteq K^n$. With $(V, \| \ \|)$ also (V, p) is complete. It follows that V is closed in (K^n, p) which implies that

$$1 \leq c_2 := p(e_n) / \inf_{v \in V} p(e_n - v) < \infty \ .$$

We set

$$c := \max(c_1 c_2, c_2 / p(e_n)) > 0 \ .$$

Let now $w \in K^n$ be any vector and write $w = v + be_n$ with $v \in V$ and $b \in K$. Since $c > c_1$ it suffices to consider those w for which $b \neq 0$. In this case we compute

$$p(w) = |b| \cdot p(b^{-1}v + e_n) \geq |b| \cdot p(e_n) \cdot c_2^{-1} = p(be_n) \cdot c_2^{-1}$$

and hence

$$p(v) = p(w - be_n) \leq \max(p(w), p(be_n)) \leq c_2 \cdot p(w) \ .$$

Finally

$$\begin{aligned} \|w\| = \max(\|v\|, |b|) &\leq \max(c_1, p(e_n)^{-1}) \cdot \max(c_1^{-1}\|v\|, |b| p(e_n)) \\ &\leq c c_2^{-1} \cdot \max(p(v), p(be_n)) \leq c \cdot p(w) \ . \end{aligned}$$

§5 Constructions and examples

In this section we will discuss various general ways to construct locally convex vector spaces. Some of them will be illustrated by concrete examples.

A. Subspaces

Let V be a locally convex vector space and let $U \subseteq V$ be a vector subspace. Then the subspace topology of U induced by V is locally convex defined by all lattices $L \cap U$ where L runs over a defining family of lattices in V, or equivalently by all restrictions $q|U$ where q runs over a defining family of seminorms on V.

B. Quotient spaces

Let V be a locally convex vector space and let $U \subseteq V$ be a vector subspace. The quotient topology on V/U is locally convex defined by all lattices $L + U$ where L runs over a defining family of lattices in V. If one wants to describe the quotient topology in terms of seminorms then one has to be a little careful. We first recall that for any seminorm q on V one has the quotient seminorm

$$\overline{q}(v + U) := \inf_{u \in U} q(v + u) \ ;$$

it satisfies

$$L^-(q) + U = L^-(\overline{q}) \ .$$

Let now $(q_i)_{i \in I}$ be a defining family of seminorms for the topology of V. Using the above identity together with Remark 4.7 it follows that the quotient topology on V/U is defined by the family of quotient seminorms $(\overline{q_F})_F$ where F runs over the finite subsets of I.

C. The finest locally convex topology

If V is any K-vector space then the family of all lattices in V, or equivalently the family of all seminorms on V, defines a locally convex topology which obviously is the finest such topology on V. If V is equipped with the finest locally convex topology then any linear map from V into any other locally convex K-vector space is continuous. Moreover, any vector subspace $U \subseteq V$ is closed; in particular, V is Hausdorff. To see this choose vectors $(v_j)_{j \in J}$ such that $(v_j + U)_j$ is a basis of V/U; then U is the intersection of the lattices $L_n := \sum_j b^n o v_j + U$ where $b \in K$ is a fixed scalar such that $0 < |b| < 1$.

In particular, the uniquely determined locally convex and Hausdorff topology on a finite dimensional vector space (Prop. 4.13) has to be the finest locally convex one.

D. Initial topologies

Let V be a K-vector space. Assume we are given a family $(V_h)_{h \in H}$ of locally convex K-vector spaces together with linear maps $f_h : V \longrightarrow V_h$. The coarsest topology on V for which all the maps f_h are continuous is called the *initial* topology on V with respect to the family $(f_h)_h$. It is locally convex defined by all the lattices which are finite intersections of lattices in the family $(f_h^{-1}(L_{hj}))_{h,j}$ where $(L_{hj})_j$ is a defining family of lattices for the topology on V_h. Equivalently it is defined by the seminorms $(q_{hi} \circ f_h)_{h,i}$ where $(q_{hi})_i$ is a defining family of seminorms for V_h.

A special case of this construction is the following. Let $V = \prod_{h \in H} V_h$ be the direct product and let $f_h : V \longrightarrow V_h$ be the projection maps. The corresponding initial topology on V is called the *direct product* topology. We recall that in this situation V is Hausdorff if and only if all the V_h are Hausdorff.

Example 1:

Let $K^{\mathbb{N}} := \prod_{n \in \mathbb{N}} K$ be the countable direct product of one dimensional K-vector spaces. This is an example of a locally convex and Hausdorff vector space whose topology cannot be defined by a single norm. Otherwise, by Cor. 4.12, there should be a bounded open lattice $L \subseteq K^{\mathbb{N}}$. By the way the direct product topology is constructed we may assume that L is of the form $L = \prod_{n \in F} o \times \prod_{n \in \mathbb{N} \setminus F} K$ for some finite subset $F \subseteq \mathbb{N}$. But the absolute value on K viewed as a continuous seminorm on $K^{\mathbb{N}}$ via the projection to a factor corresponding to some $n \notin F$ is not bounded on L.

Example 2:

Put $X := \mathbb{C}_p \setminus \mathbb{Q}_p$ and let $\mathcal{O}^{\mathrm{alg}}(X)$ denote the $\mathbb{Q}_p$-vector space of all $\mathbb{Q}_p$-rational functions in one variable all of whose poles lie in $\mathbb{Q}_p$. We will construct a countable family of norms $\| \;\|_{1/n}$, for $n \in \mathbb{N}$, on $\mathcal{O}^{\mathrm{alg}}(X)$ in the following way. For any $n \in \mathbb{N}$ define

$$X(1/n) := \{x \in X : |x| \leq n \text{ and } |x - a| \geq 1/n \text{ for any } a \in \mathbb{Q}_p\} \ .$$

We leave it to the reader to check that these sets $X(1/n)$ are infinite.

Claim: $\|R\|_{1/n} := \sup\limits_{x \in X(1/n)} |R(x)| < \infty$ for any $R \in \mathcal{O}^{\mathrm{alg}}(X)$.

Proof: Write

$$R = \frac{a_0 + a_1 T + \ldots + a_d T^d}{\prod_{j=1}^{e} (T - b_j)} \qquad \text{with } a_i, b_j \in \mathbb{Q}_p \ .$$

We then have $\|\mathrm{numerator}(R)\|_{1/n} \leq n^d \cdot \max_i(|a_i|)$ and $\|\mathrm{denominator}(R)\|_{1/n} \geq (1/n)^e$ and hence $\|R\|_{1/n} \leq n^{d+e} \cdot \max_i(|a_i|)$.

This means that $\| \;\|_{1/n}$ is a norm on $\mathcal{O}^{\mathrm{alg}}(X)$. We define the $\mathbb{Q}_p$-Banach space $\mathcal{O}_{1/n}(X)$ to be the completion of the normed vector space $(\mathcal{O}^{\mathrm{alg}}(X), \| \;\|_{1/n})$. Because of the inclusions $X(1/n) \subseteq X(1/(n+1))$ the identity maps $(\mathcal{O}^{\mathrm{alg}}(X), \| \;\|_{1/(n+1)}) \longrightarrow (\mathcal{O}^{\mathrm{alg}}(X), \| \;\|_{1/n})$ are continuous and induce therefore continuous linear maps

$$\mathcal{O}_{1/(n+1)}(X) \longrightarrow \mathcal{O}_{1/n}(X) \ .$$

We define

$$\mathcal{O}(X) := \varprojlim_n \mathcal{O}_{1/n}(X)$$

to be the corresponding projective limit equipped with the initial topology with respect to the projection maps. The space $\mathcal{O}(X)$ can be viewed as a space of certain $\mathbb{C}_p$-valued functions on X as follows.

Claim: $X = \bigcup_{n\in\mathbb{N}} X(1/n)$.

Proof: Let $x \in X$ be point and choose $n_0 \in \mathbb{N}$ such that $|x| \leq n_0$. We consider the continuous function

$$\begin{array}{ccc} \mathbb{Q}_p & \longrightarrow & \mathbb{R}_{>0} \\ a & \longmapsto & |x-a| \ . \end{array}$$

We have $|x-a| = |a| > |x|$ if $|x| < |a|$. The subset $\{a \in \mathbb{Q}_p : |a| \leq |x|\}$ on the other hand is compact so that its image in $\mathbb{R}_{>0}$ under the above function is bounded. This function therefore is bounded below by $1/n_1$ for some $n_1 \in \mathbb{N}$. It follows that $x \in X(1/n)$ for $n := \max(n_0, n_1)$.

This result in particular means that, if $F(Y, \mathbb{C}_p)$ denotes the vector space of all $\mathbb{C}_p$-valued functions on a set Y, then we have $F(X, \mathbb{C}_p) = \varprojlim_n F(X(1/n), \mathbb{C}_p)$.

The inclusion

$$(\mathcal{O}^{\mathrm{alg}}(X), \| \ \|_{1/n}) \longrightarrow BC(X(1/n), \mathbb{C}_p)$$

is by definition an isometry. Since the space of bounded continuous $\mathbb{C}_p$-valued functions on the right hand side is a Banach space it extends to an isometry

$$\mathcal{O}_{1/n}(X) \longrightarrow BC(X(1/n), \mathbb{C}_p)$$

which in the limit gives rise to a $\mathbb{Q}_p$-linear embedding

$$\mathcal{O}(X) \longrightarrow \varprojlim_n BC(X(1/n), \mathbb{C}_p) \longrightarrow \varprojlim_n F(X(1/n), \mathbb{C}_p) = F(X, \mathbb{C}_p)$$

of $\mathcal{O}(X)$ into the space of all functions on X.

The space $X = \mathbb{C}_p \setminus \mathbb{Q}_p$ is called the p-adic upper half plane and the functions in $\mathcal{O}(X)$ are called the rigid analytic or holomorphic functions on X.

E. Locally convex final topologies

Again let V be a K-vector space and $(V_h)_{h\in H}$ be a family of locally convex K-vector spaces. But this time we assume given linear maps $f_h : V_h \longrightarrow V$. Then there is a unique finest locally convex topology on V for which all the maps f_h are continuous. It is called the *locally convex final* topology on V with respect to the family $(f_h)_h$, and it is defined by the family of all lattices $L \subseteq V$ such that $f_h^{-1}(L)$ is open in V_h for every $h \in H$. In general this topology is strictly coarser than the finest topology on V making all the f_h continuous.

Lemma 5.1:

Assume that V carries the locally convex final topology with respect to a family of linear maps $f_h : V_h \longrightarrow V$; we then have:

i. a K-linear map $f : V \longrightarrow W$ into some other locally convex K-vector space W is continuous if and only if all the maps $f \circ f_h : V_h \longrightarrow W$, for $h \in H$, are continuous;

ii. a seminorm q on V is continuous if and only if the seminorm $q \circ f_h$ on V_h is continuous for any $h \in H$;

iii. assume that the topology on V_h is defined by the family of lattices $(L_{hj})_{j \in J(h)}$ and that, in addition, we have $V = \sum_{h \in H} f_h(V_h)$; then the topology on V is defined by the family of lattices $\{\sum_{h \in H} f_h(L_{hj(h)}) : j(h) \in J(h)\}$.

Proof: i. The other implication being trivial we assume that the maps $f \circ f_h$ are continuous. To see that f is continuous it suffices to show that $f^{-1}(M)$ is open in V for any open lattice $M \subseteq W$. By assumption $(f \circ f_h)^{-1}(M)$ is open in V_h. Because of the obvious identity $(f \circ f_h)^{-1}(M) = f_h^{-1}(f^{-1}(M))$ this implies that $f^{-1}(M)$ is open.

ii. Again one implication being trivial we assume that the $q \circ f_h$ are continuous. Using Lemma 4.5.i we see that on the one hand each $L(q \circ f_h)$ is open in V_h and that on the other hand it suffices to show that $L(q)$ is open in V. But this is immediate from the identity $f_h^{-1}(L(q)) = L(q \circ f_h)$.

iii. It is clear that the family of lattices in the assertion satisfies the conditions (lc1) and (lc2) and therefore defines a locally convex topology $\mathcal{T}$ on V. Because of $f_{h_0}^{-1}(\sum_{h \in H} f_h(L_{hj(h)})) \supseteq L_{h_0 j(h_0)}$ this topology $\mathcal{T}$ is coarser than the locally convex final topology. On the other hand, whenever $L \subseteq V$ is an open lattice we then find, for any $h \in H$, a $j(h) \in J(h)$ such that $f_h^{-1}(L) \supseteq L_{hj(h)}$. It follows that $\sum_{h \in H} f_h(L_{hj(h)}) \subseteq L$. This shows that the two topologies in fact are equal.

In the following we want to look more closely at two special cases of this construction.

E1. The locally convex direct sum

Let $(V_h)_{h \in H}$ be a fixed family of locally convex K-vector spaces. We form the direct sum $V := \bigoplus_{h \in H} V_h$ and equip it with the locally convex final topology with respect to the inclusion maps $V_{h_0} \longrightarrow \bigoplus_{h \in H} V_h$. This locally convex vector space V is called the *locally convex direct sum* of the V_h.

Lemma 5.2:

i. The inclusion map $\bigoplus_{h \in H} V_h \longrightarrow \prod_{h \in H} V_h$ is continuous;

ii. if the set H is finite then the identity map $\bigoplus_{h\in H} V_h \xrightarrow{\cong} \prod_{h\in H} V_h$ is a topological isomorphism.

Proof: i. By the definition of the direct product topology the inclusions $V_{h_0} \longrightarrow \prod_{h\in H} V_h$ are continuous. The assertion therefore follows from Lemma 5.1.i. ii. This follows from the definition of the direct product topology and Lemma 5.1.iii.

Lemma 5.3:

Let $U_h \subseteq V_h$, for any $h \in H$, be a vector subspace with the subspace topology; we then have:

i. The locally convex direct sum topology on $\bigoplus_{h\in H} U_h$ is the subspace topology with respect to the inclusion $\bigoplus_{h\in H} U_h \subseteq \bigoplus_{h\in H} V_h$;

ii. the quotient vector space $(\bigoplus_{h\in H} V_h)/(\bigoplus_{h\in H} U_h)$ is the locally convex direct sum of the quotients V_h/U_h;

iii. if U_h is closed in V_h for any $h \in H$ then $\bigoplus_{h\in H} U_h$ is closed in $\bigoplus_{h\in H} V_h$.

Proof: i. By Lemma 5.1.i the inclusion $\bigoplus_h U_h \subseteq \bigoplus_h V_h$ is continuous. On the other hand, by Lemma 5.1.iii the locally convex direct sum topology on $\bigoplus_h U_h$ is defined by the lattices of the form $\bigoplus_h M_h$ where M_h is an open lattice in U_h. Choose, for any $h \in H$, an open lattice $L'_h \subseteq V_h$ such that $U_h \cap L'_h \subseteq M_h$. Putting $L_h := L'_h + M_h$ we have $\bigoplus_h M_h = (\bigoplus_h U_h) \cap (\bigoplus_h L_h)$ which shows that the left hand side is open in the subspace topology.

ii. According to the universal property of the quotient topology and Lemma 5.1.i the bijection

$$\begin{array}{ccc} (\bigoplus_{h\in H} V_h)/(\bigoplus_{h\in H} U_h) & \xrightarrow{\sim} & \bigoplus_{h\in H} V_h/U_h \\ (\sum_h v_h) + (\bigoplus_h U_h) & \longmapsto & \sum_h (v_h + U_h) \end{array}$$

is continuous. Let, on the other hand, $\overline{L}$ be an open lattice in the left hand side and denote its preimage in $\bigoplus_h V_h$, resp. its image in the right hand side, by L, resp. by M. Then $L_h := L \cap V_h$ is an open lattice in V_h and $\overline{L_h} := (L_h + U_h)/U_h$ is an open lattice in V_h/U_h. We therefore see that M contains the open lattice $\bigoplus_h \overline{L_h}$ and hence is open.

iii. By Lemma 5.1.i all the projection maps $\mathrm{pr}_{h_0} : \bigoplus_h V_h \longrightarrow V_{h_0}$ are continuous. The assertion therefore is a consequence of the identity

$$\bigcap_{h\in H} \mathrm{pr}_h^{-1}(U_h) = \bigoplus_{h\in H} U_h \ .$$

In the proof of the first assertion of the above lemma we have used a simple argument which will be used over and over again and which we therefore want to point out explicitly: Let W be a locally convex vector space and let $U \subseteq W$ be a subspace; for any open lattice $M \subseteq U$ one can find an open lattice $L \subseteq V$ such that $M = U \cap L$.

Corollary 5.4:

If V_h is Hausdorff for any $h \in H$ then $\bigoplus_{h \in H} V_h$ is Hausdorff.

Proof: Apply Lemma 5.3.iii to the subspaces $U_h := \{0\}$ and use Lemma 4.6.

E2. The strict inductive limit

Let V be a K-vector space and let

$$V_1 \subseteq V_2 \subseteq \ldots \subseteq V$$

be an increasing sequence of vector subspaces such that $V = \bigcup_{n \in \mathbb{N}} V_n$. We moreover assume that each V_n is equipped with a locally convex topology $\mathcal{T}_n$ in such a way that

$$\mathcal{T}_{n+1}|V_n = \mathcal{T}_n \qquad \text{for any } n \in \mathbb{N} .$$

We equip V with the locally convex final topology $\mathcal{T}$ with respect to the inclusions $V_n \subseteq V$. In this situation V is called the *strict inductive limit* of the V_n.

Proposition 5.5:

i. $\mathcal{T}|V_n = \mathcal{T}_n$ for any $n \in \mathbb{N}$;

ii. if V_n is Hausdorff for any $n \in \mathbb{N}$ then V is Hausdorff;

iii. if V_n is closed in V_{n+1} for any $n \in \mathbb{N}$ then V_n is closed in V for any $n \in \mathbb{N}$.

Proof: i. Fix an $n \in \mathbb{N}$ and let $L_n \subseteq V_n$ be an open lattice. Because of the assumption that $\mathcal{T}_{m+1}|V_m = \mathcal{T}_m$ we inductively find open lattices $L_{n+m} \subseteq V_{n+m}$ such that $L_{n+m} = V_{n+m} \cap L_{n+m+1}$ for any $m \geq 0$. Then $L := \bigcup_{m \geq 0} L_{n+m}$ clearly is a lattice in V. Moreover, L is open in V since $L \cap V_{n+m} = L_{n+m}$ for any $m \geq 0$. In particular, $L_n = V_n \cap L$ which proves that L_n is open in the subspace topology on V_n induced by V.

ii. Let $v \in V$ be any nonzero vector. We have $v \in V_m$ for some $m \in \mathbb{N}$. Since V_m is assumed to be Hausdorff we find an open lattice $L_m \subseteq V_m$ such that $v \notin L_m$. The same inductive construction as under i. produces an open lattice $L \subseteq V$ such that $L_m = V_m \cap L$ and hence $v \notin L$. Using Lemma 4.6 we conclude that V is Hausdorff.

iii. Fix an $n \in \mathbb{N}$ and consider any vector $v \in V \setminus V_n$. We have $v \in V_m$ for some $m > n$. Since V_n is closed in V_m by assumption we find an open lattice

$L_m \subseteq V_m$ such that $(v + L_m) \cap V_n = \emptyset$. Applying our inductive construction a third time there is an open lattice $L \subseteq V$ such that $L_m = V_m \cap L$. It follows that $(v + L) \cap V_n = ((v + L) \cap V_m) \cap V_n = (v + L_m) \cap V_n = \emptyset$.

Proposition 5.6:

Assume that V_n is closed in V_{n+1} for any $n \in \mathbb{N}$; then a subset $B \subseteq V$ is bounded in V if and only if $B \subseteq V_m$ for some $m \in \mathbb{N}$ and B is bounded in V_m.

Proof: We first assume that B is bounded in V_m. If $L \subseteq V$ is any open lattice then $V_m \cap L$ is an open lattice in V_m and we find an $a \in K$ such that $B \subseteq a(V_m \cap L) \subseteq aL$.

Now let $B \subseteq V$ be any bounded subset. Fix once and for all a scalar $b \in K$ such that $0 < |b| < 1$. Arguing by contradiction we assume that B is not contained in any V_n. We then find a sequence of natural numbers $n_1 < n_2 < \ldots$ and a sequence $(v_k)_{k \in \mathbb{N}}$ in B such that

$$(*) \qquad v_k \in V_{n_{k+1}} \setminus V_{n_k} \quad \text{for any } k \in \mathbb{N} \ .$$

Note that the sequence $(b^k v_k)_k$ in V also satisfies the property $(*)$. We derive a contradiction in two steps.

Step 1: First we construct an open lattice $L \subseteq V$ which does not contain any of the vectors $b^k v_k$. We start by fixing an open lattice $L_1 \subseteq V_{n_1}$ and choosing an open lattice $L_2' \subseteq V_{n_2}$ such that $L_1 = V_{n_1} \cap L_2'$. With V_{n_1} also $bv_1 + V_{n_1}$ is closed in V_{n_2}. Since $bv_1 + V_{n_1}$ does not contain the zero vector we find an open lattice $L_2'' \subseteq V_{n_2}$ such that $L_2'' \subseteq L_2'$ and $(bv_1 + V_{n_1}) \cap L_2'' = \emptyset$. The open lattice $L_2 := L_1 + L_2''$ in V_{n_2} then satisfies

$$V_{n_1} \cap L_2 = L_1 \quad \text{and} \quad bv_1 \notin L_2 \ .$$

Repeating this construction we inductively obtain, for any $k \in \mathbb{N}$, an open lattice L_k in V_{n_k} satisfying

$$V_{n_k} \cap L_{k+1} = L_k \quad \text{and} \quad b^k v_k \notin L_{k+1} \ .$$

Then $L := \bigcup_{k \in \mathbb{N}} L_k$ is an open lattice in V not containing any $b^k v_k$.

Step 2: We now show that given any open lattice $L \subseteq V$ we have $b^k v_k \in L$ for any sufficiently big $k \in \mathbb{N}$. By Lemma 4.5.ii the gauge p_L is continuous. Since B is bounded there must be a sufficiently big $c \in \mathbb{N}$ such that $p_L(v_k) < |b|^{-c}$ for any $k \in \mathbb{N}$. It follows, using Lemma 2.2.i, that $b^k v_k \in L^-(p_L) \subseteq L$ for any $k \geq c$.

Since these two steps are in contradiction to each other there must be an $m \in \mathbb{N}$ such that $B \subseteq V_m$. Let finally $L_m \subseteq V_m$ be an open lattice. Because of

Prop. 5.5.i we find an open lattice $L \subseteq V$ such that $V_m \cap L = L_m$. The subset B being bounded in V there is an $a \in K$ such that $B \subseteq aL$. We obtain $B \subseteq V_m \cap aL = a(V_m \cap L) = aL_m$. This proves that B is bounded in V_m.

Example 3:

Let X be a locally compact topological space. In section 3 we had discussed as an example the normed K-vector space $(C_c(X), \| \|_\infty)$. In the following we will construct another natural locally convex topology on $C_c(X)$ which is finer than the norm topology.

Consider, for any compact subset $A \subseteq X$, the vector subspace

$$C_A(X) := \{\phi \in C_c(X) : \phi|(X \setminus A) = 0\} ;$$

equipped with the norm $\| \|_\infty$ this is a Banach space. It is clear that $C_c(X)$ is the union of all this subspaces $C_A(X)$. We consider on $C_c(X)$ the locally convex final topology with respect to the inclusions $C_A(X) \subseteq C_c(X)$. By Lemma 5.1.i the evaluation linear forms $\delta_x(\phi) := \phi(x)$ on $C_c(X)$ are continuous. The $C_A(X) = \bigcap_{x \notin A} \ker(\delta_x)$ as subspaces of $C_c(X)$ therefore are closed. In particular, because of $\{0\} = C_\emptyset(X)$, the locally convex vector space $C_c(X)$ is Hausdorff.

Let us assume in addition that X is σ-compact, i.e., that X has a countable covering by compact subsets. We then find an increasing sequence of compact subsets $A_1 \subseteq A_2 \subseteq \ldots \subseteq X$ such that X is covered by the interiors of the A_n. In particular, any compact subset of X is already contained in some A_m. Since the inclusions $C_{A_n}(X) \subseteq C_{A_{n+1}}(X)$ are isometries it follows that $C_c(X)$ is the strict inductive limit of the increasing sequence of Banach spaces $C_{A_n}(X)$. We will see later that therefore this new locally convex topology on $C_c(X)$ is (in contrast to the norm topology) complete.

The space $C_c(X)$ with this locally convex final topology is the starting point of measure theory. The continuous linear forms on this space are called the (K-valued) Radon measures on X.

§6 Spaces of continuous linear maps

In this section let V and W denote two locally convex K-vector spaces. Since the addition and the scalar multiplication in W are continuous, by Lemma 4.1, the continuous linear maps (or operators) from V into W form a vector subspace

$$\mathcal{L}(V, W) := \{f \in \mathrm{Hom}_K(V, W) : f \text{ is continuous}\}$$

of the K-vector space $\mathrm{Hom}_K(V, W)$ of all linear maps. We will discuss in this section a general technique to equip the vector space $\mathcal{L}(V, W)$ with a locally convex topology. As it turns out there are various ways to do this but which all follow the same pattern. The following notion will play a crucial role in this discussion.

Definition:

A subset $H \subseteq \mathrm{Hom}_K(V,W)$ is called equicontinuous *if for any open lattice $M \subseteq W$ there is an open lattice $L \subseteq V$ such that $f(L) \subseteq M$ for every $f \in H$.*

It is obvious that any equicontinuous subset already is contained in $\mathcal{L}(V,W)$.

Proposition 6.1:

Suppose that the topology on V is defined by the family of seminorms $(q_i)_{i\in I}$; for a subset $H \subseteq \mathrm{Hom}_K(V,W)$ the following assertions are equivalent:

i. H is equicontinuous;

ii. for any continuous seminorm p on W there is a continuous seminorm q on V such that

$$p(f(v)) \leq q(v) \quad \textit{for any } v \in V \textit{ and } f \in H \ ;$$

iii. for any continuous seminorm p on W there is a constant $c > 0$ and finitely many $i_1, \ldots, i_r \in I$ such that

$$p(f(v)) \leq c \cdot \max(q_{i_1}(v), \ldots, q_{i_r}(v)) \quad \textit{for any } v \in V \textit{ and } f \in H \ .$$

Proof: Let us first assume that H is equicontinuous. By definition we then have an open lattice $L \subseteq V$ such that $f(L) \subseteq L(p)$ for any $f \in H$. According to Prop's 4.3 and 4.4 we find an $\epsilon > 0$ and finitely many $i_1, \ldots, i_r \in I$ such that $V(q_{i_1}, \ldots, q_{i_r}; \epsilon) \subseteq L$. We certainly may assume that $0 < \epsilon = |b| < 1$ for some $b \in K$. Hence

$$p(f(v)) \leq 1 \quad \text{for any } f \in H, \text{ provided } \max(q_{i_1}(v), \ldots, q_{i_r}(v)) \leq |b| \ .$$

If $\max(q_{i_1}(v), \ldots, q_{i_r}(v)) = 0$ then $\max(q_{i_1}(av), \ldots, q_{i_r}(av)) = 0$ for any $a \in K$ and therefore $|a| \cdot p(f(v)) = p(f(av)) \leq 1$ for any $a \in K$. This implies $p(f(v)) = 0$. If, on the other hand, $\max(q_{i_1}(v), \ldots, q_{i_r}(v)) > 0$ then we may choose an integer m such that $|b|^{m+2} < \max(q_{i_1}(v), \ldots, q_{i_r}(v)) \leq |b|^{m+1}$. We obtain $p(f(v)) = |b|^m \cdot p(f(b^{-m}v)) \leq |b|^m < |b|^{-2} \cdot \max(q_{i_1}(v), \ldots, q_{i_r}(v))$. This means that the assertion iii. holds true with the constant $c := |b|^{-2}$. The implication from iii. to ii. is trivial by putting $q := c \cdot \max(q_{i_1}, \ldots, q_{i_r})$.

We finally assume that the assertion ii. holds true. Let $M \subseteq W$ be an open lattice. According to the Remark 4.7 we find an $\epsilon > 0$ and a continuous seminorm p on W such that $V(p;\epsilon) \subseteq M$. By assumption we have a corresponding continuous seminorm q on V such that $p(f(v)) \leq q(v)$ for any $f \in H$. Then $f(V(q;\epsilon)) \subseteq V(p;\epsilon) \subseteq M$ for any $f \in H$.

Corollary 6.2:

Suppose that the topology on V is defined by the family of seminorms $(q_i)_{i\in I}$; for an arbitrary seminorm q on V the following assertions are equivalent:

i. q is continuous;

ii. there is a constant $c > 0$ and finitely many $i_1, \ldots, i_r \in I$ such that

$$q(v) \leq c \cdot \max(q_{i_1}(v), \ldots, q_{i_r}(v)) \quad \text{for any } v \in V .$$

Proof: Let $W := V$ but equipped with the topology defined by the seminorm q. The continuity of q then amounts to the continuity of the identity map $V \xrightarrow{\mathrm{id}} W$. By Prop. 6.1 for $H := \{\mathrm{id}\}$ the latter is equivalent to the assertion ii.

If we use this corollary for the space W then it follows that in Prop. 6.1.iii the continuous seminorm p on W only needs to run over a defining family for the topology.

Corollary 6.3:

Let $H \subseteq \mathcal{L}(V, W)$ be an equicontinuous subset and let p be a continuous seminorm on W; then $q(v) := \sup_{f\in H} p(f(v))$ is a continuous seminorm on V.

The approach to define on $\mathcal{L}(V, W)$ a certain family of locally convex topologies is based on the following two parallel observations. Fix a bounded subset $B \subseteq V$.

1. For any open lattice $M \subseteq W$ the subset

$$\mathcal{L}(B, M) := \{f \in \mathcal{L}(V, W) : f(B) \subseteq M\}$$

is a lattice in $\mathcal{L}(V, W)$. It is clear that $\mathcal{L}(B, M)$ is an o-submodule. If $f \in \mathcal{L}(V, W)$ is any continuous linear map then, by the boundedness of B, there has to be an $a \in K^\times$ such that $B \subseteq af^{-1}(M)$. This means that $f(B) \subseteq aM$ or equivalently that $a^{-1}f \in \mathcal{L}(B, M)$.

2. For any continuous seminorm p on W the formula

$$p_B(f) := \sup_{v\in B} p(f(v))$$

defines a seminorm on $\mathcal{L}(V,W)$. The only point to observe is that $p \circ f$ is a continuous seminorm on V so that $p(f(B))$ is a bounded subset in $\mathbb{R}_{\geq 0}$. We have the obvious identity

$$L(p_B) = \mathcal{L}(B, L(p)) \ .$$

Let now $\mathcal{B}$ be a fixed family of bounded subsets of V. The locally convex topology on $\mathcal{L}(V,W)$ defined by the family of seminorms $\{p_B : B \in \mathcal{B}, p \text{ a continuous seminorm on } W\}$ is called the $\mathcal{B}$-topology. We write

$$\mathcal{L}_{\mathcal{B}}(V,W) := \mathcal{L}(V,W) \ \text{ equipped with the } \mathcal{B}\text{-topology.}$$

Lemma 6.4:

Assume that the family $\mathcal{B}$ is closed under finite unions; then the topology on $\mathcal{L}_{\mathcal{B}}(V,W)$ can be defined by the family of lattices $\{\mathcal{L}(B,M) : B \in \mathcal{B}, M \subseteq W \text{ an open lattice}\}$.

Proof: The family of lattices in question is nonempty and satisfies (lc1) and (lc2). We have $L^-((p_M)_B) \subseteq \mathcal{L}(B, L^-(p_M)) \subseteq \mathcal{L}(B,M)$ which shows that any lattice $\mathcal{L}(B,M)$ is open in $\mathcal{L}_{\mathcal{B}}(V,W)$. Let on the other hand $p_{1,B_1}, \ldots, p_{r,B_r}$ be finitely many of the defining seminorms for the $\mathcal{B}$-topology and let $a \in K^\times$. Setting $B := B_1 \cup \ldots \cup B_r$ and $M := a(L(p_1) \cap \ldots \cap L(p_r))$ we have $\mathcal{L}(B,M) \subseteq V(p_{1,B_1}, \ldots, p_{r,B_r}; |a|)$.

Starting from the family $\mathcal{B}$ we may, using Lemma 4.10, define the usually much larger family $\widetilde{\mathcal{B}}$ of all those bounded subsets $B \subseteq V$ for which there is an $a \in K^\times$ and finitely many $B_1, \ldots, B_m \in \mathcal{B}$ such that aB is contained in the closure of the o-submodule generated by $B_1 \cup \ldots \cup B_m$. In particular, the family $\widetilde{\mathcal{B}}$ always is closed under finite unions.

Lemma 6.5:

The $\mathcal{B}$- and $\widetilde{\mathcal{B}}$-topologies on $\mathcal{L}(V,W)$ coincide.

Proof: For trivial reasons the $\widetilde{\mathcal{B}}$-topology is finer than the $\mathcal{B}$-topology. On the other hand let $M \subseteq W$ be an open lattice and $B \in \widetilde{\mathcal{B}}$. Choose $a \in K^\times$ and $B_1, \ldots, B_m \in \mathcal{B}$ such that aB is contained in the closure of the o-submodule generated by $B_1 \cup \ldots \cup B_m$. Then $a[\mathcal{L}(B_1,M) \cap \ldots \cap \mathcal{L}(B_m,M)] \subseteq \mathcal{L}(B,M)$.

Lemma 6.6:

If W is Hausdorff and if $\bigcup_{B \in \mathcal{B}} B$ generates a dense vector subspace in V then $\mathcal{L}_{\mathcal{B}}(V,W)$ is Hausdorff.

Proof: We check the condition ii. in Lemma 4.6. Let $0 \neq f \in \mathcal{L}(V,W)$. By assumption there is a $B \in \mathcal{B}$ and a vector $v \in B$ such that $f(v) \neq 0$. Since, moreover, W is assumed to be Hausdorff there is an open lattice $M \subseteq W$ such that $f(v) \notin M$. It follows that $f \notin \mathcal{L}(B,M)$.

Examples:

1) Let $\mathcal{B}$ be the family of all one point subsets of V. The corresponding $\mathcal{B}$-topology is called the *weak topology* or the topology of pointwise convergence. We write $\mathcal{L}_s(V,W) := \mathcal{L}_{\mathcal{B}}(V,W)$. The weak topology in fact is the initial topology with respect to the evaluation maps

$$\begin{array}{rcl} \mathcal{L}(V,W) & \longrightarrow & W \\ f & \longmapsto & f(v) \end{array}$$

for $v \in V$.

2) Let $\mathcal{B}$ be the family of all bounded subsets in V. The corresponding $\mathcal{B}$-topology is called the *strong topology* or the topology of bounded convergence. We write $\mathcal{L}_b(V,W) := \mathcal{L}_{\mathcal{B}}(V,W)$.

If W is Hausdorff both locally convex vector spaces $\mathcal{L}_s(V,W)$ and $\mathcal{L}_b(V,W)$ are Hausdorff.

Remark 6.7:

If V and W are normed vector spaces then the topology on $\mathcal{L}_b(V,W)$ is defined by the operator norm $\| \ \|$.

Proof: Let $\mathcal{B}$ denote the family of all bounded subsets in V and let $\mathcal{B}_{\mathrm{o}}$ have the unit ball $B_1(0)$ in V as its single member. Since $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}_{\mathrm{o}}}$ it follows from Lemma 6.5 that the topology on $\mathcal{L}_b(V,W)$ is defined by the norm $\|f\|' := \sup_{\|v\| \leq 1} \|f(v)\|$. We trivially have $\|f\|' \leq \|f\|$. Fix a $b \in K$ such that $0 < |b| < 1$. For any $0 \neq v \in B_1(0)$ we then find an integer $m(v) \geq 0$ such that $|b^{m(v)+1}| < \|v\| \leq |b^{m(v)}|$. We compute

$$\|f\| = \sup_{0<\|v\|\leq 1} \frac{\|f(v)\|}{\|v\|} = \sup_{0<\|v\|\leq 1} \frac{\|f(b^{-m(v)}v)\|}{\|b^{-m(v)}v\|} = \sup_{|b|<\|v\|\leq 1} \frac{\|f(v)\|}{\|v\|} \leq |b|^{-1} \cdot \|f\|'$$

and see that $\| \ \|$ and $\| \ \|'$ define the same topology.

The reason for the fundamental role of equicontinuous subsets lies in the following fact.

Lemma 6.8:

Every equicontinuous subset $H \subseteq \mathcal{L}(V,W)$ is bounded in every $\mathcal{B}$-topology.

Proof: Let $\mathcal{L} \subseteq \mathcal{L}_{\mathcal{B}}(V,W)$ be any open lattice. There is an open lattice $M \subseteq W$ and a $B \in \widetilde{\mathcal{B}}$ such that $\mathcal{L} \supseteq \mathcal{L}(B,M)$. Since H is equicontinuous we find an open lattice $L \subseteq V$ such that $f(L) \subseteq M$ for any $f \in H$. Furthermore, there is an $a \in K^\times$ such that $B \subseteq aL$. Hence $f(B) \subseteq aM$ for any $f \in H$ which shows that $H \subseteq a\mathcal{L}(B,M) \subseteq a\mathcal{L}$.

There is the following analog of Lemma 4.10 for equicontinuous subsets. We begin with a trivial observation.

Remark 6.9:

If $H \subseteq \mathcal{L}(V,W)$ is equicontinuous then the o-submodule generated by H and in particular $\mathrm{Co}(H)$ *are equicontinuous.*

Concerning the passage to the closure a stronger statement holds true. To formulate it we introduce the K-vector space $\mathrm{Map}(V,W)$ of all (possibly neither continuous nor linear) maps from V into W equipped with the topology of pointwise convergence. This is a locally convex vector space whose topology is defined by the family of seminorms $p_v(f) := p(f(v))$ for any $v \in V$ and any continuous seminorm p on W.

Lemma 6.10:

If W is Hausdorff then, for any equicontinuous subset $H \subseteq \mathcal{L}(V,W)$, the closure $\overline{H}$ of H in $\mathrm{Map}(V,W)$ *is contained in $\mathcal{L}(V,W)$ and is equicontinuous.*

Proof: In a first step we show that the subspace of K-linear maps $\mathrm{Hom}_K(V,W)$ is closed in $\mathrm{Map}(V,W)$. The evaluation maps

$$\begin{array}{rcl} \mathrm{Map}(V,W) & \longrightarrow & W \\ f & \longmapsto & f(v) \ , \end{array}$$

for $v \in V$, are continuous. Since addition and scalar multiplication in V and in W are continuous the linear maps

$$\begin{array}{rcl} \mathrm{Map}(V,W) & \longrightarrow & W \\ f & \longmapsto & f(av + a'v') - af(v) - a'f(v') \ , \end{array}$$

for $a, a' \in K$ and $v, v' \in V$, are continuous as well. Hence their kernels are closed because of our assumption that W is Hausdorff. But $\mathrm{Hom}_K(V,W)$ is the intersection of all these kernels.

We therefore have $\overline{H} \subseteq \mathrm{Hom}_K(V,W)$ and it remains to show that $\overline{H}$ is equicontinuous. Let $M \subseteq W$ be an open lattice, and choose an open lattice $L \subseteq V$ such that $f(L) \subseteq M$ for any $f \in H$. Consider now any $f \in \overline{H}$. We claim that always $f(L) \subseteq M$. To see this let $v \in L$. In $\mathrm{Map}(V, W)$ we have, for any $0 < \epsilon < 1$, the open lattice

$$V((p_M)_v; \epsilon) = \{f \in \mathrm{Map}(V, W) : p_M(f(v)) \leq \epsilon\} \ .$$

Since f lies in the closure of H there must exist an $f' \in H$ such that $f - f' \in V((p_M)_v; \epsilon)$. This amounts to $p_M(f(v) - f'(v)) \leq \epsilon < 1$ which implies $f(v) - f'(v) \in M$. Since $f'(L) \subseteq M$ we obtain $f(v) \in M$.

Corollary 6.11:

Suppose that W is Hausdorff and that the $\mathcal{B}$-topology is finer than the weak topology; then the closure $\overline{H}$ in $\mathcal{L}_{\mathcal{B}}(V, W)$ of any equicontinuous subset $H \subseteq \mathcal{L}(V, W)$ is equicontinuous.

There are two important classes of locally convex vector spaces V for which Lemma 6.8 can be sharpened to a characterization of the bounded subsets in $\mathcal{L}_{\mathcal{B}}(V, W)$. For the definition of the first class the starting point is the observation that, by the definition of boundedness, a given open lattice $L \subseteq V$ satisfies the following property:

(*bor*) For any bounded subset $B \subseteq V$ there is an $a \in K$ such that $B \subseteq aL$.

Note, by the way, that any o-submodule of V which satisfies (*bor*) necessarily is a lattice.

Definition:

A locally convex vector space V is called bornological *if every lattice in V which satisfies* (*bor*) *is open.*

Proposition 6.12:

If V is bornological then a subset $H \subseteq \mathcal{L}(V, W)$ is equicontinuous if and only if it is bounded in $\mathcal{L}_b(V, W)$.

Proof: The direct implication is a special case of Lemma 6.8. For the reverse implication assume H to be bounded in $\mathcal{L}_b(V, W)$ and let $M \subseteq W$ be an open lattice. Define $L := \bigcap_{f \in H} f^{-1}(M)$. If $B \subseteq V$ is any bounded subset then, by assumption, there is an $a \in K^\times$ such that $H \subseteq a\mathcal{L}(B, M) = \mathcal{L}(B, aM)$. This means that $f(B) \subseteq aM$ for any $f \in H$ and hence that $B \subseteq aL$. In other words the o-submodule L satisfies the condition (*bor*) and therefore must be an open lattice by our assumption that V is bornological. It follows that H is equicontinuous.

Proposition 6.13:

The following assertions are equivalent:

i. V is bornological;

ii. a seminorm q on V is continuous if and only if it is bounded on bounded subsets;

iii. a K-linear map $f : V \longrightarrow W$ into any other locally convex K-vector space W is continuous if and only if it respects bounded subsets.

Proof: We first show the equivalence of i. and ii. Assume that V is bornological. We have noted already earlier that a continuous seminorm is bounded on bounded subsets. Let us therefore assume vice versa that q has this latter property. For the continuity of q it suffices, by Lemma 4.5.i, to show that the lattice $L(q)$ is open. Since V is bornological it suffices moreover to check that $L(q)$ satisfies the condition (*bor*). So let $B \subseteq V$ be bounded. There is then an $a \in K^\times$ such that $q(v) \leq |a|$ for any $v \in B$. But this amounts to $B \subseteq aL(q)$.

We now assume that the assertion ii. holds and we have to conclude from this that V is bornological. Let $L \subseteq V$ be a lattice satisfying (*bor*). As a consequence of Lemma 4.5.ii and our assumption it suffices to show that the gauge p_L is bounded on any bounded subset $B \subseteq V$. But using a scalar $a \in K$ such that $B \subseteq aL$ we have $p_L(v) \leq |a|$ for $v \in B$.

Next we establish the equivalence of i. and iii. Again we first assume that V is bornological. It is trivial that a continuous linear map respects bounded subsets. To show that f is continuous provided it has this latter property it suffices to check that, given an open lattice $M \subseteq W$, its preimage $f^{-1}(M)$ satisfies the condition (*bor*). If $B \subseteq V$ is bounded then $f(B)$, by assumption, is bounded in W. Letting $a \in K^\times$ be a scalar such that $f(B) \subseteq aM$ we have $B \subseteq af^{-1}(M)$.

Finally we assume that iii. holds true. Let Λ denote the family of all lattices in V which satisfy the condition (*bor*). It is immediate that Λ contains every open lattice and satisfies (lc1) and (lc2). Hence Λ defines a locally convex topology $\mathcal{T}'$ on V which is finer than the given topology $\mathcal{T}$. The two locally convex vector spaces $(V, \mathcal{T})$ and $(V, \mathcal{T}')$ have the same bounded subsets. Hence iii. implies that the identity map $(V, \mathcal{T}) \xrightarrow{\mathrm{id}} (V, \mathcal{T}')$ is continuous. So $\mathcal{T}' = \mathcal{T}$ which means that V is bornological.

Examples:

1) If the topology on V is defined by a single seminorm (e.g., if V is a normed vector space) then V is bornological.

Proof: According to Prop. 4.11 we have a bounded open lattice $L_\mathrm{o} \subseteq V$. Let $L \subseteq V$ be any other lattice with the property (*bor*). Then there is an $a \in K^\times$ such that $L_\mathrm{o} \subseteq aL$. This shows that aL and consequently L is open.

2) Suppose that the topology on V is the locally convex final topology with respect to a family of linear maps $f_h : V_h \longrightarrow V$; if all the V_h are bornological then so, too, is V.

Proof: Let $L \subseteq V$ be a lattice with the property (bor). To see that L is open it suffices to check that each lattice $f_h^{-1}(L)$ in V_h has the property (bor). If $B \subseteq V_h$ is bounded then $f_h(B)$ is bounded in V so that there is a scalar $a \in K^\times$ such that $f_h(B) \subseteq aL$. Hence $B \subseteq af_h^{-1}(L)$.

3) Quotient spaces, locally convex direct sums, and strict inductive limits of bornological vector spaces are bornological.

Proof: These are special cases of 2).

The following notion will be studied in more detail later. We give the definition now since it provides another class of bornological vector spaces.

Definition:

A locally convex vector space V is called <u>*metrizable*</u> *if its topology can be defined by a metric.*

Proposition 6.14:

Any metrizable vector space V is bornological.

Proof: We will check the assertion ii. in Prop. 6.13. Let q be a seminorm on V which is bounded on bounded subsets and assume that q is not continuous. By Lemma 4.5.i the latter means that $L(q)$ is not open in V. On the other hand, since V is metrizable we find a decreasing sequences $(L_n)_{n \in \mathbb{N}}$ of open lattices in V which form a fundamental system of neighbourhoods of the zero vector. Let $a \in K$ be any scalar such that $|a| > 1$. Since $a^{-n}L_n \not\subseteq L(q)$ there is a vector $v_n \in L_n$ such that $q(v_n) > |a|^n$. Hence the seminorm q is not bounded on the sequence of vectors $(v_n)_n$. But by construction this sequence converges to the zero vector and therefore is bounded. This is a contradiction.

We now come to the second class of locally convex vector spaces. Here the starting point is the fact that any open lattice in V also is closed.

Definition:

A locally convex vector space V is called <u>*barrelled*</u> *if every closed lattice in V is open.*

Proposition 6.15: (Banach-Steinhaus)

If V is barrelled then any bounded subset $H \subseteq \mathcal{L}_s(V,W)$ is equicontinuous.

Proof: Let $M \subseteq W$ be an open lattice and consider the o-submodule $L := \bigcap_{f \in H} f^{-1}(M)$ of V. We have to show that L is open. Since L obviously is closed it suffices to check that L is a lattice. Let $v \in V$ be any vector. By the boundedness of H in $\mathcal{L}_s(V,W)$ there is an $a \in K^\times$ such that $H \subseteq a\mathcal{L}(\{v\},M)$. This implies that $a^{-1}v \in L$.

As with the above proposition we sometimes follow the convention to name very basic results the same way their counterparts over the real or complex field are named. This does not mean, e.g., that Prop. 6.15 was proved by Banach-Steinhaus. But it helps a lot to remember the content of this result.

Later (in §12) we will see that over a spherically complete field the Banach-Steinhaus theorem actually characterizes barrelled vector spaces.

Corollary 6.16:

Suppose that V is barrelled; then in $\mathcal{L}_{\mathcal{B}}(V,W)$, for any $\mathcal{B}$-topology which is finer than the weak topology, the bounded subsets coincide with the equicontinuous subsets.

Proof: This is Lemma 6.8 together with Prop. 6.15.

Examples:

1) If V has no countable covering $V = \bigcup_{n \in \mathbb{N}} A_n$ by closed subsets A_n with empty interior then V is barrelled.

Proof: Let $L \subseteq V$ be a closed lattice and fix an $a \in K$ such that $|a| > 1$. Then $V = \bigcup_{n \in \mathbb{N}} a^n L$. By assumption there must therefore exist an $n \in \mathbb{N}$ such that $a^n L$ and consequently L has a nonempty interior. This means we find a vector $v \in L$ and an open lattice $L' \subseteq V$ such that $v + L' \subseteq L$. But then L also contains L' and hence is open.

2) If V is metrizable and is complete with respect to a defining metric (e.g., V is a Banach space) then V is barrelled.

Proof: By Baire's theorem ([B-GT] Chap.IX §5.3 Thm.1) we are in the situation of 1).

3) Suppose that the topology on V is the locally convex final topology with respect to a family of linear maps $f_h : V_h \longrightarrow V$; if all the V_h are barrelled then so, too, is V.

Proof: This is obvious.

4) Quotient spaces, locally convex direct sums, and strict inductive limits of barrelled vector spaces are barrelled.

Proof: These are special cases of 3).

§7 Completeness

In this section we will discuss the concepts of completeness and completion for a general locally convex K-vector space V. We recall that a directed set $(I, \leq)$ is a set I together with a partial order $\leq$ which has the additional property that for any two elements $i, j \in I$ there is a third element $k \in I$ such that $i \leq k$ and $j \leq k$.

Definition:

a) A net $(v_i)_{i \in I}$ in V is a family of vectors v_i in V where the index set I is directed;

b) a net $(v_i)_{i \in I}$ is said to converge to the vector $v \in V$ if for any open lattice $L \subseteq V$ there is an index $i \in I$ such that $v_j - v \in L$ for any $j \geq i$; we also say in this case that this net is convergent;

c) a net $(v_i)_{i \in I}$ is called a Cauchy net if for any open lattice $L \subseteq V$ there is an index $i \in I$ such that $v_j - v_k \in L$ for any $j, k \geq i$;

d) a subset $A \subseteq V$ is called complete if every Cauchy net in A converges to a vector in A.

Remark 7.1:

i. Any (convergent, Cauchy) sequence is a (convergent, Cauchy) net;

ii. any convergent net is a Cauchy net;

iii. if V is Hausdorff then a net in V converges to at most one vector;

iv. any closed subset of a complete subset is also complete;

v. if V is Hausdorff then any complete subset of V is closed;

vi. suppose that the topology on V is defined by the family of seminorms $(q_j)_{j \in J}$; then a net $(v_i)_{i \in I}$ in V converges to the vector $v \in V$ if and only if $(q_j(v_i - v))_{i \in I}$ converges to zero for any $j \in J$;

vii. let $f : V \longrightarrow W$ be a continuous linear map between locally convex K-vector spaces; if $(v_i)_{i \in I}$ converges to $v \in V$, resp. is a Cauchy net, then $(f(v_i))_{i \in I}$ converges to $f(v)$, resp. is a Cauchy net.

Remark 7.2:

If V is metrizable then we have:

i. V is complete if and only if every Cauchy sequence in V is convergent;

ii. suppose that the topology on V can be defined by a translation invariant metric d (i.e., $d(v+w, v'+w) = d(v, v')$ for any $v, v', w \in V$); then V is complete if and only if the metric space (V, d) is complete.

Proof: The second assertion is an immediate consequence of the first one and the identity $d(v, w) = d(v-w, 0)$. For the first assertion only the reverse implication has to be considered. Let $(v_i)_{i \in I}$ be a Cauchy net in V. Since V is metrizable there is a decreasing sequence $L_1 \supseteq L_2 \supseteq \ldots$ of open lattices in V which form a fundamental system of neighbourhoods of the zero vector. For any $n \in \mathbb{N}$ we choose an index $i_n \in I$ such that $v_j - v_k \in L_n$ for any $j, k \geq i_n$. We certainly may assume that $i_1 \leq i_2 \leq \ldots$ Write $w_n := v_{i_n}$; then $(w_n)_n$ is a Cauchy sequence. By assumption it converges to a vector $v \in V$. We claim that the original Cauchy net $(v_i)_i$ also converges to v. Let $L \subseteq V$ be any open lattice. It contains some L_m, and we have $v_j - v \in \overline{\{v_j - w_n\}_{n \geq m}} \subseteq \overline{L_m} \subseteq \overline{L} = L$ for $j \geq i_m$.

In particular, any Banach space is complete.

Lemma 7.3:

Let $V_{\mathrm{o}} \subseteq V$ be a dense vector subspace; any continuous (w.r.t. the subspace topology) linear map $f_{\mathrm{o}} : V_{\mathrm{o}} \longrightarrow W$ into a complete Hausdorff locally convex K-vector space W extends uniquely to a continuous linear map $f : V \longrightarrow W$.

Proof: For the uniqueness part of the assertion we may assume that $f_{\mathrm{o}} = 0$. Then $f(V) = f(\overline{V_{\mathrm{o}}}) \subseteq \overline{f_{\mathrm{o}}(V_{\mathrm{o}})} = \overline{\{0\}} = \{0\}$ which amounts to $f = 0$. For the existence of f let Λ denote the set of all open lattices in V; it is directed by the reverse of the inclusion relation. We define $f(v)$ for a given vector $v \in V$ as follows. Since V_{o} is dense in V there is, for any $L \in \Lambda$, a vector $v_L \in V_{\mathrm{o}}$ such that $v_L - v \in L$. By construction the net $(v_L)_L$ converges to v and therefore is a Cauchy net. Hence $(f_{\mathrm{o}}(v_L))_L$ is a Cauchy net in W and converges, by assumption, to a vector which we denote by $f(v)$. We proceed in three steps. Step 1: The vector $f(v)$ does not depend on the choice of the net $(v_L)_L$. Let $(v'_L)_L$ be another choice. The net $(v_L - v'_L)_L$ then converges to the zero vector in V_{o}. The continuity of f_{o} implies that the net $(f_{\mathrm{o}}(v_L) - f_{\mathrm{o}}(v'_L))_L$ converges to the zero vector in W. This shows that $v \longmapsto f(v)$ is a well defined map on V which extends f_{o}. Step 2: The map f is K-linear. Let $(v_L)_L$ and $(w_L)_L$ be nets as above converging to the vectors v and w, respectively. Given any two scalars $a, b \in K$ the net $(f_{\mathrm{o}}(av_L + bw_L))_L = (af_{\mathrm{o}}(v_L) + bf_{\mathrm{o}}(w_L))_L$ converges to $af(v) + bf(w)$ in W. On the other hand, the net $(av_L + bw_L)_L$ converges to $av + bw$. The reindexed net $(av_{cL} + bw_{cL})_L$ where $c \in K^\times$ such that $|c^{-1}| = \max(|a|, |b|, 1)$ has the property used above for the definition of $f(av + bw)$. It follows that $af(v) + bf(w) = f(av + bw)$. Step 3: The map f is continuous. Let $M \subseteq W$ be an open lattice. By the continuity of f_{o} there is an open lattice $L' \subseteq V$ such that $L' \cap V_{\mathrm{o}} = f_{\mathrm{o}}^{-1}(M)$. For any $v \in L'$ we can choose a net $(v_L)_L$ as above converging to v so that this net in fact lies in $L' \cap V_o$. Then $(f_o(v_L))_L$ is a net in M. Since M is closed it is complete. We obtain that $f(v) \in M$ and hence that $f(L') \subseteq M$.

Remark 7.4:

Let $V_o \subseteq V$ be a dense vector subspace equipped with the subspace topology; we have:

i. Any continuous seminorm q_o on V_o extends uniquely to a continuous seminorm q on V, and any continuous seminorm on V arises in this way;

ii. if L_o runs over all open lattices in V_o then the closure $\overline{L_o}$ in V runs over all open lattices in V and satisfies $\overline{L_o} \cap V_o = L_o$.

Proof: i. Using the inequality $|q_o(v) - q_o(w)|_\infty \leq q_o(v - w)$ for any $v, w \in V_o$ the existence and uniqueness proof for q is entirely analogous to the argument for Lemma 7.3. Moreover, any continuous seminorm on V, by uniqueness, is the extension of its restriction to V_o. ii. If L is an open lattice in V then we have, by the density of V_o, that $L \subseteq \overline{L \cap V_o}$. But since L is closed we also have $\overline{L \cap V_o} \subseteq L$. It remains to remark that given L_o there is an open lattice $L \subseteq V$ such that $L \cap V_o = L_o$.

Proposition 7.5:

For any locally convex K-vector space V there exists an up to a unique topological isomorphism unique complete Hausdorff locally convex K-vector space $\widehat{V}$ together with a continuous K-linear map $c_V : V \longrightarrow \widehat{V}$ such that the following universal property holds true: For any continuous K-linear map $f : V \longrightarrow W$ into a complete Hausdorff locally convex K-vector space W there is a unique continuous K-linear map $\widehat{f} : \widehat{V} \longrightarrow W$ such that $f = \widehat{f} \circ c_V$. We moreover have:

i. The image $\mathrm{im}(c_V)$ is dense in $\widehat{V}$;

ii. the map c_V induces a topological isomorphism between $V/\overline{\{0\}}$ with the quotient topology and $\mathrm{im}(c_V)$ with the subspace topology.

Proof: The unicity statement immediately follows from the universal property. For the existence proof we may assume V to be Hausdorff. Let again Λ denote the set of all open lattices in V directed by the reverse of the inclusion relation. As an o-module we define $\widehat{V}$ as the projective limit

$$\widehat{V} := \varprojlim_{L \in \Lambda} V/L$$

of the o-module quotients V/L. But V in fact is a K-vector space since multiplication by any $a \in K^\times$ induces an automorphism of Λ. Since V is Hausdorff the obvious K-linear map

$$\begin{aligned} c_V : V &\longrightarrow \widehat{V} \\ v &\longmapsto (v+L)_L \end{aligned}$$

is injective. For any open lattice $M \subseteq V$ we define the o-submodule $\widehat{M}$ of $\widehat{V}$ by

$$\widehat{M} := \varprojlim_{L \in \Lambda, L \subseteq M} M/L \ .$$

An equivalent characterization is given by

$$\widehat{M} = \{(v_L + L)_L \in \widehat{V} : v_M \in M\} \ .$$

This construction has the following straightforward properties:

1) $\widehat{M}$ is a lattice in $\widehat{V}$.

2) We have $c_V^{-1}(\widehat{M}) = M$.

3) The intersection of all $\widehat{M}$ is equal to $\{0\}$.

4) The family of all lattices $\widehat{M}$ is nonempty and satisfies (lc1) and (lc2).

The property 4) allows us to equip $\widehat{V}$ with the locally convex topology defined by the family of all the lattices $\widehat{M}$. Because of 3) this topology is Hausdorff, and because of 2) the map c_V induces a topological isomorphism between V and $\mathrm{im}(c_V)$. Next we check that $\mathrm{im}(c_V)$ is dense in $\widehat{V}$; in fact, the obvious formula $\widehat{v} - c_V(v_L) \in \widehat{L}$ for any $\widehat{v} = (v_L + L)_L \in \widehat{V}$ shows more generally that M is dense in $\widehat{M}$.

The existence and uniqueness of the map f is now an immediate consequence of Lemma 7.3. It remains to check that $\widehat{V}$ is complete. Let $(\widehat{v}_i)_{i \in I}$ with $\widehat{v}_i = (v_{i,L} + L)_L$ be a Cauchy net in $\widehat{V}$. We then find, for any $L \in \Lambda$, an index $i(L) \in I$ such that $\widehat{v}_j - \widehat{v}_k \in \widehat{L}$, i.e., $v_{j,L} - v_{k,L} \in L$ for any $j, k \geq i(L)$. It follows that the vector $\widehat{v} := (v_{i(L),L} + L)_L \in \widehat{V}$ is well defined and satisfies $\widehat{v} - \widehat{v}_i \in \widehat{L}$ for any $i \geq i(L)$ and any L. This means, of course, that the Cauchy net $(\widehat{v}_i)_i$ converges to $\widehat{v}$.

In the above proof we have seen that the open lattice $\widehat{M}$ is the closure in $\widehat{V}$ of the open lattice $M \subseteq V$. We therefore obtain from Remark 7.4.ii that the $\widehat{M}$ constitute all the open lattices in $\widehat{V}$.

Definition:

The locally convex vector space $\widehat{V}$ is called the $\underline{\textit{Hausdorff completion}}$ *of* V.

If V already is Hausdorff we call $\widehat{V}$ simply the completion of V and often identify V with a vector subspace of $\widehat{V}$.

Corollary 7.6:

Suppose that V is Hausdorff; an o-submodule $A \subseteq V$ is complete if and only if the natural map $A \longrightarrow \varprojlim A/(A \cap L)$ where L runs over the open lattices in V is surjective (and hence bijective).

Proof: Using the identifications $A/(A \cap L) \xrightarrow{\cong} (A+L)/L$ the map in the assertion corresponds to the factorization

$$A \longrightarrow \varprojlim (A+L)/L \subseteq \varprojlim V/L = \widehat{V}$$

of the canonical map c_V restricted to A. It therefore suffices to show that $\varprojlim(A+L)/L$ coincides with the closure of $c_V(A)$ in $\widehat{V}$. For $\widehat{v} = (v_L + L)_L \in \varprojlim(A+L)/L$ with $v_L \in A$ our formula $\widehat{v} - c_V(v_L) \in \widehat{L}$ shows that $\widehat{v}$ lies in the closure of A. Let, vice versa, $\widehat{w} = (w_L + L)_L$ be an element in the closure of A. For any L we then find a $v_L \in A$ such that $\widehat{w} - c_V(v_L) \in \widehat{L}$, i.e., such that $w_L - v_L \in L$. Hence $\widehat{w} = (v_L)_L \in \varprojlim(A+L)/L$.

Corollary 7.7:

Suppose that W is Hausdorff and complete; for any $\mathcal{B}$-topology the restriction map

$$\mathcal{L}_{c_V(\mathcal{B})}(\widehat{V}, W) \xrightarrow{\cong} \mathcal{L}_{\mathcal{B}}(V, W)$$

where $c_V(\mathcal{B}) := \{c_V(B) : B \in \mathcal{B}\}$ is a topological isomorphism.

It is an immediate consequence of the definitions that an arbitrary direct product of locally convex vector spaces is complete if and only if all the factors are complete.

Lemma 7.8:

Let $(V_h)_{h\in H}$ be a family of Hausdorff locally convex vector spaces; the locally convex direct sum $V := \bigoplus_{h\in H} V_h$ is complete if and only if V_h is complete for every $h \in H$.

Proof: By Lemma 5.3.iii each V_h is closed in V. Hence if V is complete then the V_h are complete as well. For the reverse implication we first of all note that

V is Hausdorff by Cor. 5.4. It therefore suffices to show that the canonical map $c_V : V \longrightarrow \widehat{V}$ is surjective. For any family $\lambda = (L_h)_{h\in H}$ of open lattices $L_h \subseteq V_h$ we have the open lattice $L_\lambda := \bigoplus_{h\in H} L_h$ in V. According to Lemma 5.1.iii the completion is the projective limit $\widehat{V} = \varprojlim_\lambda V/L_\lambda$; the map c_V corresponds to the obvious map

$$V = \bigoplus_{h\in H} V_h = \bigoplus_{h\in H} (\varprojlim_{L_h \subseteq V_h} V_h/L_h) \longrightarrow \varprojlim_{\lambda=(L_h)} (\bigoplus_{h\in H} V_h/L_h) = \widehat{V} .$$

Let now $(\sum_h v_h^\lambda + L_h)_\lambda$ with $v_h^\lambda \in V_h$ be a vector in the right hand side. Each set of indices $H(\lambda) := \{h \in H : v_h^\lambda \notin L_h\}$ is finite. We note that the coset $v_h^\lambda + L_h$ only depends on L_h and not on the family λ to which L_h belongs. This vector therefore lies in the left hand side if we can show that the union $\bigcup_\lambda H(\lambda)$ still is finite. If this union would be infinite there would have to exist a sequence of pairwise different indices $(h_n)_{n\in\mathbb{N}}$ and families $\lambda_n = (L_{n,h})_{h\in H}$ such that $v_{h_n}^{\lambda_n} \notin L_{n,h_n}$ for every $n \in \mathbb{N}$. Consider in this case the family $\lambda = (L_h)_h$ with $L_h = L_{n,h_n}$ for $h = h_n$ and $L_h = V_h$ for $h \neq h_1, h_2, \ldots$ By construction the corresponding set of indices $H(\lambda) = \{h_1, h_2, \ldots\}$ would be infinite which leads to a contradiction.

Lemma 7.9:

Let V be the strict inductive limit of the vector subspaces $V_1 \subseteq V_2 \subseteq \ldots$; if V_n is complete for every $n \in \mathbb{N}$ then V is complete, too.

Proof: Let $(v_i)_{i\in I}$ be a Cauchy net. In a first step we show that there is an $m \in \mathbb{N}$ such that for any $i \in I$ and any open lattice $L \subseteq V$ there is a $j \geq i$ such that $v_j \in V_m + L$. Assume that, for any $k \in \mathbb{N}$, there is an open lattice $L_k \subseteq V$ and an $i(k) \in I$ such that

$$v_j \notin V_k + L_k \qquad \text{for all } j \geq i(k) .$$

We certainly may assume that the $L_1 \supseteq L_2 \supseteq \ldots$ are decreasing. Consider the open lattice

$$L := \mathrm{Co}(\bigcup_{n\in\mathbb{N}} V_n \cap L_n)$$

in V. We claim that $L \subseteq V_k + L_k$ for any $k \in \mathbb{N}$. It suffices to show that $V_n \cap L_n \subseteq V_k + L_k$ for any n and k. But if $n \leq k$ then $V_n \subseteq V_k$, and if $n \geq k$ then $L_n \subseteq L_k$. It follows that $V_k + L \subseteq V_k + L_k$ for any $k \in \mathbb{N}$. Choose now an index $i \in I$ such that $v_{i_1} - v_{i_2} \in L$ for any $i_1, i_2 \geq i$. Letting $k \in \mathbb{N}$ be such that $v_i \in V_k$ we arrive at the contradiction that $v_j \in V_k + L \subseteq V_k + L_k$ for any $j \geq i$.

Denoting by Λ, as usual, the set of all open lattices in V we introduce the set $I \times \Lambda$ directed by the partial order $(i, L) \leq (j, M)$ if $i \leq j$ and $M \subseteq L$. Fix a natural number m with the property which we have established above. For any pair $(i, L) \in I \times \Lambda$ we then have an index $i'(L) \geq i$ and a vector $v_{(i,L)} \in V_m$ such that $v_{(i,L)} - v_{i'(L)} \in L$.

In the next step we show that $(v_{(i,L)})_{(i,L) \in I \times \Lambda}$ in fact is a Cauchy net in V_m. Any open lattice in V_m is of the form $V_m \cap L$ for some $L \in \Lambda$. Fix an $i \in I$ such that $v_k - v_l \in L$ for any $k, l \geq i$. Consider now any two pairs $(k, M), (l, N) \geq (i, L)$. We have $v_{(k,M)} - v_{k'(M)} \in M$ and $v_{(l,N)} - v_{l'(N)} \in N$. Since $k'(M) \geq k \geq i$, $l'(N) \geq l \geq i$, and $M + N \subseteq L$ we obtain $v_{(k,M)} - v_{(l,N)} = (v_{(k,M)} - v_{k'(M)}) + (v_{k'(M)} - v_{l'(N)}) + (v_{l'(N)} - v_{(l,N)}) \in M + L + N \subseteq L$.

Since V_m by assumption is complete the Cauchy net $(v_{(i,L)})_{(i,L)}$ converges to some vector $v \in V_m$. We conclude the proof by showing that the original Cauchy net $(v_i)_i$ in V converges to the same vector v. Let $L \subseteq V$ be any open lattice. We find a pair $(k, M) \in I \times \Lambda$ such that

(1) $v_{(l,N)} - v \in L$ for any $(l, N) \geq (k, M)$, and

(2) $v_{k_1} - v_{k_2} \in L$ for any $k_1, k_2 \geq k$.

As a special case of (1) we have $v_{(k,M \cap L)} - v \in L$. Since, by construction, $v_{(k,M \cap L)} - v_{k'(M \cap L)} \in M \cap L$ it follows that $v_{k'(M \cap L)} - v \in L$. Using (2) we finally obtain that $v_l - v \in L$ for any $l \geq k$.

There is the following important weakening of the concept of completeness.

Definition:

A locally convex vector space V is called <u>*quasi-complete*</u> *if every bounded closed subset of V is complete.*

Obviously every complete V is quasi-complete.

Lemma 7.10:

Every Cauchy sequence in V is contained in a bounded and closed subset.

Proof: Let $B = \{v_n : n \in \mathbb{N}\}$ be a Cauchy sequence in V. Then $(q(v_n))_n$, for any continuous seminorm q on V, is a Cauchy sequence in $\mathbb{R}$ and hence is bounded. It follows that B is bounded and that its closure $\overline{B}$ is bounded and closed.

Proposition 7.11:

If V is quasi-complete then we have:

i. Every Cauchy sequence in V is convergent;

ii. if V is metrizable then it is complete.

Proposition 7.12:

i. A closed vector subspace U of a quasi-complete locally convex vector space V is quasi-complete (in the subspace topology);

ii. the direct product $V = \prod_{h \in H} V_h$ of a family of quasi-complete locally convex vector spaces V_h is quasi-complete;

iii. the locally convex direct sum $V = \bigoplus_{h \in H} V_h$ of a family of quasi-complete and Hausdorff locally convex vector spaces V_h is quasi-complete;

iv. the strict inductive limit V of an increasing sequence $V_1 \subseteq V_2 \subseteq \dots$ of closed and quasi-complete vector subspaces V_n is quasi-complete.

Proof: The assertion i. is obvious. The assertion iv. is a consequence of Prop. 5.6. For ii. and iii. let $\mathrm{pr}_h : V \longrightarrow V_h$ denote the projection maps. If B is a bounded subset of V then B is contained in $\prod_h \overline{\mathrm{pr}_h(B)}$, resp. in $\bigoplus_h \overline{\mathrm{pr}_h(B)}$. In the situation of ii. this latter set obviously is complete as a direct product of complete subsets. The same holds true in the situation of iii. once we establish the fact that $\mathrm{pr}_h(B) = 0$ for all but finitely many $h \in H$. Reasoning by contradiction let us assume that there is a sequence of pairwise different indices $(h_n)_{n \in \mathbb{N}}$ in H and a sequence of vectors $(v_n)_{n \in \mathbb{N}}$ in B such that $\mathrm{pr}_{h_n}(v_n) \neq 0$ for any $n \in \mathbb{N}$. Fix a $b \in K$ such that $|b| > 1$. Since V_{h_n} is Hausdorff we find an open lattice $L_{h_n} \subseteq V_{h_n}$ such that $\mathrm{pr}_{h_n}(b^{-n} v_n) \notin L_{h_n}$. Define $L_h := V_h$ for any $h \neq h_1, h_2, \dots$ Then $L := \bigoplus_h L_h$, by Lemma 5.1.iii, is an open lattice in V. Since B is bounded we have $B \subseteq b^m L$ for some $m \in \mathbb{N}$. This leads to the contradiction that $\mathrm{pr}_{h_n}(v_n) \in b^m L_{h_n} \subseteq b^n L_{h_n}$ for all $n \geq m$.

Proposition 7.13:

Suppose that W is Hausdorff and quasi-complete; if the $\mathcal{B}$-topology is finer than the weak topology then any equicontinuous closed subset $H \subseteq \mathcal{L}_{\mathcal{B}}(V, W)$ is complete.

Proof: In a first step we consider the case of the weak topology. By Lemma 6.8 the subset H is bounded in $\mathcal{L}_s(V, W)$ and hence in $\mathrm{Map}(V, W)$, and by Lemma 6.10 it is closed in $\mathrm{Map}(V, W)$. It therefore suffices to show that $\mathrm{Map}(V, W)$ is quasi-complete. Let $(f_i)_{i \in I}$ be a Cauchy net in a bounded closed subset B of $\mathrm{Map}(V, W)$. Since the evaluation maps are continuous the subsets $\overline{B(v)} := \overline{\{f(v) : f \in B\}}$ of W, for any $v \in V$, are bounded and closed. By our assumptions on W the Cauchy net $(f_i(v))_i$ in $\overline{B(v)}$ converges to a uniquely determined vector $f(v) \in \overline{B(v)} \subseteq W$. This defines a map $f \in \mathrm{Map}(V, W)$. By construction the Cauchy net $(f_i)_i$ converges to f. Since B was closed we in fact have $f \in B$.

We now turn to the general case. According to Lemma 6.10 the closure $\overline{H}$ of H in $\mathcal{L}_s(V, W)$ is equicontinuous. By the case treated above $\overline{H}$ therefore is a complete subset of $\mathcal{L}_s(V,W)$. Since H is closed in the $\mathcal{B}$-topology it suffices to show that $\overline{H}$ is complete in $\mathcal{L}_{\mathcal{B}}(V,W)$ as well. We will in fact establish the following more general assertion:

Let $\mathcal{B}'$ and $\mathcal{B}$ be two families of bounded subsets of V such that the $\mathcal{B}'$-topology is finer than the $\mathcal{B}$-topology and the latter is finer than the weak topology; any subset $A \subseteq \mathcal{L}(V,W)$ which is complete for the $\mathcal{B}$-topology also is complete for the $\mathcal{B}'$-topology.

Let $(f_i)_{i\in I}$ be a Cauchy net in A with respect to the $\mathcal{B}'$- and hence also with respect to the $\mathcal{B}$-topology. In the $\mathcal{B}$-topology it converges, by assumption, to some $f \in A$. To check that f is the limit in the $\mathcal{B}'$-topology as well let $B \in \mathcal{B}'$ and $M \subseteq W$ be an open lattice. There is an $i \in I$ such that $f_j - f_k \in \mathcal{L}(B,M)$, or equivalently $f_j \in f_k + \mathcal{L}(B,M)$, for any $j,k \geq i$. But $\mathcal{L}(B,M) = \bigcap_{v\in B} \mathcal{L}(\{v\},M)$ is closed in the weak and hence in the $\mathcal{B}$-topology. It follows that $f \in f_k + \mathcal{L}(B,M)$, or equivalently $f_k - f \in \mathcal{L}(B,M)$, for any $k \geq i$.

Corollary 7.14:

If V is barrelled and W is Hausdorff and quasi-complete then $\mathcal{L}_{\mathcal{B}}(V,W)$ is Hausdorff and quasi-complete for any $\mathcal{B}$-topology which is finer than the weak topology.

Proof: Lemma 6.6, Cor. 6.16, and Prop. 7.13.

Corollary 7.15:

If V is bornological and W is Hausdorff and quasi-complete then $\mathcal{L}_b(V,W)$ is Hausdorff and quasi-complete.

Proof: Lemma 6.6, Prop. 6.12, and Prop. 7.13.

Proposition 7.16:

If V is bornological and W is Hausdorff and complete then $\mathcal{L}_b(V,W)$ is Hausdorff and complete.

Proof: By Lemma 6.6 the space $\mathcal{L}_b(V,W)$ is Hausdorff. Let $(f_i)_{i\in I}$ be a Cauchy net in $\mathcal{L}_b(V,W)$ and therefore in $\mathcal{L}_s(V,W) \subseteq \mathrm{Map}(V,W)$. This latter vector space visibly is Hausdorff and complete. Our net consequently has a pointwise limit $f \in \mathrm{Map}(V,W)$. In the proof of Lemma 6.10 we have seen that $\mathrm{Hom}_K(V,W)$ is closed in $\mathrm{Map}(V,W)$. Hence f is K-linear. In order to show that f indeed is the limit in $\mathcal{L}_b(V,W)$ of our Cauchy net we need three intermediate assertions.

Claim 1: Let $B \subseteq V$ be a bounded subset and $M \subseteq W$ be an open lattice; there is an $i \in I$ such that $(f_k - f)(B) \subseteq M$ for any $k \geq i$.

We certainly find an $i \in I$ such that $f_j - f_k \in \mathcal{L}(B,M)$ for any $j,k \geq i$. Since $\{g \in \mathrm{Map}(V,W) : g(B) \subseteq M\}$ is closed our claim follows by the same argument which we have used at the end of the proof of Prop. 7.13.

Claim 2: The map f respects sequences converging to the zero vector.

Let $(v_n)_{n\in\mathbb{N}}$ be such a sequence in V and let $M \subseteq W$ be an open lattice. By Lemma 7.10 the subset $B := \{v_n : n \in \mathbb{N}\}$ is bounded. Applying our first claim we find an $i \in I$ such that $f_i(v_n) - f(v_n) \in M$ for any $n \in \mathbb{N}$. Since f_i is continuous the sequence $(f_i(v_n))_n$ converges to the zero vector in W. This means that there exists an $m \in \mathbb{N}$ such that $f_i(v_n) \in M$ for any $n \geq m$. It follows that $f(v_n) \in M$ for any $n \geq m$.

Claim 3: The map respects bounded subsets.

Otherwise there exists a bounded subset $B \subseteq V$, an open lattice $M \subseteq W$, and a scalar $a \in K$ with $|a| > 1$ such that $f(B) \not\subseteq a^n M$ for any $n \in \mathbb{N}$. In other words, for any $n \in \mathbb{N}$ there is a vector $v_n \in B$ such that $a^{-n} f(v_n) \notin M$. Since B is bounded we find, given an open lattice $L \subseteq V$, an $m \in \mathbb{N}$ such that $B \subseteq a^m L$. Because of $a^{-n} v_n = (a^{-1})^{n-m}(a^{-m} v_n) \in L$ for any $n \geq m$ we see that the sequence $(a^{-n} v_n)_n$ in V converges to the zero vector. Applying our second claim we arrive at a contradiction.

Returning to the main line of the proof we now apply the last claim and Prop. 6.13 to the K-linear map f and obtain that f is continuous. The first claim then shows that the Cauchy net $(f_i)_i$ converges to f in $\mathcal{L}_b(V, W)$.

We finish this section by looking at the situation where W is arbitrary but V is quasi-complete. But first we have to introduce the following construction of general importance. Let A be any o-submodule in the locally convex K-vector space V. We let V_A denote the vector subspace of V generated by A. Since A is a lattice in V_A the gauge p_A is defined as a seminorm on V_A. We always view V_A as equipped with the locally convex topology defined by p_A. Since A is bounded in V_A it is a necessary condition for the inclusion $V_A \xrightarrow{\subseteq} V$ to be continuous that A is bounded in V.

Lemma 7.17:

If B is a bounded o-submodule in V then we have:

i. The inclusion $V_B \xrightarrow{\subseteq} V$ is continuous;

ii. if V is Hausdorff then (V_B, p_B) is a normed vector space;

iii. if V is Hausdorff and B is complete then (V_B, p_B) is a Banach space.

Proof: i. Let $L \subseteq V$ be an open lattice. Since B is bounded there is an $a \in K^\times$ such that $B \subseteq aL$ and hence that $a^{-1}B \subseteq V_B \cap L$. This shows that $V_B \cap L$ is an open lattice in V_B.

ii. Assume that $v \in V_B$ is a vector such that $0 = p_B(v) = \inf_{v \in aB} |a|$. It follows that $Kv \subseteq L^-(p_B) \subseteq B$. By the boundedness of B this implies that Kv is contained in any open lattice in V. Since V is Hausdorff we therefore must have $v = 0$.

iii. Let $(v_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in V_B. In particular, there is an $m \in \mathbb{N}$ such that $v_j - v_k \in B$ for any $j, k \geq m$. Fix a $0 \neq a \in o$ such that $av_n \in B$ for any $n \leq m$. Then $(av_n)_n$ is a Cauchy sequence in $B \subseteq V_B$ and hence, by i., in $B \subseteq V$. By assumption this sequence $(av_n)_n$ converges to some vector $v \in B \subseteq V$. Fix a scalar $c \in K$ such that $0 < |c| < 1$. Since $(av_n)_n$ is a Cauchy sequence in V_B there is an increasing sequence $n_1 < n_2 < \ldots$ of natural numbers such that

$$p_B(av_j - av_k) \leq |c|^{i+1} \quad \text{for any } j, k \geq n_i \text{ and any } i \in \mathbb{N} .$$

In particular,

$$av_{n_1} + cB \supseteq av_{n_2} + c^2 B \supseteq \ldots \supseteq av_{n_i} + c^i B \supseteq \ldots$$

With B all the convex subsets $av_{n_i} + c^i B$ are closed in V. It follows that the limit v in V of the sequence $(av_{n_i})_i$ lies in the intersection $\bigcap_i av_{n_i} + c^i B$. We consequently have $p_B(v - av_{n_i}) \leq |c|^i$ for any $i \in \mathbb{N}$. This shows that the sequence $(av_{n_i})_i$ converges to v already in V_B. Our original Cauchy sequence $(v_n)_n$ therefore converges in V_B to $a^{-1}v$.

Proposition 7.18:

Suppose that V is Hausdorff and quasi-complete; then $\mathcal{L}_s(V, W)$ and $\mathcal{L}_{\mathcal{B}}(V, W)$, for any $\mathcal{B}$-topology which is finer than the weak topology, have the same class of bounded subsets.

Proof: Since the identity map $\mathcal{L}_{\mathcal{B}}(V, W) \longrightarrow \mathcal{L}_s(V, W)$ is continuous any subset bounded for the $\mathcal{B}$-topology also is bounded for the weak topology. Let us assume vice versa that H is a bounded subset in $\mathcal{L}_s(V, W)$. To check that H is bounded in $\mathcal{L}_{\mathcal{B}}(V, W)$ as well let $B \in \mathcal{B}$ and let $M \subseteq W$ be an open lattice. We have to find an $a \in K^\times$ such that $H \subseteq a\mathcal{L}(B, M) = \mathcal{L}(B, aM)$. By Lemma 6.5 we may assume that B is a closed o-submodule of V. Since V is assumed to be quasi-complete B then is complete so that, by Lemma 7.17.iii, V_B is a Banach space and hence is barrelled (Example 2 after Cor. 6.16). Consider now the continuous linear map

$$\begin{array}{rcl} \alpha : \mathcal{L}_s(V, W) & \longrightarrow & \mathcal{L}_s(V_B, W) \\ f & \longmapsto & f|V_B \ . \end{array}$$

By Cor. 6.16 the bounded subset $\alpha(H) \subseteq \mathcal{L}_s(V_B, W)$ is equicontinuous. Since the aB for $0 \neq a \in o$ form a fundamental system of neighbourhoods of the zero vector in V_B this means that there is a $0 \neq a \in o$ such that $f(aB) \subseteq M$ for any $f \in H$. In other words we have $H \subseteq \mathcal{L}(B, a^{-1}M)$.

If we use an additional result which will be established later in the third chapter we can deduce the following consequence.

Corollary 7.19:

If the field K is spherically complete then any Hausdorff, bornological, and quasi-complete locally convex K-vector space V is barrelled.

Proof: According to Prop. 7.18 the locally convex vector spaces $\mathcal{L}_s(V,W)$ and $\mathcal{L}_b(V,W)$, for any W, have the same class of bounded subsets. Since V is bornological the bounded subsets in $\mathcal{L}_b(V,W)$ coincide, by Prop. 6.12, with the equicontinuous subsets. We obtain that the bounded subsets in $\mathcal{L}_s(V,W)$, for any W, are precisely the equicontinuous subsets. As we will prove in Prop. 13.9 this latter property characterizes, over a spherically complete field, the barrelled spaces.

§8 Fréchet spaces

In this section we will study in greater detail the properties of metrizable locally convex vector spaces.

Proposition 8.1:

For a Hausdorff locally convex K-vector space V the following assertions are equivalent:

i. V is metrizable;

ii. the topology of V can be defined by a metric d which satisfies in addition

(strict triangle inequality) $d(u,w) \le \max(d(u,v), d(v,w))$, *and*

(translation invariance) $d(u+w, v+w) = d(u,v)$ *for any* $u,v,w \in V$*;*

iii. the topology of V can be defined by a countable family of lattices;

iv. the topology of V can be defined by a countable family of seminorms.

Proof: The implications ii. $\Rightarrow$ i. $\Rightarrow$ iii. $\Rightarrow$ iv. are clear. It remains to show that iv. implies ii. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of seminorms which define the topology of V. By replacing p_n by $\max(p_1,\ldots,p_n)$ we may assume (compare Remark 4.7) that $p_1(v) \le p_2(v) \le \ldots$ for any $v \in V$. We define

$$\|v\|_F := \sup_{n\in\mathbb{N}} \frac{1}{2^n} \cdot \frac{p_n(v)}{1+p_n(v)}$$

for $v \in V$. The following properties are almost immediate:

- $\|v\|_F \ge 0$.

- $\|v\|_F = 0$ if and only if $v = 0$. Note that if $v \neq 0$ then, since V is Hausdorff, there is an $n \in \mathbb{N}$ such that $p_n(v) \neq 0$.

- $\|v + w\|_F \leq \max(\|v\|_F, \|w\|_F)$. Using the inequality $a/(1+a) \leq b/(1+b)$ for any two real numbers $0 < a \leq b$ we have

$$\frac{p_n(v+w)}{1+p_n(v+w)} \leq \frac{\max(p_n(v), p_n(w))}{1+\max(p_n(v), p_n(w))} = \max(\frac{p_n(v)}{1+p_n(v)}, \frac{p_n(w)}{1+p_n(w)}) \ .$$

We see that $d(v,w) := \|v - w\|_F$ is a translation invariant metric on V satisfying the strict triangle inequality. We claim that d defines the topology of V. Because of $p_1 \leq p_2 \leq \ldots$ the lattices

$$V(n) := \{v \in V : p_n(v) \leq 2^{-n}\}$$

for $n \in \mathbb{N}$ form a fundamental system of neighbourhoods of the zero vector. Note that for any real number $a \geq 0$ and any $m \in \mathbb{N}$ one has $a \leq 2^{-(m-1)}$ provided $a/(1+a) \leq 2^{-m}$. This implies that

$$\{v \in V : \|v\|_F \leq 2^{-(2m+1)}\} \subseteq V(m) \ .$$

On the other hand, using that $2^{-n} \cdot p_n(v)/(1+p_n(v)) \leq 2^{-m}$ for $n \geq m$ and $2^{-n} \cdot p_n(v)/(1+p_n(v)) \leq p_m(v)/(1+p_m(v)) \leq p_m(v)$ for $n \leq m$ we obtain

$$V(m) \subseteq \{v \in V : \|v\|_F \leq 2^{-m}\} \ .$$

Definition:

A locally convex K-vector space is called a K-Fréchet space if it is metrizable and complete.

Proposition 8.2:

Every Fréchet space is bornological and barrelled.

Proof: Prop. 6.14 and Example 2 after Cor. 6.16.

Proposition 8.3:

Let V be a Fréchet (resp. Banach) space, and let $U \subseteq V$ be a closed vector subspace; then V/U with the quotient topology is a Fréchet (resp. Banach) space as well.

Proof: Since U is closed the quotient V/U is Hausdorff. It follows immediately from §5.B that the quotient topology on V/U can be defined by a countable family of seminorms (resp. a single norm). In particular, V/U is metrizable by Prop. 8.1. It remains to show that V/U is complete. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in V such that $(v_n + U)_n$ is a Cauchy sequence in V/U. Moreover, let $L_1 \supseteq L_2 \supseteq \ldots$ be a descending sequence of open lattices in V which form a fundamental system of neighbourhoods of the zero vector. By passing to a subsequence we may assume that

$$v_{n+1} - v_n \in L_n + U \quad \text{ for any } n \in \mathbb{N} .$$

Choose a $w_{n+1} \in L_n$ such that $v_{n+1} - v_n - w_{n+1} \in U$ and define

$$\begin{aligned} v'_1 &:= v_1 \\ v'_2 &:= v_1 + w_2 \\ &\vdots \\ v'_n &:= v_1 + w_2 + \ldots + w_n \\ &\vdots \end{aligned}$$

Because of $v'_{n+1} - v'_n = w_{n+1} \in L_n$ the modified sequence $(v'_n)_n$ is a Cauchy sequence in V and converges therefore to some vector $v \in V$. It follows that $(v_n + U)_n = (v'_n + U)_n$ converges to $v + U$ in V/U.

Continuous linear maps between Fréchet spaces have particularly nice properties as we will see presently. If $f : V \longrightarrow W$ is any linear map between two locally convex K-vector spaces then the *graph* of f is defined to be the subset

$$\Gamma(f) := \{(v, f(v)) : v \in V\}$$

of the direct product space $V \times W$.

Remark 8.4:

If f is continuous and W is Hausdorff then the graph $\Gamma(f)$ is a closed subset of $V \times W$.

Proof: Since W is Hausdorff the diagonal $\Delta \subseteq W \times W$ is closed. But $\Gamma(f) = (f \times \mathrm{id})^{-1}(\Delta)$.

Proposition 8.5: (Closed graph theorem)

Let $f : V \longrightarrow W$ be a linear map from a barrelled locally convex K-vector space (e.g., a Fréchet space) V into a K-Fréchet space W; if the graph $\Gamma(f)$ is closed then the map f is continuous.

Proof: Let $M \subseteq W$ be an open lattice. Since V is barrelled the closed lattice $\overline{f^{-1}(M)}$ in V is open. It therefore suffices to show that $\overline{f^{-1}(M)} = f^{-1}(M)$.

Let $M = M_1 \supseteq M_2 \supseteq \ldots$ be a descending sequence of open lattices in W which form a fundamental system of neighbourhoods of the zero vector. Then

$$\overline{f^{-1}(M)} = \overline{f^{-1}(M_1)} \supseteq \overline{f^{-1}(M_2)} \supseteq \ldots$$

is a descending sequence of open lattices in V. Consider now a vector $v \in \overline{f^{-1}(M)}$. Inductively we find vectors $v_n \in f^{-1}(M_n)$ such that

$$v - (v_1 + \ldots + v_n) \in \overline{f^{-1}(M_{n+1})}$$

for any $n \in \mathbb{N}$. Put $v'_n := v_1 + \ldots + v_n$. Because of $f(v'_{n+1}) - f(v'_n) = f(v_{n+1}) \in M_{n+1}$ the sequence $(f(v'_n))_n$ is a Cauchy sequence in M and converges therefore to some vector $w \in M$. We claim that $f(v) = w$. Since the graph $\Gamma(f)$ is closed it suffices to show that for arbitrary open lattices $L \subseteq V$ and $N \subseteq W$ the intersection

$$\Gamma(f) \cap ((v + L) \times (w + N)) \neq \emptyset$$

is nonempty. But there is an $m \in \mathbb{N}$ such that

$$M_{m+1} \subseteq N \quad \text{and} \quad f(v'_m) - w \in N \ .$$

Since $v - v'_m \in \overline{f^{-1}(M_{m+1})} \subseteq f^{-1}(M_{m+1}) + L$ there is a $u \in f^{-1}(M_{m+1})$ such that $v - v'_m - u \in L$. It follows that $v'_m + u \in v + L$ and $f(v'_m + u) = f(v'_m) + f(u) \in w + N + M_{m+1} \subseteq w + N$.

Proposition 8.6: (Open mapping theorem)

Suppose that V is a Fréchet space and that W is Hausdorff and barrelled; then every surjective continuous linear map $f : V \longrightarrow W$ is open.

Proof: Since W is Hausdorff the subspace $\ker(f)$ is closed in V. By Prop. 8.3 the quotient $V/\ker(f)$ is a Fréchet space, too. Moreover, the residue class projection $V \longrightarrow V/\ker(f)$ is open. It therefore suffices to show that the induced continuous linear bijection $V/\ker(f) \longrightarrow W$ is open. In other words, we may assume that the map f is bijective. In this situation our assertion amounts to the continuity of the inverse map f^{-1}. According to the closed graph theorem Prop. 8.5 this can be checked by showing that the graph $\Gamma(f^{-1})$ is closed in $W \times V$. But by Remark 8.4 the graph $\Gamma(f)$ is closed in $V \times W$, and $\Gamma(f^{-1})$ corresponds to $\Gamma(f)$ under the topological isomorphism

$$\begin{array}{ccc} W \times V & \xrightarrow{\cong} & V \times W \\ (w, v) & \longmapsto & (v, w) \ . \end{array}$$

Corollary 8.7:

Any continuous linear bijection between two K-Fréchet spaces is a topological isomorphism.

The above results can be extended to linear maps between locally convex K-vector spaces V and W of a more general type as follows.

Proposition 8.8:

Suppose that the topology of V is the locally convex final topology with respect to a family of linear maps $g_h : V_h \longrightarrow V$, for $h \in H$, from Fréchet spaces V_h into V; suppose also that there is a countable family $(W_n)_{n\in\mathbb{N}}$ of Fréchet spaces W_n together with injective continuous linear maps $i_n : W_n \longrightarrow W$ such that $W = \bigcup_{n\in\mathbb{N}} i_n(W_n)$; we then have:

i. Any linear map $f : V \longrightarrow W$ whose graph is closed is continuous;

ii. if V is Hausdorff then any surjective continuous linear map $f : W \longrightarrow V$ is open.

Proof: i. We have to check that the maps $f \circ g_h$ are continuous. Moreover, because of

$$\Gamma(f \circ g_h) = (g_h \times \mathrm{id})^{-1}(\Gamma(f))$$

the graphs $\Gamma(f \circ g_h)$ are closed. This reduces us to proving our assertion in the case where V already is a Fréchet space. In this case we study the covering

$$V = \bigcup_{n\in\mathbb{N}} f^{-1}(i_n(W_n))$$

by freely using some notions and results from Baire's category theory (compare [B-GT] Chap.IX §5). Let $I \subseteq \mathbb{N}$ denote the subset of all those natural numbers n such that the closure of $f^{-1}(i_n(W_n))$ in V has a nonempty interior. The complement of $\bigcup_{n\in I} f^{-1}(i_n(W_n))$ in V then is a meagre subset. By Baire's theorem V is a Baire space. This has two consequences of relevance here. First of all I has to be nonempty. Secondly, being the complement of a meagre subset in a Baire space the subspace $\bigcup_{n\in I} f^{-1}(i_n(W_n))$ also is a Baire space. It follows that there exists an $m \in I$ such that the subspace $f^{-1}(i_m(W_m))$ has no countable covering by closed subsets with empty interior. (We use here the following observation: Let Y be a subspace of a topological space X; if A is closed with empty interior in Y then its closure in X has empty interior, too.) By Example 1 after Cor. 6.16 the vector subspace $f^{-1}(i_m(W_m))$ of V with the subspace topology is barrelled.

Let f_m denote the uniquely determined linear map which makes the diagram

$$\begin{array}{ccc} f^{-1}(i_m(W_m)) & \xrightarrow{f_m} & W_m \\ \subseteq \downarrow & & \downarrow i_m \\ V & \xrightarrow{f} & W \end{array}$$

commutative. Its graph $\Gamma(f_m) = (\subseteq \times i_m)^{-1}(\Gamma(f))$ is closed. By the closed graph theorem Prop. 8.5 the map f_m therefore is continuous. On the other hand, the vector subspace $\overline{f^{-1}(i_m(W_m))}$, by construction, has nonempty interior and hence is open and consequently a lattice in V. This implies that $\overline{f^{-1}(i_m(W_m))} = V$. By Lemma 7.3 the map f_m therefore extends uniquely to a continuous linear map $\overline{f_m} : V \longrightarrow W_m$. By construction the two maps $i_m \circ \overline{f_m}$ and f coincide on the dense subspace $f^{-1}(i_m(W_m))$. The continuity of $i_m \circ \overline{f_m}$ implies that $\Gamma(i_m \circ \overline{f_m}) = \overline{\Gamma(i_m \circ f_m)}$. Since $\Gamma(i_m \circ f_m)$ is contained in the closed graph $\Gamma(f)$ we obtain that $\Gamma(i_m \circ \overline{f_m}) \subseteq \Gamma(f)$. This shows that $f = i_m \circ \overline{f_m}$ and hence that f is continuous.

ii. Set $U := \ker(f)$. Since the residue class projection $W \longrightarrow W/U$ is open it suffices to show that the induced continuous linear map $W/U \longrightarrow V$ is open. We claim that W/U satisfies the same assumption as W. Define $U_n := i_n^{-1}(U)$ for any $n \in \mathbb{N}$. The induced linear maps $i'_n : W_n/U_n \longrightarrow W/U$ are injective and continuous. The assumption that V is Hausdorff implies that each U_n is closed in W_n. It therefore follows from Prop. 8.3 that each W_n/U_n is a Fréchet space. Obviously we have $W/U = \bigcup_n i'_n(W_n/U_n)$. Having established our claim we see that it suffices to treat the case where the map f is bijective. In this case the same argument as in the proof of the open mapping theorem Prop. 8.6 shows that i. implies the continuity of the inverse map f^{-1}.

The following byproduct of the above proof sometimes is useful.

Corollary 8.9:

Suppose that V is a Fréchet space and that there is a countable family $(W_n)_{n \in \mathbb{N}}$ of Fréchet spaces W_n together with injective continuous linear maps $i_n : W_n \longrightarrow W$ such that $W = \bigcup_{n \in \mathbb{N}} i_n(W_n)$; for any continuous linear map $f : V \longrightarrow W$ there is an $m \in \mathbb{N}$ and a unique continuous linear map $f_m : V \longrightarrow W_m$ such that $f = i_m \circ f_m$.

§9 The dual space

Let V be a locally convex K-vector space.

Definition:

$V' := \mathcal{L}(V, K)$ is called the dual space of V; more precisely, we call $V'_{\mathcal{B}} := \mathcal{L}_{\mathcal{B}}(V, K)$ the $\mathcal{B}$-dual of V; in particular, $V'_s := \mathcal{L}_s(V, K)$ and $V'_b := \mathcal{L}_b(V, K)$ are called the weak and strong dual, respectively.

As special cases of earlier results we have the following facts.

Proposition 9.1:

i. $V'_{\mathcal{B}}$ is Hausdorff provided the $\mathcal{B}$-topology is finer than the weak topology;

ii. if V is bornological then V'_b is Hausdorff and complete;

iii. if V is barrelled then $V'_{\mathcal{B}}$ is Hausdorff and quasi-complete for all $\mathcal{B}$-topologies which are finer than the weak topology.

Proof: i. Lemma 6.6, ii. Prop. 7.16, iii. Cor. 7.14.

The fundamental question which arises of course is whether there are any non-zero continuous linear forms on V. And indeed to obtain a positive answer we have to assume that the field K is spherically complete.

Proposition 9.2: (Hahn-Banach)

Suppose that the field K is spherically complete, and let U be a K-vector space, q a seminorm on U, and $U_o \subseteq U$ a vector subspace; for any linear form $\ell_o : U_o \longrightarrow K$ such that $|\ell_o(v)| \leq q(v)$ for any $v \in U_o$ there is a linear form $\ell : U \longrightarrow K$ such that $\ell | U_o = \ell_o$ and $|\ell(v)| \leq q(v)$ for any $v \in U$.

Proof: In a first step we fix a vector $v_1 \in U \backslash U_o$ and we will extend ℓ_o to the larger subspace $U_1 := U_o + K v_1$. Consider, for any $v \in U_o$, the subset

$$B(v) := \{a \in K : |a - \ell_o(v)| \leq q(v - v_1)\}$$

of K. For any two vectors $v, v' \in U_o$ we have

$$|\ell_o(v) - \ell_o(v')| \leq |\ell_o(v - v')| \leq q(v - v') \leq \max(q(v - v_1), q(v' - v_1))$$

and hence $\ell_o(v) \in B(v')$ or $\ell_o(v') \in B(v)$. This means that always

$$B(v) \cap B(v') \neq \emptyset \ .$$

If there is a vector $v_0 \in U_o$ such that $q(v_0 - v_1) = 0$ then $B(v_0)$ consists of one point and hence

$$\bigcap_{v \in U_o} B(v) = B(v_0) = \{\ell_o(v_0)\} \ .$$

Otherwise each $B(v)$ is a closed ball. Since K is assumed to be spherically complete Lemma 1.3 then says that the intersection

$$\bigcap_{v \in U_o} B(v) \neq \emptyset$$

is nonempty as well. We therefore find, in any case, a scalar

$$b \in \bigcap_{v \in U_o} B(v) \ .$$

We now extend ℓ_o to a linear form ℓ_1 on U_1 by $\ell_1(v_1) := b$. We have $|\ell_1| \leq q|U_1$ since

$$|\ell_1(v + av_1)| = |a| \cdot |\ell_o(a^{-1}v) + b| \leq |a| \cdot q(-a^{-1}v - v_1) = q(v + av_1)$$

for any $a \in K^\times$.

In order to prove, in a second step, our assertion we consider the set of all pairs (U_1, ℓ_1) of vector subspaces $U_o \subseteq U_1 \subseteq U$ and linear forms $\ell_1 : U_1 \longrightarrow K$ such that $\ell_1|U_o = \ell_o$ and $|\ell_1(v)| \leq q(v)$ for any $v \in U_1$. This set is inductively ordered in an obvious way. By Zorn's lemma it therefore contains a maximal element $(\widetilde{U}, \ell)$. It is immediate from the first step that necessarily $\widetilde{U} = U$.

Corollary 9.3:

Suppose that K is spherically complete and that V is Hausdorff; for any nonzero vector $v_o \in V$ there is a continuous linear form $\ell : V \longrightarrow K$ such that $\ell(v_o) = 1$.

Proof: Since V is Hausdorff there is a continuous seminorm q on V such that $q(v_o) > 0$. By scaling we may assume that $q(v_o) = 1$. Applying Prop. 9.2 to $U := V, q, U_o := Kv_o$, and $\ell_o(av_o) := a$ we obtain a linear form $\ell : V \longrightarrow K$ such that $\ell(v_o) = \ell_o(v_o) = 1$ and

$$|\ell(v)| \leq q(v) \quad \text{ for any } v \in V \ .$$

By Prop. 6.1 this latter inequality ensures the continuity of ℓ.

Corollary 9.4:

Suppose that K is spherically complete, and let $V_o \subseteq V$ be a vector subspace (with the subspace topology); every continuous linear form ℓ_o on V_o extends to a continuous linear form ℓ on V, i.e., $\ell|V_o = \ell_o$.

Proof: By Prop. 6.1 there is a continuous seminorm q on V_o such that $|\ell_o(v)| \leq q(v)$ for any $v \in V_o$. By the construction of the subspace topology we may assume that q actually is the restriction of a continuous seminorm q on V. Now apply Prop. 9.2.

Corollary 9.5:

Suppose that K is spherically complete and that V is Hausdorff; for any finite dimensional vector subspace $V_0 \subseteq V$ there is a vector subspace $V_1 \subseteq V$ such that the linear map

$$\begin{array}{ccc} V_0 \bigoplus V_1 & \xrightarrow{\cong} & V \\ (v_0, v_1) & \longmapsto & v_0 + v_1 \end{array}$$

is a topological isomorphism (w.r.t. the direct sum of the subspace topologies on the left hand side).

Proof: Since V_0 is topological isomorphic to K^n by Prop. 4.13 it is clear that the Cor. 9.4 remains valid for continuous linear maps into V_0 (instead of linear forms). There exists therefore a continuous linear map $f : V \longrightarrow V_0$ such that $f|V_0 = \mathrm{id}$. We define $V_1 := \ker(f)$. The map in the assertion obviously is a continuous bijection, and its inverse map

$$\begin{array}{ccc} V & \longrightarrow & V_0 \bigoplus V_1 \\ v & \longmapsto & (f(v), v - f(v)) \end{array}$$

is continuous as well (observe Lemma 5.2.ii).

Corollary 9.6:

Suppose that K is spherically complete, and let $A \subseteq V$ be a closed o-submodule; for any vector $v_o \in V \backslash A$ there is a continuous linear form ℓ on V such that $|\ell(v_o)| = 1$ and $|\ell(v)| \leq 1$ for any $v \in A$.

Proof: Since A is closed there is an open lattice $M \subseteq V$ such that

$$(v_o + M) \cap A = \emptyset \ .$$

The open lattice $L := A + M$ then satisfies $A \subseteq L$ and $v_o \notin L$. According to Lemma 4.5.ii the corresponding gauge p_L is a continuous seminorm on V. Since $L^-(p_L) \subseteq L$ by Lemma 2.2.ii we have $p_L(v_o) \geq 1$. Applying Prop. 9.2 we obtain a linear form ℓ on V such that $|\ell(v_o)| = 1$ and $|\ell(v)| \leq p_L(v)$ for any $v \in V$. In particular, $|\ell(v)| \leq p_L(v) \leq 1$ for any $v \in A \subseteq L$. Moreover, with p_L also ℓ is continuous.

Proposition 9.7:

Suppose that K is spherically complete and that V is Hausdorff; the linear map

$$\begin{array}{rcl} \delta : V & \longrightarrow & (V'_s)'_s \\ v & \longmapsto & \delta_v(\ell) := \ell(v) \end{array}$$

is a continuous bijection.

Proof: Because of

$$\delta_v^{-1}(o) = \{\ell \in V' : \ell(v) \in o\} = \mathcal{L}(\{v\}, o)$$

each linear form δ_v is continuous. Hence the map δ is well defined. It is continuous since

$$\delta^{-1}(\mathcal{L}(\{\ell\}, o)) = \ell^{-1}(o) \ .$$

It follows from Cor. 9.3 that δ is injective. For the surjectivity let $d \in (V_s')'$ be a fixed continuous linear form on the weak dual V_s'. Since $d^{-1}(o)$ is open in V_s' we find finitely many vectors $v_1, \ldots, v_n \in V$ such that

$$\mathcal{L}(\{v_1, \ldots, v_n\}, o) \subseteq d^{-1}(o) \ .$$

We certainly may assume that

- $v_1, \ldots, v_m$ are linearly independent and
- $v_j = \sum\limits_{i=1}^{m} a_{ij} v_i$ for each $m < j \leq n$ with $a_{ij} \in K$.

Fix a scalar $a \in K$ such that $|a| = \max(1, |a_{1m+1}|, \ldots, |a_{mn}|)$. We then have

$$\mathcal{L}(\{v_1, \ldots, v_m\}, a^{-1}o) \subseteq \mathcal{L}(\{v_1, \ldots, v_n\}, o) \subseteq d^{-1}(o) \ .$$

We consider, for any $1 \leq i \leq m$, the vector subspace

$$V_i := Kv_1 \oplus \ldots \oplus Kv_{i-1} \oplus Kv_{i+1} \oplus \ldots \oplus Kv_m$$

of V. Since V is Hausdorff each V_i (with the subspace topology) is topologically isomorphic to K^{m-1} (Prop. 4.13). The V_i therefore are complete and hence closed subspaces. By applying Cor. 9.3 to V/V_i we obtain, for each $1 \leq i \leq m$, a continuous linear form $\ell_i : V \longrightarrow K$ such that

$$\ell_i(v_i) = 1 \quad \text{and} \quad \ell_i(v_j) = 0 \ \text{ for } 1 \leq j \neq i \leq m \ .$$

We define

$$v := \sum_{i=1}^{m} d(\ell_i) v_i \ ,$$

and we claim that

$$\delta_v = d \ .$$

The vector space $V = V_0 \oplus U$ is the locally convex direct sum of $V_0 := Kv_1 \oplus \ldots \oplus Kv_m$ and the simultaneous kernel U of the linear forms $\ell_1, \ldots, \ell_m$. It follows that the dual space $V' = V_0' \oplus U'$ is the algebraic direct sum of the dual spaces V_0' and

U'. Moreover, $V_0' = K\ell_1 \oplus \ldots \oplus K\ell_m$ and $U' = \{\ell \in V' : \ell(v_1) = \ldots \ell(v_m) = 0\}$. Because of

$$\delta_v(\ell_j) = \ell_j(v) = \sum_{i=1}^{m} d(\ell_i)\ell_j(v_i) = d(\ell_j)$$

we have

$$\delta_v | V_0' = d | V_0' \ .$$

Consider on the other hand any $\ell \in U'$. For any $b \in K^\times$ we have

$$b^{-1}\ell \in \mathcal{L}(\{v_1, \ldots, v_m\}, a^{-1}o) \subseteq d^{-1}(o) \quad \text{ and hence } \quad d(\ell) \in bo \ .$$

It follows that

$$d(\ell) = 0 = \ell(v) = \delta_v(\ell) \ .$$

This last result shows that over a spherically complete field K any Hausdorff locally convex K-vector space V possesses "sufficiently many" continuous linear forms. But the map δ rarely is a topological isomorphism. We therefore introduce the *weak topology* on V as the initial topology with respect to the map δ into the locally convex vector space $(V_s')_s'$. We write V_s for V equipped with the weak topology. The weak topology on V obviously can be characterized as the weakest topology for which all linear forms $\ell \in V'$ are continuous. In particular, it is weaker than the given topology of V. The corresponding dual linear map

$$(V_s)_s' \xrightarrow{\cong} V_s'$$

is a topological isomorphism. Over a spherically complete field V_s is Hausdorff if V is.

The different $\mathcal{B}$-topologies on V' lead to different "bidual spaces". For any $\mathcal{B}$-topology which is finer than the weak topology we have the inclusions

$$(V_s')' \subseteq (V_\mathcal{B}')' \subseteq (V_b')' \ .$$

The linear map

$$\begin{aligned} \delta : V &\longrightarrow (V_\mathcal{B}')' \\ v &\longmapsto \delta_v(\ell) := \ell(v) \end{aligned}$$

is called a *duality map*.

Remark 9.8:

Suppose that K is spherically complete; the initial topology on V with respect to the duality map $\delta : V \longrightarrow (V_b')_b'$ is finer than the given topology of V.

Proof: Let $L \subseteq V$ be an open lattice. Then the subset $H := \{\ell \in V' : \ell(L) \subseteq o\}$ of V'_b is equicontinuous and hence bounded by Lemma 6.8. Fix a scalar $a \in K$ such that $0 < |a| < 1$. We claim that

$$\delta^{-1}(\mathcal{L}(H, ao)) = \{v \in V : \ell(v) \in ao \text{ for any } \ell \in H\}$$

is contained in L. For any vector $v \in V \backslash L$ we find, by Cor. 9.6, an $\ell \in V'$ such that $\ell(L) \subseteq o$ and $|\ell(v)| = 1$. In other words, there is an $\ell \in H$ such that $\ell(v) \notin ao$.

Definition:

A Hausdorff locally convex K-vector space V is called

- $\underline{semi\text{-}reflexive}$ *if the duality map* $\delta : V \longrightarrow (V'_b)'$ *is bijective,*

- $\underline{reflexive}$ *if the duality map* $\delta : V \longrightarrow (V'_b)'_b$ *is a topological isomorphism, and*

- $\underline{pseudo\text{-}reflexive}$ *if the duality map* $\delta : V \longrightarrow (V'_b)'_b$ *induces a topological isomorphism between* V *and* $\mathrm{im}(\delta)$.

Obviously a pseudo-reflexive space is reflexive if and only if it is semi-reflexive.

Lemma 9.9:

Suppose that K is spherically complete and that V is Hausdorff; if V is bornological or barrelled then V is pseudo-reflexive.

Proof: By Cor. 9.3 the duality map δ is injective. Because of Remark 9.8 it therefore remains to show that $\delta : V \longrightarrow (V'_b)'_b$ is continuous. Let $H \subseteq V'_b$ be a bounded subset. Our assumption on V guarantees that H is equicontinuous (Prop. 6.12 or Cor. 6.16). Hence $L := \bigcap_{\ell \in H} \ell^{-1}(o)$ is an open lattice in V. The continuity of δ now is a consequence of the identity

$$\delta^{-1}(\mathcal{L}(H, o)) = \{v \in V : \ell(v) \in o \text{ for any } \ell \in H\} = L \ .$$

We will investigate these notions in more detail in the third chapter. We finish this section by determining how the passage to the dual space is behaved with respect to direct products and locally convex direct sums.

Let $(V_h)_{h \in H}$ be a family of locally convex K-vector spaces. As a consequence of the universal property of the locally convex direct sum (Lemma 5.1.i) the map

$$\begin{aligned} \sigma : (\bigoplus_{h \in H} V_h)' \quad &\longrightarrow \quad \prod_{h \in H} V'_h \\ \ell \quad &\longmapsto \quad (\ell | V_h)_h \end{aligned}$$

is a well defined linear bijection.

Proposition 9.10:

The map σ induces topological isomorphisms

$$(\bigoplus_{h\in H} V_h)'_b \xrightarrow{\cong} \prod_{h\in H} (V_h)'_b$$

and

$$(\bigoplus_{h\in H} V_h)'_s \xrightarrow{\cong} \prod_{h\in H} (V_h)'_s \ .$$

Proof: First of all, because of Lemma 4.6 and Lemma 5.3 we may assume that the V_h are Hausdorff. The argument for the weak topologies being completely analogous we only discuss the case of the strong topologies. In $\prod_h (V_h)'_b$ the open lattices of the form

$$\mathcal{L} = \prod_{h\in H\setminus I} V'_h \times \prod_{h\in I} \mathcal{L}(B_h, o) \ ,$$

where $I \subseteq H$ is some finite subset and each $B_h \subseteq V_h$ is a bounded o-submodule, form a fundamental system of neighbourhoods of the zero vector. It easily follows from Lemma 5.1.iii that $\bigoplus_{h\in I} B_h$ is bounded in $\bigoplus_{h\in H} V_h$. Hence

$$\sigma^{-1}(\mathcal{L}) = \mathcal{L}(\bigoplus_{h\in I} B_h, o)$$

is an open lattice in $(\bigoplus_h V_h)'_b$.

On the other hand let $B \subseteq \bigoplus_h V_h$ be any bounded o-submodule. The projection B_h of B in V_h is bounded in V_h. In the proof of Prop. 7.12 we have seen that there is a finite subset $I \subseteq H$ such that $B_h = 0$ for $h \notin I$. Hence $B \subseteq \bigoplus_{h\in I} B_h$ and

$$\sigma(\mathcal{L}(B, o)) \supseteq \prod_{h\in H\setminus I} V'_h \times \prod_{h\in I} \mathcal{L}(B_h, o) \ .$$

This shows that $\sigma(\mathcal{L}(B, o))$ is an open lattice in $\prod_h (V_h)'_b$.

Next we consider the linear map

$$\begin{aligned} \pi : (\prod_{h\in H} V_h)' &\longrightarrow \bigoplus_{h\in H} V'_h \\ \ell &\longmapsto \sum_h \ell|V_h \ . \end{aligned}$$

To see that π is well defined let $\ell : \prod_h V_h \longrightarrow K$ be a continuous linear form. By the definition of the product topology there must exist a finite subset $I \subseteq H$ such that

$$\prod_{h \in H \setminus I} V_h \times \prod_{h \in I} \{0\} \subseteq \ell^{-1}(o) \ .$$

Since the left hand side is a vector subspace it follows that

$$\prod_{h \in H \setminus I} V_h \times \prod_{h \in I} \{0\} \subseteq \ker(\ell)$$

and hence that $\pi(\ell)$ indeed is a finite sum. The map π is bijective since an inverse map is defined by

$$\sum_h \ell_h \longmapsto [(v_h)_h \mapsto \sum_h \ell_h(v_h)] \ .$$

Proposition 9.11:

The map π induces a topological isomorphism

$$(\prod_{h \in H} V_h)'_b \xrightarrow{\cong} \bigoplus_{h \in H} (V_h)'_b \ .$$

Proof: By Lemma 5.1.iii the open lattices of the form

$$\mathcal{L} = \bigoplus_{h \in H} \mathcal{L}(B_h, o) \ ,$$

where each $B_h \subseteq V_h$ is a bounded o-submodule, form a fundamental system of neighbourhoods of the zero vector in $\bigoplus_h (V_h)'_b$. It follows immediately from the description of the product topology through defining seminorms that $\prod_h B_h$ is bounded in $\prod_h V_h$. Hence

$$\pi^{-1}(\mathcal{L}) = \mathcal{L}(\prod_h B_h, o)$$

is an open lattice in $(\prod_h V_h)'_b$.

Let on the other hand $B \subseteq \prod_h V_h$ be a bounded o-submodule. The projection B_h of B in V_h is bounded in V_h. We see that

$$\pi(\mathcal{L}(B, o)) \supseteq \pi(\mathcal{L}(\prod_h B_h, o)) = \bigoplus_h \mathcal{L}(B_h, o)$$

is an open lattice in $\bigoplus_h (V_h)'_b$.

Chap. II: The structure of Banach spaces

This is a rather brief chapter in which we establish two general structural results about Banach spaces: One for arbitrary Banach spaces over a discretely valued field K and another one for countably generated Banach spaces over an arbitrary field K. In this book a Banach space is a complete locally convex vector space whose topology can be defined by a norm. In other words the norm is not considered to be part of the structure. Correspondingly the rich metric theory of Banach spaces is outside the scope of this book. Some part of it is present in the proofs of the two structure theorems, though. A surprising and at first disappointing consequence is the fact that over a spherically complete field K there are no infinite dimensional reflexive Banach spaces. This is probably the main reason why many applications of nonarchimedean analysis focus on different and more complicated classes of locally convex vector spaces.

§10 Structure theorems

As a basic example for a Banach space we had introduced in §3, for an arbitrary set X, the Banach space

$$c_{\mathrm{o}}(X) = \begin{array}{l} \text{all functions } \phi : X \to K \text{ such that, for any} \\ \epsilon > 0, \text{ the set } \{x \in X : |\phi(x)| \geq \epsilon\} \text{ is finite} \end{array}$$

with the sup-norm

$$\|\phi\|_\infty = \sup_{x \in X} |\phi(x)| \ .$$

This Banach space has the following universal property. Let $\alpha : X \longrightarrow V$ be an arbitrary map from X into a Hausdorff and quasi-complete locally convex K-vector space V such that the image $\mathrm{im}(\alpha)$ is a bounded subset of V. We claim that there is a unique continuous linear map $f_\alpha : c_{\mathrm{o}}(X) \longrightarrow V$ such that

$$f_\alpha(1_x) = \alpha(x) \quad \text{ for any } x \in X \ .$$

Let I denote the set of all finite subsets of X directed by the inclusion relation. For any function $\phi \in c_{\mathrm{o}}(X)$ we consider the net $(\phi_i)_{i \in I}$ in V which is defined by

$$\phi_i := \sum_{x \in i} \phi(x)\alpha(x) \ .$$

Let B denote the closure of the o-submodule of V generated by the image of α. By the assumption on the map α and Lemma 4.10 the o-submodule B is bounded and closed. Hence if $L \subseteq V$ is an open lattice then there is an $a \in K^\times$ such that $aB \subseteq L$. Define $i := \{x \in X : |\phi(x)| > |a|\}$. We then have $\phi_j - \phi_k \in aB \subseteq L$

for any $j, k \geq i$. This shows that $(\phi_i)_i$ is a Cauchy net in V. More precisely, if $c \in K$ is a scalar such that $\|\phi\|_\infty \leq |c|$ then $(\phi_i)_i$ is a Cauchy net in the bounded and closed subset cB of V. Since V is assumed to be quasi-complete this Cauchy net converges to a unique vector $f_\alpha(\phi) \in V$. In this way we obtain a linear map

$$f_\alpha : c_{\mathrm{o}}(X) \longrightarrow V \quad \text{ such that } f_\alpha(1_x) = \alpha(x) \text{ for any } x \in X\ .$$

Any continuous seminorm q on V satisfies

$$q(f_\alpha(\phi)) \leq (\sup_{v \in B} q(v)) \cdot \|\phi\|_\infty\ .$$

Hence f_α is continuous (Prop. 6.1.iii). Finally, f_α is uniquely determined by α since the 1_x, for $x \in X$, generate a vector subspace which is dense in $c_{\mathrm{o}}(X)$.

Proposition 10.1:

Suppose that the field K is discretely valued; every K-Banach space V is topologically isomorphic to a K-Banach space $c_{\mathrm{o}}(X)$ for some set X.

Proof: Let $k := o/\mathfrak{m}$ denote the residue class field of K. According to Lemma 1.4 we have $|K^\times| = r^{\mathbf{Z}}$ for some real number $0 < r < 1$. We may assume that the defining norm $\|\ \|$ on V has the additional property that

$$\|V\| \subseteq |K|\ .$$

The reason is that we always can replace a given defining norm $\|\ \|'$ by the norm

$$\|v\| := \inf \{s \in |K| : s \geq \|v\|'\}$$

which, because of

$$r \leq \|v\|'/\|v\| \leq 1\ ,$$

defines the same topology.

Since $\mathfrak{m} \cdot B_1(0) \subseteq B_1^-(0)$ the o-module quotient

$$\overline{V} := B_1(0)/B_1^-(0)$$

is a k-vector space. We fix a k-basis $(\overline{v}_x)_{x \in X}$ of $\overline{V}$ and vectors $v_x \in B_1(0)$ such that $\overline{v}_x = v_x + B_1^-(0)$. We now apply the above universal property to the map $x \mapsto v_x$ into the bounded subset $B_1(0)$ of V and we obtain a continuous linear map

$$f : c_{\mathrm{o}}(X) \longrightarrow V \quad \text{ such that } f(1_x) = v_x \text{ for any } x \in X\ .$$

This map, in fact, is an isometry. To see this we first check that the norm of any finite linear combination of the vectors v_x satisfies

$$\|a_1 v_{x_1} + \ldots + a_m v_{x_m}\| = \max(|a_1|, \ldots, |a_m|) \ .$$

We may assume without loss of generality that $a_1 \neq 0$ and that $|a_1| \geq |a_i|$ for any $1 \leq i \leq m$. By the linear independence of the $\overline{v}_x$ the vector

$$v_{x_1} + \frac{a_2}{a_1} v_{x_2} + \ldots + \frac{a_m}{a_1} v_{x_m} \in B_1(0) \backslash B_1^-(0)$$

lies in $B_1(0)$ but not in $B_1^-(0)$. Hence

$$\|a_1 v_{x_1} + \ldots + a_m v_{x_m}\| = |a_1|$$

as claimed. A continuity argument now shows that we have

$$\|f(\phi)\| = \|\phi\|_\infty \quad \text{for any } \phi \in c_{\mathrm{o}}(X) \ .$$

As an isometry f in particular is injective and has a complete and hence closed image. For the surjectivity of f it therefore suffices to show that the vector subspace $V_{\mathrm{o}} := \sum_{x \in X} K v_x$ is dense in V. Let $v \in V$ be any nonzero vector. Because of our additional hypothesis on the defining norm $\| \ \|$ we find a scalar $a \in K^\times$ such that $\|av\| = 1$. By the construction of V_{o} there exists a vector $v_{\mathrm{o}} \in V_{\mathrm{o}}$ such that $\|av - v_{\mathrm{o}}\| < 1$. The vector $a^{-1} v_{\mathrm{o}} \in V_{\mathrm{o}}$ then satisfies $\|v - a^{-1} v_{\mathrm{o}}\| < \|v\|$. Repeating this argument for the vector $v - a^{-1} v_{\mathrm{o}}$ (instead of v) and using that zero is the only accumulation point of $\|V\| \subseteq |K|$ we inductively find, for any $\epsilon > 0$, a vector $w \in V_{\mathrm{o}}$ such that $\|v - w\| < \epsilon$. Hence V_{o} is dense in V.

In the above proof we in fact have established the following variant of the assertion.

Remark 10.2:

Suppose that K is discretely valued; every K-Banach space $(V, \| \ \|)$ such that $\|V\| \subseteq |K|$ is isometrically isomorphic to a K-Banach space $(c_{\mathrm{o}}(X), \| \ \|_\infty)$ for some set X.

Lemma 10:3:

Let X and Y be two sets; the K-Banach spaces $c_{\mathrm{o}}(X)$ and $c_{\mathrm{o}}(Y)$ are topologically isomorphic if and only if the sets X and Y have the same cardinality.

Proof: The other implication being trivial we assume that $c_{\mathrm{o}}(X)$ and $c_{\mathrm{o}}(Y)$ are topologically isomorphic. If one of the sets is finite then already for algebraic reasons the other set has to be finite of the same cardinality. Let us therefore assume that X and Y both are infinite, and let $f : c_o(X) \xrightarrow{\cong} c_o(Y)$ be a topological isomorphism. The sets

$$Y_x := \{y \in Y : f(1_x)(y) \neq 0\} \quad \text{for } x \in X$$

are nonempty. On the other hand, we have seen in the Example at the end of §3 that each Y_x is finite or countable. Finally, for any $y \in Y$ there is an $x \in X$ such that $y \in Y_x$. If not all the $f(1_x)$ would be contained in the complete and hence closed vector subspace $c_o(Y \backslash \{y\}) \subseteq c_o(Y)$. This would contradict the surjectivity of f. It follows that

$$|Y| \leq |\dot{\bigcup_{x \in X}} Y_x| \leq |\mathbb{N}| \cdot |X| = |X|$$

and by symmetry that $|Y| = |X|$.

Proposition 10.4:

Suppose that the K-Banach space V contains a dense vector subspace of countably infinite dimension; then V is topologically isomorphic to $c_o(\mathbb{N})$.

Proof: Choose an ascending sequence of vector subspaces

$$\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n \subseteq \ldots$$

in V such that

- $\dim_K V_n = n$ for any $n \in \mathbb{N}$, and
- $\bigcup_{n \in \mathbb{N}} V_n$ is dense in V.

In addition we fix an increasing sequence of real numbers $0 < r = r_1 < r_2 < \ldots < r_n < \ldots < 1$. We want to inductively construct a sequence of vectors $(v_n)_{n \in \mathbb{N}}$ in V with the following properties:

(a) $\{v_1, \ldots, v_n\}$ is a K-basis of V_n for any $n \in \mathbb{N}$;

(b) $\|v_n + w\| \geq \frac{r_n}{r_{n+1}} \|v_n\|$ for any $w \in V_{n-1}$ and any $n \in \mathbb{N}$.

Suppose the vectors $v_1, \ldots, v_{n-1}$ already are constructed. By Prop. 4.13 the subspace V_{n-1} is complete and hence closed in V_n. Fixing some $v \in V_n \backslash V_{n-1}$ we therefore have $\inf\{\|v + w\| : w \in V_{n-1}\} > 0$. There consequently exists a vector $w' \in V_{n-1}$ such that

$$\frac{r_n}{r_{n+1}} \leq \frac{\inf\{\|v + w\| : w \in V_{n-1}\}}{\|v + w'\|} \leq 1 .$$

The vector $v_n := v + w'$ by construction has the properties (a) and (b).

Let us have a closer look at the property (b). It follows immediately that

$$\|av_n + w\| \geq \frac{r_n}{r_{n+1}} \|av_n\| \quad \text{for any } a \in K \text{ and } w \in V_{n-1} \ .$$

In fact, we have

$$\|av_n + w\| \geq \frac{r_n}{r_{n+1}} \max(\|av_n\|, \|w\|)$$

for any $a \in K$ and $w \in V_{n-1}$. If $\|av_n\| = \|w\|$ this is a consequence of the previous inequality; if $\|av_n\| \neq \|w\|$ this follows from $\|av_n + w\| = \max(\|av_n\|, \|w\|)$. We inductively deduce from this latter inequality that

$$\|\sum_{i=1}^{n} a_i v_i\| \geq \max(\frac{r_i}{r_{n+1}} \|a_i v_i\| : 1 \leq i \leq n) \ .$$

Since $r \leq r_i$ and $r_{n+1} < 1$ we finally obtain that

(c) $$\|\sum_{i=1}^{n} a_i v_i\| \geq r \cdot \max(\|a_1 v_1\|, \ldots, \|a_n v_n\|)$$

for any $a_1, \ldots, a_n \in K$ and any $n \in \mathbb{N}$. We obviously can scale the vectors v_n without changing the properties (a) and (b) and hence (c). We therefore may assume that

$$\epsilon \leq \|v_n\| \leq 1 \quad \text{for any } n \in \mathbb{N}$$

and some $\epsilon > 0$. The inequality (c) then implies that

(d) $$\|\sum_{i=1}^{n} a_i v_i\| \geq r\epsilon \cdot \max(|a_1|, \ldots, |a_n|) \ .$$

Moreover, from the universal property of the Banach space $c_o(\mathbb{N})$ we obtain a continuous linear map

$$f : c_o(\mathbb{N}) \longrightarrow V \quad \text{such that } f(1_n) = v_n \text{ for any } n \in \mathbb{N} \ .$$

By a continuity argument we deduce from (d) that

$$\|f(\phi)\| \geq r\epsilon \cdot \|\phi\|_\infty \quad \text{for any } \phi \in c_o(\mathbb{N}) \ .$$

This means that f induces a topological isomorphism between $c_o(\mathbb{N})$ and $\mathrm{im}(f)$. In particular, $\mathrm{im}(f)$ is complete and hence closed in V. On the other hand $\mathrm{im}(f)$, by (a), is dense in V. Hence $\mathrm{im}(f) = V$.

A closed vector subspace U of a locally convex K-vector space V is called *complemented* if there is another closed vector subspace $U_1 \subseteq V$ such that the linear map

$$\begin{array}{ccc} U \bigoplus U_1 & \xrightarrow{\cong} & V \\ (v, v_1) & \longmapsto & v + v_1 \end{array}$$

is a topological isomorphism (w.r.t. the direct sum of the subspace topologies on the left hand side). To show that U is complemented amounts to finding a continuous projector onto U, i.e., a continuous endomorphism P of V such that $P^2 = P$ and $P(V) = U$ (take $U_1 := \ker(P)$). We have seen in Cor. 9.5 that, if K is spherically complete and V is Hausdorff, then every finite dimensional subspace of V is complemented. As a consequence of the above structural results we may strengthen this considerably for Banach spaces.

Proposition 10.5:

Let V be a K-Banach space and suppose that

(a) K is discretely valued or

(b) V contains a dense vector subspace of countable dimension;

then every closed vector subspace $U \subseteq V$ is complemented.

Proof: By Prop. 8.3 the quotient V/U again is a Banach space which in case (b) contains a vector subspace of countable dimension. It therefore follows from Prop. 10.1 and Prop. 10.4 that in both cases there is a topological isomorphism

$$g : V/U \xrightarrow{\cong} c_{\mathrm{o}}(X)$$

for some set X. In particular, if $\| \ \|$ is a defining norm on V then there is a constant $c > 0$ such that

$$\|g(v + U)\|_\infty > c \cdot \inf_{u \in U} \|v + u\|$$

for any $v \in V$. This shows that, for any $x \in X$, we find a vector $v_x \in V$ such that $g(v_x + U) = 1_x$ and $\|v_x\| \leq c^{-1}$. The universal property of $c_{\mathrm{o}}(X)$ therefore gives a continuous linear map $f : c_{\mathrm{o}}(X) \longrightarrow V$ such that $f(1_x) = v_x$ for any $x \in X$. The continuous linear map $f \circ g : V/U \longrightarrow V$ then is a section of the projection map $V \xrightarrow{\mathrm{pr}} V/U$, and $P := (\mathrm{id}_V - f \circ g \circ \mathrm{pr})$ is a continuous projector onto U.

§11 Non-reflexivity

It follows from Prop. 6.14 and Lemma 9.9 that, over a spherically complete field K, any K-Banach space is pseudo-reflexive.

Proposition 11.1:

Suppose that K is spherically complete and that V is a K-Banach space; then V is reflexive if and only if it is finite dimensional.

Proof: Using Prop. 4.13 it is clear that any finite dimensional Banach space is reflexive. Let us therefore assume that V is a reflexive K-Banach space. In a first step we establish that every closed vector subspace U of V is reflexive as well. Being closed U with the subspace topology again is a Banach space. According to Prop. 8.3 the quotient V/U also is a Banach space. Consider the sequence of dual Banach spaces (Cor. 3.4 and the subsequent property 1))

$$0 \longrightarrow (V/U)' \longrightarrow V' \longrightarrow U' \longrightarrow 0 \ .$$

As a consequence of Cor. 9.4 this sequence is exact. Moreover, the open mapping theorem Prop. 8.6 implies that the topology of U' as a dual Banach space coincides with the quotient topology as a quotient of V'. Dualizing once more we again obtain an exact sequence of Banach spaces

$$0 \longrightarrow U'' \longrightarrow V'' \longrightarrow (V/U)'' \longrightarrow 0 \ .$$

In addition all the vertical duality maps in the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & U & \longrightarrow & V & \longrightarrow & V/U & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & U'' & \longrightarrow & V'' & \longrightarrow & (V/U)'' & \longrightarrow 0 \end{array}$$

are homeomorphisms onto the image. It follows that with V also U and V/U are reflexive.

Let us assume now that V is not finite dimensional. By letting $U \subseteq V$ be the closure of some vector subspace of countably infinite dimension we obtain a reflexive Banach space which according to Prop. 10.4 is topologically isomorphic to $c_o(\mathbb{N})$. To arrive at a contradiction it remains to show that $c_o(\mathbb{N})$ is not reflexive.

In the Example at the end of §3 we have computed the Banach space dual of $c_o(\mathbb{N})$ as $\ell^\infty(\mathbb{N})$. On the other hand $c_o(\mathbb{N})$ is a complete and hence closed proper subspace of $\ell^\infty(\mathbb{N})$. By Cor. 9.3 there exists therefore a continuous linear form $d \neq 0$ on $\ell^\infty(\mathbb{N})$ such that $d|c_o(\mathbb{N}) = 0$. It is clear that this d cannot be in the image of the duality map, and hence that $c_o(\mathbb{N})$ is not reflexive.

Chap. III: Duality theory

Assuming that the nonarchimedean field K is spherically complete we develop in this chapter two concepts which are at the base of any deeper investigation into locally convex vector spaces. As explained in §12 the role of compact and precompact subsets is taken over by c-compact and compactoid o-submodules, respectively, in a locally convex K-vector space. Roughly speaking these are o-linear versions of the former topological properties. The concept of polarity which we describe in §13 is the fundamental tool for duality theory. The formation of the pseudo-polar passes from o-submodules in a given locally convex K-vector space V to o-submodules in the continuous linear dual V' thereby allowing for the transfer of information back and forth. It is a crucial technical detail in the definition of the pseudo-polar to only consider those continuous linear forms whose absolute value on the given o-submodule is strictly less than one. Relaxing this condition to less than or equal to one would lead to the more traditional notion of the "polar". In nonarchimedean functional analysis this latter notion does not work so well and therefore is not treated in this book at all. In §14 we introduce and study the class of admissible topologies on a given locally convex vector space. These consist of those locally convex topologies which give rise to the same continuous linear dual as the given topology. There is a weakest and a finest such topology – the weak and the Mackey topology. A particular feature of nonarchimedean functional analysis is the fact that all admissible topologies have the same class of c-compact o-submodules. In §15 all we have accumulated so far comes together and leads to a complete characterization of reflexive spaces. A locally convex vector space is called reflexive if the duality map into its strong bidual is a topological isomorphism. In classical functional analysis over the complex numbers the so called Montel spaces form an important subclass of the class of all reflexive spaces. It is a consequence of the above mentioned special feature that over a spherically complete nonarchimedean field every reflexive space already is a Montel space (if one gives this term the obvious analogous meaning).

The most important method to construct reflexive spaces is via compact inductive and projective limits of Banach spaces. This is explained in §16. The notion of a compact map which has to be introduced for this is of a very fundamental nature and will be explored more systematically in the next chapter. We conclude this section with an explicit example which illustrates the relevance of compact limits in the applications.

Throughout this chapter the field K is assumed to be spherically complete, and all occurring locally convex K-vector spaces are Hausdorff.

§12 c-compact and compactoid submodules

Since our field K in general is not locally compact the notion of compact subsets does not play a significant role in the theory of locally convex K-vector spaces. In fact, the reader may check that the field K necessarily is locally compact as soon as there is a single (Hausdorff) locally convex K-vector spaces with a nonzero compact o-submodule. On the other hand there is a certain similarity between the notions "compact" and "spherically complete". This leads to the following definition. Let V be a locally convex K-vector space.

Definition:

An o-submodule $A \subseteq V$ is called <u>c-compact</u> if, for any decreasingly filtered family $(L_i)_{i\in I}$ of open lattices $L_i \subseteq V$, the canonical map

$$A \twoheadrightarrow \varprojlim_{i\in I} A/(L_i \cap A)$$

is surjective.

Lemma 12.1:

Let $A \subseteq V$ be a c-compact o-submodule; we have:

i. A is complete and hence closed in V;

ii. for any decreasingly filtered family $(A_i)_{i\in I}$ of closed o-submodules $A_i \subseteq A$ the canonical map

$$A \twoheadrightarrow \varprojlim_{i\in I} A/A_i$$

is surjective;

iii. any closed o-submodule $B \subseteq A$ is c-compact;

iv. for any continuous linear map $f : V \longrightarrow W$ between locally convex K-vector spaces the image $f(A)$ is c-compact;

v. if A is contained in a vector subspace $V_o \subseteq V$ then A is c-compact in V_o.

Proof: i. This follows from Cor. 7.6. ii. Let $(L_j)_{j\in J}$ denote the decreasingly filtered family of all open lattices in V which contain some A_i. Consider the homomorphism

$$\begin{aligned} f: \quad \varprojlim_{i\in I} A/A_i &\longrightarrow \varprojlim_{j\in J} A/(L_j \cap A) \\ (v_i + A_i)_i &\longmapsto (w_j + (L_j \cap A))_j \end{aligned}$$

where $w_j := v_i$ provided $L_j \supseteq A_i$. By definition the canonical map from A into the target of f is surjective. In particular f is surjective. It therefore suffices to show that f is injective. For this we have to check that any given A_i is the intersection of those L_j which contain A_i. Suppose that $v \notin A_i$. Since A_i is closed we find some open lattice $L \subseteq V$ such that $(v + L) \cap A_i = \emptyset$. Hence v is not contained in the open lattice $A_i + L$. Both iii. and iv. are easy consequences of ii. The last assertion v. is immediate from the definition.

The analogy to the notion "compact" becomes more apparent if we spell out the assertion in Lemma 12.1.ii in more topological terms: The o-submodule $A \subseteq V$ is c-compact if and only if for any family $(C_i)_{i \in I}$ of closed convex subsets $C_i \subseteq A$ such that $\bigcap_{i \in I} C_i = \emptyset$ there are finitely many indices $i_1, \ldots, i_m \in I$ such that $C_{i_1} \cap \ldots \cap C_{i_m} = \emptyset$.

Since open lattices strictly contained in K (viewed as a locally convex K-vector space) are open or closed balls it follows immediately from Lemma 1.3 that K is spherically complete if and only if K is c-compact. This example shows, by the way, that c-compact o-submodules of a locally convex vector space in general will not be bounded.

Proposition 12.2:

Let $(V_h)_{h \in H}$ be a family of locally convex K-vector spaces, and let $A_h \subseteq V_h$, for any $h \in H$, be a c-compact o-submodule; then $\prod_{h \in H} A_h$ is c-compact in $\prod_{h \in H} V_h$.

Proof: Let $(L_i)_{i \in I}$ be a decreasingly filtered family of open lattices in $\prod_h V_h$, and let

$$((v_{h,i})_h)_i \in \varprojlim_{i \in I} (\prod_h A_h)/(L_i \cap \prod_h A_h) \ .$$

It is convenient in the following to think of each $(v_{h,i})_h$ as being an actual vector in $\prod_h A_h$ representing the above cosets. We have to exhibit a vector

$$(v_h)_h \in \prod_h A_h \qquad \text{such that } (v_h - v_{h,i})_h \in L_i \text{ for any } i \in I \ .$$

Consider the set $\mathcal{P}$ of all pairs $(((x_{h,j})_h)_{j \in J}, (M_j)_{j \in J})$ where $(M_j)_{j \in J}$ is a decreasingly filtered family of open lattices in $\prod_h V_h$ and

$$((x_{h,j})_h)_j \in \varprojlim_{j \in J} (\prod_h A_h)/(M_j \cap \prod_h A_h) \ .$$

The set $\mathcal{P}$ is inductively ordered by

$$(((x'_{h,j})_h)_{j \in J_1}, (M'_j)_{j \in J_1}) \leq (((x_{h,j})_h)_{j \in J_2}, (M_j)_{j \in J_2})$$

if $J_1 \subseteq J_2$ and $M'_j = M_j$ and $(x'_{h,j})_h = (x_{h,j})_h$ for any $j \in J_1$. By Zorn's lemma $\mathcal{P}$ contains a maximal pair which is $\geq (((v_{h,i})_h)_i, (L_i)_i)$. We therefore may assume that this original pair already is maximal.

Let $L_{h,i}$ denote the projection of L_i to V_h; then $L_{h,i}$ is an open lattice in V_h. Since A_h is c-compact there is a vector $v_h \in A_h$ such that $v_h - v_{h,i} \in L_{h,i}$ for any $i \in I$. We claim that this vector $(v_h)_h$ in $\prod_h A_h$ has the property we want. For the rest of the proof we may fix an index $i \in I$ and we have to show that $(v_h - v_{h,i})_h \in L_i$. By the definition of the product topology there is, for each $h \in H$, an open lattice $L'_{h,i} \subseteq V_h$ such that $L'_{h,i} = V_h$ for all but finitely many $h \in H$ and $\prod_h L'_{h,i} \subseteq L_i$.

For a fixed $k \in H$ and any $j \in I$ we have

$$\{v_k - v_{k,j}\} \times \prod_{h \neq k} V_h \ \subseteq\ (\{0\} \times \prod_{h \neq k} V_h) + L_j \ \subseteq\ (L'_{k,i} \times \prod_{h \neq k} V_h) + L_j$$

and hence $(v_h - v_{h,j})_h \in (L'_{k,i} \times \prod_{h \neq k} V_h) + L_j$. This means that

$$[(v_h)_h + L'_{k,i} \times \prod_{h \neq k} V_h] \ \cap\ [(v_{h,j})_h + L_j] \ \neq\ \emptyset \qquad \text{for any } j \in I\ .$$

The maximality of our pair then implies the existence of an index $j(k) \in I$ such that

$$L'_{k,i} \times \prod_{h \neq k} V_h = L_{j(k)} \qquad \text{and} \qquad (v_h)_h + L'_{k,i} \times \prod_{h \neq k} V_h = (v_{h,j(k)})_h + L_{j(k)}\ .$$

Let now $H_\mathrm{o} \subseteq H$ be the finite subset of those indices h for which $L'_{h,i} \neq V_h$. Since our family $(L_i)_i$ is decreasingly filtered the intersection

$$[(v_{h,i})_h + L_i] \cap \bigcap_{k \in H_\mathrm{o}} [(v_{h,j(k)})_h + L_{j(k)}] \neq \emptyset$$

is nonempty. But

$$\begin{aligned} \textstyle\bigcap_{k \in H_\mathrm{o}} [(v_{h,j(k)})_h + L_{j(k)}] &= \textstyle\bigcap_{k \in H_\mathrm{o}} [(v_h)_h + L'_{k,i} \times \prod_{h \neq k} V_h] \\ &= (v_h)_h + \textstyle\prod_h L'_{h,i} \\ &\subseteq (v_h)_h + L_i\ . \end{aligned}$$

It follows that

$$[(v_{h,i})_h + L_i] \cap [(v_h)_h + L_i] \neq \emptyset$$

which amounts to $(v_h - v_{h,i})_h \in L_i$.

Corollary 12.3:

If A and B are c-compact o-submodules of V then $A + B$ is c-compact, too.

Proof: This easily follows from Prop. 12.2 and Lemma 12.1.iv.

The corresponding analog of the notion "precompact" is defined as follows.

Definition:

An o-submodule $A \subseteq V$ is called <u>*compactoid*</u> *if for any open lattice $L \subseteq V$ there are finitely many vectors $v_1, \ldots, v_m \in V$ such that*

$$A \subseteq L + ov_1 + \ldots + ov_m \ .$$

Lemma 12.4:

Let $A \subseteq V$ be a compactoid o-submodule; we have:

i. A is bounded;

ii. the closure $\overline{A}$ of A is compactoid;

iii. every o-submodule B of A is compactoid;

iv. if B is another compactoid o-submodule of V then $A + B$ is compactoid;

v. for any continuous linear map $f : V \longrightarrow W$ between locally convex K-vector spaces the image $f(A)$ is compactoid.

Proof: For i. consider any open lattice $L \subseteq V$ and let $v_1, \ldots, v_m \in V$ be such that $A \subseteq L + ov_1 + \ldots + ov_m$. The lattice property of L ensures the existence of a $0 \neq a \in o$ such that $av_1, \ldots, av_m \in L$. Hence $A \subseteq a^{-1}L$. The arguments for the assertion ii.-v. are even easier and are left to the reader.

In order to understand the relation between the notions "c-compact" and "compactoid" we need two results from ring theory.

Definition:

An o-module M is called <u>*linearly compact*</u> *if, for any decreasingly filtered family $(M_i)_{i \in I}$ of o-submodules $M_i \subseteq M$, the canonical map*

$$M \longrightarrow\!\!\!\!\rightarrow \varprojlim_{i \in I} M/M_i$$

is surjective.

A basic example for a linearly compact o-module is the quotient module K/o. We recall that an o-module M is called *divisible* if $aM = M$ for any $0 \neq a \in o$.

Lemma 12.5:

Any linearly compact and divisible o-module M is injective.

Proof: It suffices ([McL] III.7.2) to establish the following: For any ideal $\mathfrak{a} \subseteq o$ and any homomorphism of o-modules $f : \mathfrak{a} \longrightarrow M$ there is a homomorphism of o-modules $F : o \longrightarrow M$ such that $F|\mathfrak{a} = f$.

For any $a \in \mathfrak{a}$ we define the o-submodule

$$M_a := \{m \in M : am = 0\}$$

of M. We have $M_a \subseteq M_b$ if $|b| \leq |a|$. Hence $(M_a)_{a\in\mathfrak{a}}$ is a decreasingly filtered family. Next we construct an element $(m_a)_a \in \varprojlim_{a\in\mathfrak{a}} M/M_a$. By the divisibility of M we have, for any $a \in \mathfrak{a}$, an element $m_a \in M$ such that $f(a) = am_a$. To check that the m_a form an element in the above projective limit let us suppose that $M_a \subseteq M_b$. If $|b| \leq |a|$ then $ba^{-1} \in o$, hence

$$bm_a = ba^{-1}am_a = ba^{-1}f(a) = f(b) = bm_b$$

and therefore $m_a - m_b \in M_b$. If $|a| \leq |b|$ then, by a symmetric computation, $m_b - m_a \in M_a$. But in this case we also have $M_b \subseteq M_a$ and hence $M_a = M_b$. This shows that in any case we have $m_a + M_a \subseteq m_b + M_b$.

Since M is linearly compact there must exist an element $m \in M$ such that $m - m_a \in M_a$ for any $a \in \mathfrak{a}$. This means that $am = am_a = f(a)$ for $a \in \mathfrak{a}$. The homomorphism $F : o \longrightarrow M$ defined by $F(a) := am$ therefore satisfies $F|\mathfrak{a} = f$.

Lemma 12.6:

Let M be a submodule of a finitely generated o-module; for any $a \in \mathfrak{m}$ the submodule aM is contained in a finitely generated submodule of M.

Proof: If K is discretely valued then o, by Lemma 1.5, is a principal ideal domain. Hence in this case M and consequently aM are finitely generated.

We therefore suppose in the following that $|K^\times|$ is dense in $\mathbb{R}_+^\times$ (compare Lemma 1.4). By writing the finitely generated o-module which contains M as a quotient of some free module o^n and by replacing M by its preimage under the corresponding quotient map we may assume that M is contained in o^n. We now proceed by induction with respect to n. Consider the exact columns

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
a(M \cap o^{n-1}) & \subseteq & M \cap o^{n-1} & \subseteq & o^{n-1} \\
\downarrow & & \downarrow & & \downarrow \\
aM & \subseteq & M & \subseteq & o^n \\
\downarrow & & \downarrow & & \downarrow \\
a\mathfrak{a} & \subseteq & \mathfrak{a} & \subseteq & o \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where $\mathfrak{a} := (M + o^{n-1})/o^{n-1} \subseteq o^n/o^{n-1} = o$. The first column is exact since $a(M \cap o^{n-1}) = aM \cap o^{n-1}$. We may assume that n is minimal (for M). Then $0 < r := \sup_{b\in\mathfrak{a}}|b| \leq 1$. The density of the absolute value implies the existence of $c, d \in \mathfrak{a}$ such that $|a| \cdot r \leq |c| < |d| < r \leq 1$. Put $a' := c/d$. One easily checks that $|a| \leq |a'| < 1$ and

$$a\mathfrak{a} \subseteq co \subseteq a'\mathfrak{a} \ .$$

Let $m \in a'M$ be a preimage of $c \in \mathfrak{a}$. Applying the induction hypothesis to the module $a'(M \cap o^{n-1})$ we obtain a finitely generated submodule $N \subseteq M \cap o^{n-1} \subseteq M$ such that

$$a'M \cap o^{n-1} = a'(M \cap o^{n-1}) \subseteq N \ .$$

It follows that

$$aM \subseteq N + om \ .$$

Proposition 12.7:

For any o-submodule $A \subseteq V$ the following assertions are equivalent:

i. A is c-compact and bounded;

ii. A is compactoid and complete.

Proof: We first assume that i. holds true. According to Lemma 12.1.i the subset A of V is complete. To establish that A is compactoid let $L \subseteq V$ be an open lattice. Using Zorn's lemma we find a maximal subset T of elements in the o-module $M := (A + L)/L$ such that $N := \oplus_{t\in T}\, ot \subseteq M$. That A is c-compact implies that M and hence any submodule of M is linearly compact. We apply this to the submodule N and obtain that the canonical map

$$(\underset{t\in T}{\oplus} ot) \xrightarrow{\cong} \varprojlim_{\substack{S\subseteq T\\ \text{finite}}} (\underset{s\in S}{\oplus} os) = \prod_{t\in T} ot$$

is bijective. This shows that T,in fact, is a finite set. By the maximality of T we have $om \cap N \neq \{0\}$ for any $m \in M\backslash N$. Hence the pair $N \subseteq M$ has the following property:

$(*)$ For any $m \in M\backslash N$ there is an $a \in o$ such that $0 \neq am \in N$.

(One says that M is an essential extension of N.) For any $t \in T$ we now fix a vector $v_t \in A$ such that $v_t + L = t$, and we consider the vector subspace

$$U := \sum_{t\in T} Kv_t$$

of V. The o-linear map

$$\begin{array}{ccc} N & \longrightarrow & U/(U \cap L) \\ \sum_t a_t t & \longmapsto & \sum_t a_t v_t + (U \cap L) \end{array}$$

is well defined and injective. Since U, by Prop. 4.13, is topologically isomorphic to some K^n it follows from Prop. 12.2 that U is c-compact. Hence the o-module $U/(U \cap L)$ is linearly compact and divisible. Lemma 12.5 then tells us that $U/(U \cap L)$ is an injective o-module. The above map $N \longrightarrow U/(U \cap L)$ therefore can be extended to a homomorphism of o-modules $f : M \longrightarrow U/(U \cap L)$ which, because of $(*)$, necessarily is injective, too.

Concerning the structure of the o-module $U/(U \cap L)$ we claim that there are an integer $n \geq 0$, a scalar $0 \neq a \in o$, and an o-submodule $L_{\mathrm{o}} \subseteq K^n$ such that

$$o^n \subseteq L_{\mathrm{o}} \subseteq (a^{-1}o)^n \subseteq K^n \quad \text{and} \quad U/(U \cap L) \cong K^n/L_{\mathrm{o}} \ .$$

To see this let U_{o} be a maximal vector subspace in $U \cap L$ and let n be the dimension of U/U_{o}. Then $L_{\mathrm{o}} := (U \cap L)/U_{\mathrm{o}}$ is a bounded open lattice in U/U_{o}. As such it contains a K-basis of U/U_{o}. If we use this basis to identify U/U_{o} with K^n we have $o^n \subseteq L_{\mathrm{o}}$ and, by boundedness, $L_{\mathrm{o}} \subseteq (a^{-1}o)^n$ for some $0 \neq a \in o$.

Since A is bounded there is a $0 \neq b \in o$ such that $bA \subseteq L$. This implies that $bM = \{0\}$ and a fortiori $bf(M) = \{0\}$, i.e., that

$$f(M) \subseteq b^{-1}(U \cap L)/(U \cap L) \cong b^{-1}L_{\mathrm{o}}/L_{\mathrm{o}} \subseteq ((ab)^{-1}o)^n/L_{\mathrm{o}} \ .$$

Because of the surjection $o^n \overset{(ab)^{-1}\cdot}{\twoheadrightarrow} ((ab)^{-1}o)^n/L_{\mathrm{o}}$ the latter o-module is finitely generated. Hence $f(M)$ and therefore M is contained in a finitely generated o-module and we may apply Lemma 12.6.

But instead of using Lemma 12.6 directly we first note that all of our discussion is valid for an arbitrary open lattice in V. We therefore may fix a $c \in K$ such that $0 < |c| < 1$ and apply Lemma 12.6 to cL and $M' := (A + cL)/cL$. As a result we obtain finitely many vectors $v_1, \ldots, v_m \in A$ such that $cM' \subseteq \sum_{i=1}^{m} o(v_i + cL)$. This shows that

$$A \subseteq L + c^{-1}ov_1 + \ldots + c^{-1}ov_m \ .$$

Let us now assume, vice versa, that ii. holds true. Then A is bounded by Lemma 12.4.i. That A is compactoid implies that the o-module $A/(A \cap L) = (A + L)/L$, for any open lattice $L \subseteq V$, is contained (up to isomorphism) in an o-module of the form K^n/L_{o} for some open lattice $L_{\mathrm{o}} \subseteq K^n$. Since K^n is c-compact the

o-modules K^n/L_o and hence $A/(A\cap L)$ are linearly compact. By an argument entirely analogous to the proof of Prop. 12.2 we obtain:

(+) The o-module $\widetilde{A} := \prod_L A/(A\cap L)$ equipped with the product of the discrete topologies has the property that for any decreasingly filtered family $(\widetilde{A}_i)_{i\in I}$ of closed o-submodules $\widetilde{A}_i \subseteq \widetilde{A}$ the canonical map

$$\widetilde{A} \longrightarrow\!\!\!\!\!\rightarrow \varprojlim_{i\in I} \widetilde{A}/\widetilde{A}_i$$

is surjective.

The o-module $\widehat{A} := \varprojlim_L A/(A\cap L)$ is a closed submodule of $\widetilde{A}$. Hence $\widehat{A}$ satisfies a corresponding property (+). But the completeness of A, by Cor. 7.6 and its proof, implies that the natural map $A \xrightarrow{\cong} \widehat{A}$ is a topological isomorphism. The property (+) for $\widehat{A}$ becomes, under this isomorphism, the statement that A is c-compact.

The argument in the second half of the proof of the implication i.⇒ii. above also shows the following.

Remark 12.8:

Let $A \subseteq V$ be a compactoid o-submodule, and let $a \in K$ such that $|a| > 1$ (resp., $a = 1$ if K is discretely valued); for any open lattice $L \subseteq V$ there exist finitely many vectors $v_1, \ldots, v_m \in aA$ such that $A \subseteq L + ov_1 + \ldots + ov_m$.

Example:

Let $a \in K$ be such that $0 < |a| < 1$. The o-submodule

$$A := \{\phi \in c_o(\mathbb{N}) : |\phi(n)| \leq |a|^n \text{ for any } n \in \mathbb{N}\}$$

is bounded in the K-Banach space $c_o(\mathbb{N})$. One easily checks that the o-linear bijection

$$\begin{array}{ccc} \prod_{n\in\mathbb{N}} o & \xrightarrow{\cong} & A \\ (a_n)_n & \longmapsto & [n \mapsto a^n a_n] \end{array}$$

is a homeomorphism w.r.t. the direct product topology on the left hand side and the $\|\ \|_\infty$-topology on A. By Prop. 12.2 the left hand side is c-compact in the direct product $\prod_{n\in\mathbb{N}} K$. But according to Lemma 12.1.ii c-compactness is an intrinsic property of a linearly topologized o-module. It follows that A is c-compact and compactoid in $c_o(\mathbb{N})$.

Remark 12.9:

In V_s an o-submodule A is compactoid if and only if it is bounded.

Proof: The direct implication is a special case of Lemma 12.4.i. Let us therefore assume that A is bounded, and let $L \subseteq V_s$ be an open lattice. By the definition of the weak topology there are finitely many continuous linear forms $\ell_1, \ldots, \ell_m$ on V such that

$$L \supseteq U := \ker(\ell_1) \cap \ldots \cap \ker(\ell_m) \ .$$

We may assume that the $\ell_1, \ldots, \ell_m$ are linearly independent. Consider the continuous linear surjection

$$\begin{array}{rcl} f : V_s & \twoheadrightarrow & K^m \\ v & \longmapsto & (\ell_1(v), \ldots, \ell_m(v)) \ . \end{array}$$

The image $f(A)$ is bounded and hence $f(A) \subseteq (ao)^m$ for some $a \in K$. Choosing vectors $v_1, \ldots, v_m \in V$ such that $\{f(v_1), \ldots, f(v_m)\}$ is the standard basis of K^m we have

$$A \subseteq U + oav_1 + \ldots + oav_m \subseteq L + oav_1 + \ldots + oav_m \ .$$

Lemma 12.10:

Every equicontinuous and closed o-submodule $A \subseteq V_s'$ is c-compact.

Proof: According to Prop. 7.13 the subset $A \subseteq V_s'$ is complete, and according to Lemma 6.8 it is bounded. Because of Prop. 12.7 it remains to check that A is compactoid. For a given open lattice $\mathcal{L} \subseteq V_s'$ we find finitely many linearly independent vectors $v_1, \ldots, v_m \in V$ such that $\mathcal{L} \supseteq \{\ell \in V_s' : \ell(v_1) = \ldots = \ell(v_m) = 0\}$. Consider the continuous linear map

$$\begin{array}{rcl} f : V_s' & \twoheadrightarrow & K^m \\ \ell & \longmapsto & (\ell(v_1), \ldots, \ell(v_m)) \ . \end{array}$$

It is surjective by Prop. 9.7. We therefore find $\ell_1, \ldots, \ell_m \in V_s'$ such that $\{f(\ell_1), \ldots, f(\ell_m)\}$ is the standard basis of K^m. As a consequence of the boundedness of A we have $f(A) \subseteq (ao)^m$ for some $a \in K$. It follows that

$$A \subseteq \mathcal{L} + oa\ell_1 + \ldots + oa\ell_m \ .$$

§13 Polarity

Let V be a locally convex K-vector space. The following definition will enable us to pass hence and forth between o-submodules of V and of the dual space V'.

Definition:

Let $A \subseteq V$ be an o-submodule;

a) the o-submodule

$$A^p := \{\ell \in V' : |\ell(v)| < 1 \text{ for any } v \in A\}$$

in V' is called the pseudo-polar of A;

b) the o-submodule

$$A^{pp} := \{v \in V : |\ell(v)| < 1 \text{ for any } \ell \in A^p\}$$

in V is called the pseudo-bipolar of A.

We obviously have $A \subseteq A^{pp}$. Moreover, $A^{pp} \subseteq V = (V_s')'$ (compare Prop. 9.7) can be viewed as the pseudo-polar of $A^p \subseteq V_s'$. In this section we will study the question when the identity $A = A^{pp}$ holds true.

Lemma 13.1:

Let $A \subseteq V$ be an o-submodule; we have:

i. If $A \subseteq B \subseteq V$ is another o-submodule then $B^p \subseteq A^p$;

ii. A^p is closed in V_s';

iii. if A is bounded in V_s then A^p is a lattice in V';

iv. if $A \in \mathcal{B}$ then A^p is an open lattice in the $\mathcal{B}$-dual $V_{\mathcal{B}}'$;

v. if A is a lattice in V then A^p is bounded in V_s';

vi. if A is an open lattice in V then A^p is c-compact in V_s'.

Proof: The assertion i. is trivial. ii. Since $\mathfrak{m}$ is closed in K the submodule $\delta_v^{-1}(\mathfrak{m}) = \{\ell \in V' : |\ell(v)| < 1\}$, for any $v \in V$, is closed in V_s'. Now observe that $A^p = \bigcap_{v \in A} \delta_v^{-1}(\mathfrak{m})$. iii. Let $\ell \in V'$ be given. Then $\ell^{-1}(\mathfrak{m})$ is an open lattice in V_s. We therefore find an $a \in K^\times$ such that $aA \subseteq \ell^{-1}(\mathfrak{m})$. It follows that $a\ell \in A^p$. iv. According to the definition of the $\mathcal{B}$-topology $\mathcal{L}(A, \mathfrak{m})$ is an open lattice in $V_{\mathcal{B}}'$. It remains to note that $\mathcal{L}(A, \mathfrak{m}) = A^p$. v. It suffices to show that, given a vector $v \in V$, there is an $a \in K^\times$ such that $aA^p \subseteq \mathcal{L}(\{v\}, o)$. Choose a in such a way that $av \in A$. For any $\ell \in A^p$ we then have $|a\ell(v)| = |\ell(av)| < 1$ and hence $a\ell \in \mathcal{L}(\{v\}, o)$. vi. By definition A^p is equicontinuous, and by ii. it is closed in V_s'. Apply now Lemma 12.10.

At this point we need a strengthening of Cor. 9.6. This will be a consequence of the following structural result which describes an open lattice $L \subseteq V$ in terms of its corresponding gauge p_L.

Proposition 13.2:

Let $L \subseteq V$ be an open lattice; then there is a subset $T \subseteq V$ and a family $(\ell_t)_{t\in T}$ of linear forms on V with the following properties:

1. *$p_L(t) \neq 0$ for any $t \in T$;*
2. *$\ell_t(at) = a$ for any $a \in K$ and $t \in T$;*
3. *$\ell_t(t') = 0$ for any two $t \neq t'$ in T;*
4. *$p_L(v) = \max\limits_{t\in T} |\ell_t(v)| \cdot p_L(t)$ for any $v \in V$;*
5. *there is a disjoint decomposition $T = T_0 \,\dot\cup\, T_1$ such that*

$$L = \{v \in V : |\ell_t(v)| \cdot p_L(t) \leq 1 \text{ for } t \in T_0 \text{ and } < 1 \text{ for } t \in T_1\} \ ;$$

6. *$L = \{v \in V : \ell_t(v) \in \ell_t(L) \text{ for any } t \in T\}$.*

Proof: The lattice L contains the vector subspace $U := \{v \in V : p_L(v) = 0\}$. Our assertion is an immediate consequence of the corresponding assertion for the lattice $L/U \subseteq V/U$. We therefore may assume in the following that $U = \{0\}$, i.e., that p_L is a norm.

The o-module quotient $L(p_L)/L^-(p_L)$ is a vector space over the residue class field $k := o/\mathfrak{m}$ of K. We fix a subset $T_0 \subseteq L$ which maps bijectively onto a k-basis of the k-vector subspace $L/L^-(p_L)$ in $L(p_L)/L^-(p_L)$. In particular, we have $p_L(t) = 1$ for $t \in T_0$. The subset T_0 has the property that

$$(*) \qquad p_L(\sum_{t\in T_0} a_t t) = \max_{t\in T_0} p_L(a_t t)$$

for any family $(a_t)_{t\in T_0}$ of elements $a_t \in K$ such that all but finitely many a_t are equal to zero. To see this fix a $t_0 \in T_0$ such that $|a_{t_0}| = \max_t |a_t|$. If $a_{t_0} = 0$ the identity $(*)$ is trivial. Otherwise the image of the vector $\sum_t (a_t/a_{t_0})t$ in $L/L(p_L)$ is nonzero which means that

$$p_L(\sum_t \frac{a_t}{a_{t_0}} t) = 1 \ ;$$

it follows that

$$p_L(\sum_t a_t t) = |a_{t_0}| \cdot p_L(\sum_t \frac{a_t}{a_{t_0}} t) = \max_t |a_t| = \max_t p_L(a_t t) \ .$$

The subsets $S \subseteq V\backslash\{0\}$ which contain T_0 and which satisfy $(*)$ are inductively ordered with respect to inclusion. By Zorn's lemma there exists a maximal such subset T. We put $T_1 := T\backslash T_0$. Since p_L is a norm it is immediate from $(*)$ that the set T is K-linearly independent. In particular, the property 1. holds. On the vector subspace $V_{\mathrm{o}} := \sum_{s \in T} Ks$ in V we have the linear forms

$$\ell_t(\sum_{s \in T} a_s s) := a_t$$

for any $t \in T$. The identity $(*)$ implies

$$|\ell_t(\sum_s a_s s)| \cdot p_L(t) = p_L(a_t t) \leq p_L(\sum_s a_s s) \ .$$

By the Hahn-Banach theorem Prop. 9.2 each ℓ_t therefore extends to a linear form ℓ_t on V such that

$$|\ell_t(v)| \cdot p_L(t) \leq p_L(v) \quad \text{ for any } v \in V \ .$$

The properties 2. and 3. and the inequality

$$\sup_{t \in T} |\ell_t(v)| \cdot p_L(t) \leq p_L(v) \quad \text{ for any } v \in V$$

hold by construction. To complete the proof of property 4. we reason by contradiction and assume that there is a vector $v \in V$ such that

$$p_L(v) > |\ell_t(v)| \cdot p_L(t) \quad \text{ for any } t \in T \ .$$

In particular, $p_L(v) \neq 0$, hence $v \neq 0$, and $v \notin T$. We claim that

$$(+) \qquad p_L(v - \sum_{s \in T} a_s s) \geq p_L(v)$$

for any $\sum_s a_s s \in V_{\mathrm{o}}$. Otherwise the inequality

$$|\ell_t(v) - a_t| \cdot p_L(t) = |\ell_t(v - \sum_s a_s s)| \cdot p_L(t) \leq p_L(v - \sum_s a_s s) < p_L(v)$$

implies

$$|a_t| \cdot p_L(t) \leq \max(|\ell_t(v) - a_t|, |\ell_t(v)|) \cdot p_L(t) < p_L(v)$$

for any $t \in T$ and leads consequently to the contradiction

$$p_L(v) \leq \max(p_L(v - \sum_s a_s s), \max_s |a_s| \cdot p_L(s)) < p_L(v) \ .$$

We now use (+) to show that the subset $T \cup \{v\}$ satisfies $(*)$ which is impossible by the maximality of T. Let $w \in V$ be any vector of the form $w = av + \sum_{t\in T} a_t t$ with $a, a_t \in K$ such that all but finitely many a_t are equal to zero. We distinguish two cases. First we assume that

$$p_L(av) \geq |a_t| \cdot p_L(t) \quad \text{for any } t \in T \ .$$

If $a = 0$ then $a_t = 0$ for any $t \in T$. If $a \neq 0$ then (+) implies

$$p_L(a^{-1}w) = p_L(v + \sum_t \frac{a_t}{a} t) \geq p_L(v) \ .$$

We obtain

$$\max(p_L(av), \max_t |a_t| \cdot p_L(t)) = p_L(av) \leq p_L(w) \leq \max(p_L(av), \max_t p_L(a_t t))$$

and hence

$$p_L(w) = \max(p_L(av), \max_t p_L(a_t t)) \ .$$

Secondly we assume that

$$p_L(av) < \max_t |a_t| \cdot p_L(t) \ .$$

We then have

$$p_L(av) < p_L(\sum_t a_t t)$$

and therefore

$$p_L(w) = \max(p_L(av), p_L(\sum_t a_t t)) = \max(p_L(av), \max_t p_L(a_t t)) \ .$$

In both cases $T \cup \{v\}$ satisfies $(*)$ which by contradiction establishes the property 4.

Next we verify the property 5. First let $v \in L$. Because of $L \subseteq L(p_L)$ it follows from 4. that

$$|\ell_t(v)| \cdot p_L(t) \leq 1 \quad \text{for any } t \in T \ .$$

By the construction of the subset T_0 there is a vector $\sum_{s\in T_0} a_s s \in V_o$ such that $p_L(v - \sum_{s\in T_0} a_s s) < 1$. For $t \in T_1$ we then have $\ell_t(v) = \ell_t(v - \sum_{s\in T_0} a_s s)$ and hence

$$|\ell_t(v)| \cdot p_L(t) = |\ell_t(v - \sum_{s\in T_0} a_s s)| \cdot p_L(t) \leq p_L(v - \sum_{s\in T_0} a_s s) < 1 \ .$$

Now let v be a vector in the right hand side of the identity in 5. Because of 4. we certainly have $p_L(v) \leq 1$. We claim that the subset $S := \{s \in T_0 : |\ell_s(v)| \cdot p_L(s) = 1\}$ is finite. If not we certainly would have $v \neq 0$ and (because of 3.) $v \notin T$ and for any vector $\sum_t a_t t \in V_\mathrm{o}$ there would be an $s \in S$ such that $a_s = 0$. Using 4. we would obtain

$$p_L(v - \sum_t a_t t) = \max_t |\ell_t(v) - a_t| \cdot p_L(t) \geq 1 \geq p_L(v) \ .$$

But this is the inequality (+) above which we have shown to be in contradiction to the maximality of T. The set S therefore must be finite. It then follows from 2.-4. that

$$p_L(v - \sum_{s \in S} \ell_s(v) \cdot s) < 1 \ ,$$

or equivalently that

$$v - \sum_{s \in S} \ell_s(v) \cdot s \in L^-(p_L) \subseteq L \ .$$

On the other hand, we have $|\ell_s(v)| = |\ell_s(v)| \cdot p_L(s) = 1$ and hence $\ell_s(v) \in o$ for $s \in S$. This shows that $\sum_{s \in S} \ell_s(v) \cdot s \in L$. Together we obtain that $v \in L$.

The property 6., finally, is an immediate consequence of 5.

Proposition 13.3:

Let $A \subseteq V$ be a closed o-submodule; for any vector $v_\mathrm{o} \in V \backslash A$ there is a continuous linear form ℓ on V such that $\ell(v_\mathrm{o}) = 1$ and $|\ell(v)| < 1$ for any $v \in A$.

Proof: As in the proof of Cor. 9.6 we find an open lattice $L \subseteq V$ such that $A \subseteq L$ and $v_\mathrm{o} \notin L$. We therefore may assume that $A = L$, in fact, is an open lattice. Using the notations of Prop. 13.2 it follows from the property 6. in Prop. 13.2 that

$$\ell_s(v_\mathrm{o}) \notin \ell_s(L) \quad \text{for some } s \in T \ .$$

We define

$$\ell(v) := \ell_s(v_\mathrm{o})^{-1} \cdot \ell_s(v) \ .$$

This is a linear form on V such that

$$\ell(v_\mathrm{o}) = 1 \quad \text{and} \quad L \cap \ell^{-1}(1) = \emptyset \ .$$

The image $\ell(L)$ is an o-submodule of K which does not contain 1. We therefore have $|\ell(v)| < 1$ for any $v \in L$. It remains to show that ℓ is continuous. Set $U := \ker(\ell)$. With $v_\mathrm{o} + L$ also the subsets $v_\mathrm{o} + L + U$ and then

$$\bigcup_{a \in K^\times} a(v_\mathrm{o} + L + U)$$

are open in V. Because of $L \cap (v_o + U) = \emptyset$, or equivalently $(v_o + L) \cap U = \emptyset$, this latter union is the complement of U in V. It follows that U is closed in V and hence, by Prop. 4.13, that V/U is topologically isomorphic to K. This clearly implies that ℓ is continuous.

Proposition 13.4:

For any o-submodule $A \subseteq V$ its pseudo-bipolar A^{pp} is equal to the closure of A in V, i.e., $A^{pp} = \overline{A}$.

Proof: By Lemma 13.1.ii the pseudo-bipolar A^{pp} is closed in V_s and therefore in V. Hence we have $\overline{A} \subseteq A^{pp}$. Consider, on the other hand, a vector $v \in V \backslash \overline{A}$. According to Prop. 13.3 we find an $\ell \in (\overline{A})^p \subseteq A^p$ such that $\ell(v) = 1$. Hence $v \notin A^{pp}$.

Corollary 13.5:

For an o-submodule $A \subseteq V$ the following assertions are equivalent:

i. $A^{pp} = A$;

ii. A is closed;

iii. for any $v_o \in V \backslash A$ there exists a continuous linear form ℓ on V such that $|\ell(v_o)| \geq 1$ and $|\ell(v)| < 1$ for any $v \in A$;

iv. for any $v_o \in V \backslash A$ there exists a continuous seminorm q on V such that $v_o \notin L^-(q)$ and $A \subseteq L^-(q)$.

Proof: The equivalence of i. and ii., resp. of ii. and iii., is Prop. 13.4, resp. Prop. 13.3. The assertion iv. follows trivially from iii. if we set $q(v) := |\ell(v)|$. Finally, we show that iv. implies ii. Let $v_o \in V \backslash A$ and let q be as in iv. Then $v_o + L^-(q)$ is an open neighbourhood of v_o which is disjoint from A.

Lemma 13.6:

For an o-submodule $H \subseteq V_s'$ the following assertions are equivalent:

i. H is equicontinuous;

ii. $H \subseteq L^p$ for some open lattice $L \subseteq V$;

iii. H^p is an open lattice in V.

Proof: Suppose that i. holds true. Then there is an open lattice $L \subseteq V$ such that $\ell(L) \subseteq \mathfrak{m}$ for any $\ell \in H$. This means that $H \subseteq L^p$ as stated in ii. If ii. holds true then $L \subseteq L^{pp} \subseteq H^p$ which shows that H^p is an open lattice in $V = (V_s')'$ as claimed in iii. If, finally, iii. holds true then, defining $L := H^p$, we have $H \subseteq H^{pp} = L^p$ which means that $\ell(L) \subseteq \mathfrak{m}$ for any $\ell \in H$. Hence H is equicontinuous.

Proposition 13.7:

Let $\mathcal{B}$ denote the family of all equicontinuous o-submodules in V'; we have:

i. The family $\mathcal{B}$ consists of bounded subsets in V'_s and is closed under finite sums;

ii. the duality map

$$\delta : V \xrightarrow{\cong} (V'_s)'_{\mathcal{B}}$$

is a topological isomorphism.

Proof: The assertion i. is a special case of Lemma 6.8. In Prop. 9.7 we have seen that the duality map δ is bijective. By construction the initial topology on V with respect to the map δ in the assertion ii. is defined by the lattices H^p where H runs over all equicontinuous o-submodules in V'_s. According to Lemma 13.6 these H^p are open lattices in V. Hence δ is continuous. If, vice versa, $L \subseteq V$ is an arbitrary open lattice then, again by Lemma 13.6, $H := L^p$ is an equicontinuous o-submodule in V'_s. Since $L = L^{pp} = H^p$ by Prop. 13.4 it follows that δ is a homeomorphism.

Corollary 13.8:

If V is barrelled then the duality map

$$\delta : V \xrightarrow{\cong} (V'_s)'_b$$

is a topological isomorphism.

Proof: By the Banach-Steinhaus theorem Prop. 6.15 and Lemma 4.10 any bounded subset in V'_s generates an equicontinuous o-submodule. The $\mathcal{B}$-topology in Prop. 13.7 therefore is the strong topology.

As another application of polarity theory we can show, as promised earlier, that the assertion of the Banach-Steinhaus theorem characterizes barrelled spaces.

Proposition 13.9:

The following assertions are equivalent:

i. V is barrelled;

ii. every bounded subset of V'_s is equicontinuous;

iii. every bounded o-submodule of V'_s is equicontinuous.

Prof: That i. implies ii. is the assertion of the Banach-Steinhaus theorem Prop. 6.15. The implication from ii. to iii. being trivial let us assume that iii. holds true. Let $L \subseteq V$ be a closed lattice. By Lemma 13.1.v, the pseudo-polar L^p

is bounded in V_s'. It is therefore equicontinuous. Lemma 13.6 then says that the pseudo-bipolar L^{pp} is an open lattice in V. But since L is closed we have $L^{pp} = L$ by Cor. 13.5. This proves that V is barrelled.

§14 Admissible topologies

Let V be a locally convex K-vector space. The aim of the duality theory is to reconstruct as much information as possible about V from its dual spaces $V_{\mathcal{B}}'$. We have seen already (Prop. 9.7) that the abstract vector space V can be reconstructed as the dual space of the weak dual V_s'. For the given topology of V this is not possible in general. In particular, Prop. 13.7 does not give this since the equicontinuity of a subset of V_s' cannot be characterized through the topology of V_s' alone. One reason for this difficulty is, of course, that different topologies on V can lead to the same dual space V'. This motivates the following definition.

Definition:

A locally convex topology $\mathcal{T}$ on V is called admissible if

$$\mathcal{L}((V, \mathcal{T}), K) = V' .$$

By construction the weak topology on V is the coarsest admissible topology. On the other hand, the family of pseudo-polars A^p where A runs over all c-compact and bounded o-submodules in V_s' is, by Lemma 13.1.iii, a family of lattices in V. Since the sum of two c-compact and bounded o-submodules again is c-compact (Cor. 12.3) and bounded this family of lattices A^p satisfies the conditions (lc1) and (lc2) and defines therefore a locally convex topology on V which is called the *Mackey topology*. We write V_c for V equipped with the Mackey topology.

Lemma 14.1:

The Mackey topology is the finest admissible topology on V.

Proof: Let $\mathcal{T}$ be any admissible topology on V. In the proof of Prop. 13.7 we have seen that $\mathcal{T}$ is defined by the family of lattices H^p where $H \subseteq V_s'$ runs over all with respect to $\mathcal{T}$ equicontinuous o-submodules. According to Lemmata 6.8 and 6.10 the closure $\overline{H}$ of H still is equicontinuous and bounded. By Lemma 12.10 the closure $\overline{H}$ therefore is c-compact. Since $(\overline{H})^p = H^p$ we see that $\mathcal{T}$ is coarser than the Mackey topology.

It remains to show that the Mackey topology is admissible. If we apply the above reasoning to the given topology of V we obtain that the Mackey topology is finer than the given topology which implies that $V' \subseteq (V_c)'$. For a given $\ell \in (V_c)'$ there exists a c-compact and bounded o-submodule $A \subseteq V'_s$ such that $\ell(A^p) \subseteq \mathfrak{m}$. Applying Lemma 12.1.iv to the topological inclusion $V'_s \subseteq (V_c)'_s$ we obtain that A also is c-compact and hence closed in $(V_c)'_s$. It now follows from Cor. 13.5 that $A = A^{pp}$ in $(V_c)'_s$. Since, by construction, ℓ is contained in the pseudo-bipolar of $A \subseteq (V_c)'_s$ we conclude that $\ell \in A \subseteq V'$. This shows that $V' = (V_c)'$.

Proposition 14.2:

Each of the following classes of subsets of V does not change if we replace the given topology of V by any other admissible topology:

1. *the class of all closed o-submodules;*
2. *the class of all c-compact o-submodules;*
3. *the class of all bounded subsets.*

Proof: For the first class this is a consequence of Cor. 13.5. It then also follows for the second class since the second class, through Lemma 12.1.ii, is characterized within the first class by a completely algebraic condition. We finally consider the third class. Let $\mathcal{B}$ denote the family of all closed equicontinuous o-submodules of V'_s. Since by Lemma 6.10 the closure of any equicontinuous o-submodule in V'_s again is equicontinuous it follows from Prop. 13.7 that the duality map

$$V \xrightarrow{\cong} (V'_s)'_{\mathcal{B}}$$

is a topological isomorphism. If we compare this with the topological isomorphism

$$V_s \xrightarrow{\cong} (V'_s)'_s$$

we are reduced to show that $(V'_s)'_{\mathcal{B}}$ and $(V'_s)'_s$ have the same class of bounded subsets. Since, by Prop. 7.13, any $H \in \mathcal{B}$ is complete in V'_s this exactly is what we have established in the proof of Prop. 7.18.

As an application of this result we have the following facts.

Proposition 14.3:

Let $(V_h)_{h \in H}$ be a family of barrelled locally convex K-vector spaces; then the direct product $\prod_{h \in H} V_h$ is barrelled, too.

Proof: As we have noted in §9 before Prop. 9.11 we may identify $\bigoplus_h V'_h$ and $(\prod_h V_h)'$ as abstract vector spaces. But the locally convex vector spaces $\bigoplus_h (V_h)'_s$ and $(\prod_h V_h)'_s$ in general are different. According to Prop. 9.7 and Prop. 9.10 they at least have the same dual space $\prod_h V_h$. It therefore follows from Prop. 14.2 that every bounded subset $B \subseteq (\prod_h V_h)'_s$ also is bounded in $\bigoplus_h (V_h)'_s$. We have seen in the proof of Prop. 7.12.iii that in this latter situation there are a finite subset $I \subseteq H$ and bounded subsets $B_h \subseteq (V_h)'_s$ for $h \in I$ such that

$$B \subseteq \bigoplus_{h \in I} B_h \ .$$

Since V_h is barrelled the Banach-Steinhaus theorem Prop. 6.15 implies that each B_h is equicontinuous. Hence $\bigoplus_{h \in I} B_h$ and B are equicontinuous. This argument shows that any bounded subset in $(\prod_h V_h)'_s$ is equicontinuous. Applying Prop. 13.9 we obtain that $\prod_h V_h$ is barrelled.

A straightforward reduction argument shows that the above result remains true even if the V_h are not necessarily Hausdorff.

Proposition 14.4:

If V is barrelled or bornological then the Mackey topology coincides with the given topology of V.

Proof: Let $A \subseteq V'_s$ be a c-compact and bounded o-submodule. We have to show that the lattice $A^p \subseteq V$ is open. By Lemma 13.1.ii the pseudo-polar A^p is closed in $V_s = (V'_s)'_s$ and hence in V. If V is barrelled A^p consequently is open. Let us therefore assume that V is bornological. We check that A^p satisfies the condition (*bor*). If $B \subseteq V$ is a bounded subset then, according to Prop. 14.2, B also is bounded in V_c. Since, by definition, A^p is an open lattice in V_c there exists an $a \in K$ such that $B \subseteq aA^p$.

Proposition 14.5:

Let $A \subseteq V$ be a compactoid o-submodule; the given topology and the weak topology of V induce the same topology on A.

Proof: Let $L \subseteq V$ be an open lattice. We have to show that $L \cap A$ is open in A with respect to the weak topology. Fix an $a \in K$ such that $0 < |a| < 1$. Let $(B_i)_{i \in I}$ denote the family of all o-submodules of A which

- contain $aL \cap A$ and

- are open in A with respect to the weak topology.

Since $aL \cap A$, by Prop. 14.2, is closed in A with respect to the weak topology we have

$$\bigcap_{i \in I} B_i = aL \cap A$$

(compare the argument at the end of the proof of Lemma 12.1.ii). We will show that there is an index $i_o \in I$ such that $aB_{i_o} \subseteq aL \cap A$. This is sufficient for our assertion since then $B_{i_o} \subseteq L \cap A$ which implies that $L \cap A$ is open in A with respect to the weak topology.

Since A is compactoid in V the quotient $A/(aL \cap A)$ is contained in an o-module of the form o^m/N where $(bo)^m \subseteq N \subseteq o^m$ for some $0 \neq b \in o$. The family $(B_i)_i$ corresponds to a decreasingly filtered family $(N_i)_i$ of o-submodules $N \subseteq N_i \subseteq o^m$ such that $\bigcap_i N_i = N$. We show that, given such a family, there is, for any $a \in \mathfrak{m}$, an $i_o \in I$ such that $aN_{i_o} \subseteq N$. We proceed by induction with respect to m, and, using Lemma 1.4, we distinguish two cases. First we assume that the value group $|K^\times|$ is dense in $\mathbb{R}_+^\times$. Choose an $c \in K$ such that $|a| < |c^2| < 1$. By the induction hypothesis there exists an $i_1 \in I$ such that

$$c(N_{i_1} \cap o^{m-1}) \subseteq N \ .$$

Let $\mathfrak{a}$ and $\mathfrak{a}_i$ denote the image of N and N_i, respectively, in $o^m/o^{m-1} = o$. We have $bo \subseteq \mathfrak{a} \subseteq \mathfrak{a}_i$, and $(\mathfrak{a}_i)_i$ is a (with respect to inclusion) totally ordered family of ideals in o. Consider the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & o^{m-1} & \longrightarrow & o^m & \longrightarrow & o & \longrightarrow & 0 \\
 & & f\big\downarrow & & g\big\downarrow & & h\big\downarrow & & \\
0 & \longrightarrow & \varprojlim o^{m-1}/(N_i \cap o^{m-1}) & \longrightarrow & \varprojlim o^m/N_i & \longrightarrow & \varprojlim o/\mathfrak{a}_i & &
\end{array}$$

where the vertical maps are the canonical ones. Since o^{m-1} is c-compact the map f is surjective. It follows that

$$\ker(h) = (\ker(g) + o^{m-1})/o^{m-1}$$

and hence that

$$\bigcap_i \mathfrak{a}_i = ((\bigcap_i N_i) + o^{m-1})/o^{m-1} = (N + o^{m-1})/o^{m-1} = \mathfrak{a} \ .$$

We obtain that

$$0 < \sup |\mathfrak{a}| = \inf_i \sup |\mathfrak{a}_i|$$

and consequently that there is an $i_2 \in I$ such that $\sup |\mathfrak{a}_{i_2}| < \sup |c^{-1}\mathfrak{a}|$. This implies that $\mathfrak{a}_{i_2} \subseteq c^{-1}\mathfrak{a}$ or $c\mathfrak{a}_{i_2} \subseteq \mathfrak{a}$ which amounts to

$$cN_{i_2} \subseteq N + o^{m-1} \ .$$

Choosing $i_o \in I$ such that $N \subseteq N_{i_o} \subseteq N_{i_1} \cap N_{i_2}$ we deduce that

$$cN_{i_o} \subseteq N + (N_{i_1} \cap o^{m-1})$$

and therefore that

$$aN_{i_o} \subseteq c^2 N_{i_o} \subseteq cN + c(N_{i_1} \cap o^{m-1}) \subseteq cN + N = N \ .$$

In the other case where K is discretely valued an analogous induction shows that even $N_{i_o} = N$ holds true for some $i_o \in I$.

§15 Reflexivity

In this section we will characterize the (semi-)reflexive spaces among all locally convex K-vector spaces V.

Lemma 15.1:

Every closed lattice $A \subseteq V'_s$ is open in V'_b.

Proof: By Lemma 13.1.v the pseudo-polar A^p is bounded in $V_s = (V'_s)'_s$. It follows from Prop. 14.2 that A^p is bounded in V as well. Using again Lemma 13.1.iv we obtain that A^{pp} is open in V'_b. But we have $A^{pp} = A$ by Cor. 13.5.

Proposition 15.2:

If V is semi-reflexive then V'_b is barrelled.

Proof: If V is semi-reflexive then we have $V = (V'_s)' = (V'_b)'$ as abstract vector spaces. This says that the strong topology on V' is an admissible topology for V'_s. Prop. 14.2 therefore implies that V'_s and V'_b have the same closed lattices. We then obtain from Lemma 15.1 that any closed lattice in V'_b is open.

Proposition 15.3:

The following assertions are equivalent:

i. V is semi-reflexive;

ii. every closed and bounded o-submodule of V_s is c-compact;

iii. every closed and bounded o-submodule of V is c-compact;

iv. V_s is quasi-complete;

v. V is quasi-complete, and every bounded o-submodule of V is compactoid.

Proof: i. $\Rightarrow$ ii.: We have $V_s = (V'_s)'_s = (V'_b)'_s$. Let now $A \subseteq V_s$ be a closed and bounded o-submodule. Since V'_b is barrelled by Prop. 15.2 we may apply the Banach-Steinhaus theorem Prop. 6.15 and obtain that A viewed in $(V'_b)'$ is equicontinuous. It then follows from Lemma 12.10 that A is c-compact in $(V'_b)'_s = V_s$.

ii. $\Rightarrow$ i.: Put $W := V'_s$; then $W' = V$. Because of Lemma 4.10 and Prop. 14.2 the pseudo-polars A^p where A runs over all closed and bounded o-submodules of V_s form a defining family of open lattices for the strong topology on $V' = W$. Since these A are c-compact by assumption we see that the strong topology on

W is coarser than the Mackey topology. Hence it follows from Lemma 14.1 that the strong topology on W is admissible. This implies that $(V_b')' = W' = V$.

ii. $\Leftrightarrow$ iii.: This a consequence of Prop. 14.2.

ii. $\Rightarrow$ iv.: Let $B \subseteq V_s$ be a bounded closed subset. According to Lemma 4.10 the closure A of the o-submodule generated by B is bounded in V_s. By assumption A is c-compact. Lemma 12.1.i then implies that A and hence B are complete.

iv. $\Rightarrow$ ii.: Let $A \subseteq V_s$ be a closed and bounded o-submodule. From Remark 12.9 we know that A then is compactoid. But, by assumption, A also is complete. It therefore follows from Prop. 12.7 that A is c-compact.

iii. $\Rightarrow$ v.: By the same argument as for ii. $\Rightarrow$ iv. we obtain that V is quasi-complete. Let $A \subseteq V$ be a bounded o-submodule. Then its closure $\overline{A}$ is a closed and bounded and hence c-compact o-submodule of V. It again follows from Prop. 12.7 that $\overline{A}$ and hence A are compactoid.

v. $\Rightarrow$ iii.: Let $A \subseteq V$ be a closed and bounded o-submodule. By assumption A then is complete and compactoid which, by Prop. 12.7, implies that A is c-compact.

Lemma 15.4:

If V is reflexive then V_b' is reflexive and V is barrelled.

Proof: By assumption the duality map

$$\delta_V : V \xrightarrow{\cong} (V_b')_b'$$

and hence the dual map

$$\delta_V' : ((V_b')_b')_b' \xrightarrow{\cong} V_b'$$

are topological isomorphisms. The latter map is the inverse of the duality map

$$\delta_{V_b'} : V_b' \longrightarrow ((V_b')_b')_b' \ .$$

It follows that V_b' is reflexive. Prop. 15.2 then implies that $V = (V_b')_b'$ is barrelled.

Proposition 15.5:

A locally convex vector space V is reflexive if and only if it is semi-reflexive and barrelled.

Proof: The direct implication is a consequence of Lemma 15.4. For the reverse implication we only need to recall that, by Lemma 9.9, a barrelled space is pseudo-reflexive.

Proposition 15.6:

The strong dual V'_b of a reflexive Fréchet space V is bornological.

Proof: Let $(L_n)_{n\in\mathbb{N}}$ be a countable family of open lattices in V which defines the locally convex topology on V. The o-submodules $B_n := L_n^p$ in V'_b are equicontinuous and hence bounded (Lemma 6.8). Since V'_b is barrelled by Prop. 15.2 it is straightforward from Lemma 13.6 that each bounded subset of V'_b is contained in some B_n.

We have to show that any lattice $M \subseteq V'_b$ which satisfies the condition (bor) is open. Again since V'_b is barrelled it suffices for this to find a closed lattice $N \subseteq M$. For any $n \in \mathbb{N}$ there is, by assumption, an $a_n \in K^\times$ such that $a_n B_n \subseteq M$. Because of $V' = \bigcup_n B_n$ the o-submodule

$$N := \sum_{n\in\mathbb{N}} a_n B_n$$

is a lattice contained in M. We claim that N is closed. Let us first see that, for any $r \in \mathbb{N}$, the o-submodule $C_r := \sum_{n\leq r} a_n B_n$ is closed in V'_s and hence also in V'_b. According to Lemma 13.1.vi each B_n is c-compact in V'_s. The Cor. 12.3 then implies that C_r is c-compact as well. A fortiori C_r is closed. Consider now any $\ell \notin N$. For any $r \in \mathbb{N}$ there is an open lattice $L_r \subseteq V'_b$ such that

$$(\ell + L_r) \cap C_r = \emptyset \ .$$

As we then also have $(\ell + L_r + C_r) \cap C_r = \emptyset$ we may in fact assume that

$$C_r \subseteq L_r \quad \text{ for any } r \in \mathbb{N} \ .$$

The o-submodule $L := \bigcap_{r\in\mathbb{N}} L_r$ satisfies

$$(\ell+L)\cap N = \bigcap_r (\ell+L_r)\cap(\bigcup_n C_n) = \bigcup_n (\bigcap_r (\ell+L_r)\cap C_n) \subseteq \bigcup_n (\ell+L_n)\cap C_n = \emptyset \ .$$

It therefore remains to convince ourselves that L is an open lattice in V'_b. As an intersection of open lattices it certainly is closed. For any $m \in V'$ there is an $s \in \mathbb{N}$ such that $m \in B_s$ and consequently $a_s m \in C_s \subseteq L_r$ for any $r \geq s$. On the other hand there exists a nonzero $b \in o$ such that $b a_s m \in L_r$ for any $r < s$. We therefore have $b a_s m \in L$ which shows that L is a closed lattice in V'_b. Using a third time that V'_b is barrelled we conclude that L is open.

§16 Compact limits

We have seen in §11 that there are no interesting examples of reflexive Banach spaces. We therefore finish this chapter by explaining how one can construct reflexive spaces via certain projective and locally convex inductive limits. The starting point is the following notion.

Definition:

A continuous linear map $f : V \longrightarrow W$ between two locally convex K-vector spaces V and W is called <u>*compact*</u> *if there is an open lattice $L \subseteq V$ such that the closure of the image $\overline{f(L) \text{ in } W}$ is bounded and c-compact.*

If W is quasi-complete then, as a consequence of Prop. 12.7, the map $f : V \longrightarrow W$ is compact if and only if $f(L)$, for some open lattice $L \subseteq V$, is compactoid. The basic fact for the purposes of this section is the subsequent lemma. We first make a few rather obvious remarks.

Remark 16.1:

For any continuous linear map $f : V \longrightarrow W$ between two locally convex K-vector spaces we have:

i. The map $f : V_s \longrightarrow W_s$ is continuous;

ii. the dual map $f' : W_b' \longrightarrow V_b'$ given by $f'(\ell) := \ell \circ f$ is continuous.

Proof: i. Note that $f^{-1}(\ell^{-1}(o)) = (\ell \circ f)^{-1}(o)$ for any $\ell \in W'$. ii. Here we note that $(f')^{-1}(\mathcal{L}(B, o)) = \mathcal{L}(f(B), o)$ for any bounded subset $B \subseteq V$.

Remark 16.2:

For any locally convex K-vector space V the image of the duality map $\delta : V \longrightarrow (V_b')_s'$ is dense.

Proof: If not there would exist, according to the Hahn-Banach theorem Cor. 9.3, a nonzero continuous linear form ℓ on the right hand side such that $\ell \circ \delta = 0$. This would lead to a contradiction since, by Prop. 9.7, we have $((V_b')_s')' = V'$.

Remark 16.3:

Suppose that $f : V \longrightarrow W$ is a compact map; if $B \subseteq V$ is a bounded o-submodule then $\overline{f(B)}$ is bounded and c-compact in W.

Proof: Let $L \subseteq V$ be an open lattice such that $\overline{f(L)}$ is bounded and c-compact. There is a scalar $a \in K^\times$ such that $B \subseteq aL$. It follows that $\overline{f(B)}$ is contained in $a\overline{f(L)}$. Apply now Lemma 12.1.iii.

Lemma 16.4:

For any continuous linear map $f : V \longrightarrow W$ between two K-Banach spaces V and W the following assertions are equivalent:

i. f is compact;

ii. the dual map $f' : W_b' \longrightarrow V_b'$ is compact;

iii. $f''((V_b')') \subseteq W$.

Proof: We recall first of all that V_b' and W_b' are Banach spaces with respect to the operator norm (Remark 6.7). Moreover, in the assertion iii. we view the duality map $\delta : W \longrightarrow (W_b')_b'$, since W is pseudo-reflexive by Lemma 9.9, as an inclusion. In fact, being complete W is a closed subspace of $(W_b')_b'$.

We begin by establishing the implication from i. to ii. For any bounded o-submodule $B \subseteq V$ we have

$$(f')^{-1}(\mathcal{L}(B, \mathfrak{m})) = \mathcal{L}(f(B), \mathfrak{m}) = f(B)^p \ .$$

By Remark 16.3 the right hand side contains the pseudo-polar of a bounded and c-compact o-submodule of W_s. It follows that the map

$$f' : (W_s')_c \longrightarrow V_b'$$

is continuous. Consider now the open lattice $L := \{\ell \in W_b' : \|\ell\| < 1\}$ in W_b'. We will show that $\overline{f'(L)}$ is bounded and c-compact in V_b'. Using the continuity of the above map and Lemma 12.1 it suffices for this to prove that L is contained in a bounded and c-compact o-submodule of $(W_s')_c$. Certainly L is contained in the pseudo-polar M^p of the unit ball $M := \{w \in W : \|w\| \leq 1\}$ of W. By Lemma 13.1.v/.vi the pseudo-polar M^p is bounded and c-compact in W_s'. Prop. 14.2 then says that M^p is bounded and c-compact in $(W_s')_c$ as well.

For the implication from ii. to iii. we start by noting that, by an analogous argument as above, the map

$$f'' : ((V_b')_s')_c \longrightarrow (W_b')_b'$$

is continuous. It then follows from Remark 16.1.i that

$$f'' : (V_b')_s' \longrightarrow ((W_b')_b')_s$$

is continuous. Remark 16.2 therefore implies that the image of f'' is contained in the closure of W in $((W_b')_b')_s$. But as noted at the beginning of the proof W is closed in $(W_b')_b'$ and hence, by Prop. 14.2, in $((W_b')_b')_s$.

We finally come to the implication from iii. to i. By assumption the map f'' factorizes through a map $f'' : (V_b')' \longrightarrow W$. For any $w \in W$ we have

$$\begin{aligned}(f'')^{-1}(w) &= \{\lambda \in (V_b')' : f''(\lambda)(\ell) = \ell(w) \text{ for any } \ell \in W'\} \\ &= \{\lambda \in (V_b')' : \lambda(\ell \circ f) = \ell(w) \text{ for any } \ell \in W'\}\end{aligned}$$

and hence

$$(f'')^{-1}(\ell^{-1}(o)) = \mathcal{L}(\{\ell \circ f\}, o)$$

for any $\ell \in W'$. This shows that the map

$$f'' : (V_b')_s' \longrightarrow W_s$$

is continuous. Consider now the open lattice $L := \{v \in V : \|v\| < 1\}$ in V. We want to show that $\overline{f(L)}$ is bounded and c-compact in W or equivalently, by Prop. 14.2, in W_s. Similarly as above it suffices to exhibit a bounded and c-compact o-submodule of $(V_b')_s'$ which contains the image of L under the duality map. We take the pseudo-polar of the unit ball in V_b'.

After these preparations we consider a projective system of the form

$$\ldots \longrightarrow V_{n+1} \xrightarrow{g_n} V_n \longrightarrow \ldots \longrightarrow V_2 \xrightarrow{g_1} V_1$$

with locally convex K-vector spaces V_n and continuous linear transition maps $g_n : V_{n+1} \longrightarrow V_n$ for any $n \in \mathbb{N}$. We equip the projective limit $\varprojlim_n V_n$ with the initial topology with respect to the projection maps

$$\begin{aligned} p_m : \varprojlim_n V_n &\longrightarrow V_m \\ (v_n)_n &\longmapsto v_m \end{aligned}$$

for $m \in \mathbb{N}$. Because of Remark 16.1.ii the strong duals form an inductive system

$$(V_1)_b' \xrightarrow{g_1'} (V_2)_b' \longrightarrow \ldots \longrightarrow (V_n)_b' \xrightarrow{g_n'} (V_{n+1})_b' \longrightarrow \ldots$$

of locally convex K-vector spaces. We equip the inductive limit $\varinjlim_n (V_n)_b'$ with the corresponding locally convex final topology. According to its universal property the dual maps p_m' induce a continuous linear map

$$P : \varinjlim_n (V_n)_b' \longrightarrow (\varprojlim_n V_n)_b' \ .$$

This map P always is surjective: By the Hahn-Banach theorem Cor. 9.4 any continuous linear form ℓ on $\varprojlim V_n$ extends to a continuous linear form $\widetilde{\ell}$ on $\prod_n V_n$. It follows from Prop. 9.11 that $\widetilde{\ell}$ factorizes through the projection to $V_m \times \ldots \times V_1$ for some $m \in \mathbb{N}$. We therefore have $\ell = \ell_m \circ p_m$ with $\ell_m \in (V_m)'$ defined by

$$\ell_m(v) := \widetilde{\ell}(\ldots, 0, v, g_{m-1}(v), \ldots, g_1 \circ \ldots \circ g_{m-1}(v)) \ .$$

The map P certainly is injective and hence bijective if the projection maps p_m have dense image. In particular, in this situation both sides of P are Hausdorff spaces.

Proposition 16.5:

Suppose that, for any $n \in \mathbb{N}$, V_n is a Banach space, g_n is compact, and p_n has dense image; we then have:

i. The map P is a topological isomorphism;

ii. $\varprojlim_n V_n$ is a reflexive Fréchet space.

Proof: Since $\varprojlim V_n$ is a countable projective limit of normed vector spaces its topology is Hausdorff and can be defined by a countable family of seminorms and hence is metrizable by Prop. 8.1. The direct product $\prod_n V_n$ of the Banach spaces V_n is complete. The closed vector subspace $\varprojlim V_n$ of $\prod_n V_n$ then is complete, too. We see that $\varprojlim V_n$ is a Fréchet space and in particular (Prop. 8.2) is barrelled and hence (Lemma 9.9) pseudo-reflexive. For the reflexivity it remains to prove that the duality map δ is surjective onto $((\varprojlim V_n)'_b)'$. Take any λ in this latter double dual. The linear forms $p''_n(\lambda) \in ((V_n)'_b)'$ are compatible with respect to the maps g''_n. It therefore follows from Lemma 16.4 that for each $n \in \mathbb{N}$ there is a vector $v_n \in V_n$ such that

$$p''_n(\lambda)(\ell) = \ell(v_n) \quad \text{for any } \ell \in (V_n)'_b \ .$$

We obviously have $(v_n)_n \in \varprojlim V_n$ and $\delta((v_n)_n) = \lambda$.

We already know from the preceding discussion that the map P is a continuous bijection between Hausdorff spaces. It remains to consider an open lattice $L \subseteq \varinjlim (V_n)'_b$ and to show that $P(L)$ is open. We first make the following observation. By the reflexivity assertion which we already have established the dual space of the target of P is $\varprojlim V_n$. But also for the source of P we have

$$(\varinjlim (V_n)'_b)' = \varprojlim ((V_n)'_b)' = \varprojlim V_n \ ;$$

the first identity follows from the universal property of the locally convex final topology whereas the second identity is a consequence of Lemma 16.4. This implies that the topologies on the two sides of P are admissible relative to each other. Using Prop. 14.2 we see that with L also $P(L)$ is a closed lattice. But according to Prop. 15.2 the target of P is barrelled. We see that $P(L)$ indeed is open.

The requirement in the above result that the V_n are Banach spaces is not a serious restriction. To explain this we need the following general fact about a compact map $f : V \longrightarrow W$. Let $L \subseteq V$ be an open lattice such that $B := \overline{f(L)}$ is bounded and c-compact. By Lemma 12.1.i the o-submodule B in particular is bounded and complete. In this situation we have, according to Lemma 7.17, the Banach space (W_B, p_B) together with the continuous inclusion $W_B \xrightarrow{\subseteq} W$. The map f obviously factorizes into

$$V \xrightarrow{\widetilde{f}} W_B \xrightarrow{\subseteq} W$$

where $\widetilde{f}$ is continuous because of

$$\widetilde{f}^{-1}(\{w \in W_B : p_B(w) \leq 1\}) \supseteq \widetilde{f}^{-1}(B) \supseteq L \ .$$

If we apply this to our projective system assuming that the transition maps g_n are compact we obtain a commutative diagram of the form

$$\begin{tikzcd}
\ldots \arrow[r] & V_{n+2} \arrow[rr, "g_{n+1}"] \arrow[dr, "\widetilde{g}_{n+1}"] & & V_{n+1} \arrow[rr, "g_n"] \arrow[dr, "\widetilde{g}_n"] & & V_n \arrow[r] & \ldots \\
\ldots \arrow[r] & (V_{n+2})_{B_{n+2}} \arrow[u, "\subseteq"] \arrow[rr, "h_{n+1}"'] & & (V_{n+1})_{B_{n+1}} \arrow[u, "\subseteq"] \arrow[rr, "h_n"'] & & (V_n)_{B_n} \arrow[u, "\subseteq"] \arrow[r] & \ldots
\end{tikzcd}$$

In the limit this induces a topological isomorphism

$$\varprojlim_n V_n \ \cong \ \varprojlim_n (V_n)_{B_n} \ .$$

Since B_{n+1} is an open lattice in $(V_{n+1})_{B_{n+1}}$ and since $h_n(B_{n+1}) = \widetilde{g}_n(B_{n+1})$ is bounded and c-compact in $(V_n)_{B_n}$ by Lemma 12.1.iv the map h_n is compact. We finally replace each $(V_n)_{B_n}$ by

$$W_n := \text{the closure of the image of } \widetilde{g}_n \circ p_{n+1} \text{ in } (V_n)_{B_n}$$

to obtain the projective system of Banach spaces with compact transition maps

$$\ldots \longrightarrow W_{n+1} \xrightarrow{h_n|W_{n+1}} W_n \longrightarrow \ldots \longrightarrow W_1$$

for which the images of the projection maps $\varprojlim_n W_n \longrightarrow W_m$ have dense images and such that

$$\varprojlim_n V_n \ \cong \ \varprojlim_n (V_n)_{B_n} = \varprojlim_n W_n \ .$$

All in all, starting with an arbitrary projective system with compact transition maps we have constructed, without changing the projective limit, a new projective system which satisfies all the additional requirements of Prop. 16.5.

Corollary 16.6:

For any projective system

$$\ldots \longrightarrow V_{n+1} \longrightarrow V_n \longrightarrow \ldots \longrightarrow V_1$$

of (Hausdorff) locally convex K-vector spaces with compact transition maps the projective limit $\varprojlim_n V_n$ is a reflexive Fréchet space.

At the very end of the above discussion we have implicitly made use of the following obvious consequences of Lemma 12.1 which we state for later reference.

Remark 16.7:

Let $g : V \longrightarrow W$ be a compact map; we have:

i. If $h : V_1 \longrightarrow V$ and $f : W \longrightarrow W_1$ are arbitrary continuous linear maps then the map $f \circ g \circ h : V_1 \longrightarrow W_1$ is compact;

ii. if the image of g is contained in the closed vector subspace $W_\mathrm{o} \subseteq W$ then the induced map $g : V \longrightarrow W_\mathrm{o}$ is compact.

We now turn to an analogous consideration of an inductive system

$$V_1 \xrightarrow{i_1} V_2 \longrightarrow \ldots \longrightarrow V_n \xrightarrow{i_n} V_{n+1} \longrightarrow \ldots$$

of locally convex K-vector spaces V_n and continuous linear transition maps $i_n : V_n \longrightarrow V_{n+1}$. Let $\varinjlim_n V_n$ denote the inductive limit equipped with the locally convex final topology with respect to the natural maps $j_m : V_m \longrightarrow \varinjlim_n V_n$. By Remark 16.1.ii the strong duals now form a projective system

$$\ldots \longrightarrow (V_{n+1})'_b \xrightarrow{i'_n} (V_n)'_b \longrightarrow \ldots \longrightarrow (V_2)'_b \xrightarrow{i'_1} (V_1)'_b$$

of locally convex vector spaces. We equip the projective limit $\varprojlim_n (V_n)'_b$ with the corresponding initial topology. Using Remark 16.1.ii it immediately follows from the universal properties that the linear map

$$\begin{aligned} I : (\varinjlim_n V_n)'_b &\longrightarrow \varprojlim_n (V_n)'_b \\ \ell &\longmapsto (\ell \circ j_n)_n \end{aligned}$$

is continuous and bijective.

Remark 16.8:

Let $f : V \longrightarrow W$ be a compact map and let $A \subseteq V$ be a c-compact o-submodule; for any finitely many vectors $v_1, \ldots, v_m \in V \backslash A$ there is an open lattice $L \subseteq V$ such that $f(v_1), \ldots, f(v_m) \notin f(A) + \overline{f(L)}$ and $\overline{f(L)}$ is bounded and c-compact.

Proof: By Lemma 12.1.i/.iv the image $f(A)$ is closed in W. Since f is injective we have $f(v_1), \ldots, f(v_m) \notin f(A)$. We therefore find an open lattice $M \subseteq W$ such that $f(v_1), \ldots, f(v_m) \notin f(A) + M$. Take for L any open lattice contained in $f^{-1}(M)$ such that $\overline{f(L)}$ is bounded and c-compact. Since M is closed we have $\overline{f(L)} \subseteq M$.

Lemma 16.9:

Suppose that i_n is injective and compact for any $n \in \mathbb{N}$; we then have:

i. $\varinjlim_n V_n$ is Hausdorff;

ii. for any bounded subset $B \subseteq \varinjlim_n V_n$ there are an $m \in \mathbb{N}$ and a bounded subset $B_m \subseteq V_m$ such that $j_m(B_m) = B$.

Proof: Together with the transition maps i_n also the maps j_m are injective. For simplicity we view all these maps in the following as inclusions and omit them from the notation. Moreover, for any subset $X \subseteq V_m$ we let $\overline{X}^{(m)}$ denote the closure of X in the topology of V_m.

i. Let $0 \neq v \in \varinjlim V_n$. We have to find an open lattice $L \subseteq \varinjlim V_n$ such that $v \notin L$. This will be done by an inductive construction. Suppose that $v \in V_m$. Applying Remark 16.8 to the compact map i_m and $A := \{0\}$ we obtain an open lattice $L_m \subseteq V_m$ such that $v \notin \overline{L}_m^{(m+1)}$ and $\overline{L}_m^{(m+1)}$ is c-compact in V_{m+1}. We note at once that, by Lemma 12.1.iv, $\overline{L}_m^{(m+1)}$ also is c-compact in V_n for any $n \geq m+1$. We now apply Remark 16.8 again to the map i_{m+1} and $A := \overline{L}_m^{(m+1)}$ and obtain an open lattice $L_{m+1} \subseteq V_{m+1}$ such that $v \notin \overline{L}_m^{(m+1)} + \overline{L}_{m+1}^{(m+2)}$ and $\overline{L}_{m+1}^{(m+2)}$ is c-compact in V_{m+2} and hence in V_n for any $n \geq m+2$. By Cor. 12.3 the sum $\overline{L}_m^{(m+1)} + \overline{L}_{m+1}^{(m+2)}$ is c-compact in V_n for any $n \geq m+2$. We next apply Remark 16.8 to i_{m+2} and $A := \overline{L}_m^{(m+1)} + \overline{L}_{m+1}^{(m+2)}$. Proceeding inductively in this way we find open lattices $L_n \subseteq V_n$ for any $n \geq m$ such that

$$v \notin \sum_{n \geq m} \overline{L}_n^{(n+1)} \ .$$

According to Lemma 5.1.iii the lattice $L := \sum_{n \geq m} L_n$ is open in $\varinjlim V_n$ and does not contain the vector v.

ii. Reasoning by contradiction we assume that, for any $n \in \mathbb{N}$, either $B \nsubseteq V_n$ or $B \subseteq V_n$ but B is not bounded in V_n. We fix a scalar $b \in K$ such that $0 < |b| < 1$ and a vector $0 \neq v_1 \in B$. Suppose that $v_1 \in V_{m_1}$. By an inductive procedure similar to the above one we construct an open lattice $L \subseteq \varinjlim V_n$ having properties which will give rise to a contradiction.

We start by applying Remark 16.8 to the map i_{m_1} and $A := \{0\}$ in order to obtain an open lattice $L_{m_1} \subseteq V_{m_1}$ such that $v_1 \notin \overline{L}_{m_1}^{(m_1+1)}$ and $\overline{L}_{m_1}^{(m_1+1)}$ is bounded and c-compact in V_{m_1+1}. By our assumption on B the subset bB cannot be contained in $\overline{L}_{m_1}^{(m_1+1)}$. Hence there is a vector $v_2 \in B$ such that $bv_2 \notin \overline{L}_{m_1}^{(m_1+1)}$. Suppose that $v_2 \in V_{m_2}$ for some $m_2 > m_1$. We now apply Remark 16.8 to the map i_{m_2} and $A := \overline{L}_{m_1}^{(m_1+1)}$ and find an open lattice $L_{m_2} \subseteq V_{m_2}$ such that $v_1, bv_2 \notin \overline{L}_{m_1}^{(m_1+1)} + \overline{L}_{m_2}^{(m_2+1)}$ and $\overline{L}_{m_2}^{(m_2+1)}$ is bounded and c-compact in V_{m_2+1}. Our assumption on B this time ensures that there is a vector $v_3 \in B$ such that $b^2 v_3 \notin \overline{L}_{m_1}^{(m_1+1)} + \overline{L}_{m_2}^{(m_2+1)}$. Supposing that $v_3 \in V_{m_3}$ for some $m_3 > m_2$ we next apply Remark 16.8 to i_{m_3} and so on. Inductively we obtain in this way a sequence $m_1 < m_2 < \ldots$ in $\mathbb{N}$, open lattices $L_{m_j} \subseteq V_{m_j}$, and vectors $v_j \in B \cap V_{m_j}$ such that

$$v_1, bv_2, b^2 v_3, \ldots \notin \sum_j \overline{L}_{m_j}^{(m_j+1)} .$$

Then $L := \sum_j L_{m_j}$ is an open lattice in $\varinjlim V_n$ which does not contain any member of the sequence $(b^j v_j)_j$. But since B is bounded in $\varinjlim V_n$ and $0 < |b| < 1$ this sequence converges to the zero vector in $\varinjlim V_n$. This is a contradiction.

Proposition 16.10:

Suppose that

$$V_1 \longrightarrow \ldots \longrightarrow V_n \xrightarrow{i_n} V_{n+1} \longrightarrow \ldots$$

is an inductive system of (Hausdorff) locally convex K-vector spaces with injective and compact transition maps; we then have:

i. $\varinjlim_n V_n$ *is reflexive, bornological, and complete;*

ii. $(\varinjlim_n V_n)'_b$ *is a Fréchet space;*

iii. the map $I : (\varinjlim_n V_n)'_b \xrightarrow{\cong} \varprojlim_n (V_n)'_b$ *is a topological isomorphism.*

Proof: By the same construction as before Cor. 16.6 we have a commutative diagram of the form

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & V_n & \xrightarrow{i_n} & V_{n+1} & \xrightarrow{i_{n+1}} & V_{n+2} & \longrightarrow \ldots \\
& \subseteq\uparrow & \searrow & \subseteq\uparrow & \searrow & \subseteq\uparrow & \\
\ldots \longrightarrow & (V_n)_{B_n} & \longrightarrow & (V_{n+1})_{B_{n+1}} & \longrightarrow & (V_{n+2})_{B_{n+2}} & \longrightarrow \ldots
\end{array}$$

where the bottom row consists of Banach spaces with injective and compact transition maps. In the inductive limit this induces a topological isomorphism

$$\varinjlim_n V_n \;\cong\; \varinjlim_n (V_n)_{B_n} \ .$$

Since passing to the strong dual spaces leads to a similar diagram we also have a corresponding topological isomorphism

$$\varprojlim_n (V_n)'_b \;\cong\; \varprojlim_n ((V_n)_{B_n})'_b \ .$$

This shows that without loss of generality we may assume that the V_n are Banach spaces.

By Example 2) after Prop. 6.13 and Example 3) after Cor. 6.16 we know that $\varinjlim V_n$ is bornological and barrelled. We know already from Lemma 16.9.i that $\varinjlim V_n$ is Hausdorff. For the reflexivity it remains to show, by Prop. 15.3 and Prop. 15.5, that any closed and bounded o-submodule $B \subseteq \varinjlim V_n$ is c-compact. According to Lemma 16.9.ii we find an $m \in \mathbb{N}$ and a bounded o-submodule B_m in V_m such that $B = j_m(B_m)$. The closure of $B_{m+1} := i_m(B_m)$ in V_{m+1} is c-compact by Remark 16.3. But B_{m+1} being the preimage of B was already closed. It follows that B_{m+1} is c-compact in V_{m+1} and a fortiori (Lemma 12.1.iv) that $B = j_{m+1}(B_{m+1})$ is c-compact in $\varinjlim V_n$. This settles the reflexivity of $\varinjlim V_n$.

Consider again a bounded subset $B \subseteq \varinjlim V_n$. As a consequence of Lemma 16.9.ii there are $m, k \in \mathbb{N}$ such that

$$B \subseteq B_{m,k} := j_m(\{v \in V_m : \|v\| \leq k\}) \ .$$

This shows that the strong topology on $(\varinjlim V_n)'$ is defined by the countably many lattices $\mathcal{L}(B_{m,k}, o)$. The strong dual $(\varinjlim V_n)'_b$ therefore is metrizable by Prop. 8.1. But by Prop. 9.1.ii it also is complete since $\varinjlim V_n$ is bornological. This proves the assertion ii. that $(\varinjlim V_n)'_b$ is a Fréchet space. By Lemma 16.4 and Cor. 16.6 the target $\varprojlim (V_n)'_b$ of the map I is a Fréchet space as well. The bijection I therefore is a topological isomorphism as a consequence of the open mapping theorem Cor. 8.7.

Finally, as a Fréchet space $(\varinjlim V_n)'_b$ is bornological (Prop. 6.14). Applying Prop. 9.1.ii once more and using the reflexivity we obtain that $\varinjlim V_n$ is complete.

We finish this section by an example of an inductive limit of the type considered in the above result.

Example:

Let K be an extension field of $\mathbb{Q}_p$ which is spherically complete with respect to an absolute value $|\ |$ which extends the p-adic absolute value $|\ |_p$ on $\mathbb{Q}_p$. Fix a point $a \in \mathbb{Q}_p$ and a natural number $m \in \mathbb{N}$ and consider the ball $B_{p^{-m}}(a)$ around a in $\mathbb{Q}_p$. A function $\phi : B_{p^{-m}}(a) \longrightarrow K$ is called *analytic* if ϕ can be expanded around a into a convergent power series $F(T) = \sum_{n \geq 0} c_n T^n \in K[[T]]$, i.e., if $F(x-a)$ converges to $\phi(x)$ for any $x \in B_{p^{-m}}(a)$. This notion has the following basic properties (compare [Sch] §25):

1) The power series $F(T)$ converges on $B_{p^{-m}}(0)$ if and only if $\lim\limits_{n \to \infty} |c_n| p^{-mn} = 0$.

2) (Identity theorem) The power series F is uniquely determined by ϕ and a.

3) (Invariance property) ϕ can be expanded around any other point $b \in B_{p^{-m}}(a)$, i.e., there is a power series $G(T) = \sum_{n \geq 0} d_n T^n \in K[[T]]$ such that $G(x-b)$ converges to $\phi(x)$ for any $x \in B_{p^{-m}}(b) = B_{p^{-m}}(a)$; moreover, one has $\max\limits_{n \geq 0} |c_n| p^{-mn} = \max\limits_{n \geq 0} |d_n| p^{-mn}$.

We see that

$$\|\phi\| := \max_{n \geq 0} |c_n| p^{-mn}$$

is a well defined norm on the K-vector space $\mathcal{O}(B_{p^{-m}}(a))$ of all K-valued analytic functions on the ball $B_{p^{-m}}(a)$. We leave it as an exercise to the reader to show that $\mathcal{O}(B_{p^{-m}}(a))$ in fact is a Banach space.

A function $\phi : \mathbb{Z}_p \longrightarrow K$ is called *locally analytic* if for any point $a \in \mathbb{Z}_p$ there is an $m \in \mathbb{N}$ such that $\phi | B_{p^{-m}}(a) \in \mathcal{O}(B_{p^{-m}}(a))$. Let $C^{an}(\mathbb{Z}_p, K)$ denote the K-vector space of all K-valued locally analytic functions on $\mathbb{Z}_p$. In $C^{an}(\mathbb{Z}_p, K)$ we have the increasing sequence of vector subspaces $V_1 \subseteq V_2 \subseteq \ldots$ defined by

$$V_m := \{\phi \in C^{an}(\mathbb{Z}_p, K) : \phi | B_{p^{-m}}(a) \in \mathcal{O}(B_{p^{-m}}(a)) \text{ for any } a \in \mathbb{Z}_p\}\ .$$

Since $\mathbb{Z}_p$ is compact we have

$$C^{an}(\mathbb{Z}_p, K) = \bigcup_{m \in \mathbb{N}} V_m\ .$$

Moreover, for each $m \in \mathbb{N}$, the linear map

$$\begin{array}{ccc} V_m & \xrightarrow{\cong} & \bigoplus_{a \bmod p^m} \mathcal{O}(B_{p^{-m}}(a)) \\ \phi & \longmapsto & \sum_a \phi | B_{p^{-m}}(a) \end{array}$$

is a bijection. Hence V_m with the norm

$$\|\phi\| := \max_{a \bmod p^m} \|\phi | B_{p^{-m}}(a)\|$$

is a Banach space. It is a straightforward consequence of the invariance property that the inclusion maps $V_m \hookrightarrow V_{m+1}$ are continuous. We equip $C^{an}(\mathbb{Z}_p, K)$ with the locally convex final topology with respect to the family of inclusion maps $V_m \hookrightarrow C^{an}(\mathbb{Z}_p, K)$.

Claim: The inclusion maps $V_m \hookrightarrow V_{m+1}$ are compact.

Proof: Because of Prop. 12.2 it suffices to show that the restriction map $\mathcal{O}(B_{p^{-m}}(a)) \longrightarrow \mathcal{O}(B_{p^{-(m+1)}}(a))$ is compact. By a substitution of variables this restriction map can be identified with the linear endomorphism

$$\begin{array}{ccc} c_0(\mathbb{N}) & \longrightarrow & c_0(\mathbb{N}) \\ \phi & \longmapsto & [n \mapsto p^{n-1}\phi(n)] \ . \end{array}$$

The image of the open lattice $\{\phi \in c_0(\mathbb{N}) : \|\phi\|_\infty \leq 1\}$ under this endomorphism is the o-submodule

$$A := \{\phi \in c_0(\mathbb{N}) : |\phi(n)| \leq p^{-(n-1)} \text{ for any } n \in \mathbb{N}\} \ .$$

In the Example after Remark 12.8 we have seen that A indeed is bounded and c-compact.

Hence we may apply Prop. 16.10 and obtain that $C^{an}(\mathbb{Z}_p, K)$ is a reflexive, bornological, and complete Hausdorff locally convex K-vector space whose strong dual is a Fréchet space.

Chap. IV: Nuclear maps and spaces

This final chapter is devoted to the study of a hierarchy of finiteness conditions one can impose on a continuous linear map between two locally convex K-vector spaces V and W. In increasing speciality these are the conditions of being completely continuous, compact, nuclear and of finite rank. The whole theory is intimately connected to the properties of the projective tensor product. We therefore begin in §17 by introducing the inductive and projective tensor product topologies. From §18 on it again will be a standing assumption that the field K is spherically complete. The completely continuous maps are the maps which lie in the closure of the finite rank operators with respect to the topology of bounded convergence. Compact maps were already defined in §16. Any compact map is completely continuous; the converse holds for maps between Banach spaces. Finally, continuous linear maps which factorize through a compact map between Banach spaces are called nuclear. The basic properties of these classes of maps are presented in §§18 and 20. In between in §19 we discuss nuclear spaces. In the second half of §20 these concepts are applied to the study of the continuous linear dual of a projective tensor product.

The following differences between functional analysis over the complex numbers and the nonarchimedean theory over a spherically complete field K should be pointed out. First of all every locally convex K-vector space has the so called approximation property meaning that the finite rank operators are dense in all continuous linear maps with respect to the topology of compactoid convergence. Secondly the so called injective tensor product topology coincides with the projective one. On the one hand this simplifies the theory of completely continuous maps quite a bit. In particular, there is no difference between nuclear spaces and Schwartz spaces. On the other hand it means that in contrast to classical functional analysis the nuclearity of a space cannot be characterized by its behaviour with respect to the tensor product.

In §21 we construct the trace functional for nuclear maps and establish the usual properties of a trace for it. The final §22 treats, as an application of the results in this chapter, the theory of Fredholm maps. These are continuous linear endomorphisms of a locally convex vector space whose kernel and cokernel are finite dimensional. So they are quite the opposite of compact maps. The relation between these two classes of maps consists in the fact that the sum of a compact map and the identity map is Fredholm.

§17 Topological tensor products

Let U, V, and W be locally convex K-vector spaces. In this section we will discuss a number of ways how to equip the tensor product $V \otimes_K W$ with a locally convex topology.

A. The inductive tensor product topology

A bilinear map $\beta : V \times W \longrightarrow U$ is called *separately continuous* if the linear maps

$$\begin{array}{rccc} \beta(.,w_o): & V & \longrightarrow & U \\ & v & \longmapsto & \beta(v,w_o) \end{array}$$

for any $w_o \in W$ and

$$\begin{array}{rccc} \beta(v_o,.): & W & \longrightarrow & U \\ & w & \longmapsto & \beta(v_o,w) \end{array}$$

for any $v_o \in V$ are continuous. There is a unique finest locally convex topology on $V \otimes_K W$ - the *inductive tensor product topology* - such that the canonical bilinear map

$$\begin{array}{ccc} V \times W & \longrightarrow & V \otimes_K W \\ (v,w) & \longmapsto & v \otimes w \end{array}$$

is separately continuous. It is the locally convex final topology with respect to the family of linear maps $. \otimes w_o : V \longrightarrow V \otimes_K W$, for any $w_o \in W$, and $v_o \otimes . : W \longrightarrow V \otimes_K W$, for any $v_o \in V$. We write $V \otimes_{K,\iota} W$ for $V \otimes_K W$ equipped with the inductive tensor product topology and call it the *inductive tensor product* of V and W. For any separately continuous bilinear map $\beta : V \times W \longrightarrow U$ the induced linear map

$$\begin{array}{ccc} V \otimes_{K,\iota} W & \longrightarrow & U \\ v \otimes w & \longmapsto & \beta(v,w) \end{array},$$

by construction, is continuous.

B. The projective tensor product topology

A bilinear map $\beta : V \times W \longrightarrow U$ is called *(jointly) continuous* if it is continuous as a map between topological spaces (with the product topology on the left hand side).

Lemma 17.1:

For a bilinear map $\beta : V \times W \longrightarrow U$ the following assertions are equivalent:

i. β is continuous;

ii. for any open lattice $N \subseteq U$ there are open lattices $L \subseteq V$ and $M \subseteq W$, respectively, such that $\beta(L \times M) \subseteq N$.

Proof: It is obvious that i. implies ii. Let us assume that ii. holds true. Let $v_o \in V$ and $w_o \in W$ be vectors, let $N \subseteq U$ be an open lattice, and let L and M

be as in ii. such that $\beta(L \times M) \subseteq N$. Choosing a $0 \neq a \in o$ such that $av_{\rm o} \in L$ and $aw_{\rm o} \in M$ we have

$$\begin{aligned} \beta(v_{\rm o} + aL, w_{\rm o} + aM) &\subseteq \beta(v_{\rm o}, w_{\rm o}) + \beta(v_{\rm o}, aM) + \beta(aL, w_{\rm o}) + \beta(L \times M) \\ &= \beta(v_{\rm o}, w_{\rm o}) + \beta(av_{\rm o}, M) + \beta(L, aw_{\rm o}) + \beta(L \times M) \\ &\subseteq \beta(v_{\rm o}, w_{\rm o}) + N \ . \end{aligned}$$

Let $L \subseteq V$ and $M \subseteq W$ be lattices. It follows from Lemma 1.2.iv and [B-CA] Chap.I §2.4 Prop.3(ii) that any torsionfree o-module is flat. The canonical map

$$L \otimes_o M \longrightarrow V \otimes_o W = V \otimes_K W$$

therefore is injective and we may view $L \otimes_o M$ as a lattice in $V \otimes_K W$.

The family of all lattices $L \otimes_o M$ in $V \otimes_K W$ where L and M run over the open lattices in V and W, respectively, clearly satisfies the conditions (lc1) and (lc2) and defines therefore a locally convex topology on $V \otimes_K W$ which is called the *projective tensor product topology*. We write $V \otimes_{K,\pi} W$ for $V \otimes_K W$ equipped with the projective tensor product topology and call it the *projective tensor product* of V and W. It is obvious from Lemma 17.1. that the canonical bilinear map

$$\begin{array}{ccc} V \times W & \longrightarrow & V \otimes_{K,\pi} W \\ (v, w) & \longmapsto & v \otimes w \end{array}$$

is continuous and that, for any continuous bilinear map $\beta : V \times W \longrightarrow U$, the induced linear map

$$\begin{array}{ccc} V \otimes_{K,\pi} W & \longrightarrow & U \\ v \otimes w & \longmapsto & \beta(v, w) \end{array}$$

is continuous. In particular, the projective tensor product topology is the finest locally convex topology on $V \otimes_K W$ which makes the canonical map $V \times W \longrightarrow V \otimes_K W$ continuous. By construction the projective tensor product topology is coarser than the inductive tensor product topology.

To understand this construction from the point of view of seminorms we consider any two continuous seminorms p on V and q on W, respectively. The *tensor product seminorm* $p \otimes q$ on $V \otimes_K W$ is defined by

$$p \otimes q(u) := \inf\{\max_{1 \leq i \leq r} p(v_i) \cdot q(w_i) : u = \sum_{i=1}^{r} v_i \otimes w_i, v_i \in V, w_i \in W\} \ .$$

Since obviously $L(p) \otimes_o L(q) \subseteq L(p \otimes q)$ the lattice $L(p \otimes q)$ is open in $V \otimes_{K,\pi} W$ which by Lemma 4.5.i means that the tensor product seminorm $p \otimes q$ is continuous on $V \otimes_{K,\pi} W$. In a simple way it is compatible with the formation of gauge seminorms.

Lemma 17.2

Let $L \subseteq V$ and $M \subseteq W$ be open lattices; we then have $p_L \otimes p_M = p_{L\otimes_o M}$.

Proof: Consider first arbitrary vectors $v \in V$ and $w \in W$ and recall that

$$p_L(v) \cdot p_M(w) = \inf_{v \in aL} |a| \cdot \inf_{w \in bM} |b| \quad \text{and} \quad p_{L\otimes_o M}(v \otimes w) = \inf_{v\otimes w \in cL\otimes_o M} |c| \ .$$

If $v \in aL$ and $w \in bM$ then $v \otimes w \in abL \otimes_o M$ so that $p_{L\otimes_o M}(v \otimes w) \leq |a| \cdot |b|$. We see that

$$p_{L\otimes_o M}(v \otimes w) \leq p_L(v) \cdot p_M(w) \ .$$

For an arbitrary vector $u \in V \otimes_K W$ it follows that

$$\begin{aligned} p_L \otimes p_M(u) &= \inf\{\max_i p(v_i) \cdot q(w_i) : u = \textstyle\sum v_i \otimes w_i\} \\ &\geq \inf\{\max_i p_{L\otimes_o M}(v_i \otimes w_i) : u = \textstyle\sum v_i \otimes w_i\} \\ &\geq p_{L\otimes_o M}(u) \ . \end{aligned}$$

On the other hand, if $u \in cL \otimes_o M$ we write $u = c \cdot \sum_{i=1}^r v_i \otimes w_i$ with $v_i \in L$ and $w_i \in M$ and obtain

$$p_L \otimes p_M(u) \leq |c| \cdot \max_i p_L(v_i) \cdot p_M(w_i) \leq |c| \ .$$

This shows that

$$p_L \otimes p_M(u) \leq p_{L\otimes_o M}(u) \ .$$

This lemma (together with Prop. 4.4) shows that the projective tensor product topology on $V \otimes_K W$ can equivalently be defined by the family of tensor product seminorms $p \otimes q$ where p and q run through all continuous seminorms on V and W, respectively.

All the finer information about projective tensor products is based upon the following technical fact.

Lemma 17.3:

Suppose we are given finitely many vectors $v_1, \ldots, v_m \in V$ and a constant $0 < c \leq 1$ such that

(1) $\quad p(a_1v_1 + \ldots + a_mv_m) \geq c \cdot \max_{1\leq j\leq m} |a_j| \cdot p(v_j) \quad$ *for any $a_1, \ldots, a_m \in K$;*

we then have

$$p \otimes q(v_1 \otimes w_1 + \ldots + v_m \otimes w_m) \geq c \cdot \max_{1\leq j\leq m} p(v_j) \cdot q(w_j)$$

for any $w_1, \ldots, w_m \in W$.

Proof: Consider any identity $v_1 \otimes w_1 + \ldots + v_m \otimes w_m = v'_1 \otimes w'_1 + \ldots + v'_r \otimes w'_r$. We have to show that

$$\max_{1 \leq i \leq r} p(v'_i) \cdot q(w'_i) \geq c \cdot \max_{1 \leq j \leq m} p(v_j) \cdot q(w_j) \ .$$

Let $W_\mathrm{o} \subseteq W$ be the finite dimensional vector subspace generated by the vectors $w_1, \ldots, w_m$ and $w'_1, \ldots, w'_r$. We have seen in the proof of Prop. 10.4 (formula (c) applied to the quotient space of W_o on which q becomes a norm) that, given any constant $0 < d < 1$, there is a basis $e_1, \ldots, e_n$ of W_o such that

$$(2) \qquad q(a_1e_1 + \ldots + a_ne_n) \geq d \cdot \max_{1 \leq k \leq n} |a_k| \cdot q(e_k) \qquad \text{for any } a_1, \ldots, a_n \in K \ .$$

Writing $w_j = a_{j1}e_1 + \ldots + a_{jn}e_n$ and $w'_i = b_{i1}e_1 + \ldots + b_{in}e_n$ we have

$$\sum_{k=1}^{n} \Big(\sum_{j=1}^{m} a_{jk}v_j\Big) \otimes e_k = \sum_{j=1}^{m} v_j \otimes w_j = \sum_{i=1}^{r} v'_i \otimes w'_i = \sum_{k=1}^{n} \Big(\sum_{i=1}^{r} b_{ik}v'_i\Big) \otimes e_k$$

and hence

$$(3) \qquad \sum_{j=1}^{m} a_{jk}v_j = \sum_{i=1}^{r} b_{ik}v'_i \ .$$

We now compute

$$\begin{aligned}
\max_i p(v'_i) \cdot q(w'_i) &\overset{(2)}{\geq} d \cdot \max_{i,k} p(v'_i) \cdot |b_{ik}| \cdot q(e_k) \\
&\geq d \cdot \max_k p(\textstyle\sum_i b_{ik}v'_i) \cdot q(e_k) \\
&\overset{(3)}{=} d \cdot \max_k p(\textstyle\sum_j a_{jk}v_j) \cdot q(e_k) \\
&\overset{(1)}{\geq} c \cdot d \cdot \max_{j,k} |a_{jk}| \cdot p(v_j) \cdot q(e_k) \\
&\geq c \cdot d \cdot \max_j p(v_j) \cdot q(w_j) \ .
\end{aligned}$$

Since d was arbitrary the assertion follows.

Proposition 17.4:

i. $p \otimes q(v \otimes w) = p(v) \cdot q(w)$ *for any* $v \in V$ *and* $w \in W$*;*

ii. $p \otimes q$ *is a norm if and only if* p *and* q *are norms;*

iii. suppose that $V_o \subseteq V$ and $W_o \subseteq W$ are vector subspaces; then

$$(p|V_o) \otimes (q|W_o) = (p \otimes q)|V_o \otimes_K W_o \ .$$

Proof: i. Applying Lemma 17.3 to the vector $v_1 = v$ and the constant $c = 1$ we obtain $p \otimes q(v \otimes w) \geq p(v) \cdot q(w)$. The opposite inequality is trivial.

ii. It is immediate from i. that p and q are norms if $p \otimes q$ is one. Let us therefore suppose, vice versa, that p and q are norms and let $u \in V \otimes_K W$ be an arbitrary nonzero vector. Write $u = v_1 \otimes w_1 + \ldots + v_m \otimes w_m$ with linearly independent vectors $v_1, \ldots, v_m \in V$. Since by Prop. 4.13 all norms on a finite dimensional vector space are equivalent we find a constant $0 < c \leq 1$ such that the assumption (1) in Lemma 17.3 is satisfied. It follows that $p \otimes q(u) \geq c \cdot \max\limits_j p(v_j) \cdot q(w_j) > 0$.

iii. Abbreviate $p_o := p|V_o$ and $q_o := q|W_o$ and let $u \in V_o \otimes_K W_o$. For trivial reasons we have $p \otimes q(u) \leq p_o \otimes q_o(u)$. To establish equality we then may assume that V_o is finite dimensional. We again use the construction in the proof of Prop. 10.4 to find, given any constant $0 < d < 1$, a basis $f_1, \ldots, f_n$ of V_o such that

$$p(a_1 f_1 + \ldots + a_n f_n) \geq d \cdot \max_{1 \leq k \leq n} |a_k| \cdot p(f_k) \qquad \text{for any } a_1, \ldots, a_n \in K \ .$$

Writing $u = f_1 \otimes w_1 + \ldots + f_n \otimes w_n$ with $w_k \in W_o$ we now deduce from Lemma 17.3 that

$$p \otimes q(u) \geq d \cdot \max_k p(f_k) \cdot q(w_k) = d \cdot \max_k p_o(f_k) \cdot q_o(w_k) \geq d \cdot p_o \otimes q_o(u) \ .$$

Since $d < 1$ was arbitrary it follows that $p \otimes q(u) \geq p_o \otimes q_o(u)$.

Corollary 17.5:

i. With V and W also $V \otimes_{K,\iota} W$ and $V \otimes_{K,\pi} W$ are Hausdorff;

ii. if $V_o \subseteq V$ and $W_o \subseteq W$ are vector subspaces equipped with the subspace topology then the projective tensor product topology on $V_o \otimes_K W_o$ coincides with the subspace topology induced from $V \otimes_{K,\pi} W$;

iii. given any vector $v_o \in V$, resp. $w_o \in W$, not in the closure of $\{0\}$ the linear maps $v_o \otimes . : W \to V \otimes_{K,\iota} W$ and $v_o \otimes . : W \to V \otimes_{K,\pi} W$, resp. $. \otimes w_o : V \to V \otimes_{K,\iota} W$ and $. \otimes w_o : V \to V \otimes_{K,\pi} W$, are homeomorphisms onto their image.

Proof: The assertion ii. is an immediate consequence of Prop. 17.4.iii. To see i. suppose that V and W are Hausdorff and let $u \in V \otimes_{K,\pi} W$ be a nonzero vector. We find finite dimensional vector subspaces $V_o \subseteq V$ and $W_o \subseteq W$ such that $u \in V_o \otimes_K W_o$ and continuous seminorms p on V and q on W such that $p|V_o$ and $q|W_o$ are norms. It then follows from Prop. 17.4.ii and iii that $p \otimes q|V_o \otimes_K W_o$ is a norm so that $p \otimes q(u) \neq 0$. According to Lemma 4.6 this shows that $V \otimes_{K,\pi} W$ and a fortiori $V \otimes_{K,\iota} W$ are Hausdorff. For iii. it suffices, by symmetry, to consider the maps $. \otimes w_o$. Moreover, it suffices to treat the case of the projective tensor product. By Prop. 17.4.i the tensor product seminorm $p \otimes q$ on $V \otimes_K W$ pulls back via $. \otimes w_o$ to the seminorm $q(w_o) \cdot p$ on V. If $q(w_o) \neq 0$ the latter is equivalent to p. But our assumption on w_o guarantees, by Lemma 4.6, that there exists at least one continuous seminorm q on W such that $q(w_o) \neq 0$.

We remark that, as a consequence of their universal properties, both tensor products $V \otimes_{K,\iota} W$ and $V \otimes_{K,\pi} W$ obviously are functorial: Given any continuous linear maps $f : V_1 \longrightarrow V_2$ and $g : W_1 \longrightarrow W_2$ the tensor product maps $f \otimes g : V_1 \otimes_{K,\iota} W_1 \longrightarrow V_2 \otimes_{K,\iota} W_2$ and $f \otimes g : V_1 \otimes_{K,\pi} W_1 \longrightarrow V_2 \otimes_{K,\pi} W_2$ are continuous.

There is one important situation in which the inductive and the projective tensor product topology coincide.

Proposition 17.6:

If V and W are Fréchet spaces then $V \otimes_{K,\iota} W = V \otimes_{K,\pi} W$.

Proof: By the universal properties it suffices to show that any separately continuous bilinear map $\beta : V \times W \longrightarrow U$ into any third locally convex K-vector space U already is continuous. Suppose therefore that β is separately continuous but not continuous. By Lemma 17.1 and the metrizability of V and W the latter means that we find an open lattice $N \subseteq U$ and a sequence $(v_n, w_n)_{n \in \mathbb{N}}$ in $V \times W$ converging to zero such that $\beta(v_n, w_n) \notin N$ for any $n \in \mathbb{N}$. We now look at the set of linear maps $H := \{\beta(., w_n) : n \in \mathbb{N}\} \subseteq \mathcal{L}(V, U)$. Since β is separately continuous and $(w_n)_n$ as a sequence converging to zero is bounded in W (Lemma 7.10) this set H is bounded in $\mathcal{L}_s(V, U)$. Because V as a Fréchet space is barrelled the Banach-Steinhaus theorem Prop. 6.15 then tells us that H is equicontinuous. Hence there is an open lattice $L \subseteq V$ such that $\beta(L, w_n) \subseteq N$ for any $n \in \mathbb{N}$. The sequence $(v_n)_n$ converging to zero we find, on the other hand, an $n_0 \in \mathbb{N}$ such that $v_n \in L$ for any $n \geq n_0$. This leads to the contradiction that $\beta(v_n, w_n) \in N$ for any $n \geq n_0$.

Let $V \widehat{\otimes}_{K,\pi} W$ denote the Hausdorff completion of the projective tensor product $V \otimes_{K,\pi} W$. If the topologies on V and W can be defined by countably many lattices then the same holds true, by construction, for $V \otimes_{K,\pi} W$ and then, by Remark 7.4.ii, for $V \widehat{\otimes}_{K,\pi} W$. Hence Prop. 8.1 implies that the completed projective tensor product $V \widehat{\otimes}_{K,\pi} W$ of two Fréchet spaces V and W again is a Fréchet space. On the other hand it follows directly from Prop. 17.4.ii that $V \widehat{\otimes}_{K,\pi} W$ for two Banach spaces V and W again is a Banach space.

If the field K is spherically complete there is a remarkable formula for the tensor product of two gauge seminorms in terms of linear forms. To put this into perspective we first make the following simple observations. For convenience we use the notion of the pseudo-polar even though the locally convex vector spaces under consideration are not assumed to be Hausdorff.

Remark 17.7:

i. For any seminorm p on V the pseudo-polar of the lattice $L^-(p)$ satisfies

$$L^-(p)^p = \{\ell \in V' : |\ell(v)| \le p(v) \ for\ any\ v \in V\};$$

ii. if K is spherically complete the gauge seminorm of any open lattice $L \subseteq V$ satisfies

$$p_L(v) = \sup\{|\ell(v)| : \ell \in L^-(p_L)^p\} \quad for\ any\ v \in V\ .$$

Proof: i. For ℓ in the right hand side and $v \in L^-(p)$ we have $|\ell(v)| \le p(v) < 1$. Suppose, vice versa, that $|\ell(v_o)| < 1$ provided $p(v_o) < 1$. If $p(v) < |\ell(v)|$ for some $v \in V$ we put $v_o := \ell(v)^{-1}v$ obtaining $p(v_o) < 1$. This leads to the contradiction that $|1| = |\ell(v_o)| < 1$. ii. It follows from i. that $|\ell(v)| \le p_L(v)$ for any $\ell \in L^-(p_L)^p$. On the other hand consider any fixed vector $v \in V$ and choose any $a \in K$ such that $|a| \le p_L(v)$. By the Hahn-Banach theorem Prop. 9.2 we find a linear form $\ell \in V'$ such that $\ell(v) = a$ and $|\ell(v_o)| \le p_L(v_o)$ for any $v_o \in V$. The latter means, by i., that $\ell \in L^-(p_L)^p$. We conclude that $\sup\{|\ell(v)| : \ell \in L^-(p_L)^p\} \ge \sup\{|a| : |a| \le p_L(v)\} = p_L(v)$.

Using Lemma 17.2 we obtain from Remark 17.7.ii, provided K is spherically complete, the formula

$$p_L \otimes p_M(u) = \sup\{|\lambda(u)| : \lambda \in L^-(p_L \otimes p_M)^p\} \quad \text{for any } u \in V \otimes_{K,\pi} W\ .$$

It will turn out that the pseudo-polar on the right hand side can be replaced by a much smaller o-module. It is immediate from Lemma 17.1 that for any two linear forms $\ell \in V'$ and $m \in W'$ the tensor product linear form $\ell \otimes m$ on $V \otimes_K W$ characterized by $\ell \otimes m(v \otimes w) = \ell(v) \cdot m(w)$ is continuous for the projective tensor product topology. We always view the resulting map

$$V' \otimes_K W' \longrightarrow (V \otimes_{K,\pi} W)'$$

as an inclusion.

Proposition 17.8:

Suppose that K is spherically complete and let $L \subseteq V$ and $M \subseteq W$ be open lattices; we then have

$$p_L \otimes p_M(u) = \sup\{|\ell \otimes m(u)| : \ell \in L^-(p_L)^p, m \in L^-(p_M)^p\}$$

for any $u \in V \otimes_K W$.

Proof: We abbreviate the right hand side of the asserted identity by $\epsilon_{L,M}(u)$. Whenever $u = \sum_i v_i \otimes w_i$ we have

$$\begin{aligned}
\epsilon_{L,M}(u) &= \sup\{|\textstyle\sum_i \ell(v_i) \cdot m(w_i)| : \ell \in L^-(p_L)^p, m \in L^-(p_M)^p\} \\
&\leq \max_i \sup\{|\ell(v_i)| \cdot |m(w_i)| : \ell \in L^-(p_L)^p, m \in L^-(p_M)^p\} \\
&\leq \max_i p_L(v_i) \cdot p_M(w_i)
\end{aligned}$$

where the last inequality comes from Remark 17.7.i. This shows that $\epsilon_{L,M}(u) \leq p_L \otimes p_M(u)$.

To establish the reverse inequality we choose finite dimensional vector subspaces $V_o \subseteq V$ and $W_o \subseteq W$ such that $u \in V_o \otimes_K W_o$. We also choose (by the construction in the proof of Prop. 10.4), given any $c \in K$ such that $0 < |c| < 1$, bases $e_1, \ldots, e_r$ of V_o and $f_1, \ldots, f_s$ of W_o such that

$$\begin{aligned}
&p_L(a_1e_1 + \ldots + a_re_r) \geq |c| \cdot \max_i |a_i| \cdot p_L(e_i) \quad \text{and} \\
&p_M(b_1f_1 + \ldots + b_sf_s) \geq |c| \cdot \max_j |b_j| \cdot p_M(f_j)
\end{aligned}$$

for any $a_i, b_j \in K$. Write $u = \sum_{i,j} a_{ij} e_i \otimes f_j$. By renumbering we may assume that

$$|a_{11}| \cdot p_L(e_1) \cdot p_M(f_1) = \max_{i,j} |a_{ij}| \cdot p_L(e_i) \cdot p_M(f_j) \geq p_L \otimes p_M(u) \ .$$

Now choose $a, b \in K$ such that $|a| \leq p_L(e_1)$ and $|b| \leq p_M(f_1)$ and define the linear forms $\ell_o(\sum_i a_i e_i) := a_1 ac$ on V_o and $m_o(\sum_j b_j f_j) := b_1 bc$ on W_o. Since

$$|\ell_o(\sum_i a_i e_i)| = |a_1 ac| \leq |c| \cdot |a_1| \cdot p_L(e_1) \leq p_L(\sum_i a_i e_i)$$

and similarly $|m_o| \leq p_M | W_o$ we may apply the Hahn-Banach theorem Prop. 9.2 to extend ℓ_o, resp. m_o, to a linear form $\ell \in L^-(p_L)^p$, resp. $m \in L^-(p_M)^p$. We have

$$|\ell \otimes m(u)| = |a_{11}| \cdot |abc^2| \ .$$

If the valuation of K is dense then by varying a, b, c it follows that

$$\epsilon_{L,M}(u) \geq |a_{11}| \cdot p_L(e_1) \cdot p_M(f_1) \geq p_L \otimes p_M(u) \ .$$

If the valuation of K is discrete then the above argument in fact goes through with $c = 1$ (apply Remark 10.2 to the quotient space of V_o, resp. W_o, on which p_L, resp. p_M, becomes a norm) and a and b can be chosen in such a way that $|a| = p_L(e_1)$ and $|b| = p_M(f_1)$. We therefore obtain $\epsilon_{L,M}(u) \geq p_L \otimes p_M(u)$ in this case as well.

This result has the following technical consequence which will be used later on.

Corollary 17.9:

Suppose that K is spherically complete and that the families of lattices $(L_i)_{i\in I}$ and $(M_j)_{j\in J}$ define the locally convex topologies on V and W, respectively; then the locally convex topology on $V \otimes_{K,\pi} W$ is defined by the family of seminorms

$$\epsilon_{i,j}(u) := \sup\{|\ell \otimes m(u)| : \ell \in L_i^p, m \in M_j^p\} \quad \text{for } i \in I, j \in J \ .$$

Proof: According to Prop. 4.4 the projective tensor product topology is defined by the seminorms $p_{L_i} \otimes p_{M_j}$ for $i \in I$ and $j \in J$. Fix a scalar $a \in K$ such that $|a| > 1$. By Lemma 2.2.i we have $L^-(p_{L_i}) \subseteq L_i \subseteq L^-(p_{aL_i})$ and similarly for M_j. It now follows from Prop. 17.8 that

$$p_{aL_i} \otimes p_{aM_j} \leq \epsilon_{i,j} \leq p_{L_i} \otimes p_{M_j} \ .$$

On a conceptual level we have the following consequence for the structure of equicontinuous subsets in the dual of the projective tensor product.

Proposition 17.10:

Suppose that K is spherically complete; for a subset $H \subseteq (V \otimes_{K,\pi} W)'$ the following assertions are equivalent:

i. H is equicontinuous;

ii. there are open lattices $L_o \subseteq V$ and $M_o \subseteq W$ such that H is contained in the closure of $L_o^p \otimes_o M_o^p$ in $(V \otimes_{K,\pi} W)'_s$.

Proof: By dividing out the closures of $\{0\}$ we may assume that V and W and hence, by Cor. 17.5.i, also $V \otimes_{K,\pi} W$ are Hausdorff. It is clear from the proof of Lemma 17.1 that $L_o^p \otimes_o M_o^p$ is equicontinuous. According to Cor. 6.11 this property is preserved by passing to the weak closure. This settles the implication from ii. to i. Let us therefore assume, vice versa, that H is equicontinuous, By Lemma 13.6 there is an open lattice $N \subseteq V \otimes_{K,\pi} W$ such that $H \subseteq N^p$ and by the construction of the projective tensor product topology we may assume that N is of the form $N = L(p_L \otimes p_M)$ for appropriate open lattices $L \subseteq V$ and $M \subseteq W$. Put $L_{\mathrm{o}} := L^-(p_L)$ and $M_{\mathrm{o}} := L^-(p_M)$. Prop. 17.8 implies that

$$L(p_L \otimes p_M) = \{u \in V \otimes_{K,\pi} W : |\ell \otimes m(u)| \leq 1 \text{ for any } \ell \in L_{\mathrm{o}}^p, m \in M_{\mathrm{o}}^p\} \ .$$

It follows in particular that

$$(L_{\mathrm{o}}^p \otimes_o M_{\mathrm{o}}^p)^p \subseteq L(p_L \otimes p_M) = N$$

and hence that

$$H \subseteq N^p \subseteq (L_{\mathrm{o}}^p \otimes_o M_{\mathrm{o}}^p)^{pp} \ .$$

It remains to apply Prop. 13.4 to identify the pseudo-polar on the right hand side with the weak closure.

Example:

Let X be any compact topological space and let V be a complete Hausdorff locally convex K-vector space. We put

$$C(X, V) := K\text{-vector space of all } V\text{-valued continuous functions on } X \ .$$

If L runs through all open lattices in V then the

$$C(X, L) := \{\phi \in C(X, V) : \phi(X) \subseteq L\}$$

constitute a family of lattices in $C(X, V)$ which satisfies the conditions (lc1) and (lc2). We equip $C(X, V)$ with the corresponding locally convex topology. It can alternatively be defined by the seminorms

$$p_X(\phi) := \sup_{x \in X} p(\phi(x))$$

where p runs through all continuous seminorms on V. It is obvious that with V also $C(X, V)$ is Hausdorff.

1) $C(X, V)$ is complete.

Proof: Let $(\phi_i)_{i \in I}$ be a Cauchy net in $C(X, V)$. Then $(\phi_i(x))_i$, for any $x \in X$, is a Cauchy net in V and hence converges to a unique vector $\phi(x) \in V$. Given any open lattice $L \subseteq V$ we find an $i \in I$ such that $\phi_j - \phi_k \in C(X, L)$ for $j, k \geq i$. Hence $\phi_j(x) - \phi_k(x) \in L$ and, in the limit, $\phi(x) \in \phi_k(x) + L$ for any $k \geq i$ and any $x \in X$. It easily follows that ϕ is continuous and that $\phi - \phi_k \in C(X, L)$ for $k \geq i$.

Clearly the linear map

$$\begin{array}{rcl} \Theta : C(X, K) \otimes_K V & \longrightarrow & C(X, V) \\ \sum_{i=1}^r \psi_i \otimes v_i & \longmapsto & [x \mapsto \sum_{i=1}^r \psi_i(x) \cdot v_i] \end{array}$$

is injective with image all functions whose image is contained in a finite dimensional vector subspace of V.

2) $\Theta(C(X,K)\otimes_K V)$ is dense in $C(X,V)$.

Proof: Let $\phi \in C(X,V)$ and let $L \subseteq V$ be an open lattice. Since X is compact and V/L is discrete the image $\phi(X)$ meets only finitely many cosets of L in V. This means that we find a finite disjoint open covering $X = U_1 \dot{\cup} \ldots \dot{\cup} U_m$ such that

$$\phi(U_j) \subseteq \phi(x_j) + L \qquad \text{for some (or any) } x_j \in U_j \ .$$

Let $\chi_j \in C(X,K)$ denote the characteristic function of the open and closed subset U_j. Then

$$\phi - \Theta(\sum_{i=1}^{m} \chi_j \otimes \phi(x_j)) \in C(X,L) \ .$$

3) If K is spherically complete then the topology induced via Θ by $C(X,V)$ on $C(X,K)\otimes_K V$ is the projective tensor product topology.

Proof: Let $\| \ \| = p_{C(X,o)}$ denote the sup-norm on the Banach space $C(X,K)$. For any $x \in X$ let $\delta_x \in C(X,K)'$ denote the Dirac measure $\delta_x(\psi) := \psi(x)$. The pseudo-polar $L^-(\| \ \|)^p$ is the closure in $C(X,K)'_s$ of the o-submodule generated by the δ_x for $x \in X$. Let $L \subseteq V$ be any open lattice. Using Prop. 17.8 and Remark 17.7.ii we compute

$$\begin{aligned}
\| \ \| \otimes p_L(\textstyle\sum_i \psi_i \otimes v_i) &= \sup\{|\textstyle\sum_i \delta(\psi_i)\cdot \ell(v_i)| : \delta \in L^-(\| \ \|)^p, \ell \in L^-(p_L)^p\} \\
&= \sup\{|\textstyle\sum_i \psi_i(x)\cdot \ell(v_i)| : x \in X, \ell \in L^-(p_L)^p\} \\
&= \sup\{|\ell(\Theta(\textstyle\sum_i \psi_i \otimes v_i)(x))| : x \in X, \ell \in L^-(p_L)^p\} \\
&= \sup_{x\in X} p_L(\Theta(\textstyle\sum_i \psi_i \otimes v_i)(x)) \\
&= (p_L)_X(\Theta(\textstyle\sum_i \psi_i \otimes v_i)) \ .
\end{aligned}$$

Using Lemma 7.3 we altogether obtain, assuming always that K is spherically complete, that Θ extends to an isomorphism of locally convex K-vector spaces

$$C(X,K)\,\widehat{\otimes}_{K,\pi}\, V \xrightarrow{\cong} C(X,V) \ .$$

In particular, if Y is a second compact topological space we have

$$C(X,K)\,\widehat{\otimes}_{K,\pi}\, C(Y,V) \xrightarrow{\cong} C(X,C(Y,V)) \ .$$

It is a general fact from topology ([B-GT] Chap.X §3.4 Cor.2) that

$$C(X, C(Y, V)) \cong C(X \times Y, V) .$$

The map $\psi \otimes \phi \longmapsto [(x, y) \mapsto \psi(x) \cdot \phi(y)]$ therefore extends to an isomorphism of locally convex K-vector spaces

$$C(X, K) \,\widehat{\otimes}_{K,\pi}\, C(Y, V) \xrightarrow{\cong} C(X \times Y, V) .$$

By using the one-point compactification one immediately deduces the isomorphism of K-Banach spaces

$$c_{\mathrm{o}}(X) \,\widehat{\otimes}_{K,\pi}\, c_{\mathrm{o}}(Y) \xrightarrow{\cong} c_{\mathrm{o}}(X \times Y)$$

for any two (discrete) sets X and Y.

§18 Completely continuous maps

For the rest of this chapter we again assume that the field K is spherically complete and that all occurring locally convex K-vector spaces are Hausdorff.

For any two locally convex K-vector spaces V and W the linear map

$$\begin{array}{rccc} T: & V' \otimes_K W & \longrightarrow & \mathcal{L}(V, W) \\ & \sum_{i=1}^r \ell_i \otimes w_i & \longmapsto & [v \mapsto \sum_{i=1}^r \ell_i(v) \cdot w_i] \end{array}$$

obviously is injective. Its image is the subspace of all finite rank operators, i.e., of those continuous linear maps $f : V \longrightarrow W$ for which $f(V)$ is finite dimensional. We let

$$\mathcal{CC}(V, W) := \text{closure of } T(V' \otimes_K W) \text{ in } \mathcal{L}_b(V, W)$$

viewed as a locally convex vector subspace of $\mathcal{L}_b(V, W)$. The $f \in \mathcal{CC}(V, W)$ are called *completely continuous* maps.

Let $g : V_{\mathrm{o}} \longrightarrow V$ and $h : W \longrightarrow W_{\mathrm{o}}$ be continuous linear maps between locally convex K-vector spaces. Clearly for any finite rank operator $f \in \mathcal{L}(V, W)$ the composite $h \circ f \circ g$ is of finite rank, too. On the other hand, if $B_{\mathrm{o}} \subseteq V_{\mathrm{o}}$ is bounded then $g(B_{\mathrm{o}})$ is bounded as well so that $\mathcal{L}(g(B_{\mathrm{o}}), M) \circ g \subseteq \mathcal{L}(B_{\mathrm{o}}, M)$ for any open lattice $M \subseteq W$. Equally obvious is the fact that $h \circ \mathcal{L}(B, h^{-1}(M_{\mathrm{o}})) \subseteq \mathcal{L}(B, M_{\mathrm{o}})$ for any bounded set $B \subseteq V$ and any open lattice $M_{\mathrm{o}} \subseteq W_{\mathrm{o}}$. This shows that the maps

$$\mathcal{L}_b(V, W) \xrightarrow{.\circ g} \mathcal{L}_b(V_{\mathrm{o}}, W) \quad \text{and} \quad \mathcal{L}_b(V, W) \xrightarrow{h\circ .} \mathcal{L}_b(V, W_{\mathrm{o}})$$

are continuous. Both these facts together imply that

$$h \circ \mathcal{CC}(V, W) \circ g \subseteq \mathcal{CC}(V_{\mathrm{o}}, W_{\mathrm{o}}) \ .$$

Lemma 18.1:

The map $T : V_b' \otimes_{K,\pi} W \longrightarrow \mathcal{L}_b(V, W)$ is a homeomorphism onto its image.

Proof: Let $B \subseteq V$ run over all bounded o-submodules. Then the pseudo-polars $B^p \subseteq V'$ form a defining family of open lattices for the locally convex topology on V_b'. Let $M \subseteq W$ run over all open lattices. By Cor. 17.9 the locally convex topology on $V_b' \otimes_{K,\pi} W$ is defined by the seminorms

$$\epsilon_{(B,M)}(\sum_i \ell_i \otimes w_i) := \sup\{|\sum_i \lambda(\ell_i) \cdot m(w_i)| : \lambda \in (B^p)^p, m \in L^-(p_M)^p\} \ .$$

We have to understand the iterated pseudo-polar $(B^p)^p \subseteq (V_b')_s'$. It contains the image $\widetilde{B}$ of B under the duality map $V \longrightarrow (V_b')_s'$. It is straightforward to see, making the identification $((V_b')_s')_s' = (V_b')_s$, that we have $\widetilde{B}^p = B^p$. Using Prop. 13.4 it follows that $(B^p)^p = \widetilde{B}^{pp}$ is the closure of $\widetilde{B}$ in $(V_b')_s'$. In other words, via the duality map B is dense in $(B^p)^p \subseteq (V_b')_s'$. The definition of $\epsilon_{(B,M)}$ therefore simplifies to

$$\begin{aligned} \epsilon_{(B,M)}(\textstyle\sum_i \ell_i \otimes w_i) &= \sup\{|\textstyle\sum_i \ell_i(v) \cdot m(w_i)| : v \in B, m \in L^-(p_M)^p\} \\ &= \sup\{|m(\textstyle\sum_i \ell_i(v) \cdot w_i)| : v \in B, m \in L^-(p_M)^p\} \end{aligned}$$

which by Remark 17.7.ii is equal to

$$\sup_{v \in B} p_M(T(\sum_i \ell_i \otimes w_i)(v)) = (p_M)_B(T(\sum_i \ell_i \otimes w_i)) \ .$$

It remains to recall that by the construction of the strong topology on $\mathcal{L}_b(V, W)$ the $(p_M)_B$ form a defining family of seminorms.

Proposition 18.2:

If $\mathcal{L}_b(V, W)$ is complete then T extends to an isomorphism of locally convex K-vector spaces

$$V_b' \widehat{\otimes}_{K,\pi} W \xrightarrow{\cong} \mathcal{CC}(V, W) \ .$$

Proof: In view of Lemma 7.3 this is an immediate consequence of Lemma 18.1.

We remark that according to Prop. 7.16 the assumption in the above proposition is satisfied if V is bornological and W is complete.

Lemma 18.3:

For any $f \in \mathcal{CC}(V,W)$ and any bounded o-submodule $B \subseteq V$ the image $f(B)$ is compactoid.

Proof: Let $M \subseteq W$ be an open lattice and let $(f_i)_{i\in I}$ be a net of finite rank operators converging to f in $\mathcal{L}_b(V,W)$. We find a $j \in I$ such that

$$(f - f_i)(B) \subseteq M \qquad \text{for any } i \geq j \ .$$

Since a finite rank operator obviously has the asserted property we also find finitely many $w_1, \ldots, w_m \in W$ such that

$$f_j(B) \subseteq ow_1 + \ldots ow_m + M \ .$$

It follows that

$$f(B) \subseteq (f - f_j)(B) + f_j(B) \subseteq ow_1 + \ldots + ow_m + M \ .$$

This proves that $f(M)$ is compactoid.

We will see that the property in this lemma in fact characterizes completely continuous operators. For this we first have to establish the so called approximation property. If $\mathcal{B}$ denotes the family of all compactoid o-submodules in V we put

$$\mathcal{L}_c(V,W) := \mathcal{L}_{\mathcal{B}}(V,W) \ .$$

Proposition 18.4: (Approximation property)

$T(V' \otimes_K W)$ is dense in $\mathcal{L}_c(V,W)$.

Proof: We first reduce the assertion to the case $V = W$. Consider an arbitrary $f \in \mathcal{L}(V,W)$. The map $\mathcal{L}_c(V,V) \xrightarrow{f\circ} \mathcal{L}_c(V,W)$ obviously is continuous and maps $T(V' \otimes_K V)$ into $T(V' \otimes_K W)$. If the former is dense then the closure of the latter contains the image of the map and hence in particular the element f. Exactly the same argument shows that it suffices in fact to prove that the identity map id_V on V lies in the closure of $T(V' \otimes_K V)$ in $\mathcal{L}_c(V,V)$. This means we have to establish the following: Given any compactoid o-submodule $A \subseteq V$ and any open lattice $L \subseteq V$ there is a finite rank operator $f \in T(V' \otimes_K V)$ such that $\mathrm{id}_V - f \in \mathcal{L}(A,L)$. We proceed by going through several cases of increasing generality.

Case 1: V is a Banach space isomorphic to $c_o(\mathbb{N})$.

We recall that $c_o(\mathbb{N})$ is the Banach space of all zero sequences $(a_n)_{n\in\mathbb{N}}$ in K with the norm $\|(a_n)_n\|_\infty = \max_n |a_n|$. Consider, for any $m \in \mathbb{N}$, the finite rank operator

$$\begin{array}{rcl} f_m : c_o(\mathbb{N}) & \longrightarrow & c_o(\mathbb{N}) \\ (a_n)_n & \longmapsto & (a_1, \ldots, a_m, 0, \ldots) \ . \end{array}$$

Its operator norm is ≤ 1. Hence there is an open lattice $M \subseteq L$ such that $f_m(M) \subseteq L$ for any $m \in \mathbb{N}$. Moreover, the sequence $(f_m)_m$ clearly converges pointwise to the identity map. The o-submodule A being compactoid we find finitely many vectors $v_1, \ldots, v_r \in c_o(\mathbb{N})$ such that

$$A \subseteq ov_1 + \ldots + ov_r + M \ .$$

Choose now an $m_o \in \mathbb{N}$ such that

$$v_i - f_m(v_i) \in L \qquad \text{for any } m \geq m_o \text{ and } 1 \leq i \leq r \ .$$

Then $\mathrm{id}_{c_o(\mathbb{N})} - f_m \in \mathcal{L}(A, L)$ for any $m \geq m_o$.

Case 2: V is an arbitrary Banach space.

Fix an $a \in K$ such that $0 < |a| < 1$. Since A is compactoid there is, for any $n \in \mathbb{N}$, a finite dimensional vector subspace $U_n \subseteq V$ such that

$$A \subseteq U_n + B_{|a|^n}(0) \ .$$

Then A is contained in the closure U of $\sum_n U_n$ and is compactoid in U. The Banach space U either is finite dimensional or, by Prop. 10.4, is isomorphic to $c_o(\mathbb{N})$ in which case we may apply Case 1 to it. We obtain a finite rank operator $g \in T(U' \otimes_K U)$ such that $\mathrm{id}_U - g \in \mathcal{L}(A, U \cap L)$. As a consequence of the Hahn-Banach theorem Cor. 9.4 the map g extends to a finite rank operator $f \in T(V' \otimes_K V)$ (with the same image) which then of course satisfies $\mathrm{id}_V - f \in \mathcal{L}(A, L)$.

Case 3: V is a normed vector space.

The completion $\widehat{V}$ of V is a Banach space and A can be viewed as a compactoid o-submodule of $\widehat{V}$ (Lemma 12.4.v). Applying Case 2 we obtain a finite rank operator $\widehat{f} \in T(\widehat{V}' \otimes_K \widehat{V})$ such that $\mathrm{id}_{\widehat{V}} - \widehat{f} \in \mathcal{L}(A, \widehat{L})$ where $\widehat{L}$ denotes the closure of L in $\widehat{V}$. Write

$$\widehat{f}\,|V = \ell_1 w_1 + \ldots + \ell_r w_r \qquad \text{with } \ell_i \in V' \text{ and } w_i \in \widehat{V}$$

and choose an $a \in K^\times$ such that $|\ell_i(v)| \leq |a|$ for any $v \in A$ and any $1 \leq i \leq r$. For each $1 \leq i \leq r$ there is a vector $v_i \in V$ such that $v_i - w_i \in a^{-1}\widehat{L}$. The finite rank operator $f := \ell_1 v_1 + \ldots + \ell_r v_r$ on V then satisfies

$$\begin{aligned} f(v) &= \ell_1(v)(v_1 - w_1) + \ldots + \ell_r(v)(v_r - w_r) + \widehat{f}(v) \\ &\in (a(a^{-1}\widehat{L}) + \widehat{L}) \cap V = \widehat{L} \cap V = L \end{aligned}$$

for any $v \in A$ where the last identity follows from Remark 7.4.ii.

Case 4: V is arbitrary.

Put $U := \bigcap_{a \in K^\times} aL$ and $W := V/U$. Equipped with the locally convex topology defined by the family of lattices $(aL/U)_{a \in K^\times}$ the vector space W is normed (w.r.t. the gauge of L/U). The canonical surjection $\tau : V \twoheadrightarrow W$ is continuous so that $\tau(A)$ is compactoid in W (Lemma 12.4.v). Applying Case 3 we find a finite rank operator $\overline{f} \in T(W' \otimes_K W)$ such that $\mathrm{id}_W - \overline{f} \in \mathcal{L}(\tau(A), L/U)$. By lifting a basis of the image of $\overline{f}$ to V one easily constructs a finite rank operator $f \in T(V' \otimes_K V)$ such that $\tau \circ f = \overline{f} \circ \tau$. For $v \in A$ we compute

$$\tau(v - f(v)) = \tau(v) - \overline{f}(\tau(v)) \in L/U \quad \text{and hence} \quad v - f(v) \in L\ .$$

Corollary 18.5:

For $f \in \mathcal{L}(V, W)$ the following assertions are equivalent:

i. f is completely continuous;

ii. $f(B)$ is compactoid for any bounded o-submodule $B \subseteq V$.

Proof: We have seen the implication from i. to ii. already in Lemma 18.3. Let us therefore assume that ii. holds true. Given a bounded o-submodule $B \subseteq V$ and an open lattice $M \subseteq W$ we have to find a finite rank operator $g \in \mathcal{L}(V, W)$ such that $(f - g)(B) \subseteq M$. But $f(B)$, by assumption, is compactoid. By the approximation property we therefore find a finite rank operator $\widetilde{g} : W \longrightarrow W$ such that

$$\mathrm{id}_W - \widetilde{g} \in \mathcal{L}(f(B), M)\ ,\ \text{i.e.,}\ (\mathrm{id}_W - \widetilde{g})(f(B)) \subseteq M\ .$$

The composite $g := \widetilde{g} \circ f$ then has the wanted properties.

Proposition 18.6:

If V is semi-reflexive then $\mathcal{CC}(V, W) = \mathcal{L}_b(V, W) = \mathcal{L}_c(V, W)$.

Proof: According to Prop. 15.3 in a semi-reflexive V any closed bounded o-submodule is c-compact and by Prop. 12.7 therefore compactoid. Hence

(Lemma 12.4.iii) any bounded o-submodule of V is compactoid and consequently $\mathcal{L}_c(V,W) = \mathcal{L}_b(V,W)$. The approximation property of Prop. 18.4 then implies that $\mathcal{CC}(V,W) = \mathcal{L}_c(V,W)$.

Remark 18.7:

If W is semi-reflexive then $\mathcal{CC}(V,W) = \mathcal{L}_b(V,W)$.

Proof: Let $f \in \mathcal{L}(V,W)$ and let $B \subseteq V$ be a bounded o-submodule. Then $f(B)$ is bounded in W. Since W is semi-reflexive it follows by the same argument as in the proof of Prop. 18.6 that $f(B)$ is compactoid. Hence $f \in \mathcal{CC}(V,W)$ by Cor. 18.5.

Corollary 18.8:

If V is bornological and W is complete and if either V or W is semi-reflexive then $V'_b \mathbin{\widehat{\otimes}_{K,\pi}} W \cong \mathcal{L}_b(V,W)$.

Proof: Apply Prop. 18.2, Prop. 18.6, and Remark 18.7.

In §16 we had introduced the notion of a compact map: $f \in \mathcal{L}(V,W)$ is compact if there is an open lattice $L \subseteq V$ such that the closure $\overline{f(L)}$ is bounded and c-compact in W. We note that, by Prop. 12.7, $f(L)$ then in particular is compactoid. We let

$$\mathcal{C}(V,W) := \{f \in \mathcal{L}(V,W) : f \text{ is compact}\} \ .$$

It is an easy consequence of Cor. 12.3 and Lemma 12.1.iii that $\mathcal{C}(V,W)$ is a vector subspace of $\mathcal{L}(V,W)$. Since the field K is c-compact we have

$$T(V' \otimes_K W) \subseteq \mathcal{C}(V,W) \ .$$

We recall from Remark 16.7 that

$$h \circ \mathcal{C}(V,W) \circ g \subseteq \mathcal{C}(V_{\mathrm{o}}, W_{\mathrm{o}})$$

for any two continuous linear maps between locally convex K-vector spaces $g : V_{\mathrm{o}} \longrightarrow V$ and $h : W \longrightarrow W_{\mathrm{o}}$.

It was also remarked in §16 that if W is quasi-complete then $f \in \mathcal{L}(V,W)$ is compact if and only if there is an open lattice $L \subseteq V$ such that $f(L)$ is compactoid.

Lemma 18.9:

$\mathcal{C}(V,W) \subseteq \mathcal{CC}(V,W)$.

Proof: Apply Remark 16.3, Prop. 12.7, and Cor. 18.5.

Remark 18.10:

i. If V is a Banach space and W is quasi-complete then $\mathcal{C}(V,W) = \mathcal{CC}(V,W)$;

ii. if V is a Banach space and W is semi-reflexive then $\mathcal{C}(V,W) = \mathcal{L}(V,W)$.

Proof: i. This is immediate from the fact that a Banach space contains a bounded open lattice. ii. According to Prop. 15.3 the semi-reflexive vector space W is quasi-complete. The assertion therefore follows from i. and Remark 18.7.

Proposition 18.11:

If V is a Banach space and W is complete then

$$V_b' \,\widehat{\otimes}_{K,\pi}\, W \cong \mathcal{C}(V,W) = \mathcal{CC}(V,W) \ .$$

Proof: As a Banach space V is bornological. We therefore may apply Prop. 18.2 and Remark 18.10.i.

Lemma 18.12:

Suppose that $f : V_0 \longrightarrow W_0$ and $g : V_1 \longrightarrow W_1$ are continuous linear maps between K-Banach spaces; if f and g are compact then $f \otimes g : V_0 \,\widehat{\otimes}_{K,\pi}\, V_1 \longrightarrow W_0 \,\widehat{\otimes}_{K,\pi}\, W_1$ is compact.

Proof: Since the tensor product of two finite rank operators again is a finite rank operator we have the commutative diagram

$$\begin{array}{ccc} (V_0' \otimes_K W_0) \times (V_1' \otimes_K W_1) & \longrightarrow & (V_0 \,\widehat{\otimes}_{K,\pi}\, V_1)' \otimes_K (W_0 \,\widehat{\otimes}_{K,\pi}\, W_1) \\ T \times T \big\downarrow & & \big\downarrow T \\ \mathcal{L}_b(V_0, W_0) \times \mathcal{L}_b(V_1, W_1) & \xrightarrow{\otimes} & \mathcal{L}_b(V_0 \,\widehat{\otimes}_{K,\pi}\, V_1, W_0 \,\widehat{\otimes}_{K,\pi}\, W_1) \ . \end{array}$$

By assumption (f,g) lies in the closure of the image of the left vertical arrow. If we show that the lower horizontal arrow is continuous then it follows that $f \otimes g$ lies in the closure $\mathcal{CC}(V_0 \,\widehat{\otimes}_{K,\pi}\, V_1, W_0 \,\widehat{\otimes}_{K,\pi}\, W_1)$ of the image of the right vertical arrow. Hence $f \otimes g$ is compact by Prop. 18.11. In order to establish the asserted continuity we compute the operator norm (compare §3) of the tensor product map. Note that all spaces involved are Banach spaces. For simplicity we will use $\| \ \|$ to denote all occurring norms. Letting u run over all vectors in $V_0 \otimes_K V_1$ and using Prop. 17.4.i we have

$$\begin{aligned}
\|f \otimes g\| &= \sup_{u \neq 0} \|u\|^{-1} \cdot \inf\{\| \textstyle\sum_i f(v_i^0) \otimes g(v_i^1)\| : u = \sum_i v_i^0 \otimes v_i^1\} \\
&\leq \sup_{u \neq 0} \|u\|^{-1} \cdot \inf\{\max_i \|f(v_i^0) \otimes g(v_i^1)\| : u = \textstyle\sum_i v_i^0 \otimes v_i^1\} \\
&= \sup_{u \neq 0} \|u\|^{-1} \cdot \inf\{\max_i \|f(v_i^0)\| \cdot \|g(v_i^1)\| : u = \textstyle\sum_i v_i^0 \otimes v_i^1\} \\
&\leq \|f\| \cdot \|g\| \cdot \sup_{u \neq 0} \|u\|^{-1} \cdot \inf\{\max_i \|v_i^0\| \cdot \|v_i^1\| : u = \textstyle\sum_i v_i^0 \otimes v_i^1\} \\
&= \|f\| \cdot \|g\| \cdot \sup_{u \neq 0} \|u\|^{-1} \cdot \|u\| \\
&= \|f\| \cdot \|g\| \ .
\end{aligned}$$

We therefore may apply Lemma 17.1 to conclude that the bilinear map in question is continuous.

§19 Nuclear spaces

At this point we introduce a tool which provides a systematic means of studying a general locally convex K-vector space through auxiliary Banach spaces. Before Lemma 7.17 we had introduced, for any o-submodule $A \subseteq V$, the vector subspace V_A of V generated by A and equipped with the locally convex topology defined by the gauge p_A of $A \subseteq V_A$ (or equivalently by the family of lattices bA for $b \in K^\times$). The inclusion map $V_A \xrightarrow{\subseteq} V$ is continuous if and only if A is bounded. Let

$$\widehat{V_A} := \text{ Hausdorff completion of } V_A \ .$$

This is a K-Banach space with respect to the continuous extension of p_A which we again denote by p_A. If A is bounded and closed and V is quasi-complete then $V_A = \widehat{V_A}$.

Suppose now that $A = L$ is an open lattice in V. The identity map on V then can be viewed as a continuous map $V \longrightarrow V_L$ and hence gives rise to a canonical continuous linear map with dense image

$$V \longrightarrow \widehat{V_L} \ .$$

The kernel of this latter map is equal to

$$\ker(p_L) := \{v \in V : p_L(v) = 0\} = \{v \in V : Kv \subseteq L\} \ .$$

The family of K-Banach spaces $(\widehat{V_L})_L$ has the following universal property: Let $f : V \longrightarrow W$ be a continuous linear map into some K-Banach space W. If

$L \subseteq V$ denotes the preimage of "the" unit ball in W then, by the universal property of the completion (Prop. 7.5), f extends to a continuous linear map $f_L : \widehat{V_L} \longrightarrow W$ such that the triangle

$$\begin{tikzcd} & \widehat{V_L} \arrow[dr, "f_L"] & \\ V \arrow[ur] \arrow[rr, "f"] & & W \end{tikzcd}$$

is commutative. Note that, as a locally convex vector space, $\widehat{V_L}$ is uniquely determined by f; a different choice of unit ball in W leads to a different but equivalent norm on the same vector space $\widehat{V_L}$.

Whenever $M \subseteq L$ is a second open lattice then the identity map on V can be viewed as a continuous linear map $V_M \longrightarrow V_L$. By the universal property of the completion it extends to a canonical continuous linear map

$$\widehat{V_M} \longrightarrow \widehat{V_L} \ .$$

Definition:

A locally convex K-vector space V is called <u>*nuclear*</u> *if for any open lattice $L \subseteq V$ there exists another open lattice $M \subseteq L$ such that the canonical map $\widehat{V_M} \longrightarrow \widehat{V_L}$ is compact.*

Clearly any finite dimensional V is nuclear. On the other hand, if V is a nuclear Banach space then the identity map on V is compact. Hence some open lattice in V is c-compact. We claim that this forces V to be finite dimensional. Otherwise it would contain, according to Prop. 10.4, a closed vector subspace topologically isomorphic to $c_o(\mathbb{N})$. It would follow that the unit ball $B_1(0)$ in $c_o(\mathbb{N})$ is c-compact and a fortiori that the k-vector space $B_1(0)/B_1^-(0)$ is linearly compact. But this is easily seen to contradict the fact that this vector space is of countably infinite dimension.

Remark 19.1:

V_s, for any locally convex K-vector space V, is nuclear.

Proof: This immediately follows from the fact that $\widehat{(V_s)_L}$ is finite dimensional for any open lattice $L \subseteq V_s$.

We now are able to understand the deeper reason for the validity of Remark 12.9.

Proposition 19.2:

Suppose that V is nuclear; then an o-submodule $B \subseteq V$ is compactoid if and only if it is bounded.

Proof: We know from Lemma 12.4.i that, quite generally, compactoid submodules are bounded. Let us assume therefore that B is bounded. Fix an open lattice $L \subseteq V$ and choose an open lattice $M \subseteq L$ such that the canonical map $\widehat{V_M} \longrightarrow \widehat{V_L}$ is compact. It then follows from Lemma 18.3 that the image of B in the normed vector space $(V/\ker(p_L), p_L)$ is compactoid. Hence there exist $v_1, \ldots, v_m \in V$ such that $B \subseteq ov_1 + \ldots + ov_m + L + \ker(p_L) = ov_1 + \ldots + ov_m + L$. Since L was fixed but arbitrary this proves that B is compactoid.

Corollary 19.3:

i. If V is quasi-complete and nuclear then it is semi-reflexive;

ii. any nuclear Fréchet space V is reflexive.

Proof: i. Let $B \subseteq V$ be any closed and bounded o-submodule. Since V is quasi-complete B is complete. Moreover, by Prop. 19.2, B also is compactoid. It then follows from Prop. 12.7 that B is c-compact. This implies, according to Prop. 15.3, that V is semi-reflexive. ii. As a Fréchet space V is complete and barrelled. Hence it is semi-reflexive (by i.) and barrelled. According to Prop. 15.5 this means that V is reflexive.

Proposition 19.4:

If V is nuclear then we have:

i. Any vector subspace $U \subseteq V$ with the subspace topology is nuclear;

ii. V/U, for any closed vector subspace $U \subseteq V$, is nuclear with respect to the quotient topology.

Proof: i. Let $L_o \subseteq U$ be an open lattice. We find an open lattice $L \subseteq V$ such that $L_o = U \cap L$ and then, by the nuclearity of V, another open lattice $M \subseteq L$ such that the canonical map $\widehat{V_M} \longrightarrow \widehat{V_L}$ is compact. Define $M_o := U \cap M$. The inclusion $U \subseteq V$ induces isometries $\widehat{U_{M_o}} \hookrightarrow \widehat{V_M}$ and $\widehat{U_{L_o}} \hookrightarrow \widehat{V_L}$. The canonical map $\widehat{U_{M_o}} \longrightarrow \widehat{U_{L_o}}$ therefore can be viewed as the restriction of the compact canonical map $\widehat{V_M} \longrightarrow \widehat{V_L}$ and as such is compact as well by Remark 16.7.ii.

ii. Let $L_1 \subseteq V/U$ be an open lattice. If $L \subseteq V$ denotes the preimage of L_1 we find an open lattice $M \subseteq L$ such that the canonical map $\widehat{V_M} \longrightarrow \widehat{V_L}$ is compact. Let $M_1 \subseteq V/U$ be the image of M. One checks that p_{M_1}, resp. p_{L_1}, is the quotient seminorm of p_L, resp. p_M. Using Prop. 8.3 it follows that the Banach space $\widehat{(V/U)}_{M_1}$, resp. $\widehat{(V/U)}_{L_1}$, is a quotient of the Banach space $\widehat{V_M}$, resp. $\widehat{V_L}$. Hence the canonical map $\widehat{(V/U)}_{M_1} \longrightarrow \widehat{(V/U)}_{L_1}$ can be viewed as a quotient map of the compact canonical map $\widehat{V_M} \longrightarrow \widehat{V_L}$. But it is immediate from Lemma 12.1.iv that quotient maps of compact maps are compact.

Before we continue giving more permanence properties we provide the following theoretical criterion.

Proposition 19.5:

The following assertions are equivalent:

i. V is nuclear;

ii. $\mathcal{C}(V,W) = \mathcal{L}(V,W)$ for any K-Banach space W.

Proof: To see the implication from i. to ii. let $f \in \mathcal{L}(V,W)$. By the universal property of the family of Banach spaces $(\widehat{V_L})_L$ we find an open lattice $L \subseteq V$ such that f can be factorized into $V \longrightarrow \widehat{V_L} \longrightarrow W$. But by the nuclearity of V we find another open lattice $M \subseteq L$ such that the canonical map $\widehat{V_M} \longrightarrow \widehat{V_L}$ is compact. Hence f factorizes into $V \longrightarrow \widehat{V_M} \longrightarrow \widehat{V_L} \longrightarrow W$ where the map in the middle is compact. By Remark 16.7.i the map f then is compact as well. For the reverse implication let $L \subseteq V$ be an open lattice. By assumption the canonical map $V \longrightarrow \widehat{V_L}$ is compact. Hence there is an open lattice $M \subseteq L$ such that the closure of the image of M in $\widehat{V_L}$ is bounded and c-compact. It means of course that the canonical map $\widehat{V_M} \longrightarrow \widehat{V_L}$ is compact. Since L was arbitrary this shows that V is nuclear.

We also need the following technical observation which elaborates on the Example after Remark 12.8.

Remark 19.6:

Let W be a Banach space with defining norm $\| \ \|$, and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of bounded and c-compact o-submodules in W such that $\lim_{n\to\infty} \sup \|A_n\| = 0$; then the closed o-submodule $\overline{\sum_n A_n}$ is bounded and c-compact.

Proof: By the assumption that $\lim_{n\to\infty} \sup \|A_n\| = 0$ we have the well defined o-linear map

$$\begin{array}{ccc} \prod_{n\in\mathbb{N}} A_n & \longrightarrow & W \\ (w_n)_n & \longmapsto & \sum_n w_n \end{array}$$

which is continuous with respect to the direct product topology on the left hand side. The image clearly is bounded. By Prop. 12.2 the o-module $\prod_n A_n$ is c-compact in the sense of Lemma 12.1.ii. But this property obviously is inherited by the image under any continuous o-linear map.

Proposition 19.7:

Let $(V_i)_{i\in I}$ be a family of nuclear locally convex K-vector spaces; we then have:

i. $\prod_{i\in I} V_i$ is nuclear;

ii. if I is countable then $\bigoplus_{i\in I} V_i$ is nuclear.

Proof: We first treat the locally convex direct sum and we use the criterion in Prop. 19.5. Let therefore $f : \bigoplus_{i\in I} V_i \longrightarrow W$ be any continuous linear map into a K-Banach space W and denote by f_i the restriction of f to the summand V_i. Then each f_i is compact so that we find an open lattice $L_i \subseteq V_i$ such that $A_i := \overline{f_i(L_i)}$ is bounded and c-compact in W. According to Lemma 5.1.iii the lattice $L := \bigoplus_{i\in I} L_i$ is open in $\bigoplus_{i\in I} V_i$. We certainly can arrange for the family $(A_i)_i$ to satisfy the assumption in Remark 19.6 so that $\overline{f(L)}$ is bounded and c-compact. This proves that the map f is compact.

In the direct product case we again use Prop. 19.5. In addition we observe (compare the reasoning before Prop. 9.11) that any continuous linear map into a Banach space factorizes over finitely many factors of the direct product. This reduces us to the case of a finite index set I. But then the direct product coincides with the locally convex direct sum (Lemma 5.2.ii) which was treated already.

Corollary 19.8:

Any strict inductive limit of nuclear locally convex K-vector spaces is nuclear.

Proof: This follows from Prop. 19.4.ii and Prop. 19.7.ii since strict inductive limits are quotients of countable locally convex direct sums.

Proposition 19.9:

Any compact projective or inductive limit of locally convex K-vector spaces (in the sense of Cor. 16.6 and Prop. 16.10, respectively) is nuclear.

Proof: Consider first the case where V is the projective limit (with the initial topology) of a sequence of locally convex K-vector spaces

$$\ldots \longrightarrow V_{n+1} \xrightarrow{g_n} V_n \longrightarrow \ldots \longrightarrow V_1$$

where all the transition maps are compact. As we have discussed before Cor. 16.6 we also may assume that each g_n has dense image. Using Prop. 19.5 we consider any continuous linear map $f : V \longrightarrow W$ into a K-Banach space W. It factorizes into $V \longrightarrow V_n \longrightarrow W$ for some $n \in \mathbb{N}$ and hence into $V \longrightarrow V_{n+1} \xrightarrow{g_n} V_n \longrightarrow W$. It follows that f is compact.

In the other case V is the inductive limit (with the final topology) of a sequence of locally convex K-vector spaces

$$V_1 \longrightarrow \ldots \longrightarrow V_n \xrightarrow{i_n} V_{n+1} \longrightarrow \ldots$$

where all transition maps are injective and compact. Let again $f : V \longrightarrow W$ be any continuous linear map into a K-Banach space W and put $f_n := f|V_n$. Since $f_n = f_{n+1} \circ i_n$ each f_n is compact. The argument in the proof of Prop. 19.7.ii shows that the map

$$\sum_n f_n : \bigoplus_n V_n \longrightarrow W$$

is compact. But $\sum_n f_n$ is equal to the projection from $\bigoplus_n V_n$ onto V followed by f. Hence f is compact.

Lemma 19.10:

Let L and M be open lattices in the not necessarily Hausdorff locally convex K-vector spaces V and W, respectively; we then have:

i. $V \mathbin{\widehat{\otimes}}_{K,\pi} W = \widehat{V} \mathbin{\widehat{\otimes}}_{K,\pi} \widehat{W}$;

ii. $\widehat{(V \otimes_{K,\pi} W)}_{L\otimes_o M} = \widehat{V_L} \mathbin{\widehat{\otimes}}_{K,\pi} \widehat{W_M}$.

Proof: i. Using Lemma 4.6 it is clear that the canonical bijection

$$V \otimes_{K,\pi} W/(\overline{\{0\}} \otimes_K W + V \otimes_K \overline{\{0\}}) \xrightarrow{\cong} V/\overline{\{0\}} \otimes_{K,\pi} W/\overline{\{0\}}$$

is an isomorphism of locally convex K-vector spaces. Since the right hand side is Hausdorff by Cor. 17.5.i we have

$$\overline{\{0\}} \subseteq \overline{\{0\}} \otimes_K W + V \otimes_K \overline{\{0\}}$$

where on the left the closure of $\{0\}$ in $V \otimes_{K,\pi} W$ is meant, of course. The reverse inclusion is a direct consequence of the continuity of the tensor product map $V \times W \longrightarrow V \otimes_{K,\pi} W$. We therefore have

$$(V \otimes_{K,\pi} W)/\overline{\{0\}} \xrightarrow{\cong} V/\overline{\{0\}} \otimes_{K,\pi} W/\overline{\{0\}}$$

and consequently

$$V \mathbin{\widehat{\otimes}}_{K,\pi} W \xrightarrow{\cong} V/\overline{\{0\}} \mathbin{\widehat{\otimes}}_{K,\pi} W/\overline{\{0\}} \ .$$

Hence we may assume that V and W are Hausdorff. The continuity of the tensor product map $\widehat{V} \times \widehat{W} \longrightarrow \widehat{V} \otimes_{K,\pi} \widehat{W}$ implies that $V \otimes_{K,\pi} W$ is dense in $\widehat{V} \otimes_{K,\pi} \widehat{W}$. On the other hand, by Cor. 17.5.ii, the topology on $V \otimes_{K,\pi} W$ is the subspace topology from $\widehat{V} \otimes_{K,\pi} \widehat{W}$. Hence both tensor products must have the same completion.

ii. We recall that V_L, resp. W_M, is the vector space V, resp. W, with the topology defined by the family of lattices $(aL)_{a\in K^\times}$, resp. $(aM)_{a\in K^\times}$. Hence $V_L \otimes_{K,\pi} W_M$ is $V \otimes_K W$ with the topology defined by the family of lattices $(aL \otimes_o M)_{a\in K^\times}$. But this is exactly the locally convex K-vector space $V \otimes_{K,\pi} W)_{L\otimes_o M}$. In other words we have

$$V_L \otimes_{K,\pi} W_M = (V \otimes_{K,\pi} M)_{L\otimes_o M} \ .$$

It remains to pass to the completions in this identity and to apply the first assertion.

Proposition 19.11:

With V and W also $V \otimes_{K,\pi} W$ is nuclear.

Proof: Let $N \subseteq V \otimes_{K,\pi} W$ be an open lattice. By the construction of the projective tensor product topology we find open lattices $L \subseteq V$ and $M \subseteq W$ such that $L \otimes_o M \subseteq N$. The nuclearity of V and W then implies the existence of open lattices $L_\mathrm{o} \subseteq L$ and $M_\mathrm{o} \subseteq M$ such that the canonical maps $\widehat{V_{L_\mathrm{o}}} \longrightarrow \widehat{V_L}$ and $\widehat{W_{M_\mathrm{o}}} \longrightarrow \widehat{W_M}$ are compact. According to Lemma 18.12 and Lemma 19.10.ii the tensor product of these two maps is compact as well and is the canonical map

$$(V \otimes_{K,\pi} W)\widehat{_{L_\mathrm{o}\otimes_o M_\mathrm{o}}} \longrightarrow (V \otimes_{K,\pi} W)\widehat{_{L\otimes_o M}} \ .$$

We therefore have found an open lattice $N_\mathrm{o} := L_\mathrm{o} \otimes_o M_\mathrm{o} \subseteq N$ such that the canonical map $(V \otimes_{K,\pi} W)\widehat{_{N_\mathrm{o}}} \longrightarrow (V \otimes_{K,\pi} W)\widehat{_N}$ is compact.

§20 Nuclear maps

We now introduce a relative version of nuclearity.

Definition:

A continuous linear map $f : V \longrightarrow W$ between two locally convex K-vector spaces V and W is called <u>nuclear</u> if it has a factorization $V \longrightarrow V_1 \xrightarrow{f_1} W_1 \longrightarrow W$ into continuous linear maps of which $f_1 : V_1 \longrightarrow W_1$ is a compact map between K-Banach spaces.

We let

$$\mathcal{N}(V,W) := \{f \in \mathcal{L}(V,W) : f \text{ is nuclear}\} \ .$$

Since, by Prop. 12.2, a direct sum of two compact maps between Banach spaces is a compact map between Banach spaces it is clear that $\mathcal{N}(V,W)$ is a vector subspace of $\mathcal{L}(V,W)$. We also have

$$T(V' \otimes_K W) \subseteq \mathcal{N}(V,W) \subseteq \mathcal{C}(V,W) \ .$$

For a finite rank operator f one can take f_1 to be the identity map on the image of f; this shows the first inclusion. The second one is an immediate consequence of Remark 16.7.i. It follows directly from the definition that

$$h \circ \mathcal{N}(V, W) \circ g \subseteq \mathcal{N}(V_o, W_o)$$

for any two continuous linear maps between locally convex K-vector spaces $g : V_o \longrightarrow V$ and $h : W \longrightarrow W_o$.

There is a more intrinsic characterization of nuclear maps. At the beginning of the previous section we had introduced the family of Banach spaces $(\widehat{V_L})_L$, for L running over the open lattices in V, which is universal for continuous linear maps from V into K-Banach spaces. On the other hand, let us look at the family of normed vector spaces $(W_B)_B$ for B running over all bounded o-submodules of W such that W_B is complete and hence a Banach space. It is universal for continuous linear maps from K-Banach spaces into W: Suppose that $g : U \longrightarrow W$ is such a map. If we take for B the image under g of a bounded open lattice in U then g factorizes into

$$U \twoheadrightarrow W_B \longrightarrow W.$$

In addition, the first surjective map from U onto W_B is open. It therefore follows from Prop. 8.3 that W_B is a Banach space.

If we apply these two universal properties to a factorization $V \longrightarrow V_1 \xrightarrow{f_1} W_1 \longrightarrow W$ of the nuclear map f where f_1 is a compact map between K-Banach spaces then we obtain a new factorization $V \longrightarrow \widehat{V_L} \xrightarrow{\widetilde{f}} W_B \longrightarrow W$ with $\widetilde{f}$ again being compact by Remark 16.7.i. This proves the following criterion.

Remark 20.1:

The map $f \in \mathcal{L}(V, W)$ is nuclear if and only if there is an open lattice $L \subseteq V$ and a bounded o-submodule $B \subseteq W$ for which W_B is complete such that f factorizes through a compact map $\widehat{V_L} \longrightarrow W_B$.

Lemma 20.2:

i. If W is a Banach space then $\mathcal{N}(V, W) = \mathcal{C}(V, W)$ for any locally convex K-vector space V;

ii. if V and W are Banach spaces then $\mathcal{N}(V, W) = \mathcal{CC}(V, W)$;

iii. if V is nuclear and W is a Banach space then $\mathcal{N}(V, W) = \mathcal{L}(V, W)$;

iv. if W is nuclear then $\mathcal{CC}(V, W) = \mathcal{L}(V, W)$ for any locally convex K-vector space V.

Proof: i. Let $f \in \mathcal{C}(V,W)$. There is an open lattice $L \subseteq V$ such that the closure of $f(L)$ in W is bounded and c-compact. If $M \subseteq W$ is any open lattice then we find a $b \in K^\times$ such that $f(L) \subseteq bM$ or equivalently $b^{-1}L \subseteq f^{-1}(M)$. This shows that f extends to a continuous linear map $\widehat{V_L} \longrightarrow W$ which, of course, also is compact. Hence f is nuclear. ii. Combine i. and Remark 18.10.i. iii. Combine i. and Prop. 19.5. iv. Let $f \in \mathcal{L}(V,W)$. The image $f(B)$ of any bounded o-submodule $B \subseteq V$ is bounded in the nuclear space W and hence is compactoid by Prop. 19.2. According to Cor. 18.5 this means that f is completely continuous.

Lemma 20.3:

Restricting a linear map induces a bijection $\mathcal{N}(\widehat{V},W) \xrightarrow{\cong} \mathcal{N}(V,W)$.

Proof: By the density of V in its completion $\widehat{V}$ the map in question is injective. For the surjectivity let $f : V \longrightarrow W$ be any nuclear map and let $V \longrightarrow V_1 \xrightarrow{f_1} W_1 \longrightarrow W$ be a factorization of f such that f_1 is a compact map between Banach spaces. By the universal property of the completion the first map extends to a continuous linear map $\widehat{V} \longrightarrow V_1$. The composite $\widehat{V} \longrightarrow V_1 \xrightarrow{f_1} W_1 \longrightarrow W$ then is a nuclear map which restricts to f.

Proposition 20.4:

The locally convex K-vector space V is nuclear if and only if its completion $\widehat{V}$ is nuclear.

Proof: This follows from Prop. 19.5, Lemma 20.2.i, and Lemma 20.3.

If we combine this last result with Prop. 19.11 then we see that the completed projective tensor product $V \,\widehat{\otimes}_{K,\pi}\, W$ of two nuclear spaces V and W is nuclear.

Lemma 20.5:

If V is metrizable then there exists for any bounded o-submodule $A \subseteq V$ another bounded o-submodule $A \subseteq B \subseteq V$ such that the topologies induced on A by V and by V_B coincide.

Proof: Let $L_1 \supset \ldots \supset L_n \supset \ldots$ for $n \in \mathbb{N}$ be a decreasing sequence of open lattices in V which define the locally convex topology. Since A is bounded there is, for any $n \in \mathbb{N}$, a $b_n \in K$ such that $A \subseteq b_n L_n$. We choose a sequence $(c_n)_{n\in\mathbb{N}}$ in $K^\times$ such that $\lim\limits_{n\to\infty} |b_n/c_n| = 0$ and $|c_n| \geq |b_n|$ for any $n \in \mathbb{N}$. Then

$$B := \bigcap_{n\in\mathbb{N}} c_n L_n$$

is a bounded o-submodule containing A. To see that B has the asserted property we have to find, for any given $a \in K^\times$, an open lattice $L \subseteq V$ such that $A \cap L \subseteq aB$. Letting $n(a) \in \mathbb{N}$ be such that $|b_n/c_n| \leq |a|$ for $n \geq n(a)$ we have

$$A \subseteq b_n L_n \subseteq ac_n L_n \qquad \text{for } n \geq n(a) \ .$$

Choose now an $m \in \mathbb{N}$ such that $L_m \subseteq ac_1 L_1 \cap \ldots \cap ac_{n(a)} L_{n(a)}$. Then

$$A \cap L_m \subseteq \bigcap_{n \in \mathbb{N}} ac_n L_n = aB \ .$$

Lemma 20.6:

If V is metrizable and W is a Banach space then $\mathcal{N}(V'_b, W) = \mathcal{CC}(V'_b, W)$; if V also is reflexive then $\mathcal{N}(V'_b, W) = \mathcal{L}(V'_b, W)$.

Proof: Fix a bounded open lattice $N \subseteq W$ and let $f \in \mathcal{CC}(V'_b, W)$. By the continuity of f we find a closed bounded o-submodule $A \subseteq V$ such that $A^p \subseteq f^{-1}(N)$. According to Lemma 20.5 there is another bounded o-submodule $A \subseteq B \subseteq V$ such that for any $a \in K^\times$ there is an open lattice $L \subseteq V$ with $A \cap L \subseteq aB$. We claim that the image $f(M)$ of the open lattice $M := B^p$ in V'_b is compactoid in W. By passing to the pseudo-polars we have

$$a^{-1} \cdot B^p = (aB)^p \subseteq (A \cap L)^p \ .$$

By computing in V'_s, making the identification $V_s \cong (V'_s)'_s$, and using Prop. 13.4 the right hand side becomes

$$(A \cap L)^p = (A^{pp} \cap L^{pp})^p = (A^p + L^p)^{pp} \ .$$

By Lemma 13.1 the pseudo-polar L^p, resp. A^p, is c-compact, resp. closed, in V'_s. As a quotient of L^p the submodule $(L^p + A^p)/A^p$ of the linearly topologized Hausdorff o-module V'_s/A^p therefore is c-compact in the sense that it has the property of Lemma 12.1.ii. A straightforward adaption of the proof of Cor. 7.6 implies then that this submodule is closed. Hence $A^p + L^p$ is closed in V'_s and applying Prop. 13.4 once more we obtain

$$M = B^p \subseteq a(A \cap L)^p = a(A^p + L^p)^{pp} = aA^p + aL^p$$

and consequently

$$f(M) \subseteq aN + af(L^p) \ .$$

But L^p is equicontinuous and hence bounded in V'_b and f is completely continuous. According to Cor. 18.5 the image $f(L^p)$ is compactoid which means in particular that there are vectors $w_1, \ldots, w_m \in W$ such that

$$f(L^p) \subseteq ow_1 + \ldots + ow_m + N \ .$$

It follows that

$$f(M) \subseteq oaw_1 + \ldots + oaw_m + aN \ .$$

Since $a \in K^\times$ was arbitrary this shows that $f(M)$ is compactoid and hence that f is compact. Applying finally Lemma 20.2.i we obtain that f is nuclear as claimed. If V is reflexive then, by Lemma 15.4, the strong dual V'_b also is reflexive. The second assertion therefore is a consequence of the first and Prop. 18.6.

Proposition 20.7:

For a Fréchet space V the following assertions are equivalent:

i. V is reflexive;

ii. V'_b is nuclear.

Proof: Suppose first that V is reflexive. By Lemma 20.6 we have $\mathcal{N}(V'_b, W) = \mathcal{C}(V'_b, W) = \mathcal{L}(V'_b, W)$ for any K-Banach space W. Hence Prop. 19.5 implies that V'_b is nuclear. Now let us assume, vice versa, that V'_b is nuclear. To see that V is reflexive it suffices to check, by Prop. 15.3 and Prop. 15.5 (since V is complete and barrelled), that any bounded o-submodule $B \subseteq V$ is compactoid. Fix an element $a \in K$ such that $0 < |a| < 1$. Let $L \subseteq V$ be any open lattice. The pseudo-polar B^p, resp. $(aL)^p$, is an open lattice, resp. an equicontinuous and hence bounded o-submodule, in V'_b. It follows from Prop. 19.2 that $(aL)^p$ is compactoid. Hence we find finitely many linear forms $\ell_1, \ldots, \ell_m \in V'$ such that $(aL)^p \subseteq o\ell_1 + \ldots + o\ell_m + B^p$. Using Prop. 13.4 we deduce that

$$B_\mathrm{o} := \{v \in B : |\ell_i(v)| < 1 \text{ for } 1 \leq i \leq m\} \subseteq (aL)^{pp} = aL \ .$$

Since B is bounded we find a nonzero $b \in o$ such that $bB \subseteq B_\mathrm{o}$. Hence

$$\begin{aligned} B/B_\mathrm{o} &\longrightarrow (b^{-1}o/\mathfrak{m})^m \\ v &\longmapsto (\ell_1(v), \ldots, \ell_m(v)) \end{aligned}$$

is an embedding of o-modules where the target is a finitely generated o-module. In this situation Lemma 12.6 tells us that there are vectors $v_1, \ldots, v_r \in B$ such that

$$aB \subseteq ov_1 + \ldots + ov_r + B_\mathrm{o}$$

and hence

$$B \subseteq oa^{-1}v_1 + \ldots + oa^{-1}v_r + a^{-1}B_\mathrm{o} \subseteq oa^{-1}v_1 + \ldots + oa^{-1}v_r + L \ .$$

Since L was arbitrary this shows that B is compactoid.

We note that by Cor. 19.3.ii any nuclear Fréchet space is reflexive and therefore has a strong dual which is nuclear, too. The Lemma 20.6 has the following dual counterpart. It could, in fact, be deduced from Lemma 20.6 with the help of duality considerations. Instead we will give a straightforward direct argument.

Lemma 20.8:

If V is a Banach space and W is a Fréchet space then $\mathcal{N}(V,W) = \mathcal{CC}(V,W)$; if W also is reflexive then $\mathcal{N}(V,W) = \mathcal{L}(V,W)$.

Proof: Fix a bounded open lattice $L \subseteq V$ and let $f \in \mathcal{CC}(V,W)$. Then $f(L)$ is compactoid. Since W is complete the closure A of $f(L)$ in W is a bounded and c-compact o-submodule. According to Lemma 20.5 there is another bounded o-submodule $A \subseteq B \subseteq W$ such that the topologies induced by W and by W_B on A coincide. Hence A is also bounded and c-compact in W_B. By replacing B by its closure we may in fact assume that B is complete so that W_B is a Banach space. The obvious factorization $V \longrightarrow W_B \longrightarrow W$ then shows that f is nuclear. The second assertion follows by taking Remark 18.10.ii into account.

The following result together with Prop. 20.7 will enable us to understand the bounded subsets in the completed projective tensor product $V \widehat{\otimes}_{K,\pi} W$ of two Fréchet spaces V and W provided V is reflexive.

Proposition 20.9:

If V is a reflexive Fréchet space and W is complete then

$$\begin{array}{ccc} V \widehat{\otimes}_{K,\pi} W & \longrightarrow & \mathcal{L}_b(V'_b, W) \\ v \otimes w & \longmapsto & [\ell \mapsto \ell(v) \cdot w] \end{array}$$

is an isomorphism of locally convex K-vector spaces.

Proof: According to Prop. 15.6 the strong dual V'_b is bornological. It also is reflexive by Lemma 15.4. Hence we may apply Cor. 18.8 and we obtain that

$$V \widehat{\otimes}_{K,\pi} W \cong (V'_b)'_b \widehat{\otimes}_{K,\pi} W \cong \mathcal{L}_b(V'_b, W)$$

where the first map comes from the duality map for V and the second one is induced by the map T from §18. Their composite obviously is given by the asserted formula.

We also need the following technical observation.

Remark 20.10:

Suppose that W is a metrizable locally convex K-vector space and let $(B_n)_{n\in\mathbb{N}}$ be a sequence of bounded o-submodules in W; then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $K^\times$ such that $B := \sum_{n\in\mathbb{N}} a_nB_n$ is a bounded o-submodule.

Proof: Let $L_1 \supset \ldots \supset L_n \supset \ldots$ for $n \in \mathbb{N}$ be a decreasing sequence of open lattices in W which define the locally convex topology. For each $n \in \mathbb{N}$ we fix an $a_n \in K^\times$ such that $a_nB_n \subseteq L_n$. Then

$$\sum_{n\geq m} a_nB_n \subseteq \sum_{n\geq m} L_n \subseteq L_m \qquad \text{for any } m \in \mathbb{N} .$$

Since $\sum_{n<m} a_nB_n$ is bounded it follows that there exists, for any $m \in \mathbb{N}$, a $b_m \in K$ such that $B \subseteq b_mL_m$. This proves that B is bounded.

Proposition 20.11:

Suppose that V and W are Fréchet spaces and that V is reflexive; for any bounded o-submodule $A \subseteq V \widehat{\otimes}_{K,\pi} W$ there are bounded o-submodules $B \subseteq V$ and $C \subseteq W$ such that A is contained in the closure of $B \otimes_o C$ in $V \widehat{\otimes}_{K,\pi} W$.

Proof: We will proceed in several steps. At first we apply Prop. 20.9 and view A as a bounded o-submodule in $\mathcal{L}_b(V_b', W)$.

Step 1: We show the existence of an open lattice $L \subseteq V_b'$ and of a closed bounded o-submodule $H \subseteq W$ such that A lies in the image of the injective map

$$\begin{array}{ccc} \mathcal{L}(\widehat{(V_b')_L}, W_H) & \longrightarrow & \mathcal{L}(V_b', W) \\ g & \longmapsto & \iota \circ g \circ \tau \end{array}$$

where $\tau : V_b' \longrightarrow \widehat{(V_b')_L}$ and $\iota : W_H \longrightarrow W$ denote the canonical maps.

As we saw at the beginning of the proof of Prop. 15.6 the strong dual V_b' possesses a sequence $(G_n)_{n\in\mathbb{N}}$ of bounded o-submodules with the property that any bounded subset of V_b' is contained in some G_n. We put

$$H_n := \sum_{f\in A} f(G_n) .$$

If p is a continuous seminorm on W then $p_{G_n}(f) = \sup\limits_{\ell\in G_n} p(f(\ell))$ is a continuous seminorm on $\mathcal{L}_b(V_b', W)$. Since A is bounded the seminorm p_{G_n} is bounded on A which implies that p is bounded on H_n. We see that each H_n is a bounded o-submodule in W. According to Remark 20.10 there is a sequence $(a_n)_{n\in\mathbb{N}}$ in $K^\times$ such that $\sum_n a_nH_n$ is a bounded o-submodule. We let H denote its closure. It follows from the identity $V' = \bigcup_n G_n$ that $f(V_b') \subseteq W_H$ for any $f \in A$. More precisely $f^{-1}(H)$, for $f \in A$, is a closed lattice in V_b'. But the reflexivity of V

implies that V_b' is barrelled (Prop. 15.2). Hence $f^{-1}(H)$ is an open lattice which means that any $f \in A$ factorizes through a continuous linear map

$$\widetilde{f} : V_b' \longrightarrow W_H \ .$$

Let $\widetilde{A} := \{\widetilde{f} : f \in A\} \subseteq \mathcal{L}(V_b', W_H)$. We claim that $\widetilde{A}$ is bounded in $\mathcal{L}_s(V_b', W_H)$. Consider, for any $\ell \in V'$, the open lattice $\mathcal{L}(\{\ell\}, H)$ in $\mathcal{L}_s(V_b', W_H)$. There is an $n \in \mathbb{N}$ such that $\ell \in G_n$. Then $\widetilde{f}(\ell) \in H_n \subseteq a_n^{-1}H$ or equivalently $\widetilde{A} \subseteq a_n^{-1}\mathcal{L}(\{\ell\}, H)$. It now follows from the Banach-Steinhaus theorem Prop. 6.15 that $\widetilde{A}$ is equicontinuous, i.e., that there is an open lattice $L \subseteq V_b'$ such that $\widetilde{f}(L) \subseteq H$ for any $\widetilde{f} \in \widetilde{A}$. Clearly any such $\widetilde{f}$ extends to a continuous linear map

$$\widehat{f} : \widehat{(V_b')_L} \longrightarrow W_H \ .$$

We finally obtain that A is the image under the asserted map of the o-submodule $\widehat{A} := \{\widehat{f} : f \in A\} \subseteq \mathcal{L}(\widehat{(V_b')_L}, W_H)$. Note that $\widehat{A}$ by construction is equicontinuous. Also note that W_H is a Banach space by Lemma 7.17.iii since H is closed and hence complete.

Step 2: According to Prop. 20.7 the strong dual V_b' is nuclear. Hence there is another lattice $M \subseteq L \subseteq V_b'$ such that the canonical map $\theta : \widehat{(V_b')_M} \longrightarrow \widehat{(V_b')_L}$ is compact. Introducing also the canonical map $\sigma : V_b' \longrightarrow \widehat{(V_b')_M}$ the injective map in Step 1 becomes the composite of the injective maps

$$\begin{array}{ccccc} \mathcal{L}(\widehat{(V_b')_L}, W_H) & \longrightarrow & \mathcal{C}(\widehat{(V_b')_M}, W_H) & \longrightarrow & \mathcal{L}(V_b', W) \\ g & \longmapsto & g \circ \theta & \longmapsto & \iota \circ (g \circ \theta) \circ \sigma \ . \end{array}$$

In order to pass back to the completed tensor product we use the commutative diagram

$$\begin{array}{ccc} (\widehat{(V_b')_M})_b' \,\widehat{\otimes}_{K,\pi}\, W_H & \xrightarrow{\cong} & \mathcal{C}(\widehat{(V_b')_M}, W_H) \\ (\delta^{-1} \circ \sigma') \otimes \iota \downarrow & & \downarrow \iota \circ . \circ \sigma \\ V \,\widehat{\otimes}_{K,\pi}\, W & \xrightarrow{\cong} & \mathcal{L}_b(V_b', W) \end{array}$$

where $\delta : V \xrightarrow{\cong} (V_b')_b'$ denotes the duality isomorphism (and where we recall Remark 16.1.ii for the continuity of σ'). The upper horizontal isomorphism comes from Prop. 18.11. If $A_\circ \subseteq (\widehat{(V_b')_M})_b' \,\widehat{\otimes}_{K,\pi}\, W_H$ denotes the o-submodule which corresponds to $\widehat{A} \circ \theta$ under the upper horizontal map then we have

$$A = [(\delta^{-1} \circ \sigma') \otimes \iota](A_\circ) \ .$$

For any $\widehat{f} \in \widehat{A}$ we consider the analogous commutative diagram

$$\begin{array}{ccc} ((V'_b)\widehat{_M})'_b \,\widehat{\otimes}_{K,\pi}\, (V'_b)\widehat{_L} & \xrightarrow{\cong} & \mathcal{C}((V'_b)\widehat{_M}, (V'_b)\widehat{_L}) \\ \mathrm{id} \otimes \widehat{f} \downarrow & & \downarrow \widehat{f} \circ . \\ ((V'_b)\widehat{_M})'_b \,\widehat{\otimes}_{K,\pi}\, W_H & \xrightarrow{\cong} & \mathcal{C}((V'_b)\widehat{_M}, W_H) \ . \end{array}$$

If θ_{o} in the upper left corner denotes the element corresponding to θ then the element $(\mathrm{id} \otimes \widehat{f}\ )(\theta_{\mathrm{o}})$ in the lower left corner corresponds to $\widehat{f} \circ \theta$. Hence we have $A_{\mathrm{o}} = \{(\mathrm{id} \otimes \widehat{f}\ )(\theta_{\mathrm{o}}) : \widehat{f} \in \widehat{A}\}$ and

$$A = \{[(\delta^{-1} \circ \sigma') \otimes (\iota \circ \widehat{f}\ )](\theta_{\mathrm{o}}) : \widehat{f} \in \widehat{A}\} \ .$$

With $\widehat{A}$ also the o-submodule $\{\iota \circ \widehat{f} : \widehat{f} \in \widehat{A}\} \subseteq \mathcal{L}_b((V'_b)\widehat{_L}, W)$ is equicontinuous.

Step 3: To simplify the notation we reformulate the situation we have arrived at as follows. We have constructed Banach spaces V_0 and V_1, an element $\theta_0 \in V_0 \,\widehat{\otimes}_{K,\pi}\, V_1$, a continuous linear map $g : V_0 \longrightarrow V$ and an equicontinuous and hence bounded o-submodule $\mathcal{A} \subseteq \mathcal{L}_b(V_1, W)$ such that

$$A = \{g \otimes f(\theta_0) : f \in \mathcal{A}\} \ .$$

At this point we observe that the statement of our Prop. 20.11 holds true, for rather trivial reasons, for any two Banach spaces: By the construction of the tensor product norm any ball in $V_0 \,\widehat{\otimes}_{K,\pi}\, V_1$ is contained in the closure of the tensor product (over o) of appropriate balls in V_0 and V_1. We therefore find bounded o-submodules $B_0 \subseteq V_0$ and $B_1 \subseteq V_1$ such that θ_0 lies in the closure of $B_0 \otimes_o B_1$. It follows that A is contained in the closure of $g(B_0) \otimes_o (\sum_{f \in \mathcal{A}} f(B_1))$. The o-submodule $B := g(B_0) \subseteq V$ clearly is bounded. So it remains to check that the o-submodule

$$C := \sum_{f \in \mathcal{A}} f(B_1)$$

is bounded in W. But this follows by the same argument which we used in Step 1 to see that the H_n are bounded.

This last result is important for the investigation of the dual space $(V \otimes_{K,\pi} W)'$ of the projective tensor product of two locally convex K-vector spaces V and W. By the very construction of the projective tensor product topology this dual space $(V \otimes_{K,\pi} W)'$, as a K-vector space, is naturally isomorphic to

$$\mathcal{B}(V \times W) \quad := \quad \begin{array}{l} K\text{-vector space of all continuous} \\ \text{bilinear forms } V \times W \longrightarrow K \ . \end{array}$$

We claim that the map

$$
\begin{array}{lrcl}
(1) \qquad & \mathcal{B}(V \times W) & \longrightarrow & \mathcal{L}(V, W_b') \\
& \beta & \longmapsto & f_\beta(v)(w) := \beta(v, w)
\end{array}
$$

is well defined. It is clear that each $f_\beta(v)$ is a continuous linear form on W. For the continuity of the map $v \longmapsto f_\beta(v)$, for any fixed β, we have to show that, given a bounded o-submodule $C \subseteq W$, the o-submodule $\{v \in V : f_\beta(v) \in \mathcal{L}(C, o)\} = \{v \in V : \beta(v, C) \subseteq o\}$ is an open lattice in V. Since β is continuous there are open lattices $L \subseteq V$ and $M \subseteq W$ such that $\beta(L, M) \subseteq o$. Furthermore we find an $a \in K^\times$ such that $C \subseteq aM$. Then

$$\beta(a^{-1}L, C) = \beta(L, a^{-1}C) \subseteq \beta(L, M) \subseteq o$$

which means that the o-submodule in question contains $a^{-1}L$ and therefore, indeed, is an open lattice. Obviously the map (1) is K-linear and injective. We usually write it in the form

$$
\begin{array}{lrcl}
(2) \qquad & (V \otimes_{K,\pi} W)' & \longrightarrow & \mathcal{L}(V, W_b') \\
& \beta & \longmapsto & f_\beta(v)(w) := \beta(v \otimes w) \ .
\end{array}
$$

It then fits into the commutative diagram

$$
\begin{tikzcd}
(V \otimes_{K,\pi} W)' \arrow[rr, "(2)"] & & \mathcal{L}(V, W_b') \\
& V' \otimes_K W_b' \arrow[ul, "\otimes"] \arrow[ur, "T"'] &
\end{tikzcd}
$$

where $\otimes$ is the tensor product map for linear forms (compare the paragraph before Prop. 17.8) and T is the map introduced at the beginning of §18.

Suppose now that $f : V \longrightarrow W_b'$ is a continuous linear map and define the bilinear form

$$
\begin{array}{rrcl}
\beta : & V \times W & \longrightarrow & K \\
& (v, w) & \longmapsto & f(v)(w) \ .
\end{array}
$$

For each $v \in V$ we have $\beta(v, .) = f(v)$ which is a continuous linear form on W. If we fix a $w \in W$, on the other hand, then $\beta(., w)^{-1}(o) = f^{-1}(\mathcal{L}(\{w\}, o))$ is an open lattice in V. This proves that β is separately continuous. Using Prop. 17.6 we therefore have the following fact.

Remark 20.12:

If V and W are Fréchet spaces then (2) *is a bijection.*

On both sides of (2) we have the strong topology. If $B \subseteq V$ and $C \subseteq W$ are bounded o-submodules then the open lattice $\mathcal{L}(B, \mathcal{L}(C, o))$ in the right hand side corresponds to the open lattice $\mathcal{L}(B \otimes_o C, o)$ in the left hand side. Hence we may and will view (2) as a continuous linear map

$$(V \otimes_{K,\pi} W)'_b \longrightarrow \mathcal{L}_b(V, W'_b) \ .$$

If V and W are Fréchet spaces and if in addition V is reflexive then Prop. 20.11 implies that the open lattices $\mathcal{L}(B \otimes_o C, o)$ define the strong topology on $(V \widehat{\otimes}_{K,\pi} W)'$. Under these assumptions (2) therefore is an isomorphism of locally convex K-vector spaces

$$(V \otimes_{K,\pi} W)'_b = (V \widehat{\otimes}_{K,\pi} W)'_b \xrightarrow{\cong} \mathcal{L}_b(V, W'_b) \ .$$

But the map T also induces an isomorphism of locally convex K-vector spaces

$$V'_b \widehat{\otimes}_{K,\pi} W'_b \xrightarrow{\cong} \mathcal{L}_b(V, W'_b)$$

in this situation. This follows from Cor. 18.8 since V and W as Fréchet spaces are bornological so that, in particular, W'_b is complete and since V is reflexive. Hence we have the following result.

Proposition 20.13:

If V and W are Fréchet spaces and V is reflexive then the tensor product of linear forms induces an isomorphism of locally convex K-vector spaces

$$V'_b \widehat{\otimes}_{K,\pi} W'_b \xrightarrow{\cong} (V \widehat{\otimes}_{K,\pi} W)'_b = (V \otimes_{K,\pi} W)'_b \ .$$

Corollary 20.14:

With V and W also $V \widehat{\otimes}_{K,\pi} W$ is a reflexive Fréchet space.

Proof: By Prop. 20.7 the strong duals V'_b and W'_b are nuclear. Hence $V'_b \widehat{\otimes}_{K,\pi} W'_b$ is nuclear by Prop. 19.11 and Prop. 20.4. The Prop. 20.13 then implies that $(V \widehat{\otimes}_{K,\pi} W)'_b$ is nuclear. Using Prop. 20.7 again we conclude that $V \widehat{\otimes}_{K,\pi} W$ is reflexive.

Finally it is possible to strengthen Prop. 17.10. The above discussion has shown that we cannot expect the tensor product map $V'_b \otimes_{K,\pi} W'_b \xrightarrow{\otimes} (V \otimes_{K,\pi} W)'_b$ to be continuous in general. But we can make use of the following observation.

Given any open lattices $L \subseteq V$ and $M \subseteq W$ we have the natural injective continuous linear map

$$(\widehat{V_L} \,\widehat{\otimes}_{K,\pi}\, \widehat{W_M})'_b \longrightarrow (V \,\widehat{\otimes}_{K,\pi}\, W)'_b \ .$$

As we have remarked in its proof the statement of Prop. 20.11 remains true for any two Banach spaces. Hence the corresponding version of (2) is an isomorphism of Banach spaces

$$(\widehat{V_L} \,\widehat{\otimes}_{K,\pi}\, \widehat{W_M})'_b \xrightarrow{\cong} \mathcal{L}_b(\widehat{V_L}, (\widehat{W_M})'_b) \ .$$

By Prop. 18.2 we have

$$(\widehat{V_L})'_b \,\widehat{\otimes}_{K,\pi}\, (\widehat{W_M})'_b \xrightarrow{\cong} \mathcal{CC}(\widehat{V_L}, (\widehat{W_M})'_b) \subseteq \mathcal{L}_b(\widehat{V_L}, (\widehat{W_M})'_b) \ .$$

Altogether this shows that the tensor product of linear forms does induce an injective continuous linear map

$$(\widehat{V_L})'_b \,\widehat{\otimes}_{K,\pi}\, (\widehat{W_M})'_b \xrightarrow{\otimes} (V \,\widehat{\otimes}_{K,\pi}\, W)'_b \ .$$

Remark 20.15:

For any open lattice $L \subseteq V$ we have $(\widehat{V_L})'_b \xrightarrow{\cong} (V'_b)_{L^p}$.

Proof: By Remark 16.1.ii the dual of the canonical map $\tau : V \longrightarrow \widehat{V_L}$ is an injective continuous linear map

$$\tau' : (\widehat{V_L})'_b \longrightarrow V'_b \ .$$

A continuous linear form ℓ on V extends continuously to $\widehat{V_L}$ if and only if $aL \subseteq \ell^{-1}(\mathfrak{m})$ for some $a \in K^\times$. This is equivalent to the condition that $a\ell \in L^p$. The image of τ' therefore is $(V'_b)_{L^p}$. Note that L^p is equicontinuous and hence bounded in V'_b. (Although this indirectly comes out of the present argument we also know already from Prop. 7.13, Lemma 13.1.ii, and Lemma 7.17.iii that L^p is complete and that therefore $(V'_b)_{L^p}$ is a Banach space.) If $\widehat{L}$ denotes the closure of the image of L in $\widehat{V_L}$ then the open lattice $\mathcal{L}(\widehat{L}, \mathfrak{m})$ in $(\widehat{V_L})'_b$ corresponds under τ' to the open lattice L^p in $(V'_b)_{L^p}$. It follows that τ' induces an isomorphism of Banach spaces $(\widehat{V_L})'_b \xrightarrow{\cong} (V'_b)_{L^p}$.

As a consequence of Cor. 13.5 an o-submodule $A \subseteq V'$ is of the form $A = L^p$ for some open lattice $L \subseteq V$ if and only if it is equicontinuous and *weakly closed*, i.e., closed in V'_s. In view of these remarks we may rephrase our above discussion by stating that, for any equicontinuous and weakly closed o-submodules $A \subseteq V'$

and $B \subseteq W'$, the tensor product of linear forms induces an injective continuous linear map

$$(V'_b)_A \mathbin{\widehat{\otimes}_{K,\pi}} (W'_b)_B \xrightarrow{\otimes} (V \mathbin{\widehat{\otimes}_{K,\pi}} W)'_b \ .$$

Proposition 20.16: (Kernel theorem)

Suppose that V is nuclear; for any equicontinuous subset $H \subseteq (V \mathbin{\widehat{\otimes}_{K,\pi}} W)'$ there exist equicontinuous and weakly closed o-submodules $A \subseteq V'$ and $B \subseteq W'$ such that H is the image of a bounded subset in $(V'_b)_A \mathbin{\widehat{\otimes}_{K,\pi}} (W'_b)_B$.

Proof: Since H is equicontinuous we find open lattices $L_1 \subseteq V$ and $M \subseteq W$ such that H is the image of an equicontinuous subset $G \subseteq (\widehat{V_{L_1}} \mathbin{\widehat{\otimes}_{K,\pi}} \widehat{W_M})'$. We choose another open lattice $L \subseteq L_1$ such that the canonical map $\theta : \widehat{V_L} \longrightarrow \widehat{V_{L_1}}$ is compact. Consider now the commutative diagram

$$\begin{tikzcd}
(\widehat{V_{L_1}} \mathbin{\widehat{\otimes}_{K,\pi}} \widehat{W_M})'_b \arrow[rr, "\cong"] \arrow[dd, "(\theta\otimes\mathrm{id})'"'] \arrow[dr, dashed] & & \mathcal{L}_b(\widehat{V_{L_1}}, (\widehat{W_M})'_b) \arrow[d, ".\circ\theta"] \\
& (\widehat{V_L})'_b \mathbin{\widehat{\otimes}_{K,\pi}} (\widehat{W_M})'_b \arrow[r, "\cong"] \arrow[dl, "\otimes"'] & \mathcal{CC}(\widehat{V_L}, (\widehat{W_M})'_b) \arrow[d, "\subseteq"] \\
(\widehat{V_L} \mathbin{\widehat{\otimes}_{K,\pi}} \widehat{W_M})'_b \arrow[rr, "\cong"] & & \mathcal{L}_b(\widehat{V_L}, (\widehat{W_M})'_b) \ .
\end{tikzcd}$$

It shows that the canonical map $(\theta\otimes\mathrm{id})' : (\widehat{V_{L_1}} \mathbin{\widehat{\otimes}_{K,\pi}} \widehat{W_M})'_b \longrightarrow (\widehat{V_L} \mathbin{\widehat{\otimes}_{K,\pi}} \widehat{W_M})'_b$ factorizes through $(\widehat{V_L})'_b \mathbin{\widehat{\otimes}_{K,\pi}} (\widehat{W_M})'_b$. The image of G in this latter vector space is bounded and is mapped onto H. It remains to put $A := L^p$ and $B := M^p$ and to use Remark 20.15.

Corollary 20.17:

Suppose that V is nuclear; we then have

$$(V \otimes_{K,\pi} W)' = (V \mathbin{\widehat{\otimes}_{K,\pi}} W)' = \bigcup_{A,B} (V'_b)_A \mathbin{\widehat{\otimes}_{K,\pi}} (W'_b)_B$$

where A and B run over all equicontinuous and weakly closed o-submodules in V' and W', respectively.

§21 Traces

Let V be any locally convex K-vector space. There is the obvious linear form

$$\begin{array}{rccc} tr: & V' \otimes_K V & \longrightarrow & K \\ & \ell \otimes v & \longmapsto & \ell(v) \end{array}$$

which, if we view an element in the left hand side as a finite rank operator f on V, is the usual matrix trace of f. This means that $tr(f)$ is the matrix trace of the restriction of f to any finite dimensional vector subspace of V which contains the image of f. The two basic properties of the matrix trace are:

(a) $tr(f \circ g) = tr(g \circ f)$ for any finite rank operator $f : V \longrightarrow W$ into another (locally convex) K-vector space W and any $g \in \mathcal{L}(W, V)$;

(b) $tr(f') = tr(f)$ for any finite rank operator f on V where f' denotes the dual (or transpose) finite rank operator on the dual space V'.

Suppose now that V is a Banach space. Let $L \subseteq V$ be a bounded open lattice. Then $\mathcal{L}(L, o)$ is an open lattice in the dual Banach space V'_b. Since obviously

$$tr(\mathcal{L}(L, o) \otimes_o L) \subseteq o$$

it follows from Lemma 17.1 that the linear form

$$tr : V'_b \otimes_{K,\pi} V \longrightarrow K$$

is continuous for the projective tensor product topology. Hence it extends to a continuous linear form

$$tr : V'_b \,\widehat{\otimes}_{K,\pi}\, V \longrightarrow K \ .$$

By Prop. 18.11 we may view the completed tensor product as the space $\mathcal{N}(V, V)$ of nuclear maps on V. We will continue to call the resulting linear form

$$tr : \mathcal{N}(V, V) \longrightarrow K$$

the *trace* of a nuclear map on V. Let W be another K-Banach space. Using the operator norm on the Banach space $\mathcal{L}_b(V, W)$ as described in §3 it is easy to check that the maps

$$\begin{array}{ccc} \mathcal{L}_b(V,W) \times \mathcal{L}_b(W,V) & \longrightarrow & \mathcal{L}_b(V,V) \\ (f,g) & \longmapsto & f \circ g \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{L}_b(V,V) & \longrightarrow & \mathcal{L}_b(V',V') \\ f & \longmapsto & f' \end{array}$$

are continuous. Moreover, we know already from Lemma 16.4 that the transpose respects nuclear maps. By a limit argument the properties (a) and (b) therefore continue to hold. We have:

(a') $tr(f \circ g) = tr(g \circ f)$ for any $f \in \mathcal{N}(V, W)$ and any $g \in \mathcal{L}(W, V)$;

(b') $tr(f') = tr(f)$ for any $f \in \mathcal{N}(V, V)$.

It is clear that the continuity property of tr with respect to the projective tensor product topology will not hold for any larger class of locally convex K-vector spaces. But we will show in this section that the trace can always be extended to the space $\mathcal{N}(V,V)$ of nuclear maps on V.

With V again arbitrary let (L,B) be a pair consisting of an open lattice $L \subseteq V$ and a bounded o-submodule $B \subseteq V$ such that V_B is complete; let $\tau_L : V \longrightarrow \widehat{V_L}$ and $\iota_B : V_B \longrightarrow V$ denote the canonical maps. Then

$$\begin{array}{ccc} \mathcal{N}(\widehat{V_L}, V_B) & \longrightarrow & \mathcal{N}(V,V) \\ f & \longmapsto & \iota_B \circ f \circ \tau_L \end{array}$$

is an inclusion of K-vector spaces. As a consequence of Remark 20.1 we in fact have that

$$(*) \qquad \mathcal{N}(V,V) = \varinjlim_{(L,B)} \mathcal{N}(\widehat{V_L}, V_B)$$

is the algebraic inductive limit where (L,B) runs over all such pairs. We observe that B is a bounded open lattice in V_B and that $\mathcal{L}(\tau_L \circ \iota_B(B), o)$ is an open lattice in $\widehat{V_L}$. Hence the above continuity argument based on Lemma 17.1 works here and shows that the linear form

$$\begin{array}{rccc} tr_{L,B}: & (\widehat{V_L})'_b \otimes_K V_B & \longrightarrow & K \\ & \ell \otimes v & \longmapsto & \ell \circ \tau_L \circ \iota_B(v) \end{array}$$

is continuous for the projective tensor product topology. It therefore extends to a continuous linear form

$$tr_{L,B} : \mathcal{N}(\widehat{V_L}, V_B) \cong (\widehat{V_L})'_b \widehat{\otimes}_{K,\pi} V_B \longrightarrow K\ .$$

Let us consider a second pair (M,C) with $M \subseteq L$ and $B \subseteq C$; let $\tau_{M,L} : \widehat{V_M} \longrightarrow \widehat{V_L}$ and $\iota_{B,C} : V_B \longrightarrow V_C$ denote the canonical maps. We compute

$$\begin{aligned} & tr_{M,C}(\tau'_{M,L} \circ \iota_{B,C}(\ell \otimes v)) \\ & = tr_{M,C}(\tau'_{M,L}(\ell) \otimes \iota_{B,C}(v)) = tr_{M,C}((\ell \circ \tau_{M,L}) \otimes \iota_{B,C}(v)) \\ & = \ell \circ \tau_{M,L} \circ \tau_M \circ \iota_C \circ \iota_{B,C}(v) = \ell \circ \tau_L \circ \iota_B(v) \\ & = tr_{L,B}(\ell \otimes v) \end{aligned}$$

which means that the diagram

$$\begin{tikzcd} (\widehat{V_L})'_b \otimes_K V_B \arrow[rr, "\tau'_{M,L} \otimes \iota_{B,C}"] \arrow[dr, "tr_{L,B}"'] & & (\widehat{V_M})'_b \otimes_K V_C \arrow[dl, "tr_{M,C}"] \\ & K & \end{tikzcd}$$

is commutative. Moreover, by the functoriality of the projective tensor product, the horizontal arrow is continuous for the projective tensor product topologies. By a limit argument we then obtain the commutative diagram

$$\begin{tikzcd} \mathcal{N}(\widehat{V_L}, V_B) \arrow[rr] \arrow[dr, "tr_{L,B}"'] & & \mathcal{N}(\widehat{V_M}, V_C) \arrow[dl, "tr_{M,C}"] \\ & K & \end{tikzcd}$$

where the horizontal arrow is given by $f \longmapsto \iota_{B,C} \circ f \circ \tau_{M,L}$. We therefore may pass to the algebraic inductive limit and, using $(*)$, we arrive at a well defined *trace* linear form

$$tr : \mathcal{N}(V, V) \longrightarrow K \ .$$

Proposition 21.1:

i. $tr(f)$, for any finite rank operator f on V, is the matrix trace of f;

ii. $tr(f \circ g) = tr(g \circ f)$ for any $f \in \mathcal{N}(V, W)$ and any $g \in \mathcal{L}(W, V)$;

iii. $tr(f') = tr(f)$ for any $f \in \mathcal{N}(V, V)$.

Proof: i. By the definition of $tr_{L,B}$ the diagram

$$\begin{tikzcd} (\widehat{V_L})' \otimes_K V_B \arrow[rr, "\tau_L' \otimes \iota_B"] \arrow[dr, "tr_{L,B}"'] & & V' \otimes_K V \arrow[dl, "tr"] \\ & K & \end{tikzcd}$$

is commutative.

ii. We fix a factorization

$$V \xrightarrow{\tau_L} \widehat{V_L} \xrightarrow{\widetilde{f}} W_B \xrightarrow{\iota_B} W$$

of f with $\widetilde{f}$ a nuclear map between Banach spaces and we write $\widetilde{f}$ as the uniform limit of a sequence $(f_n)_{n \in \mathbb{N}}$ of finite rank operators $f_n : \widehat{V_L} \longrightarrow W_B$.

In a first step we furthermore factorize $g \circ \iota_B$ into

$$W_B \xrightarrow{g_1} V_C \xrightarrow{\iota_C} V$$

where $C := g(B)$. We compute

$$\begin{aligned} tr(g \circ f) &= tr_{L,C}(g_1 \circ \widetilde{f}\,) = tr_{L,C}(\lim_{n\to\infty} g_1 \circ f_n) \\ &= \lim_{n\to\infty} tr_{L,C}(g_1 \circ f_n) = \lim_{n\to\infty} tr(\iota_C \circ g_1 \circ f_n \circ \tau_L) \\ &= \lim_{n\to\infty} tr(g \circ \iota_B \circ f_n \circ \tau_L) \ . \end{aligned}$$

In a second step we factorize $\tau_L \circ g$ into

$$W \xrightarrow{\tau_M} \widehat{W_M} \xrightarrow{g_0} \widehat{V_L}$$

where $M := g^{-1}(L)$ and we compute

$$\begin{aligned} tr(f \circ g) &= tr_{M,B}(\widetilde{f} \circ g_0) = tr_{M,B}(\lim_{n\to\infty} f_n \circ g_0) \\ &= \lim_{n\to\infty} tr_{M,B}(f_n \circ g_0) = \lim_{n\to\infty} tr(\iota_B \circ f_n \circ g_0 \circ \tau_M) \\ &= \lim_{n\to\infty} tr(\iota_B \circ f_n \circ \tau_L \circ g) \ . \end{aligned}$$

It remains to notice that, by (a), we have

$$tr(g \circ (\iota_B \circ f_n \circ \tau_L)) = tr((\iota_B \circ f_n \circ \tau_L) \circ g) \ .$$

iii. First of all we recall from Remark 16.1.ii and Lemma 16.4 that f' is well defined and nuclear. As above we start with a factorization

$$V \xrightarrow{\tau_L} \widehat{V_L} \xrightarrow{\widetilde{f}} V_B \xrightarrow{\iota_B} V$$

and a sequence of finite rank operators $f_n : \widehat{V_L} \longrightarrow W_B$ converging to $\widetilde{f}$. Then the sequence $(f_n')_n$ converges to $\widetilde{f}\,'$. Using (b) we have

$$\begin{aligned} tr(f) &= tr_{L,B}(\widetilde{f}\,) = \lim_{n\to\infty} tr_{L,B}(f_n) = \lim_{n\to\infty} tr(\iota_B \circ f_n \circ \tau_L) \\ &= \lim_{n\to\infty} tr(\tau_L' \circ f_n' \circ \iota_B') \ . \end{aligned}$$

Because of Remark 20.15 the dual sequence reads

$$V_b' \xrightarrow{\iota_B'} (V_B)_b' \xrightarrow{\widetilde{f}\,'} (V_b')_{L^p} \xrightarrow{\iota_{L^p}} V_b' \ .$$

It is straightforward to see that the map ι_B' factorizes into

$$V_b' \xrightarrow{\tau_{B^p}} \widehat{(V_b')_{B^p}} \xrightarrow{\sigma} (V_B)_b' \ .$$

We therefore obtain

$$\begin{aligned} tr(f') &= tr_{B^p,L^p}(\widetilde{f}\,' \circ \sigma) = \lim_{n\to\infty} tr(\iota_{L^p} \circ f'_n \circ \sigma \circ \tau_{B^p}) \\ &= \lim_{n\to\infty} tr(\tau'_L \circ f'_n \circ \iota'_B) \ . \end{aligned}$$

§22 Fredholm theory

For any locally convex K-vector space V the space $\mathcal{L}(V,V)$ of continuous linear endomorphisms of V is an algebra with respect to the composition of maps. The automorphisms of the locally convex vector space V are the invertible elements in this algebra. In this section we will study the following more general class of maps.

Definition:

A map $f \in \mathcal{L}(V,V)$ is called <u>Fredholm</u> if the kernel and the cokernel of f are finite dimensional.

In the following we will use the algebraic notations $\ker(f), \mathrm{im}(f)$, and $\mathrm{coker}(f)$ to denote the kernel, image, and cokernel of a linear map f.

If $f \in \mathcal{L}(V,V)$ is a Fredholm map then the nonnegative integer

$$\mathrm{ind}(f) := \dim_K \ker(f) - \dim_K \mathrm{coker}(f)$$

is called the *index* of f. Its subsequent basic property is an exercise in linear algebra.

Lemma 22.1:

If $f, g \in \mathcal{L}(V,V)$ are Fredholm maps then $f \circ g$ is a Fredholm map and

$$\mathrm{ind}(f \circ g) = \mathrm{ind}(f) + \mathrm{ind}(g) \ .$$

Proof: This is an immediate consequence of the two exact sequences

$$0 \longrightarrow \ker(g) \longrightarrow \ker(f \circ g) \xrightarrow{g} \mathrm{im}(g) \cap \ker(f) \longrightarrow 0$$

and

$$0 \longrightarrow \ker(f)/\ker(f) \cap \mathrm{im}(g) \longrightarrow \mathrm{coker}(g) \xrightarrow{f} \mathrm{im}(f)/\mathrm{im}(f \circ g) \longrightarrow 0 \ .$$

Lemma 22.2:

If $f \in \mathcal{L}(V,V)$ is a Fredholm map on a Fréchet space V then the vector subspace $\mathrm{im}(f)$ *is closed in V.*

Proof: By lifting a basis of $\mathrm{coker}(f)$ to V we find a finite dimensional subspace $U \subseteq V$ such that $\mathrm{im}(f) \oplus U = V$. Consider the linear map

$$\begin{array}{rccc} h: & V/\ker(f) \oplus U & \longrightarrow & V \\ & (v + \ker(f), u) & \longmapsto & f(v) + u \ . \end{array}$$

It is a continuous linear bijection between Fréchet spaces (Prop. 8.3) and as such is, by the open mapping theorem Cor. 8.7, a topological isomorphism. Since $\mathrm{im}(f)$ coincides with the kernel of h^{-1} followed by the projection onto the summand U we conclude that $\mathrm{im}(f)$ is closed.

Proposition 22.3:

Suppose that $f \in \mathcal{L}(V,V)$ is a Fredholm map on a Fréchet space V; then there exists another Fredholm map $g \in \mathcal{L}(V,V)$ such that $\mathrm{id}_V - f \circ g$ *and* $\mathrm{id}_V - g \circ f$ *are finite rank operators.*

Proof: We choose closed vector subspaces $U_0, U_1 \subseteq V$ such that

$$\ker(f) \oplus U_0 = V \quad \text{and} \quad \mathrm{im}(f) \oplus U_1 = V \ .$$

That the first choice is possible was established in Cor. 9.5. According to Lemma 22.2 the vector subspace $\mathrm{im}(f)$ is closed in V. Hence the restriction $f_1 : U_0 \longrightarrow \mathrm{im}(f)$ of f is a continuous linear bijection between Fréchet spaces and therefore is, by the open mapping theorem Cor. 8.7, a topological isomorphism. We now define the continuous linear map $g \in \mathcal{L}(V,V)$ by $g|\mathrm{im}(f) := f_1^{-1}$ and $g|U_1 := 0$. Since $\ker(g) = U_1$ and $\mathrm{im}(g) = U_0$ we see that g is a Fredholm map. Moreover, $g \circ f$, resp. $f \circ g$, is the identity on U_0, resp. on $\mathrm{im}(f)$, and is zero on $\ker(f)$, resp. on U_1. Hence $\mathrm{id}_V - g \circ f$ and $\mathrm{id}_V - f \circ g$ are finite rank operators.

This result has an interesting consequence. Recall from Remark 16.7.i that the vector subspace $\mathcal{C}(V,V)$ of compact operators is a two-sided ideal in $\mathcal{L}(V,V)$. We therefore may form the quotient algebra $\mathcal{L}(V,V)/\mathcal{C}(V,V)$. Since finite rank operators are compact the above proposition implies that on a Fréchet space V the coset of a Fredholm map is an invertible element in this quotient algebra. Much of this section is devoted to the question whether the converse is true. For this we first of all have to analyze maps of the form $\mathrm{id}_V + g$ with $g \in \mathcal{C}(V,V)$.

Lemma 22.4:

For any $g \in \mathcal{C}(V,V)$ the kernel of $\mathrm{id}_V + g$ *is finite dimensional.*

Proof: Put $U := \ker(\mathrm{id}_V + g)$. Since g is compact there is an open lattice $L \subseteq V$ such that the closure $\overline{g(L)}$ is bounded and c-compact. Because of $U \cap L = -g(U \cap L)$ it follows that $\overline{U \cap L}$ is bounded and c-compact in V. But then $\overline{U \cap L}$ is a bounded and c-compact open lattice in U (Lemma 12.1.v). This implies, by Cor. 4.12, that U is normed and also that U is nuclear. According to Prop. 20.4 the completion of U therefore is a nuclear Banach space which necessarily is finite dimensional as we have seen in §19.

Lemma 22.5:

Let $f : V \longrightarrow W$ be an injective but not surjective continuous linear map with dense image between two infinite dimensional K-Banach spaces V and W; then there exists an infinite dimensional closed vector subspace $U \subseteq V$ such that the map $f|U : U \longrightarrow W$ is compact.

Proof: We choose defining norms, both denoted by $\| \ \|$, on V and W. We also fix a $c \in K$ such that $0 < |c| < 1$. Let us begin with the following preliminary observation. Let $V_o \subseteq V$ be a closed vector subspace such that V/V_o is finite dimensional. Since the image $f(V/V_o) = f(V)/f(V_o)$ then is a finite dimensional vector subspace of $W/f(V_o)$ which is not closed it follows from Prop. 4.13 that the quotient $W/f(V_o)$ is not Hausdorff. The image $f(V_o)$ therefore is not closed in W which implies that the bijection $V_o \xrightarrow{\sim} f(V_o)$ cannot be a homeomorphism. In view of Prop. 3.1 this means that

$$\inf\{\frac{\|f(v)\|}{\|v\|} : v \in V_o \setminus \{0\}\} = 0 \ .$$

Hence, for any $\epsilon > 0$, there is a vector

$$(*) \qquad v_o \in V_o \text{ such that } |c| \leq \|v_o\| \leq 1 \text{ and } \|f(v_o)\| \leq \epsilon \ .$$

In order to prove our assertion we construct inductively a sequence $(v_n)_{n\in\mathbb{N}}$ of vectors in V such that, for any $n \in \mathbb{N}$, we have

(1_n) $\quad |c| \leq \|v_n\| \leq 1$,

(2_n) $\quad \|f(v_n)\| \leq |c|^n$,

(3_n) $\quad \|a_1v_1 + \ldots + a_nv_n\| = \max\limits_{1\leq i\leq n} |a_i| \cdot \|v_i\|$ for any $a_1, \ldots, a_n \in K$.

Suppose that $v_1, \ldots, v_n$ are constructed already. Using the Hahn-Banach theorem Prop. 9.2 we find continuous linear forms $\ell_1, \ldots, \ell_n$ on V such that

$$(**) \qquad \|\ell_i\| = \|v_i\|^{-1} \quad \text{and} \quad \ell_i(v_j) = \delta_{ij} \ .$$

Applying our initial observation $(*)$ to the vector subspace $V_o := \ker(\ell_1) \cap \ldots \cap \ker(\ell_n)$ we obtain a vector $v_{n+1} \in V_o$ such that (1_{n+1}) and (2_{n+1}) hold true. In order to establish (3_{n+1}) let $a_1, \ldots, a_{n+1} \in K$. We only need to consider the case where $\|a_{n+1}v_{n+1}\| = \|a_1v_1 + \ldots + a_nv_n\|$. Then, by $(**)$,

$$\|a_1v_1 + \ldots + a_{n+1}v_{n+1}\| \geq \|v_i\| \cdot |\ell_i(a_1v_1 + \ldots + a_{n+1}v_{n+1})| = |a_i| \cdot \|v_i\|$$

for any $1 \leq i \leq n$ and hence, in particular,

$$\|a_1v_1+\ldots+a_{n+1}v_{n+1}\| \geq \max_{1\leq i\leq n} |a_i|\cdot\|v_i\| = \|a_1v_1+\ldots+a_nv_n\| = |a_{n+1}|\cdot\|v_{n+1}\| \ .$$

Both inequalities together amount to (3_{n+1}).

Having constructed the sequence $(v_n)_n$ we deduce from (1_n) and (3_n) that the linear map

$$\begin{array}{rcl} c_{\mathrm{o}}(\mathbb{N}) & \longrightarrow & V \\ (a_n)_n & \longmapsto & \sum_{n\in\mathbb{N}} a_nv_n \end{array}$$

is a homeomorphism onto its image. Let $U \subseteq V$ denote this image which in particular is closed and has $L := \{\sum_n a_nv_n : |a_n| \leq 1 \text{ and } \lim_{n\to\infty} |a_n| = 0\}$ as a bounded open lattice. On the other hand we deduce from (2_n) that the map

$$\begin{array}{rcl} \prod_{n\in\mathbb{N}} o & \longrightarrow & W \\ (b_n)_n & \longmapsto & \sum_n b_nf(v_n) \end{array}$$

is continuous for the product topology on the left hand side and that therefore its image is a c-compact o-submodule in W (compare the Example after Remark 12.8). We see that $f(L)$ is contained in a c-compact o-submodule. Hence $f|U$ is compact.

We also need the exterior powers of a K-Banach space V. Its n-fold exterior power $\bigwedge^n V$ is a quotient of the n-fold tensor product $V\otimes_K \ldots \otimes_K V$. We always equip it with the quotient topology of the projective tensor product topology. If $\| \ \|$ is a defining norm on V then the topology on $\bigwedge^n V$ is defined by the seminorm

$$\|u\|_{(n)} := \inf\{\max_{1\leq k\leq r} \|v_1^{(k)}\| \cdot \ldots \cdot \|v_n^{(k)}\| : u = \sum_{k=1}^{r} v_1^{(k)} \wedge \ldots \wedge v_n^{(k)}, v_i^{(k)} \in V\} \ .$$

The n-fold exterior power $\bigwedge^n f$ of a map $f \in \mathcal{L}(V,V)$ lies in $\mathcal{L}(\bigwedge^n V, \bigwedge^n V)$.

Lemma 22.6:

Let V be a K-Banach space with defining norm $\| \ \|$; we then have:

i. $\| \ \|_{(n)}$ *is a norm on* $\bigwedge^n V$ *for any* $n \in \mathbb{N}$*;*

ii. for any $g \in \mathcal{C}(V,V)$ *the sequence of operator norms* $(\| \bigwedge^n g\|_{(n)})_{n\in\mathbb{N}}$ *tends to zero.*

Proof: i. We have to show that the kernel $N^n(V)$ of the natural map $V \otimes_K \ldots \otimes_K V \longrightarrow \bigwedge^n V$ is closed. We consider therefore any sequence $(u_n)_{n\in\mathbb{N}}$ in $N^n(V)$ which converges to some vector u in the normed vector space $V \otimes_K \ldots \otimes_K V$. Let $U \subseteq V$ be a finite dimensional vector subspace such that $u \in U \otimes_K \ldots \otimes_K U$. According to the Hahn-Banach theorem Cor. 9.5 the inclusion map $U \subseteq V$ has a continuous linear section $s : V \longrightarrow U$. Then $((s \otimes \ldots \otimes s)(u_n))_n$ is a sequence in $N^n(U)$ converging to $(s \otimes \ldots \otimes s)(u) = u$. But $N^n(U)$ is closed in the finite dimensional normed vector space $U \otimes_K \ldots \otimes_K U$. It follows that $u \in N^n(U) \subseteq N^n(V)$.

ii. We may assume that the defining norm $\| \ \|$ on V is the gauge of some bounded open lattice. It then has the property that

$$\text{(1)} \qquad \begin{array}{c}\text{for any } v \in V \text{ and any } \epsilon > 0 \text{ there is an } a \in K^\times \text{ such that} \\ \|v\| \leq |a| \leq \|v\| + \epsilon \ .\end{array}$$

We first establish a certain technical estimate. Suppose given some vector subspace $W \subseteq V$ of finite dimension ℓ and introduce the seminorm

$$|v|_W := \inf_{w\in W} \|v + w\|$$

on V. We claim that

$$\text{(2)} \qquad \|v_1 \wedge \ldots \wedge v_n\|_{(n)} \leq \max\{\prod_{i\notin J} |v_i|_W \cdot \prod_{i\in J} \|v_i\| : J \subseteq \{1,\ldots,n\}, |J| \leq \ell\}$$

for any $v_1, \ldots, v_n \in V$. Given any $\epsilon > 0$ we find, for any $1 \leq i \leq n$, a $w_i \in W$ such that $\widetilde{v_i} := v_i - w_i \in v_i + W$ satisfies $\|\widetilde{v_i}\| \leq |v_i|_W + \epsilon$. Then

$$\|w_i\| \leq \max(\|v_i\|, \|\widetilde{v_i}\|) \leq \|v_i\| + \epsilon$$

and hence

$$\begin{aligned}
\|v_1 \wedge \ldots \wedge v_n\|_{(n)} &= \|(\widetilde{v_1} + w_1) \wedge \ldots \wedge (\widetilde{v_n} + w_n)\|_{(n)} \\
&\leq \max_J \ \|(\textstyle\bigwedge_{i\notin J} \widetilde{v_i}) \wedge (\bigwedge_{i\in J} w_i)\|_{(n)} \\
&= \max_{|J|\leq\ell} \ \|(\textstyle\bigwedge_{i\notin J} \widetilde{v_i}) \wedge (\bigwedge_{i\in J} w_i)\|_{(n)} \\
&\leq \max_{|J|\leq\ell} \ \textstyle\prod_{i\notin J} \|\widetilde{v_i}\| \cdot \prod_{i\in J} \|w_i\| \\
&\leq \max_{|J|\leq\ell} \ \textstyle\prod_{i\notin J} (|v_i|_W + \epsilon) \cdot \prod_{i\in J} (\|v_i\| + \epsilon) \ .
\end{aligned}$$

Letting ϵ tend to zero gives the above claim.

We now choose a $c \in K^\times$ such that $|c| \geq \max(1, \|g\|)$ and we let $L := L(\| \ \|)$. Since the map g is compact the image $g(L)$ is compactoid so that there is a finite dimensional vector subspace $W \subseteq V$ such that

$$g(L) \subseteq W + c^{-1}L \ .$$

To check that in this situation we have

$$(3) \qquad |g(v)|_W \leq |c|^{-1} \cdot \|v\| \quad \text{for any } v \in V$$

we choose, by (1), an $a \in K^\times$ such that $\|v\| \leq |a| \leq \|v\| + \epsilon$. Then $a^{-1}v \in L$ so that $g(a^{-1}v) = w + c^{-1}u$ with $w \in W$ and $u \in L$. It follows that $g(v) = aw + ac^{-1}u$ and $|g(v)|_W \leq \|ac^{-1}u\| \leq |ac^{-1}| \leq |c|^{-1} \cdot \|v\| + |c|^{-1} \cdot \epsilon$.

Combining (2) and (3) we obtain

$$\begin{aligned}
\|g(v_1) \wedge \ldots \wedge g(v_n)\|_{(n)} &\leq \max_{|J| \leq \ell} \textstyle\prod_{i \notin J} |g(v_i)|_W \cdot \prod_{i \in J} \|g(v_i)\| \\
&\leq (\max_{|J| \leq \ell} |c|^{-(n-|J|)} \cdot \|g\|^{|J|}) \cdot \textstyle\prod_{i=1}^n \|v_i\| \\
&\leq |c|^{2\ell - n} \cdot \textstyle\prod_{i=1}^n \|v_i\|
\end{aligned}$$

for any $v_1, \ldots, v_n \in V$ where $\ell := \dim_K W$. It follows that

$$\| \bigwedge^n g\|_{(n)} \leq |c|^{2\ell - n} \ .$$

In order not to always have to repeat the same assumption let us fix in the following a compact map $g : V \longrightarrow V$ and put $f := \mathrm{id}_V + g$. We note that, for any $n \in \mathbb{N}$, the map f^n is of the same form $f^n = \mathrm{id}_V + g_n$ with

$$g_n := (\sum_{i=0}^{n-1} \binom{n}{i+1} g^i) \circ g \in \mathcal{C}(V, V) \ .$$

For this reason certain claims in the following about any f^n can be established by only considering the case $n = 1$. Example given, we obtain from Lemma 22.4 that $\ker(f^n)$ is finite dimensional for any $n \in \mathbb{N}$. These kernels form an ascending chain

$$\ker(f) \subseteq \ldots \subseteq \ker(f^n) \subseteq \ker(f^{n+1}) \subseteq \ldots$$

Similarly the images form a descending chain

$$\mathrm{im}(f) \supseteq \ldots \supseteq \mathrm{im}(f^n) \supseteq \mathrm{im}(f^{n+1}) \supseteq \ldots$$

Lemma 22.7:

If V is a Banach space then we have:

i. $\mathrm{im}(f^n)$ *is closed in V for any $n \in \mathbb{N}$;*

ii. there is an $m \in \mathbb{N}$ such that $\ker(f^n) = \ker(f^m)$ *and* $\mathrm{im}(f^n) = \mathrm{im}(f^m)$ *for any $n \geq m$.*

Proof: i. It suffices to consider the case $n = 1$. We also may suppose that V is infinite dimensional. The kernel of f being finite dimensional by Lemma 22.4 we find, according to Cor. 9.5, an infinite dimensional closed vector subspace $V_\mathrm{o} \subseteq V$ such that V is the topological direct sum

$$V = \ker(f) \oplus V_\mathrm{o} \ .$$

On the other hand let $W \subseteq V$ denote the closure of the image of f. Then

$$\begin{array}{rcl} f_\mathrm{o} : V_\mathrm{o} & \longrightarrow & W \\ v & \longmapsto & v + g(v) \end{array}$$

is an injective continuous linear map with dense image. Let $L \subseteq V$ be a bounded open lattice. Then $\overline{g(L)}$ is c-compact in V. Suppose that $U \subseteq V_\mathrm{o}$ is a closed vector subspace such that $f|U$ is compact. Then $\overline{f(L \cap U)}$ is c-compact as well. We have

$$L \cap U \subseteq \overline{f(L \cap U)} + \overline{g(L)} \ .$$

The right hand side is c-compact by Cor. 12.3. It follows from Lemma 12.1 that $L \cap U$ is a c-compact and bounded open lattice in the Banach space U. We have seen in the proof of Lemma 22.4 that this forces U to be finite dimensional. We therefore may apply Lemma 22.5 and conclude that f_o is bijective which means that $\mathrm{im}(f) = W$ is closed.

ii. Suppose that there is a vector $v \in \ker(f^n) \setminus \ker(f^{n-1})$. Then the vectors $v, f(v), \ldots, f^{n-1}(v)$ are linearly independent which means that

$$v \wedge f(v) \wedge \ldots \wedge f^{n-1}(v) \neq 0 \ .$$

But on the other hand we compute

$$\begin{aligned} &\| {\textstyle\bigwedge}^n g(v \wedge f(v) \wedge \ldots \wedge f^{n-1}(v)) \|_{(n)} \\ &\quad = \|(f(v) - v) \wedge (f^2(v) - f(v)) \wedge \ldots \wedge (f^n(v) - f^{n-1}(v))\|_{(n)} \\ &\quad = \|v \wedge f(v) \wedge \ldots \wedge f^{n-1}(v)\|_{(n)} \ . \end{aligned}$$

For big n this forces, according to Lemma 22.6., the vector $v \wedge f(v) \wedge \ldots \wedge f^{n-1}(v)$ to vanish. This proves the first half of our assertion.

By applying this first half to the dual map f' (recall from Lemma 16.4 that with g also g' is compact) we obtain that the sequence of vector spaces $(V/\mathrm{im}(f^n))' = \ker((f^n)')$ becomes constant. The Hahn-Banach theorem Cor. 9.3 together with the assertion i. then implies the second half of the assertion ii.

Proposition 22.8: (Riesz decomposition)

i. $\mathrm{im}(f)$ *is closed in* V*;*

ii. there is an $m \in \mathbb{N}$ *such that* $\ker(f^n) = \ker(f^m)$ *and* $\mathrm{im}(f^n) = \mathrm{im}(f^m)$ *for any* $n \geq m$*;*

iii. V *is the locally convex direct sum* $V = \mathrm{im}(f^m) \oplus \ker(f^m)$*;*

iv. $f : \mathrm{im}(f^m) \xrightarrow{\cong} \mathrm{im}(f^m)$ *is an isomorphism of locally convex vector spaces;*

v. $f|\ker(f^m)$ *is nilpotent;*

vi. f *induces an isomorphism of locally convex vector spaces* $V/\ker(f) \xrightarrow{\cong} \mathrm{im}(f)$.

Proof: The assertion v. is trivial. The assertion vi. is a consequence of iii., iv., and Lemma 22.4. Next we convince ourselves that, assuming ii., the assertions iii. and iv. at least hold algebraically.

Suppose that $v \in \mathrm{im}(f^m) \cap \ker(f^m)$. If $v = f^m(w)$ then $w \in \ker(f^{2m}) = \ker(f^m)$ which implies that $v = f^m(w) = 0$. Hence the map in iv. is injective. By ii. it also is surjective. Using ii. again we find, for any $v \in V$, a vector $w \in V$ such that $f^m(v) = f^{2m}(w)$. Then $u := v - f^m(w) \in \ker(f^m)$ and $v = f^m(w) + u \in \mathrm{im}(f^m) + \ker(f^m)$. This shows that $V = \mathrm{im}(f^m) \oplus \ker(f^m)$ algebraically.

The vector subspaces $\ker(f^n)$ in V, by Lemma 22.4, are finite dimensional and hence closed. If V is a Banach space the assertions i. and ii. are true by Lemma 22.7. All vector spaces in the other assertions, which we know to hold true algebraically, then are Banach spaces. Hence the open mapping theorem Cor. 8.7 implies that these other assertions hold true topologically as well. This establishes fully all assertions in the case of a Banach space V.

Let now V be general and let $L \subseteq V$ be an open lattice such that the closure $B := \overline{g(L)}$ is bounded and c-compact. The canonical map $\iota : V_B \longrightarrow V$ is compact and there is a continuous linear map $\widetilde{g} : V \longrightarrow V_B$ such that $g = \iota \circ \widetilde{g}$. The composite $g_B := \widetilde{g} \circ \iota$ is a compact map on the Banach space V_B, and we have the commutative diagram:

$$\begin{tikzcd}
V \arrow[r, "g"] \arrow[d, "\widetilde{g}"'] & V \arrow[d, "\widetilde{g}"] \\
V_B \arrow[r, "g_B"'] \arrow[ur, "\iota"] & V_B
\end{tikzcd}$$

If we put $f_B := \mathrm{id}_{V_B} + g_B$ then also the diagrams

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\widetilde{g}"'] & V \arrow[d, "\widetilde{g}"] \\ V_B \arrow[r, "f_B"] & V_B \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} V \arrow[r, "f"] & V \\ V_B \arrow[u, "\iota"] \arrow[r, "f_B"] & V_B \arrow[u, "\iota"'] \end{tikzcd}$$

are commutative. Hence $\widetilde{g}(\ker(f^n)) \subseteq \ker(f_B^n)$ and $\iota(\ker(f_B^n)) \subseteq \ker(f^n)$ for any $n \in \mathbb{N}$. Since g, resp. g_B, is invertible on $\ker(f^n)$, resp. on $\ker(f_B^n)$, we see that $\widetilde{g}$ in fact induces bijections $\ker(f^n) \xrightarrow{\cong} \ker(f_B^n)$ for any $n \in \mathbb{N}$. The first half of the assertion ii. therefore follows from Lemma 22.7.ii applied to the map f_B.

To get the other half of the assertion ii. we note that the above diagram restricts, for any $m \in \mathbb{N}$, to the commutative diagram:

$$\begin{tikzcd} \mathrm{im}(f^m) \arrow[r, "f"] \arrow[d, "\widetilde{g}"'] & \mathrm{im}(f^m) \arrow[d, "\widetilde{g}"] \\ \mathrm{im}(f_B^m) \arrow[r, "f_B"] & \mathrm{im}(f_B^m) \end{tikzcd}$$

We also have
$$\iota(f_B^m(V_B)) = f^m(\iota(V_B)) \subseteq f^m(V) \ .$$

We choose m in such a way that Lemma 22.7 and hence the assertion iv. holds true for f_B. Then

$$h := \mathrm{id}_V - \iota \circ f_B^{-1} \circ \widetilde{g} \ : \ \mathrm{im}(f^m) \longrightarrow \mathrm{im}(f^m)$$

is a well defined continuous linear map. We have

$$h \circ f = f - \iota \circ f_B^{-1} \circ \widetilde{g} \circ f = f - \iota \circ \widetilde{g} = f - g = \mathrm{id}_V$$

and
$$\begin{aligned} f \circ h &= (\mathrm{id}_V + g) \circ h = h + g \circ h \\ &= \mathrm{id}_V - \iota \circ f_B^{-1} \circ \widetilde{g} + g - g \circ \iota \circ f_B^{-1} \circ \widetilde{g} \\ &= \mathrm{id}_V - \iota \circ f_B^{-1} \circ \widetilde{g} + \iota \circ \widetilde{g} - \iota \circ \widetilde{g} \circ \iota \circ f_B^{-1} \circ \widetilde{g} \\ &= \mathrm{id}_V - \iota \circ (f_B^{-1} \circ \widetilde{g} + g_B \circ f_B^{-1} \circ \widetilde{g} - \widetilde{g}) \\ &= \mathrm{id}_V - \iota \circ (f_B \circ f_B^{-1} \circ \widetilde{g} - \widetilde{g}) \\ &= \mathrm{id}_V \ . \end{aligned}$$

This shows that h is a continuous inverse for the upper horizontal arrow in the last diagram and therefore proves the assertion iv. and the second half of the assertion ii. The assertion iii. also follows since

$$(h^m \circ f^m, \mathrm{id}_V - h^m \circ f^m) \ : \ V \longrightarrow \mathrm{im}(f^m) \oplus \ker(f^m)$$

is a continuous linear inverse of the continuous bijection $\mathrm{im}(f^m) \oplus \ker(f^m) \xrightarrow{+} V$. But then $\mathrm{im}(f)$ is of the form $\mathrm{im}(f) = \mathrm{im}(f^m) \oplus U$ for some vector subspace U of the finite dimensional (Lemma 22.4) vector space $\ker(f^m)$. This proves that $\mathrm{im}(f)$ is closed.

Corollary 22.9:

For any compact map $g \in \mathcal{C}(V, V)$ the map $\mathrm{id}_V + g$ is Fredholm of index zero.

Proof: This follows immediately from Lemma 22.4 and Prop. 22.8. The latter in particular says that the kernel and cokernel of $\mathrm{id}_V + g$ are isomorphic to the kernel and cokernel of a linear map on a vector space $\ker((\mathrm{id}_V + g)^m)$. But according to the former this vector space is finite dimensional.

Corollary 22.10: (Fredholm alternative)

For any compact map $g \in \mathcal{C}(V, V)$ the following assertions are equivalent:

i. $\mathrm{id}_V + g$ *is injective;*

ii. $\mathrm{id}_V + g$ *is surjective;*

iii. $\mathrm{id}_V + g$ *is an automorphism of V.*

Corollary 22.11:

If V is a Fréchet space then an $f \in \mathcal{L}(V, V)$ is Fredholm if and only if its image in $\mathcal{L}(V, V)/\mathcal{C}(V, V)$ is invertible.

Proof: We have discussed already that as a consequence of Prop. 22.3 any Fredholm map has an invertible image in the quotient algebra. Suppose now, vice versa, that the image of f is invertible. This means that there is a map $h \in \mathcal{L}(V, V)$ such that $\mathrm{id}_V - f \circ h$ and $\mathrm{id}_V - h \circ f$ are compact maps. It then follows from Cor. 22.9 that $f \circ h$ and $h \circ f$ are Fredholm maps. Since $\ker(f) \subseteq \ker(h \circ f)$ and $\mathrm{im}(f \circ h) \subseteq \mathrm{im}(f)$ this implies that f is Fredholm.

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Notations

$\mid\ \mid$	nonarchimedean absolute value,
$\Vert\ \Vert$	nonarchimedean norm,
K	nonarchimedean field,
$\mathbb{Q}_p$	field of p-adic numbers,
$\mathbb{C}_p$	completion of an algebraic closure of $\mathbb{Q}_p$,
$B_\epsilon(a), B_\epsilon^-(a)$	(open) balls in the field,
$B_\epsilon(v), B_\epsilon^-(v)$	(open) balls in a Banach space,
p_L	gauge seminorm,
$\ker(p_L)$	null space of p_L,
$L(q), L^-(q)$	lattices of a seminorm,
$V(q_{i_1}, \ldots, q_{i_r}; \epsilon)$	zero neighbourhoods,
$\mathrm{Co}(S)$	convex hull,
V_A	seminormed space associated with an o-submodule A,
$\widehat{V_A}$	Hausdorff completion of V_A,
$\ell^\infty(X)$	Banach space of bounded functions,
$c_{\mathrm{o}}(X)$	Banach space of generalized zero sequences,
$BC(X)$	Banach space of bounded continuous functions,
$C_c(X)$	vector space of continuous functions with compact support,
$C(X, V)$	vector space of vector valued continuous functions,
$\widehat{V}$	Hausdorff completion,
$\widehat{M}$	lattice in Hausdorff completion,
$\mathcal{L}(V, W)$	vector space of continuous linear maps,
$\mathcal{L}(B, M)$	lattice in $\mathcal{L}(V, W)$,
p_B	seminorm on $\mathcal{L}(V, W)$,
$\mathcal{L}_{\mathcal{B}}(V, W)$	$\mathcal{L}(V, W)$ with $\mathcal{B}$-topology,
$\mathcal{L}_b(V, W)$	$\mathcal{L}(V, W)$ with topology of bounded convergence,
$\mathcal{L}_c(V, W)$	$\mathcal{L}(V, W)$ with topology of compactoid convergence,
$\mathcal{L}_s(V, W)$	$\mathcal{L}(V, W)$ with topology of pointwise convergence,

V'	dual space,
$V'_{\mathcal{B}}$	$\mathcal{B}$-dual,
V'_b	strong dual,
V'_s	weak dual,
f'	dual or transpose map,
δ	duality map,
A^p	pseudo-polar,
A^{pp}	pseudo-bipolar
V_c	Mackey topology
V_s	weak topology
$\mathcal{C}(V, W)$	compact maps,
$\mathcal{CC}(V, W)$	completely continuous maps,
$\mathcal{N}(V, W)$	nuclear maps,
$p \otimes q$	tensor product seminorm,
$V \otimes_{K,\iota} W$	inductive tensor product,
$V \otimes_{K,\pi} W$	projective tensor product,
$V \widehat{\otimes}_{K,\pi} W$	completed projective tensor product,
$\mathcal{B}(V \times W)$	vector space of bilinear forms,
tr	trace linear form for nuclear maps,
$\mathrm{ind}(f)$	index of a Fredholm map,