λ-Rings and Adams operations in algebraic K-theory

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The purpose of this talk is to outline the construction of Adams operations on the algebraic K-theory of a quasi-projective scheme X, and to prove that $K_n(X)$ can be decomposed into the eigenspaces of these operations. These eigenspaces are the absolute cohomology of X, which will be used extensively in subsequent talks.

Adams operations are defined in terms of λ-rings, i.e. rings with maps that "behave like the exterior powers in representation rings". Unfortunately, these maps cannot be defined directly on the K-theory; therefore we have to study the general theory of λ-rings, and then to transfer the λ-structure from representation rings to K-groups. In many cases, proofs will only be sketched; a reader who wants more details, will find a complete account of the general theory of λ-rings in [A-T], and for Adams operations on the K-theory of a ring, he can consult [H] or [K]; the first announcement of a λ-structure on the K-theory of a ring can be found in [Q1]. I shall follow the terminology of [H]; what I call a λ-ring, is called pre-λ-ring in [K], and a λ-ring in the sense of [K] is called a special λ-ring here.

For a group G and a commutative ring A with unity, let

$\mathcal{P}_A$	denote the category of finitely generated projective A-modules,
$\mathcal{P}_A(G)$	the category of finitely generated projective A-modules with G-action,
$K_0(A)$	the Grothendieck group of $\mathcal{P}_A$ with respect to exact sequences,
$R(G,A)$	the Grothendieck group of $\mathcal{P}_A(G)$ with respect to exact sequences,
$R_\oplus(G,A)$	the Grothendieck group of $\mathcal{P}_A(G)$ with respect to direct sums.

Then $K_0(A)$ and $R(G,A)$ are rings (with $\oplus$ as addition and $\otimes$ as multiplication), which admit maps $\lambda^i: R \to R; M \mapsto \bigwedge^i M$. The notion of a λ-ring formalizes this situation:

Definition: *A λ-ring R is a commutative ring with unity, together with maps $\lambda^i: R \to R$, $i \in \mathbb{N}_0$, such that*

(1) $\lambda^0(x) = 1 \quad \forall x \in R$

(2) $\lambda^1(x) = x \quad \forall x \in R$

(3) $\lambda^i(x+y) = \sum_{\nu=0}^{i} \lambda^\nu(x)\lambda^{i-\nu}(y) \quad \forall x, y \in R.$

Putting

$$\lambda_t(x) = \sum_{i \geq 0} \lambda^i(x)\, t^i,$$

(1)–(3) are equivalent to saying that $\lambda_t : R \to 1 + R[[t]]^+$ is a homomorphism of abelian groups. An element $x \in R$ has λ-dimension n, if $\lambda_t(x)$ is a polynomial of degree n.

What is $\lambda^i(xy)$?

Suppose, $x = x_1 + \ldots + x_n$ and $y = y_1 + \ldots + y_m$ are sums of one-dimensional elements, whose products are one-dimensional, too. Then

$$xy = \sum_{i,j} x_i y_j, \qquad \lambda_t(x_i y_j) = 1 + x_i y_j t,$$

hence

$$\lambda_t(xy) = \prod_{i,j} (1 + x_i y_j t).$$

$\prod_{i,j}(1+X_iY_jt) \in \mathbb{Z}[X_1,\ldots X_n, Y_1, \ldots Y_m][t]$ is symmetric in the X_i and the Y_j, therefore it can be written as

$$\prod_{i,j}(1 + X_i Y_j t) = \sum_{i=0}^{nm} P_i(\sigma_1(X), \ldots, \sigma_n(X); \tau_1(Y), \ldots, \tau_m(Y)) \cdot t^i,$$

where σ_i, τ_j are the elementary symmetric functions in the X_k respectively Y_ℓ, and the P_i are universal polynomials with integer coefficients, not depending on the ring R. Because of (3), we have $\lambda^i(x) = \sigma_i(x_1, \ldots, x_n)$ and $\lambda^j(y) = \tau_j(y_1, \ldots, y_m)$, so we get

(4) $\quad \lambda^i(xy) = P_i\left(\lambda^1(x), \ldots, \lambda^n(x); \lambda^1(y), \ldots, \lambda^m(y)\right).$

Similarly, $\lambda^i \circ \lambda^j(x)$ can be computed as

(5) $\quad \lambda^i \circ \lambda^j(x) = P_{i,j}\left(\lambda^1(x), \ldots, \lambda^{ij}(x)\right),$

where the $P_{i,j}$ again are universal polynomials with integer coefficients.

Definition: *A λ-ring R is called special, if* (4) *and* (5) *are satisfied.*

In a special λ-ring, we have $\lambda_t(1) = \lambda_t(\lambda^0(x)) = 1 + t$, hence

(6) $\quad \lambda^i(1) = 0 \quad \forall i > 1.$

A trivial example of a special λ-ring is $\mathbb{Z}$ itself with the λ-operations $\lambda^i(n) = \binom{n}{i}$; it is clear from (6) and the definition of a λ-ring, that this is the only special λ-ring structure on $\mathbb{Z}$. Similarly, we can get a special λ-ring structure on other rings in which binomial coefficents exist; this leads to the

Definition: *A binomial λ-ring is a commutative ring with unity, which is torsion free as an abelian group, for every $x \in R$ and every $i \in \mathbb{N}$ contains $\binom{x}{i}$, and whose λ-operations are given by $\lambda^i(x) = \binom{x}{i}$.*

More important special λ-rings are given by the following

Example: $R(G, A)$ is a special λ-ring.

Idea of proof: For every given finite set of representations, a category $\mathcal{C}'$ is constructed, in which each of these modules is a sum of one-dimensional elements, so that the same calculations as above give (4) and (5). $\mathcal{C}'$ can be constructed inductively: It obviously suffices to construct a category $\mathcal{C}''$, in which a one-dimensional summand can be split from one given module M. Here the idea is to consider modules over the symmetric algebra $\mathcal{S}(M)$, where the module induced by M itself has a one-dimensional quotient, given by the linear functions on $\mathcal{S}(M)$. For details, see [S]. ∎

The use of the splitting principle in this proof is something very common in the theory of special λ-rings; in fact, the important thing about special λ-rings is, that they behave as if every element were a sum of one-dimensional elements. More precisely, we have the

λ-verification principle: *Let μ be an operation on the category of special λ-rings, that is a functorial family of maps $\mu_R: R \to R$. Then μ is given by a polynomial P in $\mathbb{Z}[\lambda^1, \lambda^2, \ldots]$, and in order to prove that a certain μ is given by P, it suffices to verify this for sums of one-dimensional elements.*

Idea of proof: Let $U = \mathbb{Z}[X, \lambda^2(X), \lambda^3(X), \ldots]$ be the free special λ-ring generated by one variable X. For every special λ-ring R, and every $x \in R$, there exists a unique homomorphism $\varphi: U \to R$ with $\varphi(X) = x$, and because of the functoriality of μ, the polynomial $P = \mu_U(X) \in U$ describes μ. Make $\Omega = \mathbb{Z}[\varsigma_1, \varsigma_2, \ldots]$ into a special λ-ring via $\lambda_t(\varsigma_i) = 1 + \varsigma_i t$; then truncated pieces of U can be mapped to truncated pieces of Ω by

$$\lambda^i(X) \mapsto \sigma_i^{(n)}(\varsigma_1, \ldots, \varsigma_n),$$

where $\sigma_i^{(n)}$ is the i^{th} elementary symmetric function in n variables. In Ω, every element is a sum of one-dimensional elements, so inductively one shows the verification principle. For details, see [A-T], theorem 3.2. ∎

In order to define Chern classes, we need a modification of the λ-operations, the so-called γ-operations

$$\gamma^i: R \to R; \quad x \mapsto \lambda^i(x + i - 1),$$

which can also be defined by

$$\gamma_t(x) = \sum_{i \geq 0} \gamma^i(x) t^i = \lambda_{t/(1-t)}(x).$$

We call a special λ-ring R augmented, if there is an S-linear homomorphism of λ-rings $\varepsilon: R \to S$ to a **binomial** sub-λ-ring S of R. In an augmented special λ-ring, the γ-operations define a natural filtration, the γ-filtration, whose graded pieces are

$$R_n = \left\langle \gamma^{i_1}(x_1) \cdots \gamma^{i_r}(x_r) \,\middle|\, x_\nu \in \tilde{R},\ \sum i_\nu \geq n \right\rangle_S,$$

where $< \cdots >_S$ stands for the S-module generated by the elements inside the brackets, and $\tilde{R}$ is the kernel of the augmentation ε. We call $c_n(x) = \gamma^n(x - \varepsilon(x)) \bmod R_{n+1}$ the n^{th} universal Chern class of $x \in R$.

The Adams-operations $\psi^k: R \to R$ are defined by

$$\psi_t(x) = \sum_{k\geq 1} \psi^k(x)\, t^k = -t\frac{d\log\lambda_{-t}(x)}{dt}.$$

For a one-dimensional x, this means that

$$\psi_t(x) = -t\frac{d\log(1-xt)}{dt} = -t\frac{-x}{1-xt} = \frac{xt}{1-xt} = \sum_{k\geq 1} x^k t^k,$$

hence $\psi^k(x) = x^k$. Letting $N_k(\sigma_1, \ldots \sigma_k)$ denote the polynomial in the elementary symmetric functions for which

$$N_k\Big(\sigma_1(x_1, \ldots x_n),\ \ldots,\ \sigma_k(x_1, \ldots, x_n)\Big) = x_1^k + \cdots + x_n^k,$$

the λ-verification principle shows that

$$\psi^k(x) = N_k(\lambda^1(x), \ldots, \lambda^k(x)).$$

Lemma: *The ψ^k are homomorphisms of λ-rings, and $\psi^k \circ \psi^\ell = \psi^{k\ell}$. If R is augmented with $\varepsilon: R \to S$, they are also S-linear.*

Proof: By the λ-verification principle, it suffices to consider sums of one-dimensional elements; for these we have

$$\begin{aligned}
\psi^k\Big(\sum x_i + \sum y_j\Big) &= \sum x_i^k + \sum y_j^k = \psi^k\Big(\sum x_i\Big) + \psi^k\Big(\sum y_j\Big)\\
\psi^k\Big(\Big(\sum x_i\Big)\cdot\Big(\sum y_j\Big)\Big) &= \sum x_i^k y_j^k = \Big(\sum x_i^k\Big)\cdot\Big(\sum y_j^k\Big)\\
&= \psi^k\Big(\sum x_i\Big)\cdot\psi^k\Big(\sum y_j\Big)\\
\psi^k\Big(\lambda^\ell\Big(\sum x_i\Big)\Big) &= \psi^k(\sigma_\ell(x_1, \ldots, x_r)) = \sigma_\ell(x_1^k, \ldots, x_r^k)\\
&= \lambda^\ell\Big(\sum {x_i}^k\Big) = \lambda^\ell\psi^k\Big(\sum x_i\Big)\\
\psi^k\Big(\psi^\ell\Big(\sum x_i\Big)\Big) &= \psi^k\Big(\sum x_i^\ell\Big) = \sum x_i^{k\ell} = \psi^{k\ell}\Big(\sum x_i\Big).
\end{aligned}$$

In order to show that the ring homomorphisms ψ^k are S-linear, it suffices to show that their restrictions to S are the identity map. This follows from (and is in fact equivalent to) the fact that, according to our definition, S is a binomial λ-ring: In such a ring, $\lambda_t(x) = \sum \binom{x}{k} t^k$, which we can write formally as $(1+t)^x$. Since all the usual identities for $(1+t)^x$ can be proved using purely formal properties of binomial coefficents, we have $\lambda_{-t}(x) = (1-t)^x$, and

$$\psi_t(x) = \frac{-t\frac{d}{dt}(1-t)^x}{(1-t)^x} = \frac{tx(1-t)^{x-1}}{(1-t)^x} = \frac{tx}{(1-t)} = \sum_{k\geq 1} xt^k,$$

hence $\psi^k(x) = x$ for all k. ∎

Lemma: *In an augmented special λ-ring, for $x \in R_n$, all $\psi^k(x) - k^n \cdot x$ lie in R_{n+1}.*

Proof: The ψ^k are λ-homomorphisms, and thus commute with the γ-operations; since they are S-linear, and $R_n R_m \subseteq R_{n+m}$, it suffices to show that

$$\psi^k(\gamma^n(x)) - k^n\gamma^n(x) \in R_{n+1} \quad \forall x \in \tilde{R}.$$

In complete analogy to the λ-verification principle, we have a γ-verification principle, which allows us to consider elements of γ-dimension one only. Therefore, let $x = \sum x_i$ with $\gamma_t(x_i) = 1 + x_i t$. Then $1 + x_i$ has λ-dimension one, hence $\psi^k(x_i) = (1+x_i)^k - 1$, and

$$\begin{aligned}
&\psi^k(\gamma^n(\textstyle\sum x_i)) - k^n\gamma^n(\textstyle\sum x_i)\\
&\quad = \psi^k(\sigma_n(x_1,\ldots,x_r)) - k^n\sigma_n(x_1,\ldots,x_r)\\
&\quad = \sigma_n(\psi^k(x_1),\ldots,\psi^k(x_r)) - k^n\sigma_n(x_1,\ldots,x_r)\\
&\quad = \sigma_n\Big((1+x_1)^k - 1,\ldots,(1+x_r)^k - 1\Big) - k^n\sigma_n(x_1,\ldots,x_r)\\
&\quad = k^n\sigma_n(x_1,\ldots,x_r) + \textit{higher terms} - k^n\sigma_n(x_1,\ldots,x_r)\,.
\end{aligned}$$

This is a symmetric polynomial of degree bigger than n, and thus an element of R_{n+1}. ∎

Definition: *The γ-filtration is called locally nilpotent, if for every $x \in \tilde{R}$, there exists an $N \in \mathbb{N}$, such that $\gamma^{i_1}(x)\cdots\gamma^{i_r}(x) = 0$ whenever $\sum i_\nu > N$. It is called nilpotent, if there exists an $N \in \mathbb{N}$, such that $R_n = 0$ for all $n > N$.*

Definition: $Z_n\tilde{R} = \ker\Big[(\psi^k - k^n)\cdots(\psi^k - k) : \tilde{R} \to \tilde{R}\Big]$.

Corollary: *If the γ-filtration is locally nilpotent, then $\tilde{R} = \bigcup Z_n\tilde{R}$.*

Proof: Every $x \in \tilde{R}$ generates a sub-λ-ring of R with nilpotent γ-filtration, and in such a ring the corollary is immediate from the lemma. ∎

Theorem 1: *Let R be an augmented special λ-ring with locally nilpotent γ-filtration. Then*

$$\tilde{R} \otimes \mathbb{Q} = \bigoplus_{i=1}^{\infty} V_i,$$

where V_i is the k^i-eigenspace of $\psi^k \otimes 1$, $k > 1$. V_i does not depend on k.

Proof: We show that $Z_n\tilde{R} \otimes \mathbb{Q} \cong \oplus_{i=1}^n V_i$:

$$p_n = \prod_{i \neq n} \frac{\psi^k - k^i}{k^n - k^i} : Z_n\tilde{R} \otimes \mathbb{Q} \to V_n$$

is a projection with kernel $Z_{n-1}\tilde{R}$, because $\prod_{i=1}^n (\psi^k - k^i)$ vanishes on $Z_n\tilde{R}$; continue by induction. Now let ℓ and k be different numbers; we have to show that $V_i = \ker(\psi^k - k^i)$ coincides with $\ker(\psi^\ell - \ell^i)$. Define $Z_n\tilde{R} = \ker\prod_{j=1}^n \left(\psi^{\kappa_j} - \kappa_j^j\right)$ with $\kappa_j = k$ for $j \neq i$, and $\kappa_i = \ell$. As above, we have $\bigcup Z_n\tilde{R} = \tilde{R}$, and since $\prod_{j\neq i}(\psi^k - k^j) = \prod_{j \neq i}(k^i - k^j)$ is multiplication with a non-zero scalar on V_i, $(\psi^\ell - \ell^i)$ must vanish on $V_i \cap Z_n\tilde{R}$ for all n. Therefore $V_i = \bigcup\left(V_i \cap Z_n\tilde{R}\right)$ lies in the kernel of $(\psi^\ell - \ell^i)$. ∎

$K(X,A)$ as a special λ-ring

Let A be a commutative ring with unity, and X a finite pointed CW-complex. The K-cohomology group $K(X,A)$ is defined as

$$K(X,A) = [X, K_0(A) \times \mathrm{BGL}(A)^+],$$

where $[X,Y]$ denotes the set of all homotopy classes of base point preserving continuous maps from X to Y. The reduced K-cohomology is

$$\tilde{K}(X,A) = \ker\Big(K(X,A) \to K_0(A)\Big) = [X, \mathrm{BGL}(A)^+]\,.$$

The most important examples are of course the cases $X = S^n$, when

$$K(S^n,A) = [S^n, K_0(A) \times \mathrm{BGL}(A)^+] = \begin{cases} \pi^n(\mathrm{BGL}(A)^+) & \text{for } n > 0 \\ K_0(A) & \text{for } n = 0 \end{cases} = K_n(A).$$

The main result of this talk is

Theorem 2: *$K(X,A)$ is a special λ-ring with augmentation $K(X,A) \to \mathrm{H}^0(\mathrm{Spec}\ A, \mathbb{Z})$, whose γ-filtration is locally nilpotent. There are Adams operations*

$$\psi^k : K(X,A) \to K(X,A),$$

which are ring homomorphisms, and $K(X,A) \otimes \mathbb{Q} = \oplus V_i$, where V_i is the k^i-eigenspace of ψ^k. On $K_m(A)$, the ψ^k commute with the cup product $\cup : K_m(A) \times K_n(A) \to K_{m+n}(A)$.

Here, the augmentation $K(X,A) \to \mathrm{H}^0(\mathrm{Spec}\ A, \mathbb{Z})$ is given by the canonical projection to $K_0(A)$, followed by the homomorphism $K_0(A) \to \mathrm{H}^0(\mathrm{Spec}\ A, \mathbb{Z})$ assigning to every projective module on A its (local) rank, considered as a locally constant function from Spec A to $\mathbb{Z}$.

Corollary: *Let V be a regular quasi-projective scheme over a field. Then the groups $K_m(V)$ are special λ-rings, and their Adams operations commute with the graded product on $K_*(V) = \oplus K_m(V)$.*

Proof: For affine schemes, this is the theorem, and by Jouanolou's device ([J], Lemma 1.5 and Prop. 1.6, or [Q2], §7,4.2), the K-theory of every regular quasi-projective scheme over a field is isomorphic to the K-theory of a certain affine scheme. ∎

Definition: *For $K_m(V)$, $V_i = \mathrm{H}_{\mathcal{A}}^{2i-m}\big(V, \mathbb{Q}(i)\big)$ is the absolute cohomology of V.*

The idea for the proof of theorem 2 is, to relate $K(X,A)$ to the special λ-ring $R(\pi_1(X), A)$. This must be done in a functorial way, of course, because $\pi_1(X) = 0$ for the cases in which we are mostly interested.

Definition: *Let F, G be functors from the pointed homotopy category of finite CW-complexes to the category of pointed sets. A morphism of functors $\varphi : F \to G$ is called universal with respect to the spaces in a class C, if for each $Z \in C$, each morphism of functors $F \to [\,\cdot\,, Z]$ factors in a unique way over G.*

Example: The universal property of the $+$-construction ([G], theorem 2.5 or [H], theorem 2.2) is equivalent to saying that $[\,\cdot\,, \mathrm{BGL}(A)] \to [\,\cdot\,, \mathrm{BGL}(A)^+]$ is universal with respect to H-spaces.

(Recall that an H-space is a topological space X together with a product $\mu : X \times X \to X$, such that both multiplications by constants, i.e. the maps $\{*\} \times X \to X$ and $X \times \{*\} \to X$, are homotopy equivalent to the identity map.)

We shall show that there is a morphism of functors $\varphi: R(\pi_1(\cdot), A) \to K(\cdot, A)$, which is universal with respect to those H-spaces all of whose connected components are again H-spaces. In analogy to $\tilde{K}(X,A)$, we define

$$\tilde{R}(G,A) = \ker\Big(R(G,A) \to R(<1>,A) = K_0(A)\Big),$$

and

$$\tilde{R}_\oplus(G,A) = \ker\Big(R_\oplus(G,A) \to R_\oplus(<1>,A)\Big).$$

Lemma: *There is a morphism of functors $\tilde{\psi}: \tilde{R}_\oplus(\pi_1(\cdot), A) \to \tilde{K}(\cdot, A)$, which is universal with respect to H-spaces.*

Proof: Let $\rho: \pi_1(X) \to \mathrm{Aut}(P)$ be a representation on a projective module P. By definition, there exists a projective module Q, such that $P \oplus Q \cong A^n$ is a free A-module; therefore ρ can be extended to a homomorphism $\rho': \pi_1(X) \to Gl_n(A) \hookrightarrow Gl(A)$. By [M], lemma 3.2, ρ' is determined by ρ up to conjugation by $Gl(A)$. ρ' defines a map $\mathrm{B}\pi_1(X) \to \mathrm{BGL}(A)$, which we can compose with the canonical map $\mathrm{BGL}(A) \to \mathrm{BGL}(A)^+$ and the 2-coskeleton $X \to \mathrm{B}\pi_1(X)$, to get the desired map $\tilde{\psi}(\rho): X \to \mathrm{BGL}(A)^+$. Since ρ' is defined up to conjugation by $Gl(A)$, this map is well-defined up to the action of $K_1(A) = \pi_1(\mathrm{BGL}(A)^+)$ on $\mathrm{BGL}(A)^+$. But this action is trivial modulo homotopy, because $\mathrm{BGL}(A)^+$ is an H-space with respect to the product defined by the direct sum of matrices.

For the proof of universality, note that the group $\tilde{R}_\oplus(\pi_1(X), A)$ is generated, as we have just seen, by the monoid

$$M(X) = \varinjlim \left(\mathrm{Hom}\left(\pi_1(X), Gl_n(A)\right)_{/Gl_n(A)}\right),$$

where $Gl_n(A)$ acts by conjugation. Since $\mathrm{BGL}_n(A) = K(Gl_n(A), 1)$ is an Eilenberg-MacLane space, $\mathrm{Hom}\left(\pi_1(X), Gl_n(A)\right) = [X, \mathrm{BGL}_n(A)]$, and one easily concludes that $M(X) \to [X, \mathrm{BGL}(A)^+]$ is universal with respect to H-spaces by the universal property of the $+$-construction. This implies that $\tilde{R}_\oplus(\pi_1(\cdot), A) \to [\cdot, \mathrm{BGL}(A)^+] = \tilde{K}(\cdot, A)$ is universal with respect to H-spaces, too, because $[X, \mathrm{BGL}(A)^+]$ is already a group. ∎

Using this lemma, and the fact that $R_\oplus(\pi_1(X), A)$ is a λ-ring, one can easily show that $K(X,A)$ is a λ-ring. Unfortunately, this is not yet enough, because $R_\oplus(\pi_1(X), A)$ is no special λ-ring, so we still have to consider $R(\pi_1(X), A)$.

Lemma: *$\tilde{\psi}$ factors over a morphism of functors $\tilde{\varphi}: \tilde{R}(\pi_1(\cdot), A) \to \tilde{K}(\cdot, A)$, and $\tilde{\varphi}$ is universal with respect to H-spaces.*

Proof: We must show that $\tilde{\psi}(\rho)$ only depends on the class of ρ in $\tilde{R}(\pi_1(\cdot), A)$. For this we can assume without loss of generality that $X = \mathrm{B}G$ is the classifying space of a group, the map $X \to \mathrm{B}\pi_1(X)$ causing no trouble. So we must show that $\tilde{\psi}(\mathrm{B}G)$ respects exact sequences: Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of representations ρ', ρ, ρ''. We claim that $\tilde{\psi}(\mathrm{B}G)(\rho)$ is equal to $\tilde{\psi}(\mathrm{B}G)(\rho' \oplus \rho'')$. Adding appropriate modules, we can assume that we have an exact sequence $0 \to A^p \to A^{p+q} \to A^q \to 0$ with free modules, and that

$$\rho = \begin{pmatrix} \rho' & * \\ 0 & \rho'' \end{pmatrix} \quad \text{and} \quad \rho' \oplus \rho'' = \begin{pmatrix} \rho' & 0 \\ 0 & \rho'' \end{pmatrix}.$$

In this situation, $\tilde{\psi}(\mathrm{B}G)(\rho) = \tilde{\psi}(\mathrm{B}G)(\rho' \oplus \rho'')$, because of the following

Lemma: *Let* $Gl_{p,q}(A) = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in Gl_{p+q}(A)\right\}$, *and let*

$$f: X = \mathrm{B}(Gl(A) \times Gl(A)) \to Y = \varinjlim \mathrm{B}Gl_{p,q}(A)$$

be the map defined by the system of embeddings $Gl_p(A) \times Gl_q(A) \hookrightarrow Gl_{p,q}(A)$. *Then the induced map* $f^*: [Y, \mathrm{BGL}(A)^+] \to [X, \mathrm{BGL}(A)^+]$ *is injective.*

Proof: By [Q3], theorem 2′, f is a homology isomorphism. Replacing f by its mapping cylinder, we may assume that f is a cofibration, so we have the exact Puppe sequence

$$\cdots \to [SX, \mathrm{BGL}(A)^+] \to [C_f, \mathrm{BGL}(A)^+] \to [Y, \mathrm{BGL}(A)^+] \to [X, \mathrm{BGL}(A)^+],$$

where C_f is the mapping cone of f. It suffices therefore, to show that $[C_f, \mathrm{BGL}(A)^+]$ vanishes. $\mathrm{BGL}(A)^+$ being an H-space, $[C_f, \mathrm{BGL}(A)^+] = [C_f^+, \mathrm{BGL}(A)^+]$, and since f is a homology isomorphism, C_f is acyclic, hence $\pi_1(C_f)$ is perfect, and thus $\pi_1(C_f^+) = 0$, and $[C_f, \mathrm{BGL}(A)^+] = 0$. ∎

Now define $\varphi: R(\pi_1(\cdot), A) \to K(\cdot, A) = [\cdot, K_0(A) \times \mathrm{BGL}(A)^+]$ by setting $\varphi(X)(\rho)$ to $([P], \tilde{\varphi}(X)(\rho))$ for every representation ρ on a projective module P.

Lemma: φ *is universal with respect to those* H*-spaces all of whose connected components are again* H*-spaces.*

Proof: Let Z be such a space, and $\omega: R(\pi_1(\cdot), A) \to [\cdot, Z]$ a morphism of functors. Let $Z = \coprod_{\alpha \in \pi_0(Z)} Z_\alpha$ and $X = \coprod_{\beta \in \pi_0(X)} X_\beta$ be the decompositions of Z and a test space X into connected components, and choose a base point in each X_β. We have to find a map $K(X, A) \to [X, Z]$ extending $\omega(X)$, so let f be an element of $K(X, A) = [X, K_0(A) \times \mathrm{BGL}(A)^+]$. The X_β being connected, f maps each X_β to a single component $[P_\beta] \times \mathrm{BGL}(A)^+$, hence f is given locally by elements $f_\beta \in [X_\beta, \mathrm{BGL}(A)^+]$. Because of the universality of $\tilde{\varphi}$, and since $Z_\beta = Z_{\omega(*)([P_\beta])}$ is an H-space, we get canonical maps $g_\beta: [X_\beta, \mathrm{BGL}(A)^+] \to [X_\beta, Z_\beta]$, which can be glued together to give the final map. ∎

Corollary: *Each morphism of functors* $\lambda: R(\pi_1(\cdot), A) \to R(\pi_1(\cdot), A)$ *has a unique extension* $K(\cdot, A) \to K(\cdot, A)$.

The *proof* is simple diagram chasing, because $K(X, A) = [X, K_0(A) \times \mathrm{BGL}(A)^+]$, and all connected components of $K_0(A) \times \mathrm{BGL}(A)^+$ are homeomorphic to $\mathrm{BGL}(A)^+$, and therefore are H-spaces. ∎

Similarly, each morphism of functors $\mu: R(\pi_1(\cdot), A) \times R(\pi_1(\cdot), A) \to R(\pi_1(\cdot), A)$ has a unique extension $K(\cdot, A) \times K(\cdot, A) \to K(\cdot, A)$. With this we are ready for the

Proof of theorem 2: It is clear that the property of being a special λ-ring extends from $R(\pi_1(X), A)$ to $K(X, A)$, because all axioms can be translated into existence and equality of certain maps, and these maps are functorial for $R(\pi_1(\cdot), A)$. In order to show the local nilpotency of the γ-filtration, it suffices to consider the cases $x \in \tilde{K}(X, A)$, and $x \in [X, K^0(A)]$. Let first x be an element of $\tilde{K}(X, A) = [X, \mathrm{BGL}(A)^+]$. Since X is a finite CW-complex, x already lies in some $[X, \mathrm{BGL}_n(A)^+]$. We start by showing that γ^k is trivial on $[X, \mathrm{BGL}_n(A)^+]$ for $k > n$. For this it suffices to show that γ^k is trivial on all elements of the form $[\rho] - [n]$ in $\tilde{R}(\pi_1(X), A)$, where ρ is an arbitrary, and n the trivial representation of degree n. For such an element,

$$\gamma_t([\rho]-[n]) = \gamma_t([\rho])/\gamma_t(n\cdot[1]) = \gamma_t([\rho])\cdot(1-t)^n.$$

Since $[\rho]$ is of degree n, and the λ-operations on $R(G,A)$ are exterior powers, $\lambda_t([\rho])$ is a polynomial of degree n, and $\gamma_t([\rho]) = \lambda_{t/(1-t)}([\rho])$ has $(1-t)^n$ as its denominator, hence $\gamma_t([\rho]-[n])$ is a polynomial of degree at most n. Thus $\gamma_t(x)$ is a polynomial for each x, in particular $\gamma_t(-x)$ is a polynomial, too, and $\gamma_t(x)\cdot\gamma_t(-x) = \gamma_t(0) = 1$, i.e., we have two polynomials whose product is one, and this means by [A], lemma 3.1.4, that there exists an N such that $\gamma^1(x)^{\nu_1}\cdots\gamma^n(x)^{\nu_n}$ vanishes whenever $\sum j\nu_j > N$. For the case $x\in[X,K_0(A)]$, it suffices to consider the ring $K_0(A)$ itself, because $[X,K_0(A)]$ is a (possibly empty) sum of copies of $K_0(A)$. But for $x\in K_0(A)$, $\gamma_t(x)$ is also a polynomial, because x represents an A-module of finite rank, so the same argument as above can be applied.

Therefore $K(X,A)$ is a special λ-ring, and thus has Adams operations. In order to show that the kernel of the augmentation map decomposes into a direct sum of their eigenspaces, we must show that $\mathrm{H}^0(\mathrm{Spec}\ A,\mathbb{Z})$ is a binomial λ-ring, but this is clear, because $\mathrm{H}^0(\mathrm{Spec}\ A,\mathbb{Z})$ consists of locally constant functions with values in $\mathbb{Z}$, which behave locally like integers. Since the Adams operations are trivial on a binomial λ-ring, $\mathrm{H}^0(\mathrm{Spec}\ A,\mathbb{Z})$ is a direct summand of their 1-eigenspace, hence the decomposition can be extended from the kernel of the augmentation map to the whole of $K(X,A)$.

Finally, we still have to show that the Adams operations on $K_m(A)$ commute with the cup product, that is we have to show that

$$\psi^k(x\cup y) = \psi^k(x)\cup\psi^k(y) \qquad \forall x\in K_m(A),\ y\in K_n(A).$$

This is easily seen from the diagram

$$\begin{array}{ccc}
K(S^m,A)\times K(S^n,A) & \xrightarrow{\psi^k\times\psi^k} & K(S^m,A)\times K(S^n,A)\\
\downarrow{\scriptstyle p_1^*\times p_2^*} & & {\scriptstyle p_1^*\times p_2^*}\downarrow\\
K(S^m\times S^n,A)\times K(S^m\times S^n,A) & \xrightarrow{\psi^k\times\psi^k} & K(S^m\times S^n,A)\times K(S^m\times S^n,A)\\
\downarrow{\scriptstyle \mu=\text{multiplication}} & & {\scriptstyle\mu}\downarrow\\
K(S^m\times S^n,A) & \xrightarrow{\psi^k} & K(S^m\times S^n,A)\\
\downarrow & & \downarrow\\
K(S^m\wedge S^n,A) & \xrightarrow{\psi^k} & K(S^m\wedge S^n,A)\\
\| & & \|\\
K(S^{m+n},A) & \xrightarrow{\psi^k} & K(S^{m+n},A),
\end{array}$$

where the upper and the third square commute by functoriality, and the second one, because ψ^k is a ring homomorphism. ■

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