Deligne's Conjecture

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In the following we want to recall briefly Deligne's conjecture on critical values of L-functions (already mentioned in chapter I) working in the language of motives as the original paper |D| does, and we want to point out the connection to Beilinson's version of this conjecture. Beilinson's definition of periods corresponds to the classical point of view: A period should be a determinant of a matrix, whose entries are integrals of algebraic differential forms against betti-rational cycles. This definition is not appropriate for a motivic formulation, especially if one considers motives with coefficients, since the "classical" period of a motive is related to a critical L-value of the dual motive (comp. |D|, §§ 6, 7.4). Therefore Deligne defines periods in such a way, that only the motive itself - not its dual - appears in the formulation of his conjecture.

1. Motives

1.1. We refer to |D| for possible definitions of the category of motives. In the following we will only deal with motives of the form

$$M = H^i(X)(m) \ , \quad m \in \mathbb{Z} \quad \text{and} \quad 0 \leqq i \leqq 2d \ ,$$

where $X/\mathbb{Q}$ is a d-dimensional smooth projective variety. These M's exist as motives, if we define motives via absolute Hodge cycles. But we should deal with Grothendieck motives in order to get a motivic formulation of the full Beilinson-conjecture. In both cases we can attach to M the family

$$(M_B, M_{DR}, M_\ell, I_{DR}, I_\ell)$$

of realizations and comparison isomorphisms:

- The Betti realization M_B is a $\mathbb{Q}$-vector space with a Hodge decomposition $M_B \otimes \mathbb{C} = \bigoplus_{p,q} M^{p,q}$ and an action of F_∞, where F_∞ denotes the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

- The de Rham realization M_{DR} is a $\mathbb{Q}$-vector space with a decreasing Hodge filtration: $\ldots \supset F^p M_{DR} \supset F^{p+1} M_{DR} \supset \ldots$

- The ℓ-adic realizations M_ℓ are $\mathbb{Q}_\ell$-vector spaces with an action of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, they are strictly compatible as ℓ varies.

- The comparison isomorphism $I_{DR} : M_B \otimes \mathbb{C} \xrightarrow{\sim} M_{DR} \otimes \mathbb{C}$ respects the Hodge filtration, i.e.

$$I_{DR}\Big(\bigoplus_{p'\geq p} M^{p',q'}\Big) = F^p M_{DR} \otimes \mathbb{C}.$$

- The comparison isomorphisms $I_\ell : M_B \otimes \mathbb{Q}_\ell \xrightarrow{\sim} M_\ell$ are equivariant with respect to F_∞ (we define $\overline{\mathbb{Q}}$ to be the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$, thus $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$).

1.2. We denote the trivial motive by $\mathbb{Q} := H^0(\mathrm{Spec}\ \mathbb{Q})$. The <u>Tate twist</u> $M \to M(n)$ (where $M(n) := M \otimes H^2(\mathbb{P}^1/\mathbb{Q})^{\otimes(-n)}$ if $n < 0$) acts on the realizations of a motive in the following way:

- $M_B(n)$ is the $\mathbb{Q}$-subspace $M_B \otimes_{\mathbb{Q}} \mathbb{Q}\cdot(2\pi i)^n$ of $M_B \otimes \mathbb{C}$,
- $M(n)^{p,q} = M^{p-n,q-n}$,
- $F_\infty | M_B(n) = (-1)^n \cdot F_\infty | M_B$, since F_∞ reverses the orientation of $\mathbb{P}^1(\mathbb{C})$ and therefore acts as -1 on $\mathbb{Q}_B(-1) = H^2_B(\mathbb{P}^1(\mathbb{C}),\mathbb{Q})$,
- $M_{DR}(n) = M_{DR}$,
- $F^p(M_{DR}(n)) = F^{p+n} M_{DR}$,
- $M_\ell(n) = M_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(\mathbb{G}_m)^{\otimes n}$ as $G_{\mathbb{Q}}$-module, where $T_\ell(\mathbb{G}_m) = \varprojlim_r \mu_{\ell^r}$ is the Tate module of the multiplicative group,
- $I_\ell(n)$ is essentially $I_\ell \otimes \iota^{\otimes n}$, where ι is the isomorphism $2\pi i\,\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong T_\ell(\mathbb{G}_m)$ given by the inclusion $\overline{\mathbb{Q}} \subset \mathbb{C}$,
- $I_{DR}(n) = I_{DR}$ (observe that $M_B(n) \otimes \mathbb{C} = M_B \otimes \mathbb{C}$ and $M_{DR}(n) = M_{DR}$).

1.3. For a prime p we denote by I_p the inertia group of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and by $F_p \in G_{\mathbb{Q}_p}/I_p$ the geometric Frobenius element.

The local L-function of M at p is by definition

$$L_p(M,s) = \det(\mathrm{id} - p^{-s}\cdot F_p | M_\ell^{I_p})^{-1}.$$

The L-function of the motive M is the Euler product

(1) $$L(M,s) = \prod_p L_p(M,s),$$

which converges if $\mathrm{Re}\, s \gg 0$ and is assumed to have an analytic continuation to the whole s-plane.

The relation $F_p | M_\ell(n) = p^{-n}\cdot F_p | M_\ell$ implies

(2) $$L(M(n),s) = L(M,n+s).$$

Therefore the study of the values of motivic L-functions at integers may be reduced inside the category of all motives to the L-value $L(M) := L(M,0)$.

2. Duality, functional equation, critical values

The dual of a motive M is by definition a motive $\check{M}$, whose realizations are dual (as vector spaces) resp. contragredient (as F_∞ or $G_{\mathbb{Q}}$-modules) to the realizations of M. The Hodge filtration of $\check{M}_{DR}$ is defined by:

(3) $$F^p\check{M}_{DR} = \{\phi \in \mathrm{Hom}_{\mathbb{Q}}(M_{DR},\mathbb{Q}) \mid \phi(F^{1-p}M_{DR}) = 0\}.$$

<u>2.1.Lemma:</u> If X is as above and $M = H^i(X)(m)$, then $\check{M} = H^i(X)(i-m)$.

<u>Proof:</u> $H^{2d}(X)(d) \cong H^{2d}(\mathbb{P}^d)(d) \cong H^{2d}((\mathbb{P}^1)^d)(d) \cong (H^2(\mathbb{P}^1)(1))^{\otimes d}$ is the trivial motive by the definition of the Tate twist. Therefore cup product yields a nondegenerate (Poincaré-duality) pairing

$$H^i(X)(m) \times H^{2d-i}(X)(d-m) \to H^{2d}(X)(d).$$

The characteristic class $\eta \in H^2(X)(1)$ of a hyperplane section (defined over $\mathbb{Q}$!) generates a trivial motive of rank one. Therefore the hard Lefschetz theorem implies that we have an isomorphism of motives: (we may assume $i \leq d$)

$$H^i(X)(i-m) \xrightarrow[\sim]{\cup\, \eta^{d-i}} H^{2d-i}(X)(d-m).$$

The claim follows by combining both facts.

2.2. The L-factor at the archimedean place $L_\infty(M,s)$ only depends on the Betti realization (including Hodge decomposition and F_∞-action) of the motive M. $L_\infty(H^i(X),s)$ was defined in ch. I, and the relation (2) tells us what the definition for a general $H^i(X)(m)$ has to be. If we put $\Lambda(M,s) := L_\infty(M,s) \cdot L(M,s)$, then we may restate the conjectured functional equation (ch.I) in the following form using Lemma 2.1.:

$$\Lambda(\check{M},1-s) = \varepsilon(M,s) \cdot \Lambda(M,s). \tag{4}$$

The ε-factor $\varepsilon(M,s)$ is the product of an algebraic constant and an exponential factor f^s taking rational values at integer arguments.

2.3. Definition: a) $m \in \mathbb{Z}$ is called critical for M, if neither $L_\infty(M,s)$ nor $L_\infty(\check{M},1-s)$ has a pole at $s = m$.

b) M is called critical, if 0 is critical for M.

2.4. Proposition: (|D|,1.3.) M is critical if and only if the following two conditions are fulfilled:

1. $M^{p,q} = 0$ unless $p = q$ or $p < 0 \leq q$ or $q < 0 \leq p$.
2. If $M^{p,p} \neq 0$, then F_∞ acts as (-1) if $p \geq 0$ and as $(+1)$ if $p<0$.

3. Deligne's periods

In this section we assume:

- the motive M is homogeneous of a weight w, i.e. $M^{p,q} = 0$ if $p+q \neq w$,
- F_∞ acts as a scalar on $M^{p,p}$ if $w = 2p$.

The second condition is fulfilled if M is critical. Note that $H^i(X)(m)$ has the weight $w = i - 2m$.

3.1. According to the action of the involution F_∞ we decompose the Betti realization into eigenspaces: $M_B = M_B^+ \oplus M_B^-$. The meaning of assumption 2 is that either $M^{p,p} = (M^{p,p})^+$ or $M^{p,p} = (M^{p,p})^-$.

3.2. The assumptions imply that there exist filtration steps $F^\pm M_{DR}$ of the de Rham realization satisfying

$$(F^\pm M_{DR}) \otimes \mathbb{C} = I_{DR}\Big(\bigoplus_{p>q} M^{p,q} \oplus (M^{p,p})^\pm\Big).$$

We put $M_{DR}^\pm := M_{DR} / F^\mp M_{DR}$. Since F_∞ permutes $M^{p,q}$ and $M^{q,p}$, the composite maps

$$I^\pm : \quad M_B^\pm \otimes \mathbb{C} \to M_B \otimes \mathbb{C} \xrightarrow[\sim]{I_{DR}} M_{DR} \otimes \mathbb{C} \twoheadrightarrow M_{DR}^\pm \otimes \mathbb{C}$$

are isomorphisms.

Let $c^{\pm}_{Del}(M)$ be the determinants of $I^{\pm}$ with respect to rational bases of the $\mathbb{Q}$-vector spaces $M^{\pm}_B$ and $M^{\pm}_{DR}$. They are well defined as elements of $\mathbb{C}^{\times}/\mathbb{Q}^{\times}$ and we call them Deligne's periods of the motive M.

3.3. The complex conjugation on $\mathbb{C}$ induces maps

$$k_B : \quad M_B \otimes \mathbb{C} \to M_B \otimes \mathbb{C},$$
$$k_{DR} : \quad M_{DR} \otimes \mathbb{C} \to M_{DR} \otimes \mathbb{C}.$$

Lemma: (|D|, 1.4.) If M is defined over $\mathbb{R}$, then we have

$$I_{DR} \circ F_{\infty} \circ k_B = k_{DR} \circ I_{DR}.$$

Conclusion: I^{+} (resp. $i \cdot I^{-}$) is defined over $\mathbb{R}$, i.e. $I^{+} : M^{+}_B \otimes \mathbb{R} \xrightarrow{\sim} M^{+}_{DR} \otimes \mathbb{R}$. It follows that $c^{+}_{Del}(M) \in \mathbb{R}^{\times}/\mathbb{Q}^{\times}$.

3.4. Conjecture (Deligne) If M is critical, then

$$L(M,0) \in \mathbb{Q} \cdot c^{+}_{Del}(M).$$

Remark: If $w \neq -1$, i.e. if 0 is not the central point, one conjectures $L(M,0) \in \mathbb{Q}^{\times} \cdot c^{+}_{Del}$. If $w \leqq -3$ the non-vanishing of $L(M,0)$ is a consequence of the absolute convergence of (1) at $s = 0$. The case $w \geqq 1$ may be reduced to this case by the functional equation (4) and the definition of "critical".

4. Beilinson's period

Let $M/\mathbb{Q}$ be a motive of the form $M = H^i(X)(m)$. We put $n := 1 + i - m$. We define $\check{I}$ to be the composite map

$$F^n H^i_{DR}(X/\mathbb{R}) \hookrightarrow H^i_{DR}(X/\mathbb{R}) \xrightarrow[\sim]{I_{DR}^{-1}} H^i_B(X(\mathbb{C}),\mathbb{R}(n-1)) \twoheadrightarrow$$
$$\twoheadrightarrow H^i_B(X(\mathbb{C}),\mathbb{R}(n-1))^{(-1)^{n-1}}.$$

It has been shown in chapter I, that $\check{I}$ is an isomorphism if M is critical and if M has a nonnegative weight. But it is a corollary of Proposition 4.3. below, that this weight condition is unnecessary. Therefore we may define:

4.1. Definition: If M is critical, then the determinant (with respect to rational bases) of the comparison isomorphism

$$\check{I} : \quad F^n H^i_{DR}(X/\mathbb{R}) \xrightarrow{\sim} H^i_B(X(\mathbb{C}),\mathbb{R}(n-1))^{(-1)^{n-1}}$$

is called Beilinsons's period $c_{Beil}(M)$ of the motive M. It is well defined as an element of the quotient $\mathbb{R}^{\times}/\mathbb{Q}^{\times}$.

4.2. The definition of $F^{-}M_{DR}$ allows us to restate Proposition 2.4. in the following simple form:

Proposition: Let M be a motive satisfying the conditions of § 3. Then M is critical if and only if $F^{-}M_{DR} = F^{0}M_{DR}$.

4.3. Proposition: If M is critical, then $c^{+}_{Del}(M) = c_{Beil}(M)$.

Proof: Since the determinant of a linear transformation equals the determinant of the transposed map, $c^{+}_{Del}(M)$ is as well the determinant of the transposed map

$$I^{t} : \quad (M^{+}_{DR})^{\vee} \otimes \mathbb{R} \xrightarrow{\sim} (M^{+}_{B})^{\vee} \otimes \mathbb{R}.$$

Proposition 4.2., Lemma 2.1., formula (3) and the description of the Tate-twist tell us:

$$\begin{aligned}(M^{+}_{DR})^{\vee} &= (M_{DR}/F^{-}M_{DR})^{\vee} = (M_{DR}/F^{0}M_{DR})^{\vee} = F^{1}\,\check{M}_{DR} \\ &= F^{1}(H^{i}(X)(i-m)_{DR}) = F^{1+i-m}\,H^{i}_{DR}(X) \\ &= F^{n}\,H^{i}_{DR}(X) \quad \text{and} \\ (M^{+}_{B})^{\vee} &= \check{M}^{+}_{B} = H^{i}(X)(n-1)^{+}_{B} \\ &= H^{i}_{B}(X(\mathbb{C}),\mathbb{Q}(n-1))^{(-1)^{n-1}}.\end{aligned}$$

Since the comparison isomorphisms respect the Poincaré-duality and the hard-Lefschetz-isomorphism of Lemma 2.1., the transposed map of I_{DR} can be identified with the map I^{-1}_{DR} of the dual motive. Therefore $I^{t} = \check{I}$ and the claim follows.

Remark: If we consider motives with coefficients $E \neq \mathbb{Q}$, Lemma 2.1. will no longer be true, since cup-product does not respect the E-action and $H^{2d}(X)(d)$ only has a $\mathbb{Q}$-structure. E.g. if X is an elliptic curve with complex multiplication by an order of the imaginary quadratic field E, then $H^{1}(X)(1)$ and the dual of $H^{1}(X)$ have complex conjugate E-structures. Beilinson's conjecture for motives with coefficients therefore has to relate $c_{Beil}(M)$ to a special L-value of the dual motive $\check{M}$.

Reference:

|D| P. Deligne: Valeurs de fonctions L et périodes d'intégrales. Proc. Symp. Pure Math. 33, vol. 2, p. 313 - 343.