Introduction to formal commutative algebra

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Introduction

These pages constitute the notes of a course I gave at Köln in the academic year 1991/92. They presume some basic knowledge of commutative algebra. Although it may not be apparent to the reader the material is selected with an eye towards the connections between formal commutative algebra and rigid analysis.

The following references are explicitly referred to in the text:

[Bou] Bourbaki, N.: Commutative Algebra. Paris: Hermann 1972.

[EGA] Dieudonné, J., Grothendieck, A.: Eléments de Géométrie Algébriques, vol. IV 1, Publ. Math. IHES 20 (1964)

[Mat] Matsumura, H.: Commutative ring theory. Cambridge Univ. Press 1994

List of contents

§1 Adic noetherian rings

Rings always are commutative and have a unit element, and homomorphisms of rings always respect the unit elements.

Now let A be a fixed ring and let $I \subseteq A$ be an ideal. A subset $U \subseteq A$ is called open if

$$\text{for any } a \in U \text{ there is a } n \in \mathbb{N} \text{ such that } a + I^n \subseteq U \quad .$$

In this way A is equipped with a topology with respect to which it is a topological ring. We call this topology the <u>I-adic topology</u> on A. The following properties are immediate.

– The ideals I^n for $n \geq 1$ form a fundamental system of open neighbourhoods of the zero element $0 \in A$.

– A is Hausdorff if and only if $\bigcap_{n \in \mathbb{N}} I^n = \{0\}$.

– Any open ideal in A also is closed.

– The J-adic topology on A for some other ideal $J \subseteq A$ coincides with the I-adic topology if and only if there are $m, n \in \mathbb{N}$ such that $I^m \subseteq J$ and $J^n \subseteq I$.

Definition:

An element $a \in A$ is called <u>topologically nilpotent</u> if its residue class in A/I^n is nilpotent for any $n \in \mathbb{N}$. Let $t_I(A)$ denote the subset of topologically nilpotent elements in A.

Lemma 1:

i. $t_I(A) = \{a \in A : a^m \in I$ for some $m \in \mathbb{N}\}$;
ii. $t_I(A)$ is an ideal;
iii. $t_I(A)$ is equal to the intersection of all open prime ideals in A;
iv. $t_I(A)$ is equal to the union of all ideals $J \subseteq A$ for which the J-adic topology coincides with the I-adic topology.

Proof: i. is obvious. ii. is a consequence of i. For iii. we note that any open prime ideal $\wp \subseteq A$ contains some I^n. Therefore $t_I(A)$ is contained in $\wp$. On the other hand let a be an element in the intersection of all those $\wp$. The residue class of a then lies in the intersection of all prime ideals in A/I and consequently is nilpotent. This shows that $a \in t_I(A)$. iv. It follows from i. that any such ideal J is contained in $t_I(A)$. Fix an $a \in t_I(A)$ and put $J := I + Aa$. If $a^m \in I$ then also $J^m \subseteq I$. This implies that the J-adic topology coincides with the I-adic topology.

Corollary 2:

Any open prime ideal in A contains I.

Corollary 3:

Let A be noetherian; then $t_I(A)$ is the largest ideal among all ideals $J \subseteq A$ for which the J-adic topology coincides with the I-adic topology.

Proof: By Lemma 1.iv any such J is contained in $t_I(A)$. Furthermore by Lemma 1.i the nilradical of A/I is $t_I(A)/I$. Since A/I is noetherian we have $t_I(A)^m \subseteq I$ for some $m \in \mathbb{N}$. Therefore $t_I(A)$ itself occurs among those J's.

Obviously we have a canonical homomorphism of rings

$$\varphi_A : A \longrightarrow \varprojlim_{n \in \mathbb{N}} A/I^n$$

which is injective if and only if A is Hausdorff.

Definition:

The ring A is called <u>(I-)adic</u> if φ_A is bijective.

Lemma 4:

If A is I-adic then we have:
i. I is contained in the Jacobson radical of A, i.e., any maximal ideal of A contains I and, in particular, is open;
ii. $1 + I \subseteq A^\times$;
iii. the natural map $A^\times \twoheadrightarrow (A/I)^\times$ is surjective.

Proof: ii. Given $a \in I$ the element $b := \sum_{m \geq 0} (-a)^m$ is well-defined modulo I^n with $(1 + a)b \equiv 1 \bmod I^n$ for any $n \in \mathbb{N}$. Because of the bijectivity of φ_A the same holds true in A. i. Assume that the maximal ideal $\mathfrak{m} \subseteq A$ does not contain I. We then have $I + \mathfrak{m} = A$ so that $(I + 1) \cap \mathfrak{m} \neq \emptyset$; this is a contradiction to ii. iii. If a is a unit $\bmod I$ then it follows from i. that a is not contained in any maximal ideal of A. This means that $a \in A^\times$.

Proposition 5:

Let A be I-adic; then A is noetherian if and only if A/I is noetherian and I/I^2 is finitely generated as an A/I-module.

The proof relies on the following technical fact which will be used also later on repeatedly.

Lemma 6:

Let $\alpha : G \to E$ be a homomorphism of filtered abelian groups $G = F^0 \supseteq F^1 G \supseteq \ldots$ and $E = F^0 E \supseteq F^1 E \supseteq \ldots$; we assume that
a. the associated graded homomorphism $\operatorname{gr}\alpha : \operatorname{gr} G \to \operatorname{gr} E$ is surjective,
b. the natural map $G \to \varprojlim_{n\in\mathbb{N}} G/F^n G$ is surjective, and
c. $\bigcap_{n\in\mathbb{N}} F^n E = \{0\}$;
then α is surjective.

Proof: Fix an element $e \in E$. We construct inductively a sequence $(g_n)_{n\geq 0}$ in G such that, for any $n \geq 0$,

- $g_{n+1} \equiv g_n \bmod F^n G$ and
- $\alpha(g_n) \equiv e \bmod F^n E$.

Put $g_0 := 0$ and assume that g_n already is constructed. Then $e - \alpha(g_n) \in F^n E$ so that by a. we find an element $h \in F^n G$ such that

$$\alpha(h) \equiv e - \alpha(g_n) \bmod F^{n+1} E \quad .$$

Define $g_{n+1} := g_n + h$.

Because of b. there is an element $g \in G$ such that $g \equiv g_n \bmod F^n G$ for any $n \geq 0$. We then have $\alpha(g) \equiv \alpha(g_n) \equiv e \bmod F^n E$ for any $n \geq 0$. The assumption c. now implies that $\alpha(g) = e$.

Coming to the proof of Proposition 5 we first note that the assumption that I/I^2 is a finitely generated A/I-module implies that $\operatorname{gr} A := \bigoplus_{n\geq 0} I^n/I^{n+1}$ is a finitely generated (graded) A/I-algebra. With A/I therefore also $\operatorname{gr} A$ is noetherian. We now fix an ideal $\mathfrak{a} \subseteq A$ and we have to show that $\mathfrak{a}$ is finitely generated. So far we know that the (homogeneous) ideal $\operatorname{gr}\mathfrak{a} := \bigoplus_{n\geq 0} \mathfrak{a} \cap I^n/\mathfrak{a} \cap I^{n+1} = \bigoplus_{n\geq 0} \mathfrak{a} \cap I^n + I^{n+1}/I^{n+1}$ in $\operatorname{gr} A$ is finitely generated. We choose homogeneous generators of $\operatorname{gr}\mathfrak{a}$, i.e., we fix elements $a_1 \in \mathfrak{a} \cap I^{n(1)}, \ldots, a_r \in \mathfrak{a} \cap I^{n(r)}$ with appropriate integers $n(1), \ldots, n(r) \geq 0$ such that $\operatorname{gr}\mathfrak{a}$ is generated by the cosets $a_1 + \mathfrak{a} \cap I^{n(1)+1}, \ldots, a_r + \mathfrak{a} \cap I^{n(r)+1}$. We claim that then $\mathfrak{a}$ is generated by $a_1, \ldots, a_r$ or equivalently that the homomorphism

$$\begin{array}{rccl} \alpha : & A^r & \longrightarrow & \mathfrak{a} \\ & (b_1, \ldots, b_r) & \longmapsto & \sum_{i=1}^r b_i a_i \end{array}$$

is surjective. Defining

$$F^n A^r := I^{n-n(1)} \times \ldots \times I^{n-n(r)}$$

(with the convention that $I^n := A$ if $n \le 0$) and

$$F^n \mathfrak{a} := \mathfrak{a} \cap I^n$$

the map α is a homomorphism of filtered A-modules. We clearly have

$$\bigcap_{n \in \mathbb{N}} \mathfrak{a} \cap I^n \subseteq \bigcap_{n \in \mathbb{N}} I^n = \{0\}$$

since A is Hausdorff as an adic ring. For the same reason that A is adic the natural map

$$A^r \xrightarrow{\cong} \varprojlim_{n \in \mathbb{N}} A^r / F^n A^r$$

is bijective. Our claim therefore follows from Lemma 6 once we show that the maps induced by α

$$F^n A^r / F^{n+1} A^r \longrightarrow \mathfrak{a} \cap I^n / \mathfrak{a} \cap I^{n+1} \qquad \text{for } n \ge 0$$

are surjective. Fixing an element $a \in \mathfrak{a} \cap I^n$ we find elements $\overline{b}_1, \ldots, \overline{b}_r \in \operatorname{gr} A$ such that

$$\sum_{i=1}^{r} \overline{b}_i \cdot (a_i + \mathfrak{a} \cap I^{n(i)+1}) = a + \mathfrak{a} \cap I^{n+1} \text{ in } \operatorname{gr} A \quad .$$

By passing to homogeneous components we may assume that $\overline{b}_i$ is homogeneous of degree $n - n(i)$, i.e., that $\overline{b}_i$ is the coset of an element $b_i \in I^{n-n(i)}$. We then have

$$\sum_{i=1}^{r} b_i a_i \equiv a \bmod \mathfrak{a} \cap I^{n+1} \quad . \qquad \square$$

In general

$$\hat{A} := \varprojlim_{n \in \mathbb{N}} A/I^n$$

equipped with the projective limit topology with respect to the discrete topology on the rings A/I^n is a topological ring and the homomorphism φ_A is continuous. $\hat{A}$ is called the <u>I-adic completion</u> of A. Let $\mathfrak{b} \subseteq A$ be an open ideal. Then $I^m \subseteq \mathfrak{b}$ for some $m \in \mathbb{N}$ and by construction

$$\hat{\mathfrak{b}} := \varprojlim_{n \ge m} \mathfrak{b}/I^n$$

is an open ideal in $\hat{A}$ (which is independent of the choice of m). In particular the ideals $(I^m)\hat{}\subseteq \hat{A}$ for $m \in \mathbb{N}$ form a fundamental system of open neighbourhoods of the zero element $0 \in \hat{A}$.

Lemma 7:

i. $\varphi_A(A)$ *is dense in* $\hat{A}$*; more generally we have* $\hat{\mathfrak{b}} = \overline{\varphi_A(\mathfrak{b})}$ *for any open ideal* $\mathfrak{b} \subseteq A$*;*
ii. the maps

$$\textit{open ideals in } A \overset{\longrightarrow}{\longleftarrow} \textit{open ideals in } \hat{A}$$

$$\mathfrak{b} \longmapsto \hat{\mathfrak{b}}$$

$$\varphi_A^{-1}(\mathfrak{c}) \longleftarrow\!\shortmid \mathfrak{c}$$

are inverse to each other;
iii. φ_A *induces, for any open ideal* $\mathfrak{b} \subseteq A$*, an isomorphism*

$$A/\mathfrak{b} \xrightarrow{\cong} \hat{A}/\hat{\mathfrak{b}}$$

Proof: Since $\hat{\mathfrak{b}}$ is open and therefore closed we clearly have $\overline{\varphi_A(\mathfrak{b})} \subseteq \hat{\mathfrak{b}}$. If $I^m \subseteq \mathfrak{b}$ then $\varphi_A(\mathfrak{b}) + (I^m)\hat{} = \hat{\mathfrak{b}}$. This implies that $\varphi_A(\mathfrak{b})$ is dense in $\hat{\mathfrak{b}}$ and proves i. The surjectivity of the map in iii. follows from the density of $\varphi_A(A)$ in $\hat{A}$. Its injectivity is a consequence of the obvious equality $\varphi_A^{-1}(\hat{\mathfrak{b}}) = \mathfrak{b}$. For ii. it remains to consider an open ideal $\mathfrak{c} \subseteq \hat{A}$. By the continuity of φ_A then $\mathfrak{b} := \varphi_A^{-1}(\mathfrak{c})$ is an open ideal in A. Since $\mathfrak{c}$ also is closed it follows from i. that $\hat{\mathfrak{b}} \subseteq \mathfrak{c}$. In the natural commutative diagram

$$\begin{array}{ccc} A/\mathfrak{b} & \xrightarrow{\cong} & \hat{A}/\hat{\mathfrak{b}} \\ & \searrow \quad \swarrow & \\ & A/\mathfrak{c} & \end{array}$$

the left, resp. the right, oblique arrow therefore is injective, resp. surjective. But according to iii. the horizontal arrow is bijective. This means $\hat{\mathfrak{b}} = \mathfrak{c}$.

We see that in case A is Hausdorff the topology on $\hat{A}$ induces via the injective map φ_A the I-adic topology on A.

Warning: In general the ideal $\hat{\mathfrak{b}}$ is strictly larger than the ideal $\varphi_A(\mathfrak{b}) \cdot \hat{A}$.

Proposition 8:

If A *is noetherian then we have:*
i. $\hat{A}$ *is* $\hat{I}$*-adic and noetherian;*
ii. $\hat{\mathfrak{b}} = \varphi_A(\mathfrak{b})\hat{A}$ *for any open ideal* $\mathfrak{b} \subseteq A$*.*

It is convenient to deduce this result from a more general statement about A-modules. Let M be an A-module. The submodules $I^n M$ for $n \in \mathbb{N}$ form a fundamental system of open neighbourhoods of the zero element $0 \in M$ in a unique topology which we call the <u>I-adic</u> topology on M. In this way M becomes a topological A-module. Any homomorphism of A-modules is continuous with respect to the I-adic topologies. The <u>I-adic completion</u>

$$\hat{M} := \varprojlim_{n \in \mathbb{N}} M/I^n M$$

in a natural way is an $\hat{A}$-module so that we have the functor

$$\begin{aligned} A\text{-modules} &\longrightarrow \hat{A}\text{-modules} \\ M &\longmapsto \hat{M} \quad . \end{aligned}$$

The module M is called <u>complete</u> if the obvious homomorphism

$$\varphi_M : M \longrightarrow \hat{M}$$

of A-modules is bijective. We equip $\hat{M}$ with the projective limit topology with respect to the discrete topology on the modules $M/I^n M$. It then is a topological $\hat{A}$-module and the homomorphism φ_M is continuous. For any open submodule $N \subseteq M$ we have $I^m M \subseteq N$ for some $m \in \mathbb{N}$ so that

$$\hat{N} := \varprojlim_{n \geq m} N/I^n M$$

is an open $\hat{A}$-submodule in $\hat{M}$ (which is independent of the choice of m). Note that this $\hat{N}$ also is the I-adic completion of N as an A-module in its own right: We have $I^n N \subseteq I^n M \subseteq I^{n-m} N$ for $n \geq m$. In particular the submodules $(I^m M)\hat{}$ for $m \in \mathbb{N}$ form a fundamental system of open neighbourhoods of the zero element $0 \in \hat{M}$.

Lemma 9:

i. M is Hausdorff if and only if φ_M is injective;
ii. $\varphi_M(M)$ is dense in $\hat{M}$; more generally we have $\hat{N} = \overline{\varphi_M(N)}$ for any open submodule $N \subseteq M$;
iii. the maps

$$\begin{aligned} \textit{open } A\textit{-submodules in } M &\rightleftarrows \textit{open } \hat{A}\textit{-submodules in } \hat{M} \\ N &\longmapsto \hat{N} \\ \varphi_M^{-1}(L) &\longleftarrow L \end{aligned}$$

are inverse to each other;
iv. φ_M *induces, for any open submodule* $N \subseteq M$*, an isomorphism*

$$M/N \xrightarrow{\cong} \hat{M}/\hat{N} \quad .$$

We omit the proof since it is entirely analogous to the proofs of the corresponding facts for ideals above.

Lemma 10:

If $\alpha : L \to M$ *is a surjective homomorphism of* A*-modules then the homomorphism* $\hat{\alpha} : \hat{L} \to \hat{M}$ *is surjective, too.*

Proof: We define filtrations $F^nL := I^nL$ and $F^n\hat{L} := (I^nL)\hat{}$ and similarly for M and $\hat{M}$. Then α and $\hat{\alpha}$ are filtered homomorphisms. More precisely we have $\alpha(I^nL) = I^nM$ for any $n \geq 0$ which means that $\mathrm{gr}\,\alpha$ is surjective. Since $\mathrm{gr}\,\hat{\alpha} = \mathrm{gr}\,\alpha$ by Lemma 9.iii we obtain that $\mathrm{gr}\,\hat{\alpha}$ is surjective. On the other hand it is trivial that $\bigcap_{n\in\mathbb{N}} (I^nM)\hat{} = \{0\}$ (i.e., that $\hat{M}$ is Hausdorff). The natural map

$$\hat{L} \longrightarrow \varprojlim_{n\in\mathbb{N}} \hat{L}/(I^nL)\hat{}$$

is equal to the map

$$\varprojlim_{n\in\mathbb{N}} L/I^nL \longrightarrow \varprojlim_{n\in\mathbb{N}} \hat{L}/(I^nL)\hat{}$$

induced by φ_M which is an isomorphism by Lemma 9.iii. Therefore the assertion is a consequence of Lemma 6.

Proposition 11:

If M *is a finitely generated* A*-module then we have:*
i. $(I^mM)\hat{} = (I^m)\hat{} \cdot \hat{M} = (I^m)\hat{} \cdot \varphi_M(M)$ *for any* $m \geq 0$*;*
ii. $\hat{M}$ *is finitely generated as an* $\hat{A}$*-module;*
iii. if A *is adic then* φ_M *is surjective;*
if in addition the ideal I *is finitely generated then we have:*
iv. $\hat{N} = \hat{A} \cdot \varphi_M(N)$ *for any open submodule* $N \subseteq M$*;*
v. the topology on $\hat{M}$ *is the* $\hat{I}$*-adic topology (and* $\hat{M}$ *is complete).*

Proof: i. For trivial reasons we have

$$(I^m)\hat{} \cdot \varphi_M(M) \subseteq (I^m)\hat{} \cdot \hat{M} \subseteq (I^mM)\hat{}$$

where the second inclusion is a consequence of the continuity of the multiplication by scalars in $\hat{M}$. Also the asserted equalities are obvious for a finitely generated free A-module. But by assumption we have a surjective homomorphism of A-modules

$$\alpha : A^r \longrightarrow\!\!\!\!\!\rightarrow M$$

which induces surjective A-module homomorphisms

$$\alpha : I^m A^r \longrightarrow\!\!\!\!\!\rightarrow I^m M$$

for any $m \geq 0$. By Lemma 10 the homomorphism

$$\hat{\alpha} : (I^m A^r)\hat{} \longrightarrow\!\!\!\!\!\rightarrow (I^m M)\hat{}$$

is surjective, too. We obtain

$$\begin{aligned}(I^m M)\hat{} &= \hat{\alpha}((I^m A^r)\hat{}) = \hat{\alpha}((I^m)\hat{} \cdot \varphi_{A^r}(A^r)) = (I^m)\hat{} \cdot \hat{\alpha}(\varphi_{A^r}(A^r)) \\ &= (I^m)\hat{} \cdot \varphi_M(\alpha(A^r)) = (I^m)\hat{} \cdot \varphi_M(M) \quad .\end{aligned}$$

The assertions ii. and iii. are immediate consequences; for iii., e.g., observe that

$$\hat{M} = \hat{A} \cdot \varphi_M(M) = \varphi_A(A) \cdot \varphi_M(M) = \varphi_M(M) \quad .$$

iv. We have $I^n M \subseteq N$ for some $n \in \mathbb{N}$. Applying i. to $m = 0$ and the finitely generated A-module $I^n M$ we obtain

$$(I^n M)\hat{} = \hat{A} \cdot \varphi_M(I^n M) \subseteq \hat{A} \cdot \varphi_M(N) \quad .$$

On the other hand Lemma 9.iii implies that

$$\hat{N} = (I^n M)\hat{} + \varphi_M(N) \quad .$$

v. Applying iv. to $N := I^m N$ we obtain

$$(I^m M)\hat{} = \hat{A} \cdot \varphi_M(I^m M) = \varphi_A(I)^m \cdot \hat{M} = (\hat{I})^m \cdot \hat{M} \quad .$$

The completeness of $\hat{M}$ was shown already in the proof of Lemma 10.

Proof of Proposition 8: Everything except the noetherian property of $\hat{A}$ is a special case of Prop. 11. But $\hat{I}/\hat{I}^2 = \hat{I}/(I^2)\hat{} = I/I^2$ is a finitely generated module for the noetherian ring $\hat{A}/\hat{I} = A/I$. It therefore follows from Prop. 5 that $\hat{A}$ is noetherian.

Corollary 12:

If A is noetherian and M is a finitely generated A-module then we have

$$(I^m M)\hat{} = I^m \cdot \hat{M} \quad \textit{for any } m \in \mathbb{N} \quad .$$

Proof: The argument was given already in the proof of Prop. 11.v.

As in the previous statement we will in the following usually omit the maps φ_A and φ_M in the formulas if the meaning is clear from the context.

Lemma of Artin-Rees:

Let A be noetherian and let N be a submodule of the finitely generated A-module M; then there is a $m_0 \in \mathbb{N}$ such that

$$I(I^m M \cap N) = I^{m+1} M \cap N \quad \textit{for any } m \geq m_0 \quad .$$

Proof: Obviously the left hand side is contained in the right hand side for any $m \geq 0$. In order to establish the reverse inclusion we consider the subring

$$B := \sum_{m \geq 0} I^m T^m \subseteq A[T]$$

of the ring of polynomials over A and the B-submodule

$$N' := \sum_{m \geq 0} (I^m M \cap N) \underset{A}{\otimes} AT^m \subseteq \sum_{m \geq 0} I^m M \underset{A}{\otimes} AT^m = M \underset{A}{\otimes} B$$

in $M \underset{A}{\otimes} B$. If $a_1, \ldots, a_s$ are generators of the ideal I then $a_1 T, \ldots, a_s T$ are generators of B as an A-algebra. Therefore B is noetherian and N' is a finitely generated B-module. Let

$$x_1 \otimes T^{m(1)}, \ldots, x_r \otimes T^{m(r)} \quad \text{with} \quad x_i \in I^{m(i)} M \cap N$$

be generators of N' as a B-module and put

$$m_0 := \max(m(1), \ldots, m(r)) \quad .$$

Fixing an integer $m \geq m_0$ and an element $y \in I^m M \cap N$ we have

$$y \otimes T^m = \sum_{i=1} t_i (x_i \otimes T^{m(i)})$$

with appropriate elements $t_i \in B$. By passing to homogeneous components we may assume that t_i is of the form

$$t_i = b_i T^{m-m(i)} \quad \text{with} \quad b_i \in I^{m-m(i)} \quad .$$

Then

$$y \otimes T^m = \left(\sum_{i=1}^{r} b_i x_i \right) \otimes T^m$$

and hence

$$y = \sum_{i=1}^{r} b_i x_i \quad .$$

This shows that

$$I^m M \cap N \subseteq \sum_{i=1}^{r} I^{m-m(i)} (I^{m(i)} M \cap N) \subseteq I^{m-m_0} (I^{m_0} M \cap N)$$

and proves the assertion in the equivalent form

$$I^m M \cap N = I^{m-m_0} (I^{m_0} M \cap N) \quad \text{for any} \quad m \geq m_0 \quad .$$

Proposition 13: (Krull)

Let A be noetherian and let N be a submodule of the finitely generated A-module M; then the topology on N induced by the I-adic topology of M is equal to the I-adic topology of N.

Proof: With m_0 being the constant in the Lemma of Artin-Rees we have

$$I^m M \cap N = I^{m-m_0} (I^{m_0} M \cap N) \subseteq I^{m-m_0} N$$

for $m \geq m_0$.

Corollary 14: (Krull)

Let A be noetherian; for any ideal $\mathfrak{a} \subseteq A$ and any finitely generated A-module M we have:
i. $\bigcap_{n \in \mathbb{N}} \mathfrak{a}^n M = \{x \in M : (1-a)x = 0$ *for some* $a \in \mathfrak{a}\}$;
ii. if $1 + \mathfrak{a}$ *contains no zero divisors then* $\bigcap_{n \in \mathbb{N}} \mathfrak{a}^n = \{0\}$.

Proof: i. If we have $x = ax$ for some $a \in \mathfrak{a}$ then clearly $x = a^n x$ lies in $\mathfrak{a}^n M$ for any $n \in \mathbb{N}$. On the other hand let x be contained in the intersection $\bigcap_{n \in \mathbb{N}} \mathfrak{a}^n M$. The $\mathfrak{a}$-adic topology on M then induces the trivial topology on the submodule Ax. By the above Prop. 13 the $\mathfrak{a}$-adic topology on Ax therefore is equal to the trivial topology. Since $\mathfrak{a}x$ is open in Ax with respect to the $\mathfrak{a}$-adic topology we obtain $\mathfrak{a}x = Ax$. This means that we find an element $a \in \mathfrak{a}$ such that $ax = x$.
ii. This is an obvious consequence of the first assertion.

Proposition 15:

Let A be noetherian and let M be a finitely generated A-module equipped with the I-adic topology; assume that I is contained in the Jacobson radical of A; then we have:
i. M is Hausdorff;
ii. every submodule of M is closed;
iii. A is Hausdorff and every ideal in A is closed.

Proof: Our assumption implies that $1 + I$ consists of units in A. Therefore M is Hausdorff as a consequence of Cor. 14.i. Moreover it is clear that, for any submodule $N \subseteq M$, the I-adic topology on M/N is the quotient topology of the I-adic topology on M. Hence M/N is Hausdorff in the quotient topology which means that N is closed in M.

Corollary 16:

Let A be I-adic and noetherian; then every finitely generated A-module M is complete and any submodule in M is closed (w.r.t. the I-adic topology).

Proof: According to Lemma 4.i the ideal I is contained in the Jacobson radical of A so that Prop. 15 applies: The canonical homomorphism φ_M is injective and every submodule of M is closed. The surjectivity of φ_M was shown already in Prop. 11.iii.

If A is noetherian the completion functor $M \mapsto \hat{M}$ (always w.r.t. the I-adic topologies) has very nice properties. Let

$$\alpha_M : \hat{A} \otimes_A M \longrightarrow \hat{M}$$
$$\hat{a} \otimes x \longmapsto \hat{a}x$$

be the obvious homomorphism of $\hat{A}$-modules.

Proposition 17:

If A is noetherian then we have:
i. The functor $M \mapsto \hat{M}$ is exact for finitely generated A-modules M;
ii. if M is a finitely generated A-module then α_M is bijective;
iii. $\hat{A}$ is a flat A-algebra.

Proof: i. Let

$$0 \longrightarrow L \xrightarrow{\beta} M \xrightarrow{\gamma} N \longrightarrow 0$$

be a short exact sequence of finitely generated A-modules. It was shown in Lemma 10 that $\hat{\gamma}$ is surjective. On the other hand we have

$$\ker(L/I^n L \longrightarrow M/I^n M) = I^n M \cap L/I^n L \quad .$$

The Lemma of Artin-Rees implies that, for n large, the natural map

$$\begin{array}{ccc} I^n(I^n M \cap L)/I^{2n} L & = & I^{2n} M \cap L/I^{2n} L \\ & & \downarrow \\ & & I^n M \cap L/I^n L \end{array}$$

is the zero map. Hence $\hat{\beta}$ is injective. By functoriality we have $\operatorname{im} \hat{\beta} \subseteq \ker \hat{\gamma}$. Fix now an element $\hat{x} = (x_n + I^n M)_{n \in \mathbb{N}} \in \ker \hat{\gamma}$ so that $\gamma(x_n) \in I^n N$ for any $n \in \mathbb{N}$. Because of $\gamma(I^n M) = I^n N$ we may assume that $\gamma(x_n) = 0$ for any $n \in \mathbb{N}$. Let $y_n \in L$ be the unique element such that $\beta(y_n) = x_n$. Then

$$(y_n + I^n M \cap L)_{n \in \mathbb{N}} \in \varprojlim_{n \in \mathbb{N}} L/I^n M \cap L \quad .$$

But according to the Lemma of Artin-Rees (compare the proof of Prop. 13) there is a $m_0 \in \mathbb{N}$ such that

$$\hat{y} := (y_{n+m_0} + I^n L)_{n \in \mathbb{N}} \in \hat{L} \quad .$$

Obviously $\hat{\beta}(\hat{y}) = \hat{x}$.
ii. The assertion is trivial for $M = A$ and therefore for any finitely generated free A-module. For general M we then consider a presentation

$$A^r \longrightarrow A^s \longrightarrow M \longrightarrow 0 \quad .$$

It gives rise to the commutative exact diagram

$$\begin{array}{ccccccc} \hat{A} \otimes_A A^r & \longrightarrow & \hat{A} \otimes_A A^s & \longrightarrow & \hat{A} \otimes_A M & \longrightarrow & 0 \\ \cong \downarrow \alpha_{A^r} & & \cong \downarrow \alpha_{A^s} & & \downarrow \alpha_M & & \\ (A^r)\hat{} & \longrightarrow & (A^s)\hat{} & \longrightarrow & \hat{M} & \longrightarrow & 0 \end{array}$$

where the exactness of the lower row is a consequence of the previous assertion. With α_{A^s} also α_M is bijective.
iii. Since any A-module is the filtered direct limit of its finitely generated submodules and since the tensor product commutes with filtered direct limits the exactness of $\hat{A} \underset{A}{\otimes} .$ follows from the first two assertions.

Proposition 18:

Let A be noetherian and assume that I is contained in the Jacobson radical of A; then we have:
i. $\hat{A}$ is a faithfully flat A-algebra;
ii. if M is a finitely generated A-module then $\hat{M}$ is free over $\hat{A}$ if and only if M is free over A.

Proof: i. Since $\hat{A}$ is flat over A according to the previous Prop. 17 it remains to show that the natural map

$$M \xrightarrow{1 \otimes \mathrm{id}} \hat{A} \underset{A}{\otimes} M$$

is injective for any A-module M. By a direct limit argument we again may assume that M is finitely generated. We have the commutative diagram

$$\begin{array}{ccc} & & \hat{A} \underset{A}{\otimes} M \\ & \overset{1\otimes\mathrm{id}}{\nearrow} & \\ M & & \cong \Big\downarrow \alpha_M \\ & \underset{\varphi_M}{\searrow} & \\ & & \hat{M} \end{array}$$

The map α_M being an isomorphism by Prop. 17 we are reduced to prove that φ_M is injective, i.e., that M is Hausdorff. This was established in Prop. 15.
ii. By assumption $\hat{M}/\hat{I}\hat{M}$ is free over $\hat{A}/\hat{I}$. But using Lemma 9 and Prop. 11 we obtain

$$M/IM = \hat{M}/(IM)\hat{} = \hat{M}/\hat{I}\hat{M} \quad .$$

Therefore M/IM is free over A/I. Let $x_1 + IM, \ldots, x_r + IM$ be an A/I-basis of M/IM and consider the homomorphism

$$\begin{array}{rccc} \alpha: & L := A^r & \longrightarrow & M \\ & (a_1, \ldots, a_r) & \longmapsto & \sum\limits_{i=1}^{r} a_i x_i \quad . \end{array}$$

The Nakayama lemma immediately implies that α is surjective. For the injectivity we look at the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & I \underset{A}{\otimes} \ker\alpha & \longrightarrow & I \underset{A}{\otimes} L & \longrightarrow & I \underset{A}{\otimes} M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\alpha & \longrightarrow & L & \xrightarrow{\alpha} & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & A/I \underset{A}{\otimes} L & \xrightarrow{\cong} & A/I \underset{A}{\otimes} M & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}$$

We claim that the map $I \underset{A}{\otimes} M \to M$ is injective. Then the map $I \underset{A}{\otimes} \ker\alpha \to \ker\alpha$ is surjective so that $\ker\alpha = I \cdot \ker\alpha$. Applying again the Nakayama lemma we obtain that $\ker\alpha = 0$, i.e., that α is injective. Since $\hat{A}$ is faithfully flat over A the claimed injectivity can be checked after tensoring by $\hat{A}$. But in the commutative diagram

$$\begin{array}{ccc}
\hat{A} \underset{A}{\otimes} (I \underset{A}{\otimes} M) & \longrightarrow & \hat{A} \underset{A}{\otimes} M \\
\cong \downarrow \alpha_I \otimes \alpha_M & & \cong \downarrow \alpha_M \\
\hat{I} \underset{\hat{A}}{\otimes} \hat{M} & \longrightarrow & \hat{M}
\end{array}$$

the perpendicular maps are isomorphisms by Prop. 17 and the lower horizontal map is injective since $\hat{M}$ is free over $\hat{A}$. Therefore the upper horizontal map is injective, too.

§2 Adic algebras

Throughout this section we fix an I-adic and noetherian ring A. Let B be an A-algebra. Because of $I^n B = (IB)^n$ the I-adic topology of B as an A-module coincides with the IB-adic topology on the ring B. Therefore B equipped with the I-adic topology is a topological ring such that the ring homomorphism $A \to B$ which defines the algebra structure is continuous.

Definition:

The A-algebra B is called adic if B as an A-module is complete.

The following properties of this notion are immediate.

– If B is an adic A-algebra then B is IB-adic as a ring.

– If B is noetherian then its completion $\hat{B}$ (with respect to the IB-adic topology) is a noetherian adic A-algebra (Prop. 1.8).

– If B is finitely generated as an A-module then B is an adic A-algebra (Cor. 1.16).

– If B is an adic A-algebra then B is noetherian if and only if B/IB is noetherian (Prop. 1.5).

Applying the second property to the ring of polynomials $A[T_1, \ldots, T_r]$ we see that its completion

$$A\{T_1, \ldots, T_r\} := A[T_1, \ldots, T_r]\hat{}$$

with respect to the I-adic topology is a noetherian adic A-algebra such that

$$A\{T_1, \ldots, T_r\}/I^n \cdot A\{T_1, \ldots, T_r\} \cong (A/I^n)[T_1, \ldots, T_r] \quad .$$

Moreover, since the ring of polynomials is flat over A we obtain from Prop. 1.17 that $A\{T_1, \ldots, T_r\}$ is flat over A. There is the following explicit description in terms of power series.

Definition:

A formal power series

$$\sum_{m_1, \ldots, m_r \geq 0} a_{m_1, \ldots, m_r} T_1^{m_1} \cdot \ldots \cdot T_r^{m_r} \in A[[T_1, \ldots, T_r]]$$

is called <u>*restricted*</u> *if for any* $n \in \mathbb{N}$ *there is an* $m \geq 0$ *such that* $a_{m_1, \ldots, m_r} \in I^n$ *whenever* $\max(m_1, \ldots, m_r) \geq m$.

It is easily seen that

$$A[[T_1, \ldots, T_r]]^{res} := \text{ set of all restricted formal power series}$$

is a subring of $A[[T_1, \ldots, T_r]]$. Reducing the coefficients of a power series modulo I^n defines a homomorphism of rings

$$A[[T_1, \ldots, T_r]]^{res} \longrightarrow (A/I^n)[T_1, \ldots, T_r] \quad .$$

In the projective limit with respect to n these homomorphisms give rise to a natural isomorphism of rings

$$A[[T_1, \ldots, T_r]]^{res} \xrightarrow{\cong} \varprojlim_{n \in \mathbb{N}} (A/I^n)[T_1, \ldots, T_r] \cong A\{T_1, \ldots, T_r\}$$

which we always will view as an identification. For this reason $A\{T_1,\ldots,T_r\}$ is called the ring of restricted formal power series in $T_1,\ldots,T_r$ over A. We have

$$I^n \cdot A\{T_1,\ldots,T_r\} = \text{set of all restricted formal power series whose coefficients are contained in } I^n.$$

Proposition 1: (Universal property)

Let C be any adic ring and let $\varphi : A \to C$ be a continuous homomorphism of rings; for any choice of elements $c_1,\ldots,c_r \in C$ there is a unique continuous homomorphism of rings $\tilde{\varphi} : A\{T_1,\ldots,T_r\} \to C$ such that

$$\tilde{\varphi} \mid A = \varphi \quad \textit{and} \quad \tilde{\varphi}(T_i) = c_i \quad \textit{for} \quad 1 \le i \le r \quad .$$

Proof: Certainly there is a unique homomorphism of rings $\varphi' : A[T_1,\ldots,T_r] \to C$ such that $\varphi' \mid A = \varphi$ and $\varphi'(T_i) = c_i$. We claim that φ' is continuous with respect to the I-adic topology on $A[T_1,\ldots,T_r]$. If the topology on C is the J-adic one for some ideal $J \subseteq C$ then the continuity of φ implies that $\varphi(I^m) \subseteq J$ for some $m \in \mathbb{N}$. It follows that $\varphi'(I^{mn}A[T_1,\ldots,T_r]) \subseteq J^n$ for any $n \in \mathbb{N}$. Therefore φ' induces a continuous homomorphism of rings

$$\begin{array}{ccc} A\{T_1,\ldots,T_r\} & = & \varprojlim_{n\in\mathbb{N}} A[T_1,\ldots,T_r]/I^{mn}A[T_1,\ldots,T_r] \\ & & \big\downarrow \tilde{\varphi} \\ C & = & \varprojlim_{n\in\mathbb{N}} C/J^n \end{array}$$

such that $\tilde{\varphi} \mid A[T_1,\ldots,T_r] = \varphi'$. Since C is Hausdorff the unicity of $\tilde{\varphi}$ follows from the fact that $A[T_1,\ldots,T_r]$ is dense in $A\{T_1,\ldots,T_r\}$.

In the situation of the above Proposition we usually will write

$$F(c_1,\ldots,c_r) := \tilde{\varphi}(F)$$

for any restricted formal power series $F \in A\{T_1,\ldots,T_r\}$. One useful application of this universal property is the following. First note that for $s \ge r$ we have

$$(A\{T_1,\ldots,T_r\})\{T_{r+1},\ldots,T_s\} = A\{T_1,\ldots,T_s\} \quad .$$

Therefore any $F \in A\{T_1,\ldots,T_s\}$ can be written as

$$F = \sum_{m_{r+1},\ldots,m_s \ge 0} G_{m_{r+1},\ldots,m_s}(T_1,\ldots,T_r) \cdot T_{r+1}^{m_{r+1}} \cdot \ldots \cdot T_s^{m_s}$$

with $G_{m_{r+1},\dots,m_s} \in A\{T_1,\dots,T_r\}$. Applying the universal property to the natural homomorphism $A \to A\{T_{r+1},\dots,T_s\}$ and the s elements $a_1,\dots,a_r \in A$ and $T_{r+1},\dots,T_s$ we obtain the identity

$$F(a_1,\dots,a_r,T_{r+1},\dots,T_s) = \sum_{m_{r+1},\dots,m_s \geq 0} G_{m_{r+1},\dots,m_s}(a_1,\dots,a_r) \cdot T_{r+1}^{m_{r+1}} \cdot \ldots \cdot T_s^{m_s}$$

in $A\{T_{r+1},\dots,T_s\}$.

Of basic importance is the following characterization of the residue class algebras of restricted formal power series rings.

Proposition 2:

For any A-algebra B the following assertions are equivalent:
i. B is a (noetherian) adic A-algebra such that B/IB is an A/I-algebra of finite type;
ii. B (with the I-adic topology) is as an A-algebra (topologically) isomorphic to a residue class algebra of $A\{T_1,\dots,T_r\}$ for some $r \geq 0$.

Proof: First let us assume that ii. holds true. Then B is of the form

$$B \cong A\{T_1,\dots,T_r\}/\mathfrak{a}$$

for some ideal $\mathfrak{a} \subseteq A\{T_1,\dots,T_r\}$. Obviously B is noetherian. By Cor. 1.16 the ideal $\mathfrak{a}$ is closed so that B is Hausdorff. Consider now the commutative diagram

$$\begin{array}{ccc} & & B \\ & \overset{\varphi}{\nearrow} & \big\downarrow \varphi_B \\ A\{T_1,\dots,T_r\} & & \\ & \underset{\hat{\varphi}}{\searrow} & \hat{B} \end{array}$$

where φ denotes the obvious projection map and where the completions are formed with respect to the I-adic topologies. We have seen already that φ_B is injective. But by Lemma 1.10 with φ also $\hat{\varphi}$ is surjective. Therefore φ_B has to be bijective, i.e., B is an adic A-algebra. Moreover φ induces a surjection

$$A/I[T_1,\dots,T_r] \cong A\{T_1,\dots,T_r\}/IA\{T_1,\dots,T_r\} \twoheadrightarrow B/IB$$

which shows that B/IB is an A/I-algebra of finite type. This proves i.
Now assume that i. holds true. We then find elements $b_1,\dots,b_r \in B$ such that the homomorphism

$$A/I[T_1,\dots,T_r] \longrightarrow B/IB$$
$$T_i \longmapsto b_i + IB$$

is surjective. By the universal property in Prop. 1 this homomorphism lifts to a continuous homomorphism of adic A-algebras

$$\varphi : A\{T_1, \ldots, T_r\} \longrightarrow B \text{ such that } \varphi(T_i) = b_i \text{ for } 1 \leq i \leq r \ .$$

We claim that φ is surjective. For that it suffices by Lemma 1.6 to show that the induced maps

$$gr^n\varphi : I^n A\{T_1, \ldots, T_r\}/I^{n+1}A\{T_1, \ldots, T_r\} \longrightarrow I^n B/I^{n+1}B$$

are surjective. This is the case for $n = 0$ by construction and then follows for any $n \geq 0$ since $gr\, B$ is generated by $gr^0 B = B/IB$ as a $gr\, A$-module. The surjectivity implies that

$$\varphi(I^n A\{T_1, \ldots, T_r\}) = I^n B \ \text{ for any } n \geq 0$$

which means that φ is open. Therefore φ induces the topological isomorphism of A-algebras

$$A\{T_1, \ldots, T_r\}/\ker\varphi \xrightarrow{\cong} B$$

required in ii.

Definition:

The A-algebra B is called <u>topologically of finite type</u> if B is a noetherian adic A-algebra such that B/IB is an A/I-algebra of finite type. Let $\mathrm{Alg}_{tft}(A)$ denote the category of all A-algebras topologically of finite type.

Note that any A-algebra homomorphism between two algebras in $\mathrm{Alg}_{tft}(A)$ is continuous. Also it is clear that if $A \to B$ and $B \to C$ are topologically of finite type then $A \to C$ is topologically of finite type, too.

Proposition 3:

Let B be a noetherian adic A-algebra; then B is flat over A if and only if $B/I^n B$ is flat over A/I^n for all $n \in \mathbb{N}$.

Proof: The other implication being obvious we assume that $B/I^n B$ is flat over A/I^n for all $n \in \mathbb{N}$. We have to show that for any injective homomorphism $\alpha : N \hookrightarrow M$ of A-modules the homomorphism

$$\alpha \otimes \mathrm{id} : N \underset{A}{\otimes} B \longrightarrow M \underset{A}{\otimes} B$$

is injective, too. A direct limit argument reduces us to the case where M and N are finitely generated A-modules. In a first step we now claim that

$$\ker(\alpha \otimes \mathrm{id}) \subseteq I^n(N \underset{A}{\otimes} B)$$

for any $n \in \mathbb{N}$. By the Lemma of Artin-Rees we find a $m_0 \in \mathbb{N}$ such that

$$I^{n+m_0} M \cap N \subseteq I^n N \quad \text{for any } n \in \mathbb{N} \quad .$$

Our assumption implies that the map

$$\begin{array}{ccc} (N/I^{n+m_0} M \cap N) \underset{A}{\otimes} B = (N/I^{n+m_0} M \cap N) & \underset{A/I^{n+m_0}}{\otimes} & (B/I^{n+m_0} B) \\ & \downarrow & \\ (M/I^{n+m_0} M) \underset{A}{\otimes} B \quad = \quad (M/I^{n+m_0} M) & \underset{A/I^{n+m_0}}{\otimes} & (B/I^{n+m_0} B) \end{array}$$

is injective. Considering the commutative diagram

$$\begin{array}{ccc} N \underset{A}{\otimes} B & \longrightarrow & (N/I^{n+m_0} M \cap N) \underset{A}{\otimes} B \\ \downarrow{\scriptstyle \alpha \otimes \mathrm{id}} & & \downarrow \\ M \underset{A}{\otimes} B & \longrightarrow & (M/I^{n+m_0} M) \underset{A}{\otimes} B \end{array}$$

we then see that

$$\begin{aligned} \ker(\alpha \otimes \mathrm{id}) &\subseteq \ker(N \underset{A}{\otimes} B \longrightarrow (N/I^{n+m_0} M \cap N) \underset{A}{\otimes} B) \\ &\subseteq \ker(N \underset{A}{\otimes} B \longrightarrow (N/I^n N) \underset{A}{\otimes} B) \\ &= I^n (N \underset{A}{\otimes} B) \end{aligned}$$

holds true as we have claimed. But by Cor. 1.16 the finitely generated B-module $N \underset{A}{\otimes} B$ is Hausdorff so that $\bigcap_{n \in \mathbb{N}} I^n (N \underset{A}{\otimes} B) = 0$. Therefore $\alpha \otimes \mathrm{id}$ is injective.

§3 Complete tensor products

We fix a continuous homomorphism $\varphi : A \to C$ of noetherian adic rings. Let A, resp. C, be I-adic, resp. J-adic. Then there is a $m_0 \in \mathbb{N}$ such that $\varphi(I^{m_0}) \subseteq J$. Let now B be an A-algebra topologically of finite type. Equipping the tensor product $B \underset{A}{\otimes} C$ considered as a C-algebra with the J-adic topology the natural homomorphisms

$$\begin{array}{ccc} B & \longrightarrow & B \underset{A}{\otimes} C \\ & & \uparrow \\ & & C \end{array}$$

are continuous. The completion

$$B \underset{A}{\hat{\otimes}} C := (B \underset{A}{\otimes} C)^{\wedge}$$

is a topological ring called the complete tensor product. Composing with $\varphi_{B \underset{A}{\otimes} C}$ gives rise to the commutative diagram of continuous homomorphisms of rings

$$\begin{array}{ccc} B & \longrightarrow & B \underset{A}{\hat{\otimes}} C \\ \uparrow & & \uparrow \\ A & \xrightarrow{\varphi} & C \end{array} \quad .$$

Because of

$$B \underset{A}{\otimes} C / J^n (B \underset{A}{\otimes} C) = B \underset{A}{\otimes} (C/J^n) = (B/I^{m_0 n} B) \underset{A}{\otimes} (C/J^n)$$

we have

$$B \underset{A}{\hat{\otimes}} C = \varprojlim_{n \in \mathbb{N}} (B/I^{m_0 n} B) \underset{A}{\otimes} (C/J^n) \quad .$$

Lemma 1:

$A\{T_1, \ldots, T_r\} \underset{A}{\hat{\otimes}} C = C\{T_1, \ldots, T_r\}$; moreover, for any $m \in \mathbb{N}$, we have

$$(J^m (A\{T_1, \ldots, T_r\} \underset{A}{\otimes} C))\hat{} = J^m C\{T_1, \ldots, T_r\} \quad .$$

Proof: Using that $A\{T_1, \ldots, T_r\}$ is flat over A we have, for any $m \geq 0$, that

$$\begin{aligned} &(J^m (A\{T_1, \ldots, T_r\} \underset{A}{\otimes} C))\hat{} \\ &\quad = \varprojlim_{n \geq m} J^m (A\{T_1, \ldots, T_r\} \underset{A}{\otimes} C) / J^n (A\{T_1, \ldots, T_r\} \underset{A}{\otimes} C) \\ &\quad = \varprojlim_{n \geq m} A\{T_1, \ldots, T_r\} \underset{A}{\otimes} (J^m/J^n) \\ &\quad = \varprojlim_{n \geq m} (A\{T_1, \ldots, T_r\} / I^{m_0 n} A\{T_1, \ldots, T_r\}) \underset{A}{\otimes} (J^m/J^n) \\ &\quad = \varprojlim_{n \geq m} A/I^{m_0 n}[T_1, \ldots, T_r] \underset{A}{\otimes} (J^m/J^n) \\ &\quad = \varprojlim_{n \geq m} J^m C[T_1, \ldots, T_r] / J^n C[T_1, \ldots, T_r] \\ &\quad = J^m C\{T_1, \ldots, T_r\} \quad . \end{aligned}$$

Proposition 2:

$B \hat{\otimes}_A C$ is a C-algebra topologically of finite type.

Proof: Let $\psi : A\{T_1, \ldots, T_r\} \twoheadrightarrow B$ be a surjective homomorphism of A-algebras. According to Lemma 1.10

$$(\psi \otimes \mathrm{id})\hat{} : A\{T_1, \ldots, T_r\} \hat{\otimes}_A C \twoheadrightarrow B \hat{\otimes}_A C$$

then is surjective, too. The same argument more precisely shows that

$$(\psi \otimes \mathrm{id})\hat{}((J^m(A\{T_1, \ldots, T_r\} \otimes_A C))\hat{}) = (J^m(B \otimes_A C))\hat{} \text{ for any } m \geq 0 .$$

Using Lemma 1 we obtain

$$(J^m(B \otimes_A C))\hat{} = (\psi \otimes \mathrm{id})\hat{}(J^m C\{T_1, \ldots, T_r\}) = J^m(B \hat{\otimes}_A C)$$

which means that the topology on $B \hat{\otimes}_A C$ is the J-adic one. Hence $B \hat{\otimes}_A C$ is topologically isomorphic to a residue class algebra of $C\{T_1, \ldots, T_r\}$.

Corollary 3:

With B and B' also $B \hat{\otimes}_A B'$ is an A-algebra topologically of finite type.

Proposition 4: (Universal property)

Let D be any adic ring and let

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \xrightarrow{\varphi} & C \end{array}$$

be any commutative diagram of continuous homomorphisms of rings; then there exists a unique continuous homomorphism of rings $\psi : B \hat{\otimes}_A C \to D$ such that the diagram

$$\begin{array}{ccccc} B & \longrightarrow & B \hat{\otimes}_A C & \longleftarrow & C \\ & \searrow & \downarrow{\scriptstyle \psi} & \swarrow & \\ & & D & & \end{array}$$

is commutative.

Proof: Since the proof is entirely analogous to the proof of Prop. 2.1 we only indicate the argument. The universal property of the usual tensor product gives rise to a homomorphism $B \underset{A}{\otimes} C \to D$ which is easily seen to be continuous with respect to the J-adic topology on $B \underset{A}{\otimes} C$. The wanted homomorphism ψ then is obtained by completion.

The complete tensor product of course is functorial. In particular

$$\begin{aligned} \mathrm{Alg}_{tft}(A) &\longrightarrow \mathrm{Alg}_{tft}(C) \\ B &\longmapsto B \underset{A}{\hat{\otimes}} C \end{aligned}$$

is a functor. It follows from the formula before Lemma 1 that the complete tensor product is commutative and associative and that we have natural topological isomorphisms

$$B' \underset{B}{\hat{\otimes}} (B \underset{A}{\hat{\otimes}} C) = B' \underset{A}{\hat{\otimes}} C$$

for any B' in $\mathrm{Alg}_{tft}(B)$ and

$$(B \underset{A}{\hat{\otimes}} C) \underset{C}{\hat{\otimes}} C' = B \underset{A}{\hat{\otimes}} C'$$

for any continuous homomorphism $C \to C'$ of noetherian adic rings.

Lemma 5:

If B is finitely generated as an A-module then $B \underset{A}{\hat{\otimes}} C = B \underset{A}{\otimes} C$.

Proof: $B \underset{A}{\otimes} C$ is a finitely generated C-module and therefore is complete for the J-adic topology by Cor. 1.16.

Proposition 6:

If B is flat over A then $B \underset{A}{\hat{\otimes}} C$ is flat over C.

Proof: By assumption $B/I^{m_0 n}B$ is flat over $A/I^{m_0 n}$ for any $n \in \mathbb{N}$. By tensoring with C/J^n we obtain that $B \underset{A}{\hat{\otimes}} C/J^n(B \underset{A}{\hat{\otimes}} C)$ is flat over C/J^n for all $n \in \mathbb{N}$. Now apply Prop. 2 and Prop. 2.3.

§4 Complete localization

Let A be a noetherian I-adic ring and let $S \subseteq A$ be a multiplicative subset. The localization $S^{-1}A$ again is noetherian. Therefore its completion

$$A\{S^{-1}\} := (S^{-1}A)\hat{}$$

with respect to the I-adic topology is a noetherian adic A-algebra called the complete localization of A in S. Because of

$$I^n \cdot S^{-1}A = S^{-1}I^n$$

we have

$$A\{S^{-1}\} = \varprojlim_{n \in \mathbb{N}} S^{-1}(A/I^n) \quad .$$

Proposition 1: (Universal property)

Let C be any adic ring and let $\varphi : A \to C$ be a continuous homomorphism of rings such that $\varphi(S) \subseteq C^\times$; then there exists a unique continuous homomorphism of rings $\tilde{\varphi} : A\{S^{-1}\} \to C$ such that $\tilde{\varphi} \mid A = \varphi$.

Corollary 2:

Let $\varphi : A \to C$ be a continuous homomorphism of noetherian adic rings and let $T \subseteq C$ be a multiplicative subset such that $\varphi(S) \subseteq T$; then there exists a unique continuous homomorphism of rings $\tilde{\varphi} : A\{S^{-1}\} \to C\{T^{-1}\}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & C \\ \downarrow & & \downarrow \\ A\{S^{-1}\} & \xrightarrow{\tilde{\varphi}} & C\{T^{-1}\} \end{array}$$

is commutative.

Remark 3:

$A\{S^{-1}\} = 0$ if and only if the zero element 0 is contained in the closure $\overline{S}$ of S in A.

Proof: If $0 \in \overline{S}$ then $I^n \cap S \neq \emptyset$ for any $n \in \mathbb{N}$ which implies $S^{-1}(A/I^n) = 0$ for any n and therefore $A\{S^{-1}\} = 0$. Now assume that $A\{S^{-1}\} = 0$. Then $\{0\}$ is a dense subset in $S^{-1}A$ which means $I^n \cdot S^{-1}A = S^{-1}A$ for any $n \in \mathbb{N}$. We therefore have $1 \in I^n \cdot S^{-1}A$ and hence $I^n \cap S \neq \emptyset$ for any $n \in \mathbb{N}$. The latter implies $0 \in \overline{S}$.

Lemma 4:

$A\{S^{-1}\}$ is flat over A.

Proof: Since $S^{-1}A$ is flat over A this follows from Prop. 1.17.

Lemma 5:

If S is finitely generated then $A\{S^{-1}\}$ is an A-algebra topologically of finite type.

Proof: We have

$$A\{S^{-1}\}/I\cdot A\{S^{-1}\} = S^{-1}A/I\cdot S^{-1}A = S^{-1}(A/I) \quad .$$

This is an A/I-algebra of finite type if S is finitely generated.

For any element $b \in A$ put $S_b := \{1, b, b^2, \ldots\}$ and

$$A_{\{b\}} := A\{S_b^{-1}\} \ .$$

This is an A-algebra topologically of finite type.

Proposition 6:

The maps

$$\begin{array}{ccc} \textit{set of all open prime} & \longrightarrow & \textit{set of all open prime} \\ \textit{ideals in } A \textit{ disjoint from } S & \longleftarrow & \textit{ideals in } A\{S^{-1}\} \\ & & \\ \mathfrak{p} & \longmapsto & \mathfrak{p}\cdot A\{S^{-1}\} \\ \textit{preimage of } \mathfrak{q} \textit{ in } A & \longleftarrow\!\shortmid & \mathfrak{q} \end{array}$$

are inverse to each other; moreover if $\mathfrak{p} \subseteq A$ is an open prime ideal such that $\mathfrak{p}\cap S = \emptyset$ then we have the isomorphism

$$\mathrm{Quot}(A/\mathfrak{p}) \xrightarrow{\cong} \mathrm{Quot}(A\{S^{-1}\}/\mathfrak{p}A\{S^{-1}\})$$

between the corresponding quotient fields.

Proof: Since the analogous maps for $S^{-1}A$ are inverse to each other the first assertion follows from Lemma 1.7 and Prop. 1.8.ii. It also follows that

$$A\{S^{-1}\}/\mathfrak{p}A\{S^{-1}\} = S^{-1}A/\mathfrak{p}\cdot S^{-1}A = S^{-1}(A/\mathfrak{p})$$

which implies the second assertion.

§5 Complete blowing up

First let A be an arbitrary ring. The localization $A_b := S_b^{-1}A$ of A in an element $b \in A$ solves the universal problem of making the element b invertible. It is natural to pose the more general universal problem of making the element

b generate a given ideal in A. Let $J \subseteq A$ be an ideal which contains b. In the localization A_b we have the subring

$$A_{J,b} := \{u \in A_b : u = \frac{a}{b^m} \text{ for some } m \geq 0 \text{ and some } a \in J^m\} \quad .$$

It is called a generalized localization or <u>blowing up</u> of A. The natural homomorphism

$$\begin{aligned} A &\longrightarrow A_{J,b} \subseteq A_b \\ a &\longmapsto \ \frac{a}{1} \end{aligned}$$

into the localization factorizes through $A_{J,b}$. Moreover:

– $J \cdot A_{J,b} = \frac{b}{1} \cdot A_{J,b}$, and

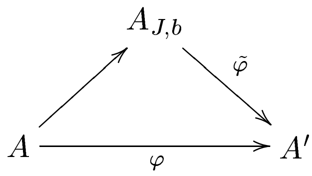
– $\frac{b}{1}$ is not a zero divisor in $A_{J,b}$ (since it is invertible in A_b).

Lemma 1: (Universal property)

Let $\varphi : A \to A'$ be any homomorphism of rings such that

$$- \varphi(J)A' = \varphi(b)A' \quad , \quad \textit{and}$$
$$- \varphi(b) \quad \textit{is not a zero divisor in } A' \quad ;$$

then there is a unique homomorphism of rings $\tilde{\varphi} : A_{J,b} \to A'$ such that the diagram



is commutative.

Proof: Define $\tilde{\varphi}\left(\frac{a}{b^m}\right) := a'$ if $\varphi(a) = \varphi(b^m) \cdot a'$.

Lemma 2:

If A is noetherian then $A_{J,b}$ is an A-algebra of finite type and, in particular, is noetherian.

Proof: If $a_1, \ldots, a_r$ generate the ideal J then $\frac{a_1}{b}, \ldots, \frac{a_r}{b}$ generate $A_{J,b}$ as an A-algebra.

Let $\varphi : A \to A'$ be a homomorphism of rings and put $J' := \varphi(J)A'$ and $b' := \varphi(b) \in J'$. Then φ induces the homomorphism of rings

$$\begin{aligned} A_{J,b} &\longrightarrow A'_{J',b'} \\ \frac{a}{b^m} &\longmapsto \frac{\varphi(a)}{b'^m} \end{aligned}$$

which we usually again denote by φ.

Lemma 3:

If φ is flat then

$$A_{J,b} \underset{A}{\otimes} A' \xrightarrow{\cong} A'_{J',b'}$$

$$u \otimes a' \longmapsto \varphi(u) \cdot \frac{a'}{1}$$

is an isomorphism.

Proof: The surjectivity is obvious. For the injectivity consider the commutative diagram

$$\begin{array}{ccc} A_b \underset{A}{\otimes} A' & \xrightarrow{\cong} & A'_{b'} \\ \uparrow & & \uparrow \subseteq \\ A_{J,b} \underset{A}{\otimes} A' & \longrightarrow & A'_{J',b'} \end{array} \quad .$$

The upper horizontal arrow is an isomorphism by one of the basic properties of localization. Our assumption that φ is flat guarantees that the left perpendicular arrow is injective.

If we apply this Lemma to the natural homomorphism $A \to A_c$ for some $c \in J$ we obtain that

$$(A_{J,b})_{\frac{c}{1}} = A_{bc} \quad .$$

Now we assume again that A is a noetherian I-adic ring. By Lemma 2 the completion

$$A_{\{J,b\}} := (A_{J.b})\hat{}$$

with respect to the I-adic topology is an A-algebra topologically of finite type called a <u>complete blowing up</u> of A. The natural homomorphisms

$$A \longrightarrow A_{J,b} \xrightarrow{\subseteq} A_b$$

by completion induce (continuous) homomorphisms of rings

$$A \longrightarrow A_{\{J,b\}} \longrightarrow A_{\{b\}}$$

but the second one in general is no longer injective.

Remark 4:

We have $J \cdot A_{\{J,b\}} = \frac{b}{1} \cdot A_{\{J,b\}}$, and $\frac{b}{1}$ is not a zero divisor in $A_{\{J,b\}}$.

Proof: The first assertion is clear. Concerning the second one we observe that $\frac{b}{1}$ being not a zero divisor in $A_{J,b}$ means that the map

$$\frac{b}{1}\cdot : A_{J,b} \longrightarrow A_{J,b}$$

is injective. By Prop. 1.17.i the map

$$\varphi_{A_{J,b}}\left(\frac{b}{1}\right)\cdot : A_{\{J,b\}} \longrightarrow A_{\{J,b\}}$$

then is injective, too. This amounts to $\frac{b}{1}$ (or more precisely $\varphi_{A_{J,b}}\left(\frac{b}{1}\right)$) not being a zero divisor in $A_{\{J,b\}}$.

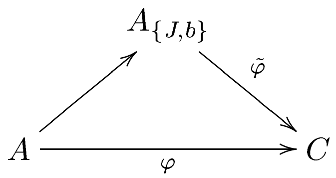
Proposition 5: (Universal property)

Let C be any adic ring and let $\varphi : A \to C$ be a continuous homomorphism of rings such that

– $\varphi(J)C = \varphi(b)C$, *and*

– $\varphi(b)$ *is not a zero divisor in* C ;

then there exists a unique continuous homomorphism of rings $\tilde{\varphi} : A_{\{J,b\}} \to C$ such that the diagram



is commutative.

Lemma 6:

Let $\varphi : A \to C$ be a flat continuous homomorphism of noetherian adic rings and put $J' := \varphi(J)C$ and $b' := \varphi(b)$; we then have

$$A_{\{J,b\}} \hat{\underset{A}{\otimes}} C = C_{\{J',b'\}} \quad .$$

Proof: Let C be $\tilde{I}$-adic and let $m_0 \in \mathbb{N}$ be such that $\varphi(I^{m_0}) \subseteq \tilde{I}$. The isomorphism

$$A_{J,b} \underset{A}{\otimes} C = C_{J',b'}$$

in Lemma 3 induces by completion an isomorphism

$$\begin{aligned} C_{\{J',b'\}} &= \lim_{\substack{\longleftarrow \\ n\in\mathbb{N}}} A_{J,b} \underset{A}{\otimes} C/\tilde{I}^n(A_{J,b} \underset{A}{\otimes} C) \\ &= \lim_{\substack{\longleftarrow \\ n\in\mathbb{N}}} A_{J,b} \underset{A}{\otimes} (C/\tilde{I}^n) \\ &= \lim_{\substack{\longleftarrow \\ n\in\mathbb{N}}} (A_{J,b}/I^{m_0 n} A_{J,b}) \underset{A}{\otimes} (C/\tilde{I}^n) \\ &= A_{\{J,b\}} \underset{A}{\hat{\otimes}} C \quad . \end{aligned}$$

As an application we obtain

$$(A_{\{J,b\}})_{\{\frac{c}{1}\}} = A_{\{bc\}} \quad \text{for any } c \in J \quad .$$

§6 Weierstraß theory

Let A be a noetherian I-adic ring and let $\mathfrak{a} \subseteq A$ be an ideal which is contained in the ideal $t_I(A)$ of topologically nilpotent elements (e.g., $\mathfrak{a} = I$). We view $A/\mathfrak{a}$ as an adic A-algebra; in particular $\mathfrak{a}$ is closed. Given $F \in A\{T\}$ we write $F \bmod \mathfrak{a}$ for its canonical image in $A/\mathfrak{a}\{T\}$.
Additional results about the structure of restricted formal power series rings can be obtained by using Hensel's lemma. Since the proof of the latter given in [Bou] III §4.3 is completely elementary we only give the statement without repeating the proof.

Hensel's lemma:

Let $F \in A\{T\}$ and $\overline{G} \in A/\mathfrak{a}\{T\}$ be restricted formal power series and let $\overline{P} \in A/\mathfrak{a}[T]$ be a monic polynomial such that

- $F \bmod \mathfrak{a} = \overline{P} \cdot \overline{G}$, *and*
- $\overline{P}$ *and* $\overline{G}$ *generate* $A/\mathfrak{a}\{T\}$ *as an ideal* ;

then there exists a uniquely determined restricted formal power series $G \in A\{T\}$ and a uniquely determined monic polynomial $P \in A[T]$ such that

- $F = P \cdot G$, *and*
- $P \bmod \mathfrak{a} = \overline{P}$, $G \bmod \mathfrak{a} = \overline{G}$;

moreover we have:
i. P and G generate $A\{T\}$ as in ideal;
ii. if F is a polynomial so, too, is G.

In order to make the proof of the Weierstraß theorems more transparent it is useful to establish some simple technical facts first.

Remark 1:

i. Let $P = Q \cdot G$ *be an identity in* $A\{T\}$*; if* P *and* Q *are monic polynomials so, too, is* G *and we have* $\deg G = \deg P - \deg Q$*;*
ii. an element $a \in A$ *is a unit if and only if* $a + \mathfrak{a}$ *is a unit in* $A/\mathfrak{a}$*;*
iii. an element $F \in A\{T\}$ *is a unit if and only if* $F \bmod \mathfrak{a}$ *is a unit in* $A/\mathfrak{a}\{T\}$*.*

Proof: i. This assertion clearly holds true in $A\{T\}/I^n A\{T\} = A/I^n[T]$ for any $n \in \mathbb{N}$. Since A is Hausdorff it then also holds true in $A\{T\}$. ii. This is proved in the same way as Lemma 1.4 using, of course, the assumption that $\mathfrak{a} \subseteq t_I(A)$. iii. The kernel of the projection map $A\{T\} \to A/\mathfrak{a}\{T\}$ is the ideal $\mathfrak{a}\{T\}$ of all restricted formal power series whose coefficients are contained in $\mathfrak{a}$. By ii. (applied to $A\{T\}$ and $\mathfrak{a}\{T\}$) it suffices to show that any power series in $\mathfrak{a}\{T\}$ is topologically nilpotent. Let $F(T) = \sum_{m \geq 0} a_m T^m$ be in $\mathfrak{a}\{T\}$ and fix an $n \in \mathbb{N}$. Since F is a restricted formal power series there is some $m_0 \geq 0$ such that $a_m \in I^n$ for $m \geq m_0$. Since any a_m is topologically nilpotent we find some $n_0 \in \mathbb{N}$ such that $a_m^{n_0} \in I^n$ for all $0 \leq m < m_0$. We then have $F^{m_0(n_0-1)+1} \in I^n \cdot A\{T\}$.

Weierstraß preparation theorem:

Let $F \in A\{T\}$ *be a restricted formal power series such that* $F \bmod \mathfrak{a}$ *is a unitary polynomial of degree* $d \geq 0$*; then there exist a uniquely determined monic polynomial* $P \in A[T]$ *and a uniquely determined unit* $U \in A\{T\}^\times$ *such that*

$$F = P \cdot U \quad ;$$

moreover P *has degree* d*.*

Proof: (Recall that a polynomial is called unitary if its highest coefficient is a unit.) Because of Remark 1.ii we may multiply F by a unit in A in such a way that $F \bmod \mathfrak{a}$ becomes a monic polynomial. We therefore are reduced to prove the assertion under the additional assumption that $F \bmod \mathfrak{a}$ is monic. Applying Hensel's lemma to $\overline{P} := F \bmod \mathfrak{a}$ and $\overline{G} := 1$ we obtain a uniquely determined monic polynomial $P \in A[T]$ and a uniquely determined $U \in A\{T\}$ such that

$$F = P \cdot U$$

and
a. $P \bmod \mathfrak{a} = F \bmod \mathfrak{a}$, and
b. $U \bmod \mathfrak{a} = 1$.
It immediately follows from a. that P has degree d. On the other hand U is

a unit in $A\{T\}$ by b. and Remark 1.iii. In order to establish the unicity let (P', U') be another pair with the properties in the assertion. We then have the identity

$$F \bmod \mathfrak{a} = P' \bmod \mathfrak{a} \cdot U' \bmod \mathfrak{a}$$

in $A/\mathfrak{a}\{T\}$ where $F \bmod \mathfrak{a}$ and $P' \bmod \mathfrak{a}$ are monic polynomials. The Remark 1.i then says that $U' \bmod \mathfrak{a}$ is a monic polynomial, too. On the other hand $U' \bmod \mathfrak{a}$ is a unit in $A/\mathfrak{a}\{T\}$. Therefore the same Remark 1.i applied to the identity $1 = U' \bmod \mathfrak{a} \cdot U'^{-1} \bmod \mathfrak{a}$ implies that $U' \bmod \mathfrak{a} = 1$ and hence that $P' \bmod \mathfrak{a} = F \bmod \mathfrak{a}$. Now the unicity statement in Hensel's lemma says that $P = P'$ and $U = U'$.

Corollary 2:

Let $F \in A\{T_1, \ldots, T_r\}$ be a restricted formal power series such that after reducing its coefficients $\bmod\ \mathfrak{a}$ *it becomes a polynomial in $A/\mathfrak{a}\{T_1, \ldots, T_{r-1}\}[T_r]$ unitary of degree $d \geq 0$ in T_r; then there exist a uniquely determined polynomial $P \in A\{T_1, \ldots, T_{r-1}\}[T_r]$ monic in T_r and a uniquely determined unit $U \in A\{T_1, \ldots, T_r\}^\times$ such that*

$$F = P \cdot U \quad ;$$

moreover P has degree d in T_r.

Proof: In the proof of Remark 1.iii we have seen that the ideal $\mathfrak{a}\{T_1, \ldots, T_{r-1}\}$ of all restricted formal power series in $T_1, \ldots, T_{r-1}$ whose coefficients are contained in $\mathfrak{a}$ consists of topologically nilpotent elements in the noetherian adic ring $A\{T_1, \ldots, T_{r-1}\}$. We therefore may apply the preparation theorem to $A\{T_1, \ldots, T_{r-1}\}$ and the ideal $\mathfrak{a}\{T_1, \ldots, T_{r-1}\}$.

Weierstraß division theorem:

Let $G \in A\{T\}$ be a restricted formal power series such that $G \bmod \mathfrak{a}$ is a unitary polynomial of degree $d \geq 0$; for any $F \in A\{T\}$ there exist a uniquely determined $H \in A\{T\}$ and a uniquely determined polynomial $R \in A[T]$ of degree $< d$ such that

$$F = G \cdot H + R \quad .$$

Proof: In a first step we assume that G is a monic polynomial of degree $d \geq 0$. By the usual division algorithm we then find, for each $n \in \mathbb{N}$, uniquely determined polynomials $H_n, R_n \in A/I^n[T]$ such that

$$F \bmod I^n = (G \bmod I^n) \cdot H_n + R_n \quad \text{and} \quad \deg R_n < d \quad .$$

It follows from the unicity that

$$H_{n+1} \bmod I^n = H_n \quad \text{and} \quad R_{n+1} \bmod I^n = R_n \quad .$$

Therefore

$$H := (H_n)_{n\in\mathbb{N}} \quad \text{and} \quad R := (R_n)_{n\in\mathbb{N}}$$

in $A\{T\} = \varprojlim_{n\in\mathbb{N}} A/I^n[T]$ have the required properties. For general G we first apply the preparation theorem which allows us to write G as

$$G = P \cdot U$$

with $U \in A\{T\}^\times$ and a monic polynomial $P \in A[T]$ of degree d. According to our first step we have

$$F = P \cdot H' + R$$

with $H' \in A\{T\}$ and $R \in A[T]$ of degree $< d$. We obtain

$$F = G \cdot H + R \quad \text{with} \quad H := U^{-1} \cdot H' \quad .$$

The unicity of H and R follows from the unicity statements in the first step and in the preparation theorem.

Corollary 3:

Let $G \in A\{T_1, \ldots, T_r\}$ be a restricted formal power series such that after reducing its coefficients $\bmod\, \mathfrak{a}$ *it becomes a polynomial in $A/\mathfrak{a}\{T_1, \ldots, T_{r-1}\}[T]$ unitary of degree $d \geq 0$ in T_r; for any $F \in A\{T_1, \ldots, T_r\}$ there exist a uniquely determined $H \in A\{T_1, \ldots, T_r\}$ and a uniquely determined polynomial $R \in A\{T_1, \ldots, T_{r-1}\}[T_r]$ of degree $< d$ in T_r such that*

$$F = G \cdot H + R \quad .$$

Corollary 4:

Let $P = Q \cdot H$ be an identity in $A\{T\}$ where P and Q are polynomials; if Q is unitary then H is a polynomial.

Proof: By the division algorithm in $A[T]$ we find polynomials H' and R in $A[T]$ such that

$$P = Q \cdot H' + R \quad \text{and} \quad \deg R < \deg Q \quad .$$

The division theorem says that this decomposition even is unique in $A\{T\}$. We therefore must have $H = H'$ and $R = 0$.

In order to make use of these results we have to understand to which power series they actually can be applied. From now on we restrict to the case $\mathfrak{a} = I$.

Definition:

A restricted formal power series

$$F(T_1, \ldots, T_r) = \sum_{m \geq 0} G_m(T_1, \ldots, T_{r-1}) T_r^m \in A\{T_1, \ldots, T_r\}$$

is called $\underline{T_r\text{-special}}$*, resp.* $\underline{T_r\text{-distinguished}}$*, of degree* $d \geq 0$ *if*

$$- \, G_m \in I \cdot A\{T_1, \ldots, T_r\} \text{ for any } m > d \quad , \quad \text{and}$$
$$- \, G_d \bmod I \in (A/I) \backslash \{0\} \quad , \quad \text{resp. } \in (A/I)^\times \quad .$$

Lemma 5:

For any $F \in A\{T_1, \ldots, T_r\} \backslash IA\{T_1, \ldots, T_r\}$ *there is an* A*-algebra automorphism* φ *of* $A\{T_1, \ldots, T_r\}$ *such that* $\varphi(F)$ *is* T_r*-special.*

Proof: First of all we observe that for any natural numbers $\ell_1, \ldots, \ell_{r-1}$ we have the A-algebra automorphism

$$\begin{aligned} A\{T_1, \ldots, T_r\} &\xrightarrow{\sim} A\{T_1, \ldots, T_r\} \\ T_i &\longmapsto \begin{cases} T_i + T_r^{\ell_i} & \text{if } i \neq r \quad , \\ T_r & \text{if } i = r \quad . \end{cases} \end{aligned}$$

It is well-defined by Prop. 2.1 and it is bijective sine $T_i \mapsto T_i - T_r^{\ell_i}$ for $1 \leq i < r$ and $T_r \mapsto T_r$ defines an inverse. Write now

$$F = \sum_{\underline{m} \geq 0} a_{\underline{m}} T_1^{m_1} \cdot \ldots \cdot T_r^{m_r} \quad .$$

Let $\underline{n} = (n_1, \ldots, n_r)$ be the maximal (w.r.t. the lexicographic ordering) tuple such that $a_{\underline{n}} \notin I$ and put

$$t := \text{ total degree of } F \bmod I \quad .$$

We recursively define natural numbers

$$\begin{aligned} \ell_{r-1} &:= 1 + t \quad , \\ \ell_{r-2} &:= 1 + t \cdot (1 + \ell_{r-1}) \quad , \\ &\vdots \\ \ell_1 &:= 1 + t \cdot (1 + \ell_{r-1} + \ldots + \ell_2) \end{aligned}$$

and put

$$d := \ell_1 n_1 + \ldots + \ell_{r-1} n_{r-1} + n_r \quad .$$

The reason for these definitions is the following property. The function

$$\begin{aligned} M := \{\underline{m} : m_1 + \ldots + m_r \leq t\} &\to \mathbb{N} \cup \{0\} \\ \underline{m} &\mapsto d(\underline{m}) := \ell_1 m_1 + \ldots + \ell_{r-1} m_{r-1} + m_r \end{aligned}$$

is strictly increasing with respect to the lexicographical ordering on the left hand side. In order to see this let $\underline{m}, \underline{m}' \in M$ be tuples such that $\underline{m}'$ is strictly larger than $\underline{m}$. We then have

$$m_1 = m'_1, \ldots, m_{i-1} = m'_{i-1} \quad , \quad \text{and} \quad m_i < m'_i \quad \text{for some} \quad 1 \leq i \leq r$$

and we compute

$$\begin{aligned} d(\underline{m}) &\leq \ell_1 m'_1 + \ldots + \ell_{i-1} m'_{i-1} + \ell_i (m'_i - 1) + t \cdot (\ell_{i+1} + \ldots + \ell_{r-1} + 1) \\ &= \ell_1 m'_1 + \ldots + \ell_{i-1} m'_{i-1} + \ell_i m'_i - 1 \\ &< d(\underline{m}') \quad . \end{aligned}$$

By construction the set $M' := \{\underline{m} : a_{\underline{m}} \notin I\}$ is contained in M so that the function $d(.)$ is strictly increasing on M' with maximum value $d(\underline{n}) = d$. Consider now the A-algebra automorphism φ of $A\{T_1, \ldots, T_r\}$ given by

$$\varphi(T_i) := T_i + T_r^{\ell_i} \quad \text{for} \quad 1 \leq i < r \quad \text{and} \quad \varphi(T_r) := T_r \quad .$$

We claim that $\varphi(F)$ is T_r-special of degree d. We have

$$\varphi(F) = \sum_{\underline{m} \geq 0} a_{\underline{m}} (T_1 + T_r^{\ell_r})^{m_1} \cdot \ldots \cdot (T_{r-1} + T_r^{\ell_{r-1}})^{m_{r-1}} \cdot T_r^{m_r} =$$

$$\sum_{\underline{m} \geq 0} a_{\underline{m}} \sum_{\substack{\mu_1, \ldots, \mu_{r-1} \\ 0 \leq \mu_i \leq m_i}} \binom{m_1}{\mu_1} \ldots \binom{m_{r-1}}{\mu_{r-1}} T_1^{m_1 - \mu_1} \ldots T_{r-1}^{m_{r-1} - \mu_{r-1}} T_r^{d(\mu_1, \ldots, \mu_{r-1}, m_r)}.$$

Since $d(\mu_1, \ldots, \mu_{r-1}, m_r) \leq d(\underline{m}) \leq d$ for $\underline{m} \in M'$ it follows that $\varphi(F) \bmod I$ is a polynomial of degree $\leq d$ in T_r. It remains to consider those $\underline{m} \in M'$ such that $d(\mu_1, \ldots, \mu_{r-1}, m_r) = d$ occurs. Since d is the maximum value of $d(.)$ on M' it follows that $\mu_i = m_i = n_i$ for $1 \leq i < r$ and that $m_r = n_r$. We conclude that

$$(\text{coefficient of } T_r^d \text{ in } \varphi(F)) \bmod I = a_{\underline{n}} \bmod I \neq 0 \quad .$$

Let B be a complete discrete valuation ring. We will discuss this notion in more detail in the next section. Here we work with the following definition: B is a noetherian local integral domain such that

— the maximal ideal of B is a nonzero principal ideal πB, and

— B is πB-adic.

Note that we have $\bigcap_{n\in\mathbb{N}} \pi^n B = \{0\}$ by Cor. 1.14.

Lemma 6:

For any nonzero $F \in B\{T_1,\dots,T_r\}$ *there is a* B*-algebra automorphism* ψ *of* $B\{T_1,\dots,T_r\}$ *such that*

$$F = \pi^m \cdot \psi(P) \cdot U$$

holds true for some $m \geq 0$*,* $U \in B\{T_1,\dots,T_r\}^\times$*, and some polynomial* $P \in B\{T_1,\dots,T_{r-1}\}[T_r]$ *monic in* T_r*.*

Proof: First we find a $m \geq 0$ such that

$$\pi^{-m}F \in B\{T_1,\dots,T_r\} \backslash \pi B\{T_1,\dots,T_r\} \quad .$$

Applying Lemma 5 we then find a B-algebra automorphism φ of $B\{T_1,\dots,T_r\}$ such that $\varphi(\pi^{-m}F)$ is T_r-distinguished. Finally Cor. 2 provides the polynomial P and a unit U' such that

$$\varphi(\pi^{-m}F) = P \cdot U' \quad .$$

Set $\psi := \varphi^{-1}$ and $U := \psi(U')$.

Proposition 7:

Let C *be an adic* B*-algebra such that* C *is torsion free as a* B*-module and let* $\xi : B\{T_1,\dots,T_r\} \to C$ *be a finite homomorphism of* B*-algebras; then there exist a* B*-algebra automorphism* ψ *of* $B\{T_1,\dots,T_r\}$ *and an integer* $0 \leq d \leq r$ *such that the composite homomorphism*

$$B\{T_1,\dots,T_d\} \xrightarrow{\subseteq} B\{T_1,\dots,T_r\} \xrightarrow{\psi} B\{T_1,\dots,T_r\} \xrightarrow{\xi} C$$

is finite and injective.

Proof: We will prove the assertion by induction with respect to r. By the torsion freeness of C the homomorphism $B \to C$ is injective which settles the case $r = 0$. Let us assume therefore that $r \geq 1$ and that $F \neq 0$ is in the kernel of ξ. Applying

Lemma 6 we find a B-algebra automorphism ψ of $B\{T_1, \ldots, T_r\}$, an $m \geq 0$, and a polynomial $P \in B\{T_1, \ldots, T_{r-1}\}[T]$ monic in T_r such that

$$\pi^m \cdot P \in \ker(\xi \circ \psi) \quad .$$

Since C is torsion free it follows that even

$$P \in \ker(\xi \circ \psi) \quad .$$

On the other hand Cor. 3 implies that the natural homomorphism

$$B\{T_1, \ldots, T_{r-1}\} \longrightarrow B\{T_1, \ldots, T_r\}/P \cdot B\{T_1, \ldots, T_r\}$$

is finite. Therefore the composite homomorphism

$$B\{T_1, \ldots, T_{r-1}\} \longrightarrow B\{T_1, \ldots, T_r\}/P \cdot B\{T_1, \ldots, T_r\} \xrightarrow{\xi \cdot \psi} C$$

is finite, too. Apply now the induction hypothesis.

Normalization lemma:

Let C be a B-algebra topologically of finite type which is torsion free as a B-module; then there exists a finite and injective homomorphism

$$B\{T_1, \ldots, T_d\} \hookrightarrow C$$

of B-algebras for some $d \geq 0$.

Proof: Apply Prop. 7 to some surjective homomorphism $B\{T_1, \ldots, T_r\} \longrightarrow\!\!\!\!\rightarrow C$ of B-algebras.

§7 Discrete valuation rings

Let A be a noetherian I-adic integral domain with quotient field K.

Definition:

A discrete valuation v of A is a surjective homomorphism $v : K^\times \to \mathbb{Z}$ such that
1) $v(a+b) \geq \min(v(a), v(b))$ for all $a, b \in K^\times$ with $a + b \neq 0$,
2) $v(a) \geq 0$ for all $a \in A \backslash \{0\}$, and
3) $v(a) > 0$ for all $a \in I \backslash \{0\}$.

Fixing a discrete valuation v of A let

$$o_{(v)} := \{a \in K^\times : v(a) \geq 0\} \cup \{0\}$$

be the valuation ring of v and

$$\mathfrak{m}_{(v)} := \{a \in K^\times : v(a) > 0\} \cup \{0\}$$

its maximal ideal. Then 2) says that $A \subseteq o_{(v)}$ and 3) says that the inclusion map $A \overset{\subseteq}{\hookrightarrow} o_{(v)}$ is continuous with respect to the I-adic, resp. $\mathfrak{m}_{(v)}$-adic, topology on A, resp. $o_{(v)}$. Put

$$\begin{aligned} o_v &:= \mathfrak{m}_{(v)}\text{-adic completion of } o_{(v)} \quad \text{and} \\ \mathfrak{m}_v &:= \text{maximal ideal in } o_v \quad . \end{aligned}$$

Then o_v is a complete discrete valuation ring; the corresponding discrete valuation of o_v will again be denoted by v. We have a natural continuous injective homomorphism of noetherian adic rings $A \hookrightarrow o_v$. The following easy observation is sometimes useful in proofs.

Remark 1:

The map $v : A \backslash \{0\} \to \mathbb{Z}$ is locally constant.

Proof: Fix an element $a \in A\backslash\{0\}$ and put $m := v(a) \geq 0$. For any $b \in I^{m+1}$ we have $v(b) > m$. From the well-known formula

$$v(a+b) = \min(v(a), v(b)) \quad \text{whenever} \quad v(a) \neq v(b)$$

we therefore obtain $v(a + I^{m+1}) = \{m\}$.

Let $A \hookrightarrow B$ be a continuous injective homomorphism of noetherian adic integral domains. A discrete valuation w of B is said to <u>extend</u> (or to be an extension of) a discrete valuation v of A if the corresponding valuation rings are mapped into each other and the induced injective homomorphism $o_{(v)} \hookrightarrow o_{(w)}$ is continuous. In terms of the maps v and w this means that there is a natural number $e \in \mathbb{N}$ such that $w \mid K^\times = e \cdot v$; this number e is called the <u>ramification index</u> of w over v. By completion we obtain in this case a commutative diagram of continuous injective homomorphisms of rings

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ o_v & \longrightarrow & o_w \end{array} \quad .$$

A crucial definition for later purposes is the following.

Definition:

Let v be a discrete valuation of A; a discrete valuation w of B extending v is called <u>*finite over v*</u> *if the homomorphism $o_v \hookrightarrow o_w$ is finite.*

If w is finite over v then o_w is a finitely generated free o_v-module and the $\mathfrak{m}_w$-adic and $\mathfrak{m}_v o_w$-adic topologies on o_w coincide (note that $\mathfrak{m}_w^e = \mathfrak{m}_v o_w$ if $w \mid K^\times = e \cdot v$).

In this section we want to study the set

$$V := \text{set of all discrete valuations of } A \quad .$$

Theorem 2:

V is nonempty if and only if $A \neq K$.

Clearly V is empty if $A = K$. We therefore assume that $A \neq K$. First of all we need to know how to recognize a discrete valuation ring.

Lemma 3:

Let $o \subseteq K$ be a subring such that

– K is the quotient field of o,
– o is a noetherian local ring whose maximal ideal $\mathfrak{n}$ is nonzero and principal,
– $A \subseteq o$, and $I \subseteq A \cap \mathfrak{n}$;

then there exists a (unique) discrete valuation v of A such that $o_{(v)} = o$.

Proof: From Cor. 1.14 we know that $\bigcap_{m \in \mathbb{N}} \mathfrak{n}^m = \{0\}$. Since $\mathfrak{n} \neq 0$ we furthermore have $\mathfrak{n}^{m+1} \underset{\neq}{\subset} \mathfrak{n}^m$ for any $m \geq 0$. Therefore there is, for any nonzero element $a \in o$, a unique integer $v(a) \geq 0$ such that

$$a \in \mathfrak{n}^{v(a)} \quad \text{and} \quad a \notin \mathfrak{n}^{v(a)+1} \quad .$$

If π is a generator of $\mathfrak{n}$ this means that

$$a = \pi^{v(a)} \cdot u \quad \text{for some} \quad u \in o^\times \quad .$$

For an arbitrary element $\frac{a}{b} \in K^\times$ with $a, b \in o$ we then put $v\left(\frac{a}{b}\right) := v(a) - v(b)$. This defines the required discrete valuation.

Lemma 4:

Let $o \subseteq K$ be a subring such that
— K is the quotient field of o,
— o is an integrally closed noetherian local ring of dimension 1,
— $A \subseteq o$, and $I \subseteq A \cap \mathfrak{n}$ where $\mathfrak{n}$ denotes the maximal ideal in o;
then there exists a (unique) discrete valuation v of A such that $o_{(v)} = o$.

Proof: All we have to show, by Lemma 3, is that $\mathfrak{n}$ is a principal ideal. Since $\mathfrak{n} \neq 0$ we know from Cor. 1.14 that $\mathfrak{n}^2 \underset{\neq}{\subset} \mathfrak{n}$. Fix an element $\pi \in \mathfrak{n} \backslash \mathfrak{n}^2$. We want to show that $\mathfrak{n} = \pi o$. In order to do that we consider the o-subalgebra

$$\mathfrak{n}^{-1} := \{a \in K : a \cdot \mathfrak{n} \subseteq o\}$$

in K and the ideal

$$\mathfrak{n}^{-1} \cdot \mathfrak{n} \subseteq o \quad .$$

Let us assume for a moment that

$$\mathfrak{n}^{-1} \cdot \mathfrak{n} = o$$

holds true. Because of $\pi \cdot \mathfrak{n}^{-1} \cdot \mathfrak{n} = \pi o \underset{\neq}{\subset} \mathfrak{n}^2$ the ideal $\pi \cdot \mathfrak{n}^{-1} \subseteq o$ cannot be contained in $\mathfrak{n}$ then and we must have $\pi \cdot \mathfrak{n}^{-1} = o$. This implies

$$\pi o = \pi \cdot \mathfrak{n}^{-1} \cdot \mathfrak{n} = \mathfrak{n} \quad .$$

Concerning our assumption we certainly have $o \subseteq \mathfrak{n}^{-1}$ and therefore $\mathfrak{n} \subseteq \mathfrak{n}^{-1} \cdot \mathfrak{n}$ which means that there are only the two possibilities $\mathfrak{n}^{-1} \cdot \mathfrak{n} = \mathfrak{n}$ or $= o$. We have to exclude the first possibility. For this we first construct an element $d \in \mathfrak{n}^{-1} \backslash o$. For any $b \in o \backslash \pi o$ we have the proper ideal

$$\mathfrak{a}_b := \{a \in o : ab \in \pi o\}$$

in o. Since o is noetherian there is a largest ideal $\mathfrak{a}_c$ among the $\mathfrak{a}_b$'s . This $\mathfrak{a}_c$ has to be a prime ideal: If $a'ac \in \pi o$ but $ac \notin \pi o$ then $\mathfrak{a}_c \subseteq \mathfrak{a}_{ac}$ and $a' \in \mathfrak{a}_{ac}$; the maximality of $\mathfrak{a}_c$ then implies $\mathfrak{a}_c = \mathfrak{a}_{ac}$ and hence $a' \in \mathfrak{a}_c$. On the other hand $\mathfrak{a}_c \neq \{0\}$ since $\pi \in \mathfrak{a}_c$. We consequently obtain

$$\mathfrak{a}_c = \mathfrak{n} \quad .$$

Now define $d := c\pi^{-1} \in K$. Then $d \notin o$ since $c \notin \pi o$ but $d \in \mathfrak{n}^{-1}$ since $d\mathfrak{n} = d\mathfrak{a}_c \subseteq o$. Assuming $\mathfrak{n}^{-1} \cdot \mathfrak{n} = \mathfrak{n}$ we would have $d \cdot \mathfrak{n} \subseteq \mathfrak{n}$. It would mean that $\mathfrak{n}$ is a faithful $o[d]$-module which as an o-module is finitely generated. But this is an equivalent way of saying that d is integral over o. Since o is integrally closed we would be led to the contradiction that $d \in o$.

In addition we need some deeper results from commutative algebra for the proofs of which we have to refer to the standard literature.

Proposition 5: (Krull-Akizuki)

Let o be a noetherian local integral domain of dimension 1 with quotient field K, let L/K be a finite extension of fields, and let $o \subseteq C \subseteq L$ be any intermediate ring; then C is noetherian of dimension ≤ 1 and $C/\mathfrak{c}$, for any nonzero ideal $\mathfrak{c} \subseteq C$, is of finite length as an o-module.

Proof: [Bou] VII §2.5 Prop. 5.

Proposition 6:

Let o be a noetherian local integral domain of dimension 1 with maximal ideal $\mathfrak{n}$ and with quotient field K, let L/K be a finite extension of fields, and let C be the integral closure of o in L; then the map

$$\left\{\begin{array}{l}\textit{maximal ideals}\\ \mathfrak{m} \subseteq C\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{l}\textit{discrete valuation rings } o_{(v)} \subseteq L\\ \textit{such that } o \subseteq o_{(v)} \textit{ and } \mathfrak{n} = o \cap \mathfrak{m}_{(v)}\end{array}\right\}$$

$$\mathfrak{m} \longmapsto C_{\mathfrak{m}}$$

is a bijection between two finite sets.

Proof: Let $\mathfrak{m} \subseteq C$ be a maximal ideal. We claim that $\mathfrak{m} \cap o$ is a maximal ideal in o: Let $\overline{c} \in o/\mathfrak{m} \cap o$ be any nonzero element; then $\overline{c}^{-1} \in C/\mathfrak{m}$ is integral over $o/\mathfrak{m} \cap o$ so that we have an equation

$$\overline{c}^{-n} + \overline{a}_1 \overline{c}^{-(n-1)} + \ldots + \overline{a}_n = 0$$

with appropriate elements $\overline{a}_i \in o/\mathfrak{m} \cap o$; it follows that

$$\overline{c}^{-1} = -\overline{a}_1 - \ldots - \overline{a}_n \overline{c}^{n-1} \in o/\mathfrak{m} \cap o$$

which means that $o/\mathfrak{m} \cap o$ is a field. We then necessarily have $\mathfrak{m} \cap o = \mathfrak{n}$, resp. $\mathfrak{n}C \subseteq \mathfrak{m}$. By the same argument C cannot be a field since it is integral over o which is not a field. We deduce from Prop. 5 that the integral domain C is noetherian of dimension 1 and that $C/\mathfrak{n}C$ is a finite dimensional $o/\mathfrak{n}$-vector space. The latter implies that the set of maximal ideals in C is finite. And it follows from the former that $C_{\mathfrak{m}}$ is a noetherian local integral domain of dimension 1 with quotient field L. As a localization of an integrally closed domain $C_{\mathfrak{m}}$ is integrally closed, too. Therefore, by Lemma 4, $C_{\mathfrak{m}}$ is a discrete valuation ring in L. Note also that $o \subseteq C_{\mathfrak{m}}$ and $o \cap \mathfrak{m}C_{\mathfrak{m}} = o \cap (C \cap \mathfrak{m}C_{\mathfrak{m}}) = o \cap \mathfrak{m} = \mathfrak{n}$. We see that the map in the assertion is well-defined. It clearly is injective. Now let $o_{(v)} \subseteq L$ be a discrete valuation ring such that $o \subseteq o_{(v)}$ and $\mathfrak{n} = o \cap \mathfrak{m}_{(v)}$. Discrete valuation rings are easily seen to be integrally closed. We obtain $C \subseteq o_{(v)}$ and then also $C_{\mathfrak{m}} \subseteq o_{(v)}$ for $\mathfrak{m} := C \cap \mathfrak{m}_{(v)}$. Since $\mathfrak{n}C \subseteq \mathfrak{m}$ and C has dimension 1 the prime ideal $\mathfrak{m}$ has to be maximal. We have seen that $C_{\mathfrak{m}}$ itself is a discrete valuation ring in L. But any discrete valuation ring is maximal among all proper subrings of its quotient field. This shows that $C_{\mathfrak{m}} = o_{(v)}$ and proves the surjectivity of our map.

Principal ideal theorem: (Krull)

Let o be a noetherian integral domain, let $a \in o$ be a nonzero element, and let $\mathfrak{p} \subseteq o$ be a prime ideal which is minimal with respect to containing a; then $o_{\mathfrak{p}}$ is a noetherian local integral domain of dimension 1.

Proof: [Mat] Thm. 13.5.

<u>Proof of Theorem 2</u>: Assume that $A \neq K$. Since also $I \neq A$ we find an open ideal $J \subseteq A$ such that $\{0\} \underset{\neq}{\subset} J \underset{\neq}{\subset} A$. For any nonzero element $b \in J$ the blowing up $A' := A_{J,b}$ by Lemma 5.2 is a noetherian integral domain with quotient field K such that the ideal $J' := J \cdot A_{J,b} = \frac{b}{1} \cdot A_{J,b}$ is principal. Let us assume for a moment that we can choose b in such a way that $J' \neq A'$. Fixing then a prime ideal $\mathfrak{p}' \subseteq A'$ which is minimal with respect to containing J' the principal ideal theorem says that $A'_{\mathfrak{p}'}$ is a noetherian local integral domain of dimension 1 with quotient field K. In this situation Prop. 6 ensures the existence of a discrete valuation ring $o_{(v)} \subseteq K$ such that $A'_{\mathfrak{p}'} \subseteq o_{(v)}$ and $\mathfrak{p}'A'_{\mathfrak{p}'} = A'_{\mathfrak{p}'} \cap \mathfrak{m}_{(v)}$. In particular we have $A \subseteq o_{(v)}$ and $I \subseteq A \cap \mathfrak{p}' \subseteq A \cap \mathfrak{m}_{(v)}$. This means that $v \in V$. It remains to justify the above choice of b. First of all we note that by Cor. 1.14 we have $J^{m+1} \underset{\neq}{\subset} J^m$ for all $m \in \mathbb{N}$. We claim that there exists a $b \in J$ such that $b^{m-1} \notin J^m$ for any $m \in \mathbb{N}$: Otherwise we find, for a given system of generators $b_1, \ldots, b_r$ of the ideal J, natural numbers m_i such that $b_i^{m_i-1} \in J^{m_i}$ for any $1 \leq i \leq r$; but this implies $J^{r(m-1)+1} = J^{r(m-1)}$ for $m := \max(m_1, \ldots, m_r)$. If with this choice of b we assume that $\frac{b}{1} \cdot A_{J,b} = A_{J,b}$ then there exist a $m \in \mathbb{N}$ and an $a \in J^m$ such that $\frac{b}{1} \cdot \frac{a}{b^m} = 1$, i.e., such that $b^{m-1} = a \in J^m$. This is a contradiction and it therefore follows that b has the property that $J' \neq A'$.

Remark 7:

If $I \neq \{0\}$ then the map

$$\begin{array}{rcl} V & \longrightarrow & \textit{set of open prime ideals in } A \\ v & \longmapsto & A \cap \mathfrak{m}_v \end{array}$$

is surjective.

Proof: In the proof of Theorem 2 the assumption that A is adic was actually not used at all. Therefore if $\mathfrak{p} \subseteq A$ is any open prime ideal we can carry out the constructions in that proof starting from the local ring $A_{\mathfrak{p}}$ equipped with the $\mathfrak{p}A_{\mathfrak{p}}$-adic topology in order to obtain a discrete valuation $v \in V$ such that $A \cap \mathfrak{m}_v = A \cap (A_{\mathfrak{p}} \cap \mathfrak{m}_v) = A \cap \mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}$. Note that $A_{\mathfrak{p}} \neq K$ since $\mathfrak{p}$ contains the nonzero ideal I.

The set V carries a natural topology which is defined as follows. For any A-subalgebra $A' \subseteq K$ which is of finite type over A we put

$$V(A') := \{v \in V : A' \subseteq o_{(v)}\} \quad .$$

We obviously have $V(A) = V$ and

$$V(A') \cap V(A'') = V(\langle A', A'' \rangle)$$

whenever $A', A'' \subseteq K$ are two A-subalgebras of finite type over A and $\langle A', A'' \rangle \subseteq K$ denotes the A-subalgebra generated by $A' \cup A''$. Calling a subset of V open if it is the union of subsets of the form $V(A')$ therefore defines a topology on V which we call the Zariski topology.

Remark 8:

i. Any point in V is closed;
ii. any A-subalgebra $A' \subseteq K$ which is of finite type over A is a blowing up of A;
iii. the subsets $V(A_{J,b})$ where $J \subseteq A$ is a nonzero open ideal and b is a nonzero element in J form a basis of the topology of V;
iv. if $J \subseteq A$ is a nonzero ideal and $b_1, \ldots, b_r \in A\backslash\{0\}$ is a system of generators of J then we have

$$V = V(A_{J,b_1}) \cup \ldots \cup V(A_{J,b_r}) \quad .$$

Proof: i. Let $v' \neq v$ be two different points in V. Since $o_{(v)}$ is not contained in $o_{(v')}$ (remember that $o_{(v)}$ is maximal among all proper subrings of K) we find an element $c \in o_{(v)} \backslash o_{(v')}$. Then $V(A[c])$ is an open neighbourhood of v which does not contain v'. ii. Let $c_1, \ldots, c_r \in K$ be generators of A' as an A-algebra. We find elements $b \in A\backslash\{0\}$ and $a_1, \ldots, a_r \in A$ such that

$$c_i = \frac{a_i}{b} \quad \text{for any} \quad 1 \leq i \leq r \quad .$$

If J denotes the ideal in A generated by $b, a_1, \ldots, a_r$ then we have

$$A' = A_{J,b} \quad .$$

iii. We have

$$V(A_{J,b}) = \bigcup_{v \in V(A_{J,b})} V(A_{J+I^{v(b)},b}) \quad .$$

iv. Fix a discrete valuation $v \in V$. Since J is nonzero we have $J \cdot o_{(v)} = \mathfrak{m}_{(v)}^n$ for some $n \geq 0$. Of course the exponent n is given as

$$n = \min(v(b_1), \ldots, v(b_r)) \quad ;$$

in particular there is a $1 \leq i \leq r$ such that $v(b_i) = n$. We claim that $A_{J,b_i} \subseteq o_{(v)}$. Any element in A_{J,b_i} is of the form $\frac{a}{b_i^m}$ for some $m \geq 0$ and some $a \in J^m$. Since $J^m \subseteq \mathfrak{m}_{(v)}^{nm}$ we have $v(a) \geq nm$ and therefore $v\left(\frac{a}{b_i^m}\right) = v(a) - nm \geq 0$.

There is another important fact about extensions of discrete valuations.

Lemma 9:

Let o be a complete discrete valuation ring with quotient field K and let L/K be a finite extension of fields; then the integral closure C of o in L is a finitely generated o-module.

Proof: Let π be a generator of the maximal ideal in o. As a consequence of Prop. 5 we obtain that C is noetherian and that $C/\pi C$ is a finitely generated o-module. Since o is integrally closed we have $\pi^{-1} \notin C$ and hence $\pi C \neq C$. It then follows from Cor. 1.14 that $\bigcap\limits_{n \in \mathbb{N}} \pi^n C = \{0\}$. Moreover $gr C := \bigoplus\limits_{n \geq 0} \pi^n C/\pi^{n+1} C$ as a module over $gr\, o := \bigoplus\limits_{n \geq 0} \pi^n o/\pi^{n+1} o$ is generated by $gr^0 C = C/\pi C$ and therefore is finitely generated. Since o is complete we have together all ingredients needed to apply Lemma 1.6 in the same manner as we did in the proof of Prop. 1.5 and to conclude that C is a finitely generated o-module.

Lemma 10:

Let o be a noetherian local integral domain which is adic with respect to its maximal ideal; let $o \to C$ be a finite homomorphism of integral domains; then C is a local ring, too.

Proof: Let $\mathfrak{n} \subseteq o$ be the maximal ideal. Then C is $\mathfrak{n}C$-adic. By the same argument as at the beginning of the proof of Prop. 6 we see that any maximal ideal in C lies above $\mathfrak{n}$. Therefore assuming that C is not local the artinian $o/\mathfrak{n}$-algebra $C/\mathfrak{n}C$ is not local so that we find a nontrivial idempotent $\overline{c} \in C/\mathfrak{n}C$. Consider now the polynomial factorization

$$T^2 - T \bmod \mathfrak{n}C = (T - \overline{c})(T - 1 + \overline{c}) \quad .$$

Because of

$$(1 - 2\overline{c})(T - \overline{c}) + (2\overline{c} - 1)(T - 1 + \overline{c}) = 1$$

the two factors generate $C/\mathfrak{n}C\{T\}$ as an ideal. We therefore may apply Hensel's lemma in order to obtain a monic polynomial $P \in C[T]$ which divides $T^2 - T$ and such that $P \bmod \mathfrak{n}C = T - \overline{c}$. But this means that $P(T) = T - c$ where $c \in C$ is an element such that $c^2 = c$ and $c + \mathfrak{n}C = \overline{c}$. Hence c is a nontrivial idempotent in C. This is a contradiction since C was assumed to be an integral domain.

Proposition 11:

Let $o = o_{(v)}$ be a complete discrete valuation ring and let $o \hookrightarrow C$ be an injective and finite homomorphism of integral domains so that, in particular, C is an o-algebra topologically of finite type; then there exists precisely one discrete valuation w of C which extends v; moreover w is finite over v.

Proof: According to Lemma 9 the integral closure of C still is finite over o. We therefore may assume that C is integrally closed. Lemma 10 implies that C is local. In view of Prop. 6 this means that C is the valuation ring of the unique discrete valuation w extending v.

The assertion of Lemma 9 should be seen in the following context.

Definition:

A noetherian integral domain o with quotient field K is called <u>Japanese</u> if for any extension of fields L/K the integral closure of o in L is a finitely generated o-module.

We can now rephrase Lemma 9 by saying that any complete discrete valuation ring is Japanese. Actually a much stronger result holds true.

Theorem 12: (Nagata)

Any noetherian local integral domain which is adic with respect to its maximal ideal is Japanese.

Proof: [EGA] IV 1 (0.23.1.5).

§8 Affinoid spectra

As a base ring we fix a noetherian $\mathfrak{m}$-adic integral domain o where $\mathfrak{m}$ is some nonzero proper ideal in o. In particular o is not a field; we denote by K the quotient field of o. Moreover we let $V = V(o)$ be the nonempty (Thm. 7.2) set of all discrete valuations of o.

For any o-algebra A topologically of finite type we define

$$Sp_o(A) := \text{set of all pairs } (\mathfrak{p}, w) \text{ where } \mathfrak{p} \subseteq A \text{ is a prime ideal such that } o \to A/\mathfrak{p} \text{ is injective and } w \text{ is a discrete valuation of } A/\mathfrak{p} \text{ which is finite over some } v \in V.$$

The set $Sp_o(A)$ is called the affinoid spectrum of A. If it is clear from the context which base ring we refer to we simply write $Sp(A)$.

Remark 1:

The prime ideal $\mathfrak{p}$, for any point $(\mathfrak{p}, w) \in Sp(A)$, is not open in A.

Proof: If $\mathfrak{p}$ is open in A the injectivity of $o \hookrightarrow A/\mathfrak{p}$ implies that the zero ideal $\{0\}$ is open in o. This means that $\mathfrak{m}$ is nilpotent and therefore that $\mathfrak{m} = 0$ which is a contradiction.

Forming the affinoid spectrum of course is functorial. For any homomorphism $\varphi : A \to B$ in $\mathrm{Alg}_{tft}(o)$ we have the induced map

$$\begin{aligned} Sp(\varphi) : Sp(B) &\longrightarrow Sp(A) \\ (\mathfrak{q}, w) &\longmapsto (\varphi^{-1}(\mathfrak{q}), w \mid (A/\varphi^{-1}(\mathfrak{q}))) \quad ; \end{aligned}$$

here and in the following the restriction of a discrete valuation to a subring has to be understood in the sense that the restricted map is renormalized again (by dividing it by the corresponding ramification index) to be onto $\mathbb{Z}$ (so possible). In particular, under the obvious identification

$$\begin{aligned} Sp(o) &\overset{\sim}{\longleftrightarrow} V \\ (\{0\}, v) &\longmapsto v \end{aligned}$$

the natural map $Sp(A) \to Sp(o)$ becomes the map

$$\begin{aligned} Sp(A) &\longrightarrow V \\ (\mathfrak{p}, w) &\longmapsto v \quad \text{if } w \text{ extends } v \quad . \end{aligned}$$

Let us fix for a moment a discrete valuation $v \in V$. It was shown in Prop. 3.2 that $A \hat{\otimes}_o o_v$ is an o_v-algebra topologically of finite type. Let

$$\psi_v : A \longrightarrow A \hat{\otimes}_o o_v$$

be the natural continuous homomorphism of rings. It induces the map

$$\begin{aligned}\psi_v^* : Sp_{o_v}(A \hat{\underset{o}{\otimes}} o_v) &\longrightarrow Sp_o(A)\\ (\mathfrak{q}, w) &\longmapsto (\psi_v^{-1}(\mathfrak{q}), w \mid (A/\psi_v^{-1}(\mathfrak{q}))) \quad .\end{aligned}$$

Proposition 2:

The map ψ_v^* induces a bijection

$$Sp_{o_v}(A \hat{\underset{o}{\otimes}} o_v) \xrightarrow{\sim} \{(\mathfrak{p}, w) \in Sp_o(A) : w \text{ extends } v\} \quad .$$

Proof: Let $(\mathfrak{p}, w)$ be a point in $Sp_o(A)$ such that w extends v. We then have the commutative diagram of continuous injective homomorphisms of rings

$$\begin{array}{ccc} o & \longrightarrow & A/\mathfrak{p} \\ \downarrow & & \downarrow \\ o_v & \longrightarrow & o_w \end{array} \quad .$$

By the universal property of the complete tensor product (Prop. 3.4) it extends uniquely to a commutative diagram of continuous homomorphisms

$$\begin{array}{ccccc} o_v & \longrightarrow & (A/\mathfrak{p}) \hat{\underset{o}{\otimes}} o_v & \longleftarrow & A/\mathfrak{p} \\ & \searrow & \downarrow & \swarrow & \\ & & o_w & & \end{array} \quad .$$

On the other hand we have the commutative diagram

$$\begin{array}{ccc} A & \twoheadrightarrow & (A/\mathfrak{p}) \\ {\scriptstyle \psi_v}\downarrow & & \downarrow \\ A \hat{\underset{o}{\otimes}} o_v & \twoheadrightarrow & (A/\mathfrak{p}) \hat{\underset{o}{\otimes}} o_v \end{array}$$

where the lower horizontal arrow is surjective by Lemma 1.10. Consider the prime ideal

$$\mathfrak{q} := \ker(A \hat{\underset{o}{\otimes}} o_v \twoheadrightarrow (A/\mathfrak{p}) \hat{\underset{o}{\otimes}} o_v \longrightarrow o_w) \quad .$$

It is now a simple diagram chase to see that $(\mathfrak{q}, w) \in Sp_{o_v}(A \hat{\underset{o}{\otimes}} o_v)$ is the unique preimage of $(\mathfrak{p}, w)$ under the map ψ_v^*.

Corollary 3:

$Sp_o(A) = \dot{\bigcup\limits_{v \in V}} \operatorname{im} \psi_v^*$ *where the union is disjoint.*

Lemma 4:

Let B be an o_v-algebra topologically of finite type; then the map

$$\begin{aligned} Sp_{o_v}(B) &\xrightarrow{\sim} \{\mathfrak{p} \in \mathrm{Spec(B)} : \mathrm{o_v} \hookrightarrow \mathrm{B}/\mathfrak{p} \text{ is injective and finite}\} \\ (\mathfrak{p}, w) &\longmapsto \mathfrak{p} \end{aligned}$$

is a bijection.

Proof: Given the point $(\mathfrak{p}, w) \in Sp_{o_v}(B)$ we have the injective homomorphisms $o_v \hookrightarrow B/\mathfrak{p} \hookrightarrow o_w$ such that the composition $o_v \hookrightarrow o_w$ is finite. Therefore $o_v \hookrightarrow B/\mathfrak{p}$ is finite, too. This shows that the map is well-defined. Its bijectivity follows from Prop. 7.11.

Proposition 5:

Let B be an o_v-algebra topologically of finite type and let K_v denote the quotient field of o_v; then the map

$$\begin{aligned} Sp_{o_v}(B) &\xrightarrow{\sim} \text{ set of maximal ideals in } B \underset{o_v}{\otimes} K_v \\ (\mathfrak{p}, w) &\longmapsto \mathfrak{p} \underset{o_v}{\otimes} K_v \end{aligned}$$

is a bijection.

Proof: Because of Lemma 4 we may replace the left hand side in the assertion by the set

$$\{\mathfrak{p} \in \mathrm{Spec(B)} : \mathrm{o_v} \hookrightarrow \mathrm{B}/\mathfrak{p} \text{ is injective and finite}\} \quad .$$

For each prime ideal $\mathfrak{p}$ in this set

$$(B/\mathfrak{p}) \underset{o_v}{\otimes} K_v = B \underset{o_v}{\otimes} K_v / \mathfrak{p} \underset{o_v}{\otimes} K_v$$

is an integral domain finite dimensional over K_v and therefore is a field. This shows that $\mathfrak{p} \underset{o_v}{\otimes} K_v$ is a maximal ideal in $B \underset{o_v}{\otimes} K_v$. Now let $\mathfrak{q} \subseteq B \underset{o_v}{\otimes} K_v$ be any prime ideal. Then $\mathfrak{p} := \mathfrak{q} \cap B$ is the unique prime ideal in B such that $\mathfrak{p} \underset{o_v}{\otimes} K_v = \mathfrak{q}$; moreover $o_v \hookrightarrow B/\mathfrak{p}$ clearly is injective. It remains to show that if $\mathfrak{q}$ is maximal then $o_v \hookrightarrow B/\mathfrak{p}$ is finite. According to the Normalization lemma in §6 there is a finite and injective homomorphism of o_v-algebras

$$o_v\{T_1, \ldots, T_d\} \hookrightarrow B/\mathfrak{p}$$

for some $d \geq 0$. Then

$$o_v\{T_1, \ldots, T_d\} \underset{o_v}{\otimes} K_v \hookrightarrow (B/\mathfrak{p}) \underset{o_v}{\otimes} K_v = B \underset{o_v}{\otimes} K_v/\mathfrak{q}$$

is injective and finite, too, and the right hand side is a field. Therefore (compare the beginning of the proof of Prop. 7.6) the left hand side also has to be a field which only can happen if $d = 0$.

Remark 6:

If $A \neq 0$ then there exists a $v \in V$ such that $A \hat{\underset{o}{\otimes}} o_v \neq 0$.

Proof: It suffices to show that $A \hat{\underset{o}{\otimes}} o_v \neq \mathfrak{m}_v \cdot (A \hat{\underset{o}{\otimes}} o_v)$ for some $v \in V$. We fix an open prime ideal in A (e.g., a maximal ideal by Lemma 1.4) and we let $\mathfrak{p}$ denote its preimage in o. According to Remark 7.7 there exists a discrete valuation $v \in V$ such that $\mathfrak{p} = o \cap \mathfrak{m}_v$. We then obtain

$$(A \hat{\underset{o}{\otimes}} o_v)/\mathfrak{m}_v(A \hat{\underset{o}{\otimes}} o_v) = (A/\mathfrak{p}A) \underset{o/\mathfrak{p}}{\otimes} (o_v/\mathfrak{m}_v) \quad .$$

Since $o/\mathfrak{p} \hookrightarrow A/\mathfrak{p}A$ is injective and since $o_v/\mathfrak{m}_v$ being a field extension of the quotient field of $o/\mathfrak{p}$ is flat over $o/\mathfrak{p}$ the right hand side certainly is nonzero.

Proposition 7:

If $A \neq 0$ is flat over o then $Sp_o(A) \neq \emptyset$.

Proof: The Prop. 3.6 implies that $A \hat{\underset{o}{\otimes}} o_v$ is flat over o_v for any $v \in V$. If we choose $v \in V$ in such a way that $A \hat{\underset{o}{\otimes}} o_v \neq 0$ which is possible by the previous Remark 6 then

$$o_v \hookrightarrow A \hat{\underset{o}{\otimes}} o_v$$

is injective. For this v we then obtain $(A \hat{\underset{o}{\otimes}} o_v) \underset{o_v}{\otimes} K_v \neq 0$ where K_v again denotes the quotient field of o_v. It follows now from Prop. 5 that $Sp_{o_v}(A \hat{\underset{o_v}{\otimes}} o_v) \neq \emptyset$ and therefore that $Sp_o(A) \neq \emptyset$.

As a next step we will see that our affinoid spectra $Sp_o(A)$ carry a natural topology. This comes from the following crucial observation.

Lemma 8:

For any open ideal $J \subseteq A$ and any element $b \in J$ the map

$$Sp_o(A_{\{J,b\}}) \hookrightarrow Sp_o(A)$$

induced by the complete blowing up $A \to A_{\{J,b\}}$ is injective; its image is

$$D_{J,b} := \{(\mathfrak{p}, w) \in Sp_o(A) : b \notin \mathfrak{p} \text{ and } w(b+\mathfrak{p}) = \min_{a \in J} w(a+\mathfrak{p})\} \quad .$$

Proof: (We adopt from now on the usual convention that $w(0) := \infty$.) First we fix a point $(\mathfrak{q}, \tilde{w}) \in Sp(A_{\{J,b\}})$. Let $\mathfrak{p} \subseteq A$ denote the preimage of $\mathfrak{q}$, set $w := \tilde{w} \mid (A/\mathfrak{p})$, and let $e \geq 1$ be the ramification index of $\tilde{w}$ over w. By Remark 1 the prime ideal $\mathfrak{q}$ is not open in $A_{\{J,b\}}$ and therefore cannot contain the open ideal $JA_{\{J,b\}} = \frac{b}{1} \cdot A_{\{J,b\}}$; hence $b \notin \mathfrak{p}$. On the other hand, for any $a \in J$, we have

$$\begin{aligned} e \cdot w(a+\mathfrak{p}) = \tilde{w}\left(\frac{a}{1} + \mathfrak{q}\right) &= \tilde{w}\left(\frac{b}{1} \cdot \frac{a}{b} + \mathfrak{q}\right) = \tilde{w}\left(\frac{b}{1} + \mathfrak{q}\right) + \tilde{w}\left(\frac{a}{b} + \mathfrak{q}\right) \\ &\geq \tilde{w}\left(\frac{b}{1} + \mathfrak{q}\right) = e \cdot w(b+\mathfrak{p}) \quad . \end{aligned}$$

This shows that the image of the map under consideration is contained in $D_{J,b}$. Now we fix a point $(\mathfrak{p}, w) \in D_{J,b}$. Since $b \notin \mathfrak{p}$ the image $\tilde{b} \in o_w$ of b under the associated continuous homomorphism $A \twoheadrightarrow A/\mathfrak{p} \hookrightarrow o_w$ is not a zero divisor in o_w. The second requirement in the definition of $D_{J,b}$ guarantees that the ideal $\tilde{J} \subseteq o_w$ generated by the image of J is the principal ideal $\tilde{J} = \tilde{b} \cdot o_w$. By the universal property of a complete blowing up the homomorphism $A \to o_w$ extends therefore in a unique way to a continuous homomorphism $A_{\{J,b\}} \to o_w$. If $\mathfrak{q}$ denotes the kernel of the latter then $(\mathfrak{q}, w) \in Sp(A_{\{J,b\}})$ is the unique preimage of $(\mathfrak{p}, w)$.

Remark 9:

The proof of Lemma 8 shows that for any point $(\mathfrak{q}, \tilde{w}) \in Sp_o(A_{\{J,b\}})$ with image $(\mathfrak{p}, w) \in Sp_o(A)$ we have $o_{\tilde{w}} = o_w$.

In the situation of Lemma 8 let $a_1, \ldots, a_r \in J$ be a system of generators of that ideal. We then of course have

$$D_{J,b} = \{(\mathfrak{p}, w) \in Sp_o(A) : b \notin \mathfrak{p} \text{ and } w(b+\mathfrak{p}) \leq \min_{1 \leq i \leq r} w(a_i + \mathfrak{p})\} \; .$$

Lemma 10:

Let $J, J' \subseteq A$ be open ideals and let b and b' be elements in J and J', respectively; we then have

i. $D_{J,b} \cap D_{J',b'} = D_{JJ',bb'}$;

ii. $Sp(\varphi)^{-1}(D_{J,b}) = D_{\varphi(J),B,\varphi(b)}$ *for any homomorphism* $\varphi: A \to B$ *in* $\mathrm{Alg}_{tft}(o)$*;*

iii. $(\psi_v^*)^{-1}(D_{J,b}) = D_{\psi(J)A\hat{\otimes}_o o_v, \psi_v(b)}$ *for any* $v \in V$*.*

Proof: Easy exercise. (Observe that with J and J' also JJ' is an open ideal.)

The subsets $D_{J,b} \subseteq Sp_o(A)$ for $J \subseteq A$ an open ideal and $b \in J$ are called rational subsets. According to Lemma 10.i they form a basis for the open sets in a unique topology on $Sp_o(A)$ which we call the canonical topology. The other two assertions of Lemma 10 then say that the maps $Sp(\varphi)$ for any φ in $\mathrm{Alg}_{tft}(o)$ and ψ_v^* for any $v \in V(o)$ are continuous. On $Sp_o(o) = V(o)$ the canonical topology coincides, by Remark 7.8, with the Zariski topology.

Remark 11:

i. The image of ψ_v^**, for any* $v \in V(o)$*, is a closed subset in* $Sp_o(A)$*;*

ii. for any surjective homomorphism $\varphi : A \twoheadrightarrow B$ *in* $\mathrm{Alg}_{tft}(o)$ *the map*

$$Sp(\varphi) : Sp_o(B) \hookrightarrow Sp_o(A)$$

is a closed immersion.

Proof: i. Remark 7.8.i and Prop. 2. ii. Set $\mathfrak{a} := \ker \varphi$. The map $Sp(\varphi)$ clearly is injective and its image is $\{(\mathfrak{p}, w) \in Sp(A) : \mathfrak{a} \subseteq \mathfrak{p}\}$. For any open ideal $J' \subseteq B$ and any element $b' \in J'$ we put $J := \varphi^{-1}(J')$ and we choose an element $b \in J$ such that $\varphi(b) = b'$. We then have

$$D_{J',b'} = Sp(\varphi)^{-1}(D_{J,b}) \quad .$$

This shows that the canonical topology on $Sp(A)$ induces via the injection $Sp(\varphi)$ the canonical topology on $Sp(B)$. It remains to prove that the image of $Sp(\varphi)$ is closed in $Sp(A)$. Consider a point $(\mathfrak{p}, w) \in Sp(A)$ not in this image, i.e., such that $\mathfrak{a} \not\subseteq \mathfrak{p}$. We choose an element $b \in \mathfrak{a} \backslash \mathfrak{p}$ and we put $J := b \cdot A + \mathfrak{m}^{w(b)} A$. The rational subset $D_{J,b}$ then is an open neighbourhood of $(\mathfrak{p}, w)$ which is disjoint from the image of $Sp(\varphi)$.