The 1999 Britton Lectures at McMaster University on
"p-adic representation theory"
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Lecture 1: The Lazard ring

We fix a field K with a complete nonarchimedean absolute value $|\ |$. The unique extension of $|\ |$ to an algebraic closure $\overline{K}$ of K again will be denoted by $|\ |$. Let $G_K := \mathrm{Gal}(\overline{K}|K)$ be the absolute Galois group of K.

Disk algebras

For $r \in |\overline{K}^\times|$ let

$$X_r(\overline{K}) := \{a \in \overline{K} : |a| \leq r\}$$

be the closed disk around 0 of radius r and

$$X_r := \text{ space of } G_K\text{-orbits in } X_r(\overline{K})$$

Definition:

$$\begin{aligned}\mathcal{O}(X_r) := &\ \text{all power series } f(T) = \sum_{n\geq 0} a_n T^n \quad \text{with } a_n \in K \text{ which converge} \\ &\ \text{on } X_r(\overline{K}) \\ = &\ \text{all power series } f(T) = \sum_{n\geq 0} a_n T^n \quad \text{with } \lim_{n\to\infty} |a_n| r^n = 0\end{aligned}$$

is called the ring of holomorphic functions on X_r.

Facts:

1. $\mathcal{O}(X_r)$ is an integral domain;
2. $\mathcal{O}(X_r)$ is a K-Banach algebra w.r.t. the (multiplicative) norm

$$||\sum_{n\geq 0} a_n T^n||_r := \max_{n\geq 0} |a_n| r^n.$$

An effective divisor on X_r is a map $D : X_r \longrightarrow \mathbb{Z}_{\geq 0}$ with finite support; the effective divisors form a semigroup $\mathrm{Div}^+(X_r)$ which is partially ordered by

$$D \leq D' \text{ iff } D(x) \leq D'(x) \text{ for any } x \in X_r\ .$$

Fact:

$$\begin{array}{ccccc} \text{all monic polynomials } f & = & \text{all monic polynomials} & \xrightarrow{\sim} & \mathrm{Div}^+(X_r) \\ \in K[T] \text{ with } \|f\|_r = r^{\deg f} & & \text{in } K[T] \text{ with all zeros} & & \\ & & \text{in } X_r(\overline{K}) & & \\ & & f & \mapsto & (f)(x) := \\ & & & & \text{multiplicity} \\ & & & & \text{of } a \in x \\ & & & & \text{as a zero of } f \end{array}$$

is a bijection.

Weierstrass division:

Fix a $0 \neq g(T) = \sum_{n \geq 0} a_n T^n \in \mathcal{O}(X_r)$ *and let* $m \geq 0$ *be the largest index such that* $|a_m| r^m = ||g||_r$*; then for any* $f \in \mathcal{O}(X_r)$ *there are uniquely determined elements* $q \in \mathcal{O}(X_r)$ *and* $R \in K[T]$ *of degree* $< m$ *such that*

$$f = qg + R$$

moreover: i. $||f||_r = \max(||qg||_r, ||R||_r)$*;*

ii. if f *and* g *are polynomials and* g *has degree* m *then* q *is a polynomial.*

Weierstrass preparation:

Let g *and* m *as above; then there is a unique monic polynomial* $g_0 \in K[T]$ *of degree* m *and a unique* $e \in \mathcal{O}(X_r)^\times$ *such that*

$$g = e \cdot g_0;$$

moreover: i. $||g_0||_r = r^m$*;*

ii. if g *is a polynomical so, too, is* e*.*

(Proof: Apply division to $T^m = e'g + R$ and put $g_0 := T^m - R$.)

Reference: U. Güntzer, Modellringe in der nichtarchimedischen Funktionentheorie, Indagationes math. 29, 334-342 (1967)

In particular we may define the divisor of any $0 \neq g \in \mathcal{O}(X_r)$ by

$$(g) := (g_0) \in \mathrm{Div}^+(X_r).$$

Consequences:

1. $g|g'$ if and only if $(g) \leq (g')$;
2. $\mathcal{O}(X_r)$ is a principal ideal domain;
3.

$$\begin{array}{rcl} \text{nonzero ideals in } \mathcal{O}(X_r) & \xrightarrow{\sim} & \mathrm{Div}^+(X_r) \\ g\mathcal{O}(X_r) & \leftrightarrow & (g) \end{array}$$

is an order reversing bijection; it restricts to a bijection

$$\text{maximal ideals in } \mathcal{O}(X_r) \overset{\sim}{\leftrightarrow} X_r \ ;$$

4. for $r \leq r'$ in $|\overline{K}^\times|$ the diagram

$$\begin{array}{ccc} \mathcal{O}(X_{r'}) \setminus \{0\} & \xrightarrow{(\ )} & \mathrm{Div}^+(X_{r'}) \\ \subseteq\downarrow & & \downarrow \text{ restriction} \\ \mathcal{O}(X_r) \setminus \{0\} & \xrightarrow{(\ )} & \mathrm{Div}^+(X_r) \end{array}$$

is commutative.

Note: Since $\|\ \|_r$ is multiplicative the map $\mathcal{O}(X_r) \xrightarrow{g\cdot} \mathcal{O}(X_r)$, for any $0 \neq g \in \mathcal{O}(X_r)$, is a homeomorphism onto its image. It follows that any principal ideal and hence any ideal in $\mathcal{O}(X_r)$ is closed.

<u>The open unit disk</u>

Let

$$X(\overline{K}) := \{a \in \overline{K} : |a| < 1\}$$

be the open unit disk and

$$X := G_K\text{-orbits in } X(\overline{K}) \ .$$

Definition:

$\mathcal{O}(X) :=$ all power series $f(T) = \sum_{n\geq 0} a_n T^n$ with $a_n \in K$ which converge on $X(\overline{K})$

is called the ring of holomorphic functions on X.

Since
$$\mathcal{O}(X) = \varprojlim_{r<1} \mathcal{O}(X_r) = \bigcap_{r<1} \mathcal{O}(X_r)$$

the ring $\mathcal{O}(X)$ is an integral domain as well as a K-Fréchet algebra w.r.t. the family of norms $\{\|\ \|_r\}_{r<1}$. We note that this family of norms is increasing, i.e., $\|f\|_r \leq \|f\|_{r'}$ for $f \in \mathcal{O}(X)$ and $r \leq r'$. This leads to the following technical observation which will be used later on.

Remark: An element $f \in \mathcal{O}(X)$ lies in the closure $\overline{A}$ of a subset $A \subseteq \mathcal{O}(X)$ if and only if for any $r < 1$ in $|\overline{K}^\times|$ and any $\varepsilon > 0$ we find a $g \in A$ such that $\|f - g\|_r < \varepsilon$.

We define

$$\begin{aligned} \mathrm{Div}^+(X) := &\ \varprojlim_{r<1} \mathrm{Div}^+(X_r) \\ = &\ \text{all maps } D : X \longrightarrow \mathbb{Z}_{\geq 0} \text{ such that, for any } r<1, \text{ the} \\ &\ \text{restriction } D|X_r \text{ has finite support} \end{aligned}$$

to be the partially ordered semigroup of effective divisors on X and obtain the divisor map
$$\begin{aligned} \mathcal{O}(X) \setminus \{0\} &\longrightarrow \mathrm{Div}^+(X) \\ f &\longmapsto (f) \end{aligned}$$
as the projective limit of the divisor maps for the X_r with $r < 1$. It is immediately clear that, for any $0 \neq f, g \in \mathcal{O}(X)$, we have

$$f|g \text{ if and only if } (f) \leq (g) ;$$

in particular:

$$(f) = (g) \text{ if and only if } f = ug \text{ for some } u \in \mathcal{O}(X)^\times .$$

Theorem: (Lazard)

If K is spherically complete then the divisor map $\mathcal{O}(X) \setminus \{0\} \longrightarrow \mathrm{Div}^+(X)$ is surjective.

Proof: (M. Lazard, Les zéros des fonctions analytiques d'une variable sur un corps valué complet, Publ. Math. IHES 14, 47-75 (1962); we follow the argument in J. Fresnel/M. van der Put, Géométrie Analytique Rigide et Applications, Progress in Math. vol. 18, Birkhäuser 1981, p. 44.)

We fix a strictly increasing sequence $r_1 < r_2 < \ldots < 1$ in $|\overline{K}^\times|$ tending towards 1 and put $X_n := X_{r_n}$. Given a divisor $D \in \mathrm{Div}^+(X)$ we may choose, for any $n \in \mathbb{N}$, an element $f_n \in \mathcal{O}(X_n) \setminus \{0\}$ such that $(f_n) = D|X_n$. We then have

$$f_{n+1}|X_n = u_n \cdot f_n \ \text{ with appropriate units } \ u_n \in \mathcal{O}(X_n)^\times .$$

We claim that

$$(*) \qquad \text{there are } v_n \in \mathcal{O}(X_n)^\times \text{ such that } v_n \cdot (v_{n+1}|X_n)^{-1} = u_n \ .$$

If so we modify our original lifts f_n by posing $\tilde{f}_n := v_n \cdot f_n$ and obtain $(\tilde{f}_n) = D|X_n$ and

$$\tilde{f}_{n+1}|X_n = (v_{n+1}|X_n) \cdot u_n \cdot f_n = v_n \cdot f_n = \tilde{f}_n \ .$$

This shows that there is a (unique) $f \in \mathcal{O}(X) \setminus \{0\}$ such that $f|X_n = \tilde{f}_n$ for any $n \in \mathbb{N}$. By construction we have $(f) = D$ which proves our assertion.

Before we start establishing the claim $(*)$ we remark that, by a standard Čech covering argument, it amounts to the vanishing of the Picard group $H^1(X, \mathcal{O}_X^\times)$ of the rigid analytic variety X.

As part of Weierstrass preparation we know that any unit $u_n \in \mathcal{O}(X_n)^\times$ is of the form

$$u_n = c_n \cdot h_n \ \text{ with } \ c_n \in K^\times, \ h_n(0) = 1, \ \text{ and } \ ||h_n - 1||_{r_n} < 1 \ .$$

We would like to define

$$v_n := (c_1 \cdot \ldots \cdot c_{n-1})^{-1} \cdot \prod_{m \geq n} h_m | X_n$$

but the infinite product may not converge. Our assumption on the field K of being spherically complete will enable us to "correct" this expression in such a way that it becomes well defined.

We first consider, for any $n, m \in \mathbb{N}$, the finite product

$$h_{n,m} := \prod_{l=n+1}^{m} h_l | X_n = \sum_{i \geq 0} a_i^{(n,m)} T^i$$

in $\mathcal{O}(X_n)^\times$ (in particular $h_{n,m} = 1$ if $n \geq m$). We have

$$\begin{aligned} |a_i^{(n,m)}| \cdot r_n^i = \ & |a_i^{(n,m)}| \cdot r_{n+1}^i \cdot (\tfrac{r_n}{r_{n+1}})^i \\ \leq \ & (\tfrac{r_n}{r_{n+1}})^i \cdot \prod_{l=n+1}^{m} ||h_l|X_{n+1}||_{r_{n+1}} \\ \leq \ & (\tfrac{r_n}{r_{n+1}})^i \end{aligned}$$

and in particular

$$|a_i^{(n,m)}| \leq r_{n+1}^{-i} \quad \text{for any} \quad n, m \geq 1 \quad \text{and} \quad i \geq 0 \ .$$

The sequence $(a_i^{(n,1)}, a_i^{(n,2)}, a_i^{(n,3)}, \ldots)$ then is, for any fixed $n \geq 1$ and $i \geq 0$, an element in the K-Banach space $\ell^\infty(K)$ of all bounded sequences $(b_1, b_2, \ldots)$ in K with norm $||(b_1, b_2, \ldots)|| := \sup_{m \in \mathbb{N}} |b_m|$. On the subspace of converging sequences we have the continuous linear form $\lambda(b_1, b_2, \ldots) := \lim_{m \to \infty} b_m$ of norm 1. Since over a spherically complete K the Hahn-Banach theorem is valid we may extend λ to a continuous linear form λ of norm 1 on all of $\ell^\infty(K)$. We now define

$$a_i^{(n)} := \lambda(a_i^{(n,1)}, a_i^{(n,2)}, \ldots)$$

and

$$H_n(T) := a_0^{(n)} + a_1^{(n)} T + a_2^{(n)} T^2 + \ldots$$

Clearly $a_0^{(n)} = \lambda(1, 1, \ldots) = 1$. Moreover

$$|a_i^{(n)}| \cdot r_n^i \leq \sup_{m \in \mathbb{N}} |a_i^{(n,m)}| \cdot r_n^i \leq (\frac{r_n}{r_{n+1}})^i < 1 \quad \text{for } i \geq 1$$

which shows that

$$H_n \in \mathcal{O}(X_n)^\times .$$

We also see that, for any $a \in K$ with $|a| \leq r_n$, the sequence $\{(a_i^{(n,1)} a^i, a_i^{(n,2)} a^i, \ldots)\}_{i \geq 0}$ in $\ell^\infty(K)$ is a zero sequence. By continuity we therefore have

$$\begin{aligned}
&\lambda(h_{n,1}(a), h_{n,2}(a), \ldots) \\
&= \lambda(\sum_{i \geq 0} a_i^{(n,1)} a^i, \sum_{i \geq 0} a_i^{(n,2)} a^i, \ldots) = \lambda(\sum_{i \geq 0} a^i \cdot (a_i^{(n,1)}, a_i^{(n,2)}, \ldots)) \\
&= \sum_{i \geq 0} a^i \cdot \lambda(a_i^{(n,1)}, a_i^{(n,2)}, \ldots) = \sum_{i \geq 0} a_i^{(n)} a^i \\
&= H_n(a)
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{H_n(a)}{H_{n+1}(a)} &= \frac{\lambda(1, \ldots, 1, h_{n,n+1}(a), h_{n,n+2}(a), \ldots)}{\lambda(1, \ldots, 1, 1, h_{n+1,n+2}(a), \ldots)} \\
&= h_{n+1}(a) \cdot \frac{\lambda(1, \ldots, 1, h_{n+1,n+2}(a), \ldots)}{\lambda(1, \ldots, 1, h_{n+1,n+2}(a), \ldots)} \\
&= h_{n+1}(a) \ .
\end{aligned}$$

Here we have used that, by construction, λ has the same value on any two bounded sequences which differ only in finitely many entries. Since any nonzero power series in $\mathcal{O}(X_n)$ can have at most countably many zeros it follows that we have the identity

$$H_n \cdot H_{n+1}^{-1} = h_{n+1} \quad \text{in} \ \ \mathcal{O}(X_n) \ .$$

One easily verifies that the units

$$v_n := (c_1 \cdot \ldots \cdot c_{n-1})^{-1} \cdot h_n \cdot H_n \quad \text{in} \ \ \mathcal{O}(X_n)^\times$$

satisfy the claim

$$v_n \cdot (v_{n+1}|X_n)^{-1} = u_n$$

Complement: (Lazard, loc. cit, Prop. 5)

If K is arbitrary then there is, for any $D \in \mathrm{Div}^+(X)$, an element $0 \neq f \in \mathcal{O}(X)$ such that $(f) \geq D$.

Proposition:

i. The map

$$\begin{aligned} \mathrm{Div}^+(X) \ &\xrightarrow{\sim} \ \text{all nonzero closed ideals in } \mathcal{O}(X) \\ D \ &\longmapsto \ I_D := \{f \in \mathcal{O}(X) : (f) \geq D\} \cup \{0\} \end{aligned}$$

is a well defined bijection;

ii. any principal ideal in $\mathcal{O}(X)$ is closed;

iii. if K is spherically complete then any closed ideal in $\mathcal{O}(X)$ is principal.

Proof: First of all we note that, for any nonzero ideal $I \subseteq \mathcal{O}(X)$, the divisor

$$D(I)(x) := \min_{0 \neq f \in I} (f)(x) \quad \text{for} \ \ x \in X$$

is well defined.

Given a divisor $D \in \mathrm{Div}^+(X)$ we find, according to the Complement, a $0 \neq f \in \mathcal{O}(X)$ such that $(f) \geq D$. It follows that I_D is a nonzero ideal. For trivial reasons we have $D(I_D) \geq D$. To see the opposite inequality we fix a $0 \neq f \in I_D$. For any $r < 1$ in $|\overline{K}^\times|$ let $P_r \in K[T]$ be the unique monic polynomical with all zeros in $X_r(\overline{K})$ and

$$(P_r) + D|X_r = (f|X_r) \ .$$

Then $fP_r^{-1} \in I_D$ with $(fP_r^{-1}|X_r) = D|X_r$. This shows that $D(I_D) \leq D$. The ensuing equality $D(I_D) = D$ implies that the map $I \longmapsto I_D$ is injective. If K is spherically complete then, by the Theorem, $D = (f)$ for some $0 \neq f \in \mathcal{O}(X)$; in this case the ideal $I_D = f\mathcal{O}(X)$ is principal. In general, being the intersection

$$I_D = \bigcap_{r<1} I_D \mathcal{O}(X_r)$$

of closed ideals in the disk algebras $\mathcal{O}(X_r)$ the ideal I_D is closed in $\mathcal{O}(X)$.

Given an arbitrary nonzero ideal I in $\mathcal{O}(X)$ we claim that $I_{D(I)}$ is the closure of I. We clearly have $\overline{I} \subseteq I_{D(I)}$. Consider therefore an $f \in I_{D(I)}$. In order to show that f lies in $\overline{I}$ we fix an $r < 1$ in $|\overline{K}^\times|$ and an $\varepsilon > 0$ and we have to find a $g \in I$ such that $||f - g||_r < \varepsilon$ (compare the earlier Remark). Since $I\mathcal{O}(X_r) = I_{D(I)}\mathcal{O}(X_r)$ we find $g_1, \ldots, g_m \in I$ and $h_1, \ldots, h_m \in \mathcal{O}(X_r)$ such that

$$f = g_1 h_1 + \ldots + g_m h_m \ .$$

We now choose polynomials $R_i \in K[T]$ such that $||g_i||_r \cdot ||h_i - R_i||_r < \varepsilon$ for $1 \leq i \leq m$ and set $g := g_1 R_1 + \ldots + g_m R_m \in I$. Then $||f - g||_r < \varepsilon$.

Applying this claim to a nonzero closed ideal I we obtain that $I = I_{D(I)}$ lies in the image of our map. If, on the other hand, $I = f\mathcal{O}(X)$ is nonzero principal then $D(I) = (f)$ and $I_{D(I)} = I_{(f)} = f\mathcal{O}(X) = I$ is closed.

Remark:

In the above as well as the subsequent proofs we repeatedly use the simple observation that for any nonzero ideal $I \subseteq \mathcal{O}(X)$ and any $r < 1$ in $|\overline{K}^\times|$ we have

$$I\mathcal{O}(X_r) = \{f \in \mathcal{O}(X_r) : (f) \geq D(I)|X_r\} \cup \{0\} \ .$$

Proof: Fixing r we find finitely many nonzero $g_1, \ldots, g_m \in I$ such that

$$D(I)(x) = \min_{1 \leq i \leq m} (g_i)(x) \quad \text{for } x \in X_r \ .$$

Since $\mathcal{O}(X_r)$ is a principal ideal domain the g_i generate the ideal on the right hand side of the asserted identity.

Lemma:

Given a closed ideal $I \subseteq \mathcal{O}(X)$ and elements $g_r \in \mathcal{O}(X_r)$ such that

$$g_{r'} - g_r \in I\mathcal{O}(X_r) \quad \textit{for any } r \leq r' < 1 \ \textit{ in } |\overline{K}^\times|$$

there is a $g \in \mathcal{O}(X)$ such that

$$g - g_r \in I\mathcal{O}(X_r) \quad \textit{for any } r < 1 \textit{ in } |\overline{K}^\times| \ .$$

Proof: If $I = \{0\}$ the assertion is a consequence of the fact that the X_r form an admissible covering of the rigid analytic variety X. Otherwise we fix a $0 \neq f \in I$ and put $D := (f) - D(I)$. Then $\{g_r f^{-1}\}_r$ is a family of principal parts with respect to D on X. Since X is a rigid analytic Stein space the invertible sheaf $\mathcal{O}(D)$ on X has no cohomology which implies that any such family is the family of principal parts with respect to D of a meromorphic function on X, i.e., there is a meromorphic function h on X such that

$$h - g_r f^{-1} \in \mathcal{O}(D)(X_r) \quad \text{for any } r < 1 \text{ in } |\overline{K}^\times| \ .$$

We set $g := fh$. Then

$$g - g_r = f(h - g_r f^{-1}) \in f \cdot \mathcal{O}(D)(X_r) \subseteq I_{D(I)}\mathcal{O}(X_r) = I\mathcal{O}(X_r)$$

and $g \in g_r + \mathcal{O}(X_r) \subseteq \mathcal{O}(X_r)$ for any $r < 1$ which shows that $g \in \mathcal{O}(X)$.

Proposition:

The sum $I + J$ of any two closed ideals I and J in $\mathcal{O}(X)$ is closed.

Proof: We may assume that both I and J are nonzero so that, by the previous Proposition $I = I_{D_0}$ and $J = I_{D_1}$ for two divisors $D_0, D_1 \in \mathrm{Div}^+(X)$. From the proof of that Proposition we also know that the closure of $I + J$ is the ideal I_D where the divisor D is defined by

$$D(x) := \min(D_0(x), D_1(x)) \quad \text{for } x \in X \ .$$

We therefore consider any $g \in I_D$. Since $I\mathcal{O}(X_r) + J\mathcal{O}(X_r) = I_D\mathcal{O}(X_r)$ we find $g_{0,r} \in I\mathcal{O}(X_r)$ and $g_{1,r} \in J\mathcal{O}(X_r)$ such that

$$g = g_{0,r} + g_{1,r} \ .$$

For $r < r'$ we then have

$$g_{i,r'} - g_{i,r} \in I\mathcal{O}(X_r) \cap J\mathcal{O}(X_r) = (I \cap J)\mathcal{O}(X_r)$$

so that, by the above Lemma, there are $g_0, g_1 \in \mathcal{O}(X)$ such that

$$g_i - g_{i,r} \in (I \cap J)\mathcal{O}(X_r)$$

and, in particular, $g_i \in I_{D_i}\mathcal{O}(X_r)$ for any $r < 1$. Hence $g_0 \in I$ and $g_1 \in J$. Since

$$g - (g_0 + g_1) = (g_{0,r} - g_0) + (g_{1,r} - g_1) \in (I \cap J)\mathcal{O}(X_r)$$

for any $r < 1$ we have $g - (g_0 + g_1) \in I \cap J$ and consequently $g \in I + J$.

Corollary: (Lazard)

Any finitely generated ideal in $\mathcal{O}(X)$ is closed.

Lecture 2: The Fourier transform

The character group of $\mathbb{Z}_p$

Let $\mathbb{C}_p$ be the completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We will determine the character group $\hat{G}$ of the compact group $G := \mathbb{Z}_p$. The character group

$$\hat{G} := \mathrm{Hom}_{an}(G, \mathbb{C}_p^\times)$$

is defined to be the group of all locally $\mathbb{C}_p$-analytic group homomorphisms $\kappa : G \longrightarrow \mathbb{C}_p^\times$. Let $o_p \subseteq \mathbb{C}_p$ be the ring of integers and $\mathfrak{m}_p \subseteq o_p$ be the maximal ideal. We begin with a few very simple observations. Since $\mathbb{C}_p^\times / o_p^\times = \mathbb{Q}$ is discrete and torsionfree whereas $\mathbb{Z}_p$ is compact we must have $\kappa(\mathbb{Z}_p) \subseteq o_p^\times$. But $o_p^\times / 1 + \mathfrak{m}_p = \mathbb{F}_p^\times$ is a discrete torsion group without p-torsion; so we actually have $\kappa(\mathbb{Z}_p) \subseteq 1 + \mathfrak{m}_p$. The group $\mathbb{Z}_p$ being topologically cyclic generated by $1 \in \mathbb{Z}_p$ any continuous character κ of $\mathbb{Z}_p$ is determined by its value $\kappa(1)$. All this together means that

$$\begin{array}{ccc} \hat{G} & \longrightarrow & \mathfrak{m}_p \\ \kappa & \longmapsto & \kappa(1) - 1 \end{array}$$

is a well defined injective map. On the other hand, for any $z \in \mathfrak{m}_p$ and any $a \in \mathbb{Z}_p$ the series

$$\kappa_z(a) := (1+z)^a := \sum_{n \geq 0} \binom{a}{n} z^n$$

converges. The notation $(1+z)^a$ is justified by the fact that

$$(1+z)^a = \lim_{i \to \infty} (1+z)^{m_i}$$

for any sequence of integers $m_i \in \mathbb{Z}$ converging to a. The latter formula shows that $\kappa_z : \mathbb{Z}_p \longrightarrow \mathbb{C}_p^\times$ is a homomorphism of groups. Later on we will show that κ_z is locally $\mathbb{C}_p$-analytic. (For more details see W. Schikhof, Ultrametric calculus, Cambridge Univ. Press 1984, §§32 and 47). Since $\kappa_z(1) - 1 = z$ we obtain that the above map

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\sim} & \mathfrak{m}_p \\ \kappa & \longmapsto & \kappa(1) - 1 \end{array}$$

is a bijection. Actually we can be a little bit more precise. Let us fix a complete intermediate field $\mathbb{Q}_p \subseteq K \subseteq \mathbb{C}_p$ and let X denote, as in Lecture 1, the rigid analytic open unit disk over K. Its K-points are

$$X(K) := \{z \in K : |z| < 1\} = \mathfrak{m}_p \cap K .$$

We will see that κ_z, for $z \in X(K)$, in fact is locally K-analytic. This means we have the bijection

$$\begin{array}{rcl} \mathrm{Hom}_{an}(G, K^\times) & \xrightarrow{\sim} & X(K) \\ \kappa & \longmapsto & \kappa(1) - 1 \ . \end{array}$$

The Fourier transform

This last fact is the starting point for the theory of the Fourier transform from (the dual of) locally K-analytic functions on G to rigid K-analytic functions on X. The locally convex K-vector space

$$C^{an}(\mathbb{Z}_p, K) := \text{ all locally } K\text{-analytic functions on } \mathbb{Z}_p$$

is constructed in the following way. For any $b \in K$ and any $r \in |\overline{K}^\times|$ let $X_r(b)$ denote the closed rigid K-analytic disk of radius r around b and let $\mathcal{O}_{b,r} := \mathcal{O}(X_r(b))$ be the K-Banach space (with respect to the multiplicative spectral norm $|| \, ||_{b,r}$) of holomorphic functions on $X_r(b)$. Then

$$C^{an}(\mathbb{Z}_p, K) := \varinjlim_{j \geq 1} \prod_{b \in \mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}_{b,p^{-j}}$$

is the locally convex inductive limit of the Banach spaces $\prod_{b \in \mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}_{b,p^{-j}}$ with respect to the obvious restriction maps. We define the K-Fréchet space

$$\begin{aligned} D(\mathbb{Z}_p, K) :=& \text{ continuous dual } C^{an}(Z_p, K)' \text{ of } C^{an}(Z_p, K) \\ =& \varprojlim_{j \geq 1} \prod_{b \in \mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}'_{b,p^{-j}} \end{aligned}$$

of K-valued distributions on $\mathbb{Z}_p$ as the projective limit of the dual Banach spaces $\prod_{b \in \mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}'_{b,p^{-j}}$. Without proof we state the following result.

Proposition:

If K is spherically complete then $D(\mathbb{Z}_p, K) = C^{an}(\mathbb{Z}_p, K)'_b$ is the strong dual of $C^{an}(\mathbb{Z}_p, K)$ and is reflexive and nuclear.

Using the embedding

$$\begin{array}{rcl} X(K) & \hookrightarrow & C^{an}(\mathbb{Z}_p, K) \\ z & \mapsto & \kappa_z \end{array}$$

any linear form $\lambda \in D(\mathbb{Z}_p, K)$ gives rise to the function

$$F_\lambda(z) := \lambda(\kappa_z)$$

which is called the Fourier transform of λ. It will turn out that F_λ in fact is a holomorphic function on X. The main aim of this lecture is to prove the following result.

Theorem: (Amice)

The Fourier transform

$$\begin{array}{rcl} D(\mathbb{Z}_p, K) & \xrightarrow{\cong} & \mathcal{O}(X) \\ \lambda & \longmapsto & F_\lambda \end{array}$$

is an isomorphism of locally convex K-vector spaces.

The technical heart of the proof consists in the computation of the spectral norms $||P||_{b,p^{-j}}$ of certain concrete polynomials P. We fix integers $j \geq 1$ and $0 \leq b < p^j$ and consider the polynomials

$$P_{n,l}(T) := \prod_{i=0}^{n-1} (T - lp^j - i) \quad \text{for } n, l \in \mathbb{N}_0$$

over K viewed as elements in $\mathcal{O}_{b,p^{-j}}$. By definition we have

$$||P_{n,l}||_{b,p^{-j}} = \sup\{|P_{n,l}(z)| : z \in b + p^j o_p\}.$$

Given a $z \in b + p^j o_p$ and a $0 \leq i < n$ we distinguish two cases. If $i \notin b + p^j o_p$ then $|z - lp^j - i| = |b - i|$. If $i \in b + p^j o_p$ then $|z - lp^j - i| \leq p^{-j}$. Moreover, since the residue class field $\overline{\mathbb{F}}_p$ of o_p is infinite we certainly find a $z \in b + p^j o_p$ such that $|z - lp^j - i| = p^{-j}$ whenever $i \in b + p^j o_p$. This shows that, setting

$$\nu(b, n, k) := \#\{0 \leq i < n : i \in b + p^k o_p\} \quad \text{for } 1 \leq k \leq j \ ,$$

we have

$$\begin{aligned} ||P_{n,l}||_{b,p^{-j}} &= p^{-j \cdot \nu(b,n,j)} \cdot \prod_{\substack{0 \leq i < n \\ i \notin b + p^j o_p}} |b - i| \\ (0) \qquad &= p^{-j \cdot \nu(b,n,j)} \cdot p^{-(j-1)[\nu(b,n,j-1) - \nu(b,n,j)]} \cdot \ldots \cdot p^{-[\nu(b,n,1) - \nu(b,n,2)]} \\ &= p^{-\sum_{k=1}^{j} \nu(b,n,k)} \ . \end{aligned}$$

Writing

$$n = m_k p^k + n_k \quad \text{with } 0 \leq n_k < p^k \text{ and}$$

$$b = c_k p^k + b_k \qquad \text{with } 0 \leq b_k < p^k$$

we find that the numbers $b_k, b_k + p^k, \ldots, b_k + (m_k - 1)p^k$ lie between 0 and $n-1$. This implies that

$$(1) \qquad \nu(b, n, k) \geq m_k = [\frac{n}{p^k}] \ .$$

If $b = n_j$ then $b_k = n_k$ and hence $b_k + m_k p^k = n$. Therefore

$$(2) \qquad \nu(n_j, n, k) = [\frac{n}{p^k}] \quad \text{for any } 1 \leq k \leq j \ .$$

If $b < n_j$ then $b_j + m_j p^j < n$ and consequently

$$(3) \qquad \nu(b, n, j) = [\frac{n}{p^j}] + 1 \quad \text{for } b < n_j \ .$$

A standard calculation for the absolute value of any factorial $m!$ (Schikhof, loc. cit., 25.5) shows that

$$|n!/\left[\frac{n}{p^j}\right]!| = p^{-\sum\limits_{k=1}^{j}[\frac{n}{p^k}]} \ .$$

Inserting these computations into (0) we obtain the following formulas:

$$(1') \qquad ||P_{n,l}||_{b,p^{-j}} \leq |n!/[\frac{n}{p^j}]!|$$

$$(2') \qquad ||P_{n,l}||_{n_j,p^{-j}} = |n!/[\frac{n}{p^j}]!| \ , \quad \text{and}$$

$$(3') \qquad ||P_{n,l}||_{b,p^{-j}} < |n!/[\frac{n}{p^j}]!| \quad \text{for } \ b < n_j \ .$$

We in particular have

$$(4) \qquad ||P_{n,l}||_{(j)} = |n!/[\frac{n}{p^j}]!|$$

where $|| \ ||_{(j)} = \max\limits_{b \in \mathbb{Z}/p^j\mathbb{Z}} || \ ||_{b,p^{-j}}$ denotes the norm on the Banach space $\prod\limits_{b \in \mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}_{b,p^{-j}}$.

Proposition:

The polynomials $\binom{T}{n} := \frac{T(T-1)\cdot\ldots\cdot(T-n+1)}{n!}$ *for* $n \geq 0$ *form an orthogonal basis of the* K*-Banach space* $\prod\limits_{b \in \mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}_{b,p^{-j}}$ *with* $||\binom{T}{n}||_{(j)} = |[\frac{n}{p^j}]!|^{-1}$.

Proof: The assertion about the norm is a special case of (4). Let $\mathbb{F}$ denote the residue class field of K and define $\mathcal{O}^0_{b,p^{-j}}$, resp. $\mathcal{O}^{00}_{b,p^{-j}}$, to be the ball of elements of norm ≤ 1, resp. < 1, in $\mathcal{O}_{b,p^{-j}}$. Then

$$E := \prod_{b\in\mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}^0_{b,p^{-j}}/\mathcal{O}^{00}_{b,p^{-j}}$$

in a natural way is an $\mathbb{F}$-algebra. For any $f \in \prod_b \mathcal{O}_{b,p^{-j}}$ with $||f||_{(j)} \leq 1$ let $\overline{f}$ denote its class in E. Our assertion about the $\binom{T}{n}$ is equivalent to the statement that the classes $\overline{Q}_n \in E$ of the normalized polynomials

$$\overline{Q}_n := \left[\frac{n}{p^j}\right]! \cdot \binom{T}{n}$$

form a basis of E as an $\mathbb{F}$-vector space (van Rooij, Non-Archimedean Functional Analysis, M. Dekker 1978, 5.A). But E has the obvious basis consisting of the classes $\overline{\chi}_{k,c}$ of the elements

$$\chi_{k,c} := \prod_{b\in\mathbb{Z}/p^j\mathbb{Z}} \chi_{k,c,b} \quad \text{with} \quad \chi_{k,c,b} := \begin{cases} p^{-j}(T-c)^k & \text{if } b = c, \\ 0 & \text{otherwise} \end{cases}$$

for $k \geq 0$ and $c \in \mathbb{Z}/p^j\mathbb{Z}$. We let $E_l \subseteq E$, for $l \geq 0$, denote the subspace of dimension $(l+1)p^j$ spanned by the $\overline{\chi}_{k,c}$ for $0 \leq k \leq l$ and $c \in \mathbb{Z}/p^j\mathbb{Z}$; put $E_{-1} := \{0\}$. One has $E_l \cdot E_{l'} \subseteq E_{l+l'}$. We will show that

$$(5) \qquad E_l = E_{l-1} + \sum_{m=0}^{p^j-1} \mathbb{F}\overline{Q}_{lp^j+m} \quad \text{for any } l \geq 0 .$$

For $0 \leq m < p^j$ one has

$$Q_{lp^j+m} = Q_{m,l} \cdot Q_{lp^j} \;\; \text{with} \;\; Q_{m,l} := \frac{(lp^j)!}{(lp^j+m)!} P_{m,l} \; .$$

Since $|\frac{(lp^j)!}{(lp^j+m)!}| = |m!|^{-1}$ we may use (1')-(3') and obtain, for $0 \leq b, m < p^{-j}$, that

$$||Q_{m,l}||_{b,p^{-j}} \leq 1, \;\; ||Q_{m,l}||_{m,p^{-j}} = 1, \text{ and } \;\; ||Q_{m,l}||_{b,p^{-j}} < 1 \;\text{ if }\; b < m \; .$$

Moreover, if $||Q_{m,l}||_{b,p^{-j}} = 1$ then, because of $b \geq m$, $Q_{m,l}$ has no zeros in $b + p^{-j}o_p$; in this situation it is an immediate consequence of Weierstrass preparation in the ring $\mathcal{O}_{b,p^{-j}}$ that $Q_{m,l} \in \mathcal{O}^0_{b,p^{-j}}$ is a constant modulo $\mathcal{O}^{00}_{b,p^{-j}}$. It follows that

$$(6) \qquad \overline{Q}_{m,l} = \sum_{m\leq c<p^j} \gamma_{m,c} \cdot \overline{\chi}_{0,c} \quad \text{with } \gamma_{m,c} \in \mathbb{F} \text{ and } \gamma_{m,m} \neq 0 \; .$$

In particular, for any $l \geq 0$, the $\overline{Q}_{m,l}$ with $0 \leq m < p^j$ span E_0. Our claim (5) follows from (6) by a straightforward induction argument with respect to l. This finishes the proof of the Proposition.

The Proposition says that $\prod_{b \in \mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}_{b,p^{-j}}$ is the Banach space of all series

$$\varphi = \sum_{n \geq 0} c_n \binom{T}{n} \quad \text{with} \;\; c_n \in K \;\; \text{and} \;\; \lim_{n \to \infty} |c_n / \left[\frac{n}{p^j}\right]!| = 0$$

and that

$$||\varphi||_{(j)} = \max_{n \geq 0} |c_n / [\frac{n}{p^j}]!|$$

(Schikhof, loc. cit., §50). Hence the dual Banach space $D_j := \prod_{b \in \mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}'_{b,p^{-j}}$ can be identified with the K-Banach space of all sequences

$$\underline{a} = (a_0, a_1, a_2, \ldots) \;\; \text{in} \;\; K \;\; \text{such that} \;\; \sup_{n \geq 0} |a_n \cdot \left[\frac{n}{p^j}\right]!| < \infty$$

with the norm

$$||\underline{a}|| := \sup_{n \geq 0} |a_n \cdot \left[\frac{n}{p^j}\right]!| < \infty \; .$$

The duality is given by the continuous pairing $(\varphi, \underline{a}) \longmapsto \sum_{n \geq 0} a_n c_n$. We put

$$r_j := |p|^{1/p^j(p-1)} \in |\overline{K}^\times| \; .$$

Consider any $n \geq 1$ and write $n = lp^j + m$ with $0 \leq m < p^j$. Then (using again 25.5 in Schikhof)

$$|\left[\frac{n}{p^j}\right]!|^{1/n} = |l!|^{1/(lp^j+m)} \geq |l!|^{1/lp^j} > |p|^{1/p^j(p-1)}$$

which means that $|[\frac{n}{p^j}]!|/r_j^n \longrightarrow \infty$ with $n \longrightarrow \infty$. It follows that

$$\begin{array}{ccc} D_j & \longrightarrow & \mathcal{O}(X_{r_j}) \\ \underline{a} & \longmapsto & \sum_{n \geq 0} a_n T^n \end{array}$$

is an injective continuous linear map. Passing to the projective limit with respect to j gives rise to an injective continuous linear map

$$\mathcal{F} : D(\mathbb{Z}_p, K) \longrightarrow \mathcal{O}(X) \; .$$

On the other hand one checks that

$$|\left[\frac{n}{p^j}\right]!| < r_{j+1}^n \ \text{ for } \ n \geq p^{j+2} \ .$$

Hence

$$\begin{array}{ccc} \mathcal{O}(X_{r_{j+1}}) & \longrightarrow & D_j \\ \sum\limits_{n\geq 0} a_n T^n & \longmapsto & (a_0, a_1, \ldots) \end{array}$$

also is an injective continuous linear map. By a cofinality argument $\mathcal{F}$ therefore must be bijective. Since both sides are Fréchet spaces the open mapping theorem finally implies that $\mathcal{F}$ is a topological isomorphism. Let $z \in K$ with $|z| < r_j$. We then have $\lim\limits_{n\to\infty} |z^n/|[\frac{n}{p^j}]!| = 0$ so that $\sum\limits_{n\geq 0} z^n \binom{T}{n}$ is a convergent expansion in the Banach space $\prod\limits_{b\in\mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}_{b,p^{-j}}$. This shows that κ_z is locally K-analytic, as was claimed earlier, and in fact lies in $\prod\limits_{b\in\mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}_{b,p^{-j}}$. Moreover, for any $\lambda \in D(\mathbb{Z}_p, K)$ and $\mathcal{F}(\lambda) = \sum\limits_{n\geq 0} a_n T^n$ we obtain

$$\lambda(\kappa_z) = \sum_{n\geq 0} a_n z^n = \mathcal{F}(\lambda)(z).$$

This establishes the identity $F(\lambda)(z) = F_\lambda(z)$ for any $z \in X(K)$ and completes the proof of Amice's theorem.

As to be expected the multiplication of functions in $\mathcal{O}(X)$ corresponds to a "convolution" of linear forms in $D(\mathbb{Z}_p, K)$.

Lecture 3: The principal series

We fix a spherically complete intermediate field $\mathbb{Q}_p \subseteq K \subseteq \mathbb{C}_p$. In this lecture we will construct (rather in the form of a survey) certain topologically irreducible representations of the compact group $G := GL_2(\mathbb{Z}_p)$ in locally convex K-vector spaces.

Distributions

In a completely analogous way as for the group $\mathbb{Z}_p$ one can construct the locally convex K-vector space

$$C^{an}(G, K) := \quad \text{all locally } K\text{-analytic functions on } G$$

as a countable inductive limit of Banach spaces. Its strong dual

$$D(G, K) := C^{an}(G, K)'_b$$

- the space of K-valued distributions on G - is a reflexive and nuclear K-Fréchet space. In fact, the usual convolution product of distributions makes $D(G, K)$ into a K-Fréchet algebra. This algebra should be considered as some kind of analytic version of the group algebra of G. And, indeed, sending a group element $g \in G$ to the Dirac distribution δ_g defines an injective ring homomorphism $K[G] \hookrightarrow D(G, K)$ whose image is dense in $D(G, K)$.

The Lie algebra $\mathfrak{g} := M_2(\mathbb{Q}_p)$ of G is another source for explicit elements in $D(G, K)$. Any $\mathfrak{x} \in \mathfrak{g}$ acts on $C^{an}(G, K)$ via

$$(\mathfrak{x}f)(g) := \frac{d}{dt} f(\exp(-t\mathfrak{x})g)_{|t=0}$$

where $\exp : \mathfrak{g} - - - > G$ denotes the exponential map defined locally around 0. This induces the embedding

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & D(G, K) \\ \mathfrak{x} & \longmapsto & [f \mapsto (-\mathfrak{z}(f))(1)] \end{array}$$

which by the universal property of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ extends to a homomorphism of K-algebras

$$U(\mathfrak{g}) \underset{\mathbb{Q}_p}{\otimes} K \longrightarrow D(G, K) \ .$$

It can be shown to be injective. Technically more important is that $D(G, K)$ even contains a certain kind of explicit completion of $U(\mathfrak{g})$. To make this precise we fix an ordered basis $\mathfrak{x}_1, \ldots, \mathfrak{x}_4$ of $\mathfrak{g}$ as a K-vector space. A power series in 4 variables over K is called entire if it converges on the whole affine space $\mathbb{C}_p^4$; this amounts to the requirement that the coefficients $b_{\underline{n}}$ of such a power series satisfy $\sup\limits_{\underline{n}} |b_{\underline{n}}| r^{-|\underline{n}|} < \infty$ for any $r > 0$.

Lemma:

Any entire power series in the $\mathfrak{x}_1, \ldots, \mathfrak{x}_4$ converges in $D(G, K)$.

The group G acts continuously on $C^{an}(G, K)$ via left translation giving rise to the (left) regular representation of G in the present context. If the philosophy about $D(G, K)$ being some kind of group algebra is correct then this G-action should extend to a $D(G, K)$-module structure. And indeed it does: For $\mu \in D(G, K)$ and $f \in C^{an}(G, K)$ define

$$(\mu * f)(h) := \int_G f(g^{-1}h) d\mu(g) \ .$$

The principal series

The principal series of G consists of certain subrepresentations of the regular representation which are defined as follows. Let $P \subseteq G$ denote the subgroup of lower triangular matrices and $T \subseteq G$ the subgroup of diagonal matrices. We fix a locally K-analytic character

$$\chi : T \to K^\times$$

and consider the space

$$\mathrm{Ind}_P^G(\chi) := \{ f \in C^{an}(G, K) : f(gp) = \chi(p^{-1}) f(g) \text{ for any } g \in G, p \in P \} \ .$$

This clearly is a closed G-invariant subspace of $C^{an}(G, K)$ and hence is a continuous G-representation as well as a $D(G, K)$-module in its own right. It will turn out to be much preferable to study the strong dual

$$M_\chi := \mathrm{Ind}_P^G(\chi)'_b$$

which (by functoriality) carries a continuous $D(G, K)$-module structure. Since all these spaces are reflexive this certainly will not mean any loss of information.

The most important numerical invariant of a principal series representation is the number $c(\chi) \in K$ defined by the expansion

$$\chi(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}) = \exp(c(\chi) \log(t))$$

for t sufficiently close to 1 in $\mathbb{Z}_p$.

Main Theorem:

If $c(\chi) \notin -\mathbb{N}_0$, then M_χ is an (algebraically) simple $D(G,K)$-module. In particular, $\mathrm{Ind}_P^G(\chi)$ is a topologically irreducible G-representation.

In a first step one reduces the assertion to a similar assertion for a certain subgroup of G. This is the so called Iwahori subgroup B consisting of all matrices which are lower triangular modulo p. We set $w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G$ and $P^- := B \cap wPw$. Using the Bruhat decomposition

$$G = BP \,\dot{\cup}\, BwP$$

we obtain by the Mackey formula the B-equivariant decomposition

$$\mathrm{Ind}_P^G(\chi) = \mathrm{Ind}_P^B(\chi) \oplus \mathrm{Ind}_{P^-}^B(w\chi) \ .$$

Defining

$$M_\chi^+ := \mathrm{Ind}_P^B(\chi) \quad \text{and} \quad M_\chi^- := \mathrm{Ind}_{P^-}^B(\chi)$$

this dualizes into a decomposition

$$M_\chi = M_\chi^+ \oplus M_{w\chi}^-$$

as $D(B,K)$-modules. Since the action of w is easily seen not to respect this decomposition the Main Theorem is a consequence of the following slightly stronger result.

Theorem:

i. If $c(\chi) \notin \pm\mathbb{N}_0$, then $M_\chi^\mp$ is an (algebraically) simple $D(B,K)$-module;

ii. M_χ^+ and $M_{\chi'}^-$ never are isomorphic as $D(B,K)$-modules.

The second assertion reduces via a Frobenius reciprocity argument to the fact that there are no p-adic valued Haar measures.

The two cases in the first assertion are completely analogous. So let us look at M_χ^-. The reason why it is easier to deal with the group B (and not G directly) is the fact that the map

$$\begin{aligned} \iota : \mathbb{Z}_p \quad &\longrightarrow \quad B \\ b \quad &\longmapsto \quad \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \end{aligned}$$

is a section of the projection map $B \to B/P^- = \mathbb{Z}_p$. Pushing forward distributions via ι therefore yields a linear isomorphism

$$\iota_* : D(\mathbb{Z}_p, K) \xrightarrow{\cong} M_\chi^- \ .$$

Under this isomorphism the $D(\mathbb{Z}_p, K)$-module structure on M_χ^- coming from the ring inclusion

$$\iota_* : D(\mathbb{Z}_p, K) \to D(B, K)$$

becomes simply the ring multiplication in $D(\mathbb{Z}_p, K)$. Thus M_χ^- is a copy of $D(\mathbb{Z}_p, K)$ with the additional structure corresponding to the action of $D(B, K)$. One then calculates this additional structure explicitly. The most important formula one gets is the following. Identifying $D(\mathbb{Z}_p, K)$ with $\mathcal{O}(X)$ under the Fourier transform the action of the Lie algebra element $\mathfrak{u}^+ := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g}$ satisfies

$$(\mathfrak{u}^+)^m F \equiv [\prod_{i=0}^{m-1} (c(\chi) - i)] \cdot \left((1+T)\frac{d}{dT}\right)^m F \pmod{\log(1+T)\mathcal{O}(X)}$$

for any $m \geq 0$ and $F \in \mathcal{O}(X)$. Under our assumption on $c(\chi)$ this congruence means we have control over the divisor of $(\mathfrak{u}^+)^m F$ in terms of the divisor of F. Now, each nonzero $D(B, K)$-submodule of M_χ^- corresponds to a (not necessarily closed) nonzero ideal in $\mathcal{O}(X)$ with extra invariance properties. In particular it is invariant under all the operators $(\mathfrak{u}^+)^m$ and hence under all operators which are given as an entire series in $\mathfrak{u}^+$. This information then is enough to conclude that such an ideal necessarily has to be the full ring.

No two of the simple $D(G, K)$-modules in the Main Theorem are isomorphic. This follows from the next result. In order to formulate it we introduce the character ϵ defined by $\epsilon(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}) := t_2/t_1$.

Theorem:

For any two locally K-analytic characters $\chi \neq \chi'$ of T the vector space $\mathrm{Hom}_{D(G,K)}(M_{\chi'}, M_\chi)$ of all $D(G, K)$-module homomorphisms is 1-dimensional if $c(\chi) \in -\mathbb{N}_0$ and $\chi' = \epsilon^{1-c(\chi)}\chi$ and is zero otherwise.

We close with the following remark. Let us consider an algebraic character $\chi(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}) = t_1^{m_1} t_2^{m_2}$ with $m_i \in \mathbb{Z}$ such that $c(\chi) = m_2 - m_1 \leq 0$. In this case we have the finite dimensional K-rational representation of GL_2 of highest weight χ (w.r.t. the Borel subgroup of upper triangular matrices). It can be described as an "algebraic" induction and as such is a G-invariant closed subspace of $\mathrm{Ind}_P^G(\chi)$.

(The details of this Lecture 3 can be found in the forthcoming joint paper with J. Teitelbaum on "Locally analytic distributions and p-adic representation theory, with applications to GL_2".)