# On the Blumberg–Mandell Künneth theorem for TP

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**Abstract**

We give a new proof of the recent Künneth theorem for periodic topological cyclic homology (TP) of smooth and proper dg categories over perfect fields of characteristic $p > 0$ due to Blumberg and Mandell. Our result is slightly stronger and implies a finiteness theorem for topological cyclic homology (TC) of such categories.

**Key Words.** Künneth theorems, the Tate construction, topological Hochschild homology, periodic topological cyclic homology.

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## 1 Introduction

Let $k$ be a perfect field and let $A$ be a commutative $k$-algebra. In [Hes16], Hesselholt studies the *periodic topological cyclic homology* $\mathrm{TP}(A)$, defined as the Tate construction $\mathrm{THH}(A)^{tS^1}$ of the $S^1$-action on the topological Hochschild homology spectrum $\mathrm{THH}(A)$. In characteristic 0, the analogous construction, periodic cyclic homology $\mathrm{HP}(\cdot/k) = \mathrm{HH}(\cdot/k)^{tS^1}$, is related to (2-periodic) de Rham cohomology. In characteristic $p > 0$, which we will assume henceforth, $\mathrm{TP}(A)$ is of significant arithmetic interest: by forthcoming work of Bhatt, Morrow, and Scholze [BMS], the construction $\mathrm{TP}(A)$ is closely related to the *crystalline cohomology* of $A$ over $k$. For instance, one has $\pi_*(\mathrm{TP}(k)) \simeq W(k)[x^{\pm 1}]$ with $|x| = 2$, a 2-periodic form of the coefficient ring of crystalline cohomology.

The construction $A \mapsto \mathrm{TP}(A)$ globalizes to schemes, and in fact TP of a scheme $X$ is determined entirely by the dg category of perfect complexes on $X$. Let $\mathcal{C}$ be a $k$-linear dg category. In this case, one similarly defines the spectrum $\mathrm{TP}(\mathcal{C}) = \mathrm{THH}(\mathcal{C})^{tS^1}$. Thus TP defines a functor from $k$-linear dg categories to $\mathrm{TP}(k)$-module spectra. In a similar way that crystalline cohomology is a lift to characteristic zero of de Rham cohomology, $\mathrm{TP}(\mathcal{C})$ is an integral lift of the periodic cyclic homology $\mathrm{HP}(\mathcal{C}/k)$ (see Theorem 3.4, which is due to [BMS]). With respect to the tensor product on $k$-linear dg categories, the construction $\mathcal{C} \mapsto \mathrm{TP}(\mathcal{C})$ is a lax symmetric monoidal functor.

Many cohomology theories for schemes, such as the crystalline theory mentioned above, satisfy a Künneth formula. In [BM17], Blumberg and Mandell prove the following result.

**Theorem 1.1** (Blumberg–Mandell)**.** *Let $k$ be a perfect field of characteristic $p > 0$.*

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1. *If $\mathcal{C}$ is a smooth and proper $k$-linear dg category, then* $\mathrm{TP}(\mathcal{C})$ *is compact as a* $\mathrm{TP}(k)$*-module spectrum.*

2. *(Künneth formula) If $\mathcal{C}$ and $\mathcal{D}$ are smooth and proper $k$-linear dg categories, then the natural map*
$$\mathrm{TP}(\mathcal{C}) \otimes_{\mathrm{TP}(k)} \mathrm{TP}(\mathcal{D}) \to \mathrm{TP}(\mathcal{C} \otimes_k \mathcal{D})$$
*is an equivalence.*

This result is the key ingredient in Tabuada's proof [Tab17] of the fact that the category of noncommutative numerical motives is abelian semi-simple. This is a generalization of Jannsen's theorem [Jan92] to the noncommutative case. With this application in mind the above theorem was suggested by Tabuada.

In this paper, we give a short proof of a stronger form of Theorem 1.1. Let $(\mathcal{A}, \otimes, \mathbb{1})$ be a symmetric monoidal, stable $\infty$-category with biexact tensor product (in short: *stably symmetric monoidal $\infty$-category*). Let Sp be the $\infty$-category of spectra and let $F\colon \mathcal{A} \to \mathrm{Sp}$ be a lax symmetric monoidal, exact functor. Note that $F(\mathbb{1})$ is naturally an $\mathbb{E}_\infty$-ring and $F(X)$ is an $F(\mathbb{1})$-module for any $X \in \mathcal{A}$. For any $X, Y \in \mathcal{A}$, we have a natural map
$$F(X) \otimes_{F(\mathbb{1})} F(Y) \to F(X \otimes Y). \tag{1}$$

We now use the following definition, which agrees with the usual definition for modules over a ring spectrum.

**Definition 1.2.** An object $X \in \mathcal{A}$ is *perfect* if it belongs to the thick subcategory generated by the unit.

If $X$ is perfect, then, by a thick subcategory argument, (1) is an equivalence for every $Y \in \mathcal{A}$ and $F(X)$ is perfect as an $F(\mathbb{1})$-module.

Note that for $\mathcal{A} \simeq \mathrm{Mod}_R$, the $\infty$-category of modules over an $\mathbb{E}_\infty$-ring spectrum $R$, an object is perfect if and only if it is dualizable if and only if it is compact. For general $\mathcal{A}$, this is not typically the case. We will be interested in the following example: let $\mathrm{Sp}^{BS^1}$ denote the $\infty$-category of spectra equipped with an $S^1$-action. Then $\mathrm{THH}(k)$ naturally defines a commutative algebra object in $\mathrm{Sp}^{BS^1}$ and we take $\mathcal{A} = \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{BS^1})$. When $\mathcal{C}$ is any $k$-linear dg category, $\mathrm{THH}(\mathcal{C})$ defines an object of $\mathcal{A}$. We then consider the functor $F\colon \mathcal{A} \to \mathrm{Sp}$ sending $X \mapsto X^{tS^1}$. With this in mind, Theorem 1.1 is implied by the following result.

**Theorem 1.3.** *Let $k$ be a perfect field of characteristic $p > 0$. Any dualizable object in the symmetric monoidal $\infty$-category* $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{BS^1})$ *is perfect. In particular, if $\mathcal{C}$ is a smooth and proper dg category over $k$, then* $\mathrm{THH}(\mathcal{C}) \in \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{BS^1})$ *is perfect.*

We note also that our result implies that in Theorem 1.1, one only needs one of $\mathcal{C}, \mathcal{D}$ to be smooth and proper, and one can replace TP with $\mathrm{THH}^{tH}$ or with $\mathrm{THH}^{hH}$ for any closed subgroup $H$ of $S^1$. In particular, taking $H = S^1$, we obtain a Künneth theorem for $\mathrm{TC}^- \overset{\mathrm{def}}{=} \mathrm{THH}^{hS^1}$. We deduce below Theorem 1.3 directly from the *regularity* of $\pi_*\mathrm{THH}(k)$. The argument also works for dg categories over the localization $A$ of a ring of integers in a number field at a prime over $p$ but with THH replaced with THH relative to $S^0[q]$ where $S^0[q] \to HA$ sends $q$ to a chosen uniformizing parameter $\pi$ (see Corollary 3.7).

We show, using the formula of Nikolaus–Scholze [NS17], that our strengthened version of the Blumberg–Mandell theorem has an application to a finiteness theorem for topological cyclic homology. When one works with smooth and proper schemes, stronger results are known by work of Geisser and Hesselholt [GH99], but our results seem to be new in the generality of smooth and proper dg categories.

**Theorem 1.4.** *If $\mathcal{C}$ is a smooth and proper dg category over a finite field $k$ of characteristic $p$, then $\mathrm{TC}(\mathcal{C})$ is perfect as an $H\mathbb{Z}_p$-module.*[1]

Finally, given the theorems above and the fact that for $\mathcal{C}$ smooth and proper $\mathrm{THH}(\mathcal{C})$ is dualizable as a $\mathrm{THH}(k)$-module spectrum in the $\infty$-category CycSp of cyclotomic spectra, a very natural question is to ask whether $\mathrm{THH}(\mathcal{C})$ is perfect as a $\mathrm{THH}(k)$-module in CycSp. We give an example to show that this is not the case.

**Theorem 1.5.** *If $X$ is a supersingular K3 surface over a perfect field $k$ of characteristic $p > 0$, then $\mathrm{THH}(X)$ is not perfect in $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$.*

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## 2 Symmetric monoidal ∞-categories

As indicated in the introduction, our approach is based on the notion of *perfectness* (Definition 1.2) in a stably symmetric monoidal $\infty$-category. Any perfect object is dualizable, but in general the converse need not hold. In this section, we give a criterion (Theorem 2.15) for when dualizable objects *are* perfect in symmetric monoidal $\infty$-categories of spectra with the action of a compact Lie group.

We will need some preliminaries on presentably symmetric monoidal stable $\infty$-categories[2]. We refer to [MNN17] for more details. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be one such. In this case, one has an adjunction

$$\mathrm{Mod}_{\mathrm{End}_{\mathcal{C}}(\mathbb{1})} \rightleftarrows \mathcal{C}, \tag{2}$$

where the left adjoint $\cdot \otimes_{\mathrm{End}_{\mathcal{C}}(\mathbb{1})} \mathbb{1} \colon \mathrm{Mod}_{\mathrm{End}_{\mathcal{C}}(\mathbb{1})} \to \mathcal{C}$ is symmetric monoidal and the right adjoint $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \cdot) \colon \mathcal{C} \to \mathrm{Mod}_{\mathrm{End}_{\mathcal{C}}(\mathbb{1})}$ is lax symmetric monoidal. Here, $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is an $\mathbb{E}_\infty$-ring spectrum and $\mathrm{Mod}_{\mathrm{End}_{\mathcal{C}}(\mathbb{1})}$ is the $\infty$-category of $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$-modules in Sp. One knows (cf. [SS03] and [Lur14, Sec. 7.1.2]) that the adjunction is an equivalence precisely when $\mathbb{1} \in \mathcal{C}$ is a compact generator. We need a definition which applies in some situations when $\mathbb{1}$ is a generator, but is no longer required to be compact.

**Definition 2.1** (Cf. [MNN17, Def. 7.7])**.** The presentably symmetric monoidal stable $\infty$-category $\mathcal{C}$ is *unipotent* if (2) is a localization, i.e., if the counit map $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X) \otimes_{\mathrm{End}_{\mathcal{C}}(\mathbb{1})} \mathbb{1} \to X$ is an equivalence for every $X \in \mathcal{C}$.

**Remark 2.2.** If $\mathcal{C}$ is a symmetric monoidal localization (see [Lur14, Definition 2.2.1.6 and Example 2.2.1.7]) of the module category of an $\mathbb{E}_\infty$-ring, then $\mathcal{C}$ is unipotent. It follows that if $\mathcal{C}$ is unipotent and if $A \in \mathrm{CAlg}(\mathcal{C})$, then $\mathrm{Mod}_A(\mathcal{C})$ is unipotent too.

[1] Note that $\mathrm{TC}(\mathcal{C})$ is an $H\mathbb{Z}_p$-module since $H\mathbb{Z}_p \simeq \tau_{\geqslant 0}\mathrm{TC}(k)$.

[2] By *presentably symmetric monoidal* we mean that the underlying $\infty$-category is presentable and the tensor products preserves colimits in both variables seperately.

We will apply this definition in the following scenario. Let $X$ be a space. We can then form the presentable stable $\infty$-category $\mathrm{Sp}^X = \mathrm{Fun}(X, \mathrm{Sp})$ of spectra parametrized over $X$. We regard $\mathrm{Sp}^X$ as a symmetric monoidal $\infty$-category with the pointwise tensor product. Given an $\mathbb{E}_\infty$-algebra $R$ in $\mathrm{Sp}^X$, we can form the presentably symmetric monoidal stable $\infty$-category $\mathrm{Mod}_R(\mathrm{Sp}^X)$ of $R$-modules in $\mathrm{Sp}^X$. When $R$ is given by a constant diagram, then there is a natural equivalence $\mathrm{Mod}_R(\mathrm{Sp}^X) \simeq \mathrm{Fun}(X, \mathrm{Mod}_R)$ and the endomorphisms of the unit are given by the function spectrum $F(X_+; R)$. In this case, one has the following unipotence criterion.

**Theorem 2.3** ([MNN17, Theorem 7.35])**.** *Let $R$ be an $\mathbb{E}_\infty$-ring such that $\pi_*(R)$ is concentrated in even degrees. Let $G$ be a compact, connected Lie group such that the cohomology $\mathrm{H}^*(BG; \pi_0(R))$ is a polynomial algebra. In this case, the presentably symmetric monoidal $\infty$-category $\mathrm{Fun}(BG, \mathrm{Mod}_R)$ is unipotent. Furthermore, there are classes $x_1, \ldots, x_t \in \pi_*(F(BG_+; R))$ such that*

1. *$x_1, \ldots, x_t$ is a regular sequence in $\pi_*(R^{hG})$;*
2. *there is an equivalence $R \simeq F(BG_+; R)/(x_1, \ldots, x_t)$ and $\pi_*(R) \simeq \pi_*(R^{hG})/(x_1, \ldots, x_t)$;*
3. *$\mathrm{Fun}(BG, \mathrm{Mod}_R)$ is identified with the $\infty$-category of $(x_1, \ldots, x_t)$-complete $F(BG_+; R)$-modules (i.e., those that are Bousfield local with respect to the iterated cofiber $F(BG_+; R)/(x_1, \ldots, x_t)$).*

The existence of the classes $x_1, \ldots, x_t$ with the desired properties appears in the proof, rather than the statement, of [MNN17, Theorem 7.35]. Note that the unipotence assertion implies that if $M \in \mathrm{Fun}(BG, \mathrm{Mod}_R)$, then one has an equivalence of $R$-modules $M \simeq M^{hG} \otimes_{F(BG_+;R)} R \simeq M^{hG}/(x_1, \ldots, x_t)$.

**Remark 2.4.** We refer also to [MNN17, Theorem 8.13] for an example (essentially due to Hodgkin, Snaith, and McLeod) where $\mathrm{Fun}(BG, \mathrm{Mod}_R)$ is unipotent although the polynomiality condition above is far from satisfied.

We will need a version of this result when $R$ has a nontrivial $G$-action.

**Corollary 2.5.** *Let $G$ be a compact, connected Lie group. Let $R \in \mathrm{CAlg}(\mathrm{Sp}^{BG})$ be an $\mathbb{E}_\infty$-ring with $G$-action such that $\pi_*(R)$ is concentrated in even degrees and such that the cohomology $\mathrm{H}^*(BG; \pi_0(R))$ is a polynomial algebra over $\pi_0(R)$ on even-dimensional generators. Then $\mathrm{Mod}_R(\mathrm{Sp}^{BG})$ is unipotent.*

*Proof.* Note that by Remark 2.2 if $\mathcal{C}$ is unipotent and $A \in \mathrm{CAlg}(\mathcal{C})$, then $\mathrm{Mod}_A(\mathcal{C})$ is also unipotent. Therefore, this result follows from Theorem 2.3 with $R$ replaced by $R^{hG}$ by using the equivalence $\mathrm{Mod}_R(\mathrm{Fun}(BG, \mathrm{Mod}_{R^{hG}})) \simeq \mathrm{Mod}_R(\mathrm{Sp}^{BG})$. One checks from the assumptions that $R^{hG}$ satisfies the conditions of Theorem 2.3. In fact, the homotopy fixed point spectral sequence shows that $\pi_*(R^{hG})$ has cohomology in even degrees. In addition, $\pi_0 R^{hG}$ has a complete filtration whose quotients are all $\pi_0 R$-modules. Using this filtration, one sees that $\mathrm{H}^*(BG; \pi_0(R^{hG}))$ is a polynomial algebra on even dimensional classes (i.e., by lifting the polynomial generators from $\mathrm{H}^*(BG; \pi_0 R)$ to $\mathrm{H}^*(BG; \pi_0(R^{hG}))$). ☐

Let $\mathcal{C}$ be a unipotent presentably symmetric monoidal stable $\infty$-category. Suppose $X \in \mathcal{C}$ is a *dualizable* object. In this case, one wants a criterion in order for $X$ to be perfect. The next result follows from the statement that $X \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X) \otimes_{\mathrm{End}_{\mathcal{C}}(\mathbb{1})} \mathbb{1}$ for any $X \in \mathcal{C}$ when $\mathcal{C}$ is unipotent.

**Proposition 2.6.** *If $\mathcal{C}$ is a unipotent presentably symmetric monoidal stable $\infty$-category, then an object $X \in \mathcal{C}$ is perfect if and only if $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X)$ is perfect (i.e., dualizable) as an $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$-module.*

Equivalently, $\mathcal{C}$ is a localization of $\mathrm{Mod}_R$ for $R = \mathrm{End}_{\mathcal{C}}(\mathbb{1})$, and we need to know when an object dualizable in the localization is dualizable in $\mathrm{Mod}_R$. In general, dualizable objects need not be perfect without some type of regularity. We illustrate this with two examples.

**Example 2.7.** Let $X \in \mathrm{Mod}_{H\mathbb{Z}_p}$ be $p$-complete. Then the following are equivalent:

1. $X$ is perfect in $\mathrm{Mod}_{H\mathbb{Z}_p}$.
2. $X \otimes_{H\mathbb{Z}_p} H\mathbb{F}_p \simeq X/p \in \mathrm{Mod}_{H\mathbb{F}_p}$ is perfect.
3. $X$ is dualizable in the $\infty$-category of $p$-complete $H\mathbb{Z}_p$-modules.
4. The direct sum $\bigoplus_{i\in\mathbb{Z}} \pi_i(X)$ is a finitely generated $\mathbb{Z}_p$-module.

Clearly 1 implies 3 which implies 2, and 4 implies 1. Thus, it suffices to see that 2 implies 4, which follows because $\pi_*(X)$ is derived $p$-complete and $\pi_*(X)/p\pi_*(X) \subseteq \pi_*(X/p)$ is finitely generated (cf. Lemma 2.12 below for a more general argument).

**Example 2.8.** Let $\mathcal{C} = L_{K(n)}\mathrm{Sp}$ be the $\infty$-category of $K(n)$-local spectra, which is unipotent as a symmetric monoidal localization of Sp (see Remark 2.2). There are numerous invertible (hence, dualizable) objects in $\mathcal{C}$ [HMS94] which are not perfect.

Next, we need to review some facts about completion in the derived context, and with an extra grading. Let $A_0$ be a commutative ring and let $I = (x_1, \dots, x_t) \subseteq A_0$ be a finitely generated ideal. In this case, one has the notion of an *$I$-complete* object of the derived category $D(A_0)$ (see [DG02]): an object $M$ in $D(A_0)$ is $I$-complete if it is local for the object $A_0/x_1 \otimes_{A_0} \cdots \otimes_{A_0} A_0/x_t$ (the iterated cofiber, where all tensor products are derived). When $I = (x)$ is a principal ideal, we will often write $x$-complete instead of $(x)$-complete. Given a (discrete) $A_0$-module $M_0$, we will say that $M_0$ is *derived $I$-complete* if $M_0$, considered as an object of $D(A_0)$, is $I$-complete. Note the following fact about $I$-complete objects in the derived category.

**Proposition 2.9** (Compare [DG02, Prop 5.3])**.** *Let $A_0$ be a commutative ring and let $I = (x_1, \dots, x_t)$ be a finitely generated ideal. If $M \in D(A_0)$, then $M$ is $I$-complete if and only if each homology group of $M$ is derived $I$-complete.*

We will need an analog in the graded context. Let $A_*$ be a commutative, graded ring. Let $I = (x_1, \dots, x_t) \subseteq A_*$ be a finitely generated homogeneous ideal generated by elements $x_1, \dots, x_t \in A_*$ of degrees $d_1, \dots, d_t$. Consider the derived category $D(A_*)^{\mathrm{gr}}$ of *graded* $A_*$-modules. Let $(d)$ denote the operation of shifting the grading by $d$, so that if $M \in D(A_*)^{\mathrm{gr}}$, then one has maps $x_i \colon M_{(d_i)} \to M$.

**Definition 2.10.** Given $M \in D(A_*)^{\mathrm{gr}}$, we say that $M$ is *$I$-complete* if the natural map

$$M \to \lim_n \mathrm{cofib}(x_i^n \colon M_{(nd_i)} \to M)$$

is an equivalence for $i = 1, 2, \dots, t$. Equivalently, for $i = 1, 2, \dots, t$, one requires that the limit of multiplication by $x_i$ on $M$ vanishes. We say that a (discrete) graded $A_*$-module $M_*$ is *derived $I$-complete* if it is $I$-complete when considered as an object of $D(A_*)^{\mathrm{gr}}$.

As in [DG02, Prop. 5.3] (see also the treatment in [Lur11, Sec. 4]), one shows that an object of $D(A_*)^{\mathrm{gr}}$ is $I$-complete if and only if the homology groups (which are graded $A_*$-modules) are derived $I$-complete. This implies that the collection of derived $I$-complete graded $A_*$-modules is abelian and contains every classically $I$-complete graded $A_*$-module.

Let $A$ be an $\mathbb{E}_\infty$-ring. Suppose that $\pi_*(A)$ is concentrated in even degrees. Let $I = (x_1, \ldots, x_t) \subseteq \pi_*(A)$ be a finitely generated ideal generated by elements $x_i \in \pi_{d_i}(A)$. Let $M$ be an $A$-module. We say that $M$ is *$I$-complete* if $M$ is Bousfield local with respect to the iterated cofiber $A/(x_1, \ldots, x_t)$. Equivalently, $M$ is $I$-complete if and only if for $i = 1, \ldots, t$, the (homotopy) inverse limit of multiplication by $x_i$ on $M$ is null, i.e., $\lim(\cdots \xrightarrow{x_i} \Sigma^{d_i} M \xrightarrow{x_i} M) = 0$.

**Proposition 2.11.** *If $A$ is an $\mathbb{E}_\infty$-ring and $M$ is an $A$-module, then $M$ is $I$-complete if and only if $\pi_*(M)$ is derived $I$-complete as a graded $\pi_*(A)$-module.*

*Proof.* In fact, for $i = 1, \ldots, t$, the homotopy groups of $\lim(\cdots \xrightarrow{x_i} \Sigma^{d_i} M \xrightarrow{x_i} M)$ are computed by the Milnor short exact sequence

$$\lim{}^1(\cdots \xrightarrow{x_i} \pi_{*-d_i} M \xrightarrow{x_i} \pi_* M) \hookrightarrow \pi_*\left(\lim(\cdots \xrightarrow{x_i} \Sigma^{d_i} M \xrightarrow{x_i} M)\right) \twoheadrightarrow \lim(\cdots \xrightarrow{x_i} \pi_{*-d_i} M \xrightarrow{x_i} \pi_* M).$$

If $M$ is $I$-complete, then the middle term vanishes for each $i = 1, 2, \ldots, t$; thus the two outer terms vanish, which implies precisely that $\pi_*(M)$ is derived $I$-complete. The converse follows similarly. $\square$

**Lemma 2.12.** *Let $A_*$ be a commutative, graded ring which is derived $x$-complete for some homogeneous element $x \in A_d$. Let $M_*$ be a (discrete) graded $A_*$-module which is derived $x$-complete. Suppose $M_*/(x)M_*$ is a finitely generated $A_*$-module. Then $M_*$ is a finitely generated $A_*$-module.*

*Proof.* Let $F_*$ be a finitely generated graded free $A_*$-module together with a map $f\colon F_* \to M_*$ which induces a surjection $F_*/(x)F_* \to M_*/(x)M_*$. We claim that $f$ itself is a surjection. In fact, form the cofiber $Cf$ of $f$ in the derived category $D(A_*)^{\mathrm{gr}}$; it suffices to see that $Cf$ has homology concentrated in (homological, not graded) degree 1. Note first that $\mathrm{cofib}(x\colon Cf_{(d)} \to Cf)$ is concentrated in degree 1. But $Cf$ is $x$-complete, so $Cf \simeq \lim_n \mathrm{cofib}(x^n\colon Cf_{(nd)} \to Cf)$. The cofibers $\mathrm{cofib}(x^n\colon Cf_{(nd)} \to Cf)$ are concentrated in degree one by induction on $n$ and the transition maps are surjective; thus the homotopy limit $Cf$ is concentrated in degree one by the Milnor exact sequence, as desired. $\square$

**Proposition 2.13.** *Let $A$ be an $\mathbb{E}_\infty$-ring such that $\pi_*(A)$ is a regular noetherian ring of finite Krull dimension. Then an $A$-module $M$ is perfect if and only if $\pi_*(M)$ is a finitely generated $\pi_*(A)$-module.*

*Proof.* We show that finite generation implies perfectness; the other direction follows from a thick subcategory argument. Note that the homological dimension of $\pi_*(M)$ as a $\pi_*(A)$-module is finite by regularity. We thus use induction on the homological dimension $\mathrm{h.dim}(\pi_*(M))$. If $\pi_*(M)$ is projective as a $\pi_*(A)$-module, then it is also projective as a graded $\pi_*(A)$-module by a simple argument [NvO82, Cor. I.2.2] and $M$ itself is a retract of a finitely generated free $A$-module. In general, choose a finitely generated free $A$-module $N$ with a map $N \to M$ inducing a surjection on $\pi_*$, and form a fiber sequence $N' \to N \to M$. We have that $\mathrm{hdim}(\pi_*(N')) = \mathrm{hdim}(\pi_*(M)) - 1$. Since $M$ is perfect if and only if $N'$ is, this lets us conclude by the inductive hypothesis. $\square$

**Proposition 2.14.** *Let $A$ be an $\mathbb{E}_\infty$-ring such that $\pi_*(A)$ is a regular noetherian ring of finite Krull dimension. Suppose that $x_1, \ldots, x_t \in \pi_*(A)$ form a regular sequence and $A$ is $(x_1, \ldots, x_t)$-complete.*[3] *Let $M \in \mathrm{Mod}_A$ be an $(x_1, \ldots, x_t)$-complete module such that $M/(x_1, \ldots, x_t)$ is perfect as an $A$-module. Then $M$ is perfect as an $A$-module.*

[3] By regularity, this is equivalent to the condition that $\pi_*(A)$ should be classically $(x_1, \ldots, x_t)$-complete.

*Proof.* By Proposition 2.13, it suffices to prove that $\pi_*(M)$ is finitely generated as an $\pi_*(A)$-module. By induction on $t$, one can reduce to the case where $t = 1$ where we let $x = x_1$. In this case, $M/x$ is perfect as an $A$-module by assumption. It follows that $\pi_*(M)/(x)\pi_*(M) \subset \pi_*(M/x)$ is finitely generated as a $\pi_*(A)$-module. Note that $\pi_*(M)$ is a derived $x$-complete $\pi_*(A)$-module (Proposition 2.11). By Lemma 2.12, it follows that $\pi_*(M)$ is a finitely generated $\pi_*(A)$-module, completing the proof. □

We can now state and prove our main theorem about perfectness.

**Theorem 2.15.** *Let $G$ be a compact, connected Lie group and let $R \in \mathrm{CAlg}(\mathrm{Sp}^{BG})$. Suppose that $\pi_*(R)$ is a regular noetherian ring of finite Krull dimension concentrated in even degrees and that $\mathrm{H}^*(BG; \pi_0 R)$ is a polynomial algebra over $\pi_0 R$ on even-dimensional classes. Then any dualizable object in $\mathrm{Mod}_R(\mathrm{Sp}^{BG})$ is perfect.*

*Proof.* We first show that in the above situation, the graded ring $\pi_*(R^{hG})$ is a regular noetherian ring concentrated in even degrees.[4] The fact that $\pi_*(R^{hG})$ is concentrated in even degrees follows from the homotopy fixed point spectral sequence, which degenerates for degree reasons. It follows as in the proof of [MNN17, Theorem 7.35] that one has classes $x_1, \dots, x_t \in \pi_*(R^{hG})$ such that the $x_i$'s form a regular sequence in $\pi_*(R^{hG})$, the $x_i$'s map to zero under $\pi_*(R^{hG}) \to \pi_*(R)$, and such that one has equivalences

$$R^{hG}/(x_1, \dots, x_t) \simeq R, \quad \pi_*(R^{hG})/(x_1, \dots, x_t) \simeq \pi_*(R).$$

The $(x_1, \dots, x_t)$-adic filtration on $\pi_*(R^{hG})$ is complete by the convergence of the spectral sequence and it has noetherian associated graded, so $\pi_*(R^{hG})$ is noetherian at least as a graded ring. Therefore, $\pi_*(R^{hG})$ is noetherian as an ungraded ring as well [GY83, Th. 1.1].

By completeness, it follows that any maximal graded ideal of $\pi_*(R^{hG})$ contains $(x_1, \dots, x_t)$. Given a maximal graded ideal $\mathfrak{m} \in \pi_*(R^{hG})$, it follows that the (ungraded) localization $\pi_*(R^{hG})_{\mathfrak{m}}$ is a regular local ring: in fact, the quotient $\pi_*(R^{hG})_{\mathfrak{m}}/(x_1, \dots, x_t)$ by the regular sequence $x_1, \dots, x_t$ is a regular local ring since it is a local ring of $\pi_*(R)$. By [Mat75, Th. 2.1], this implies that $\pi_*(R^{hG})$ itself is a regular ring. Finally, we argue that $\pi_*(R^{hG})$ has finite Krull dimension. Let $\mathfrak{p} \subseteq \pi_*(R^{hG})$ be a prime ideal of height $h$; up to replacing $h$ by $h-1$, we can assume that $\mathfrak{p}$ is homogeneous by [Mat75, Prop. 1.3]. Up to enlarging $\mathfrak{p}$, we can then assume $(x_1, \dots, x_t) \in \mathfrak{p}$. But then $\dim \pi_*(R^{hG})_{\mathfrak{p}} \leqslant \dim \pi_*(R)_{\overline{\mathfrak{p}}} + t$ for $\overline{\mathfrak{p}}$ the image of $\mathfrak{p}$ in $\pi_*(R)$. This shows that $h$ is globally bounded and thus that $\pi_*(R^{hG})$ has finite Krull dimension.

Recall now that $\mathrm{Mod}_R(\mathrm{Sp}^{BG})$ is unipotent by Corollary 2.5. If $M \in \mathrm{Mod}_R(\mathrm{Sp}^{BG})$ is dualizable, then unipotence implies that one has an equivalence of $R$-module spectra $M \simeq M^{hG} \otimes_{R^{hG}} R \simeq M^{hG}/(x_1, \dots, x_t)$. Note that $M^{hG}$ is an $(x_1, \dots, x_t)$-complete $R^{hG}$-module (e.g., by writing $M^{hG}$ as an inverse limit of a tower over the skeletal filtration of $EG$). In addition, $M^{hG}/(x_1, \dots, x_t) \simeq M^{hG} \otimes_{R^{hG}} R \simeq M$ is perfect as an $R$-module and hence as an $R^{hG}$-module. By Proposition 2.14, it follows that $M^{hG}$ is perfect as an $R^{hG}$-module. Thus, by Proposition 2.6, $M$ is perfect in $\mathrm{Mod}_R(\mathrm{Sp}^{BG})$ as desired. □

# 3 Applications to THH

In this section, we give our main applications to THH and TP. After some preliminary remarks we prove our main theorem (Theorem 3.2) and give the application to Künneth theorems (Theorem 3.3).

[4] In the example of interest below, one can simply compute the ring explicitly and thus the first two paragraphs become redundant.

These results immediately imply Theorem 1.1 of Blumberg and Mandell. Then, we give a slightly different, second proof of the Künneth theorem based on the observation that TP is an integral lift of HP (Theorem 3.4). Finally, we go on to discuss a similar result in mixed characteristic.

Let $k$ be a perfect field of positive characteristic. One needs the following fundamental calculation of Bökstedt (cf. [HM97, Sec. 5]) of the homotopy ring of $\mathrm{THH}(k)$.

**Theorem 3.1** (Bökstedt)**.** *There is an isomorphism of graded rings $\pi_*\mathrm{THH}(k) \simeq k[\sigma]$ where $|\sigma| = 2$.*

Here $\mathrm{THH}(k)$ is naturally an $\mathbb{E}_\infty$-ring spectrum equipped with an $S^1$-action. One can thus consider modules in $\mathrm{Sp}^{BS^1}$ over $\mathrm{THH}(k)$. One has $\pi_*(\mathrm{THH}(k)^{hS^1}) \simeq W(k)[x,\sigma]/(x\sigma = p)$, a regular noetherian ring[5]. Here $\sigma$ is a lift of the Bökstedt element and $x \in \pi_{-2}(\mathrm{THH}(k)^{hS^1})$ is a generator which is detected in filtration two in the homotopy fixed point spectral sequence. Using Theorem 2.15, one obtains the following result, stated as Theorem 1.3 of the introduction.

**Theorem 3.2.** *Let $k$ be a perfect field of characteristic $p > 0$. There is an equivalence of symmetric monoidal $\infty$-categories between $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{BS^1})$ and $x$-complete $\mathrm{THH}(k)^{hS^1}$-modules. Moreover any dualizable object in $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{BS^1})$ is perfect.*

*Proof.* With Bökstedt's calculation in hand, this result is a special case of Theorem 2.15. □

We now give the application to the Künneth formula. We will work with small, idempotent-complete $k$-linear stable $\infty$-categories (which can be modeled as dg categories). These are naturally organized into an $\infty$-category $\mathrm{Cat}^{\mathrm{perf}}_{\infty,k}$. For any such $\mathcal{C}$, we can consider the topological Hochschild homology $\mathrm{THH}(\mathcal{C})$, together with its natural $S^1$-action. We recall that THH defines a symmetric monoidal functor of $\infty$-categories

$$\mathrm{Cat}^{\mathrm{perf}}_{\infty,k} \to \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{BS^1}).$$

Compare the discussion in [BGT14, Sec. 6] for stable $\infty$-categories, and we refer to [BM17, Th. 14.1] for a proof for $k$-linear $\infty$-categories (at least those with a compact generator, to which the general result reduces).

Recall (cf. [Toë12, Prop. 1.5], [BGT13, Th. 3.7]) that the dualizable objects in $\mathrm{Cat}^{\mathrm{perf}}_{\infty,k}$ are precisely the smooth and proper $k$-linear stable $\infty$-categories.

**Theorem 3.3.** *Let $k$ be a perfect field of characteristic $p > 0$. Let $\mathcal{C}, \mathcal{D}$ be $k$-linear dg categories and suppose $\mathcal{C}$ is smooth and proper. Then for any closed subgroup $H \subseteq S^1$, $\mathrm{THH}(\mathcal{C})^{tH}$ is a perfect $\mathrm{THH}(k)^{tH}$-module and the natural map*

$$\mathrm{THH}(\mathcal{C})^{tH} \otimes_{\mathrm{THH}(k)^{tH}} \mathrm{THH}(\mathcal{D})^{tH} \to \mathrm{THH}(\mathcal{C} \otimes_k \mathcal{D})^{tH}$$

*is an equivalence. The same holds with $tH$ replaced by $hH$.*

*Proof.* Since the functor $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{BS^1}) \to \mathrm{Mod}_{\mathrm{THH}(k)^{tH}}, X \mapsto X^{tH}$ is lax symmetric monoidal and exact, it follows (as in the discussion in the introduction after Theorem 1.1) that it suffices to prove that $\mathrm{THH}(\mathcal{C}) \in \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{BS^1})$ is perfect. However, the construction $\mathcal{C} \mapsto \mathrm{THH}(\mathcal{C})$ is symmetric monoidal, and $\mathcal{C}$ is dualizable in $\mathrm{Cat}^{\mathrm{perf}}_{\infty,k}$. Since symmetric monoidal functors preserve dualizable objects, it follows from Theorem 3.2 that $\mathrm{THH}(\mathcal{C}) \in \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{BS^1})$ is perfect, completing the proof. □

[5]We refer to [NS17, Sec. IV-4] for a treatment of this calculation.

As a complement, we give a slightly different proof of Theorem 1.1 based on the result (Theorem 3.4) that TP is an integral lift of periodic cyclic homology $\mathrm{HP}(\cdot/k)$. This result will also appear in forthcoming work of Bhatt-Morrow-Scholze [BMS] on integral $p$-adic Hodge theory, and plays a role in various key steps there. We include a proof for the convenience of the reader.

Note to begin that one has an $S^1$-equivariant map of $\mathbb{E}_\infty$-rings

$$HW(k) \to \mathrm{THH}(k)$$

inducing an equivalence $(HW(k))^{tS^1} \simeq \mathrm{TP}(k)$. When $k = \mathbb{F}_p$, the map arises from the cyclotomic trace $\widehat{K(\mathbb{F}_p)}_p \simeq H\mathbb{Z}_p \to \mathrm{THH}(\mathbb{F}_p)$ (the first equivalence by Quillen [Qui72]); it can also be constructed by computing $\mathrm{TC}(\mathbb{F}_p)$ ([HM97], [NS17, Sec. IV-4]). In general, the same argument gives a map $H\mathbb{Z}_p \to \mathrm{THH}(k)^{hS^1}$ and the calculation of $\pi_0(\mathrm{THH}(k)^{hS^1}) = W(k)$ yields an extension on $\pi_0$. The existence of the spectrum level extension to a map from $HW(k)$ follows by obstruction theory from the $p$-adic vanishing of the cotangent complex of $W(k)$ over $\mathbb{Z}_p$. Since the map $(HW(k))^{tS^1} \to \mathrm{TP}(k)$ induces a $W(k)$-linear map on homotopy rings $W(k)[t^{\pm 1}] \to W(k)[x^{\pm 1}]$, we see it is an equivalence.

**Theorem 3.4** (Compare [BMS])**.** *Let $k$ be a perfect field of characteristic $p > 0$. For any $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\infty,k}$, the natural map $\mathrm{TP}(\mathcal{C}) \otimes_{\mathrm{TP}(k)} Hk^{tS^1} \simeq \mathrm{TP}(\mathcal{C})/p \to \mathrm{HP}(\mathcal{C}/k)$ is an equivalence.*

*Proof.* In fact, one has an equivalence in $\mathrm{Sp}^{BS^1}$

$$\mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(k)} Hk \simeq \mathrm{HH}(\mathcal{C}/k), \quad \mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\infty,k}.$$

If $\mathcal{C} = \mathrm{Perf}(A)$ for an $\mathbb{E}_1$-algebra, then the equivalence arises from the equivalence of cyclic objects

$$N^{\mathrm{cy}}(A) \otimes_{N^{\mathrm{cy}}(k)} Hk \simeq N^{\mathrm{cy},k}(A),$$

where $N^{\mathrm{cy}}$ denotes the cyclic bar construction in spectra and $N^{\mathrm{cy},k}$ in $Hk$-module spectra.

In $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{Sp}^{BS^1})$, we observe that $Hk$ is perfect as it is the cofiber of the map $\Sigma^2\mathrm{THH}(k) \to \mathrm{THH}(k)$ given by multiplication by $\sigma$, which can be made $S^1$-equivariant by the degeneration of the homotopy fixed point spectral sequence. It follows thus that

$$\mathrm{HH}(\mathcal{C}/k)^{tS^1} \simeq \left(\mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(k)} Hk\right)^{tS^1} \simeq \mathrm{TP}(\mathcal{C}) \otimes_{\mathrm{TP}(k)} Hk^{tS^1},$$

as desired. □

*Second proof of Theorem 1.1.* Let $\mathcal{C}$ be smooth and proper over $k$. Since $\mathrm{THH}(\mathcal{C})$ is bounded below and $p$-torsion, $\mathrm{TP}(\mathcal{C})$ is automatically $p$-complete. To see that $\mathrm{TP}(\mathcal{C})$ is a perfect $\mathrm{TP}(k)$-module, it suffices to show that the homotopy groups of $\mathrm{TP}(\mathcal{C})$ are finitely generated $W(k)$-modules (Proposition 2.13). However, the homotopy groups $\pi_*(\mathrm{TP}(\mathcal{C})/p) \simeq \mathrm{HP}_*(\mathcal{C}/k)$ are finite-dimensional $k$-vector spaces, which forces the homotopy groups of $\mathrm{TP}(\mathcal{C})$ to be finitely generated $W(k)$-modules by Lemma 2.12. (Compare with the argument in the proof of [BM17, Theorem 16.1].)

Similarly, if $\mathcal{D}$ is another dg category over $k$ such that $\mathrm{THH}(\mathcal{D})$ is bounded below (e.g., if $\mathcal{D}$ is smooth and proper) then $\mathrm{TP}(\mathcal{D})$ and the tensor product $\mathrm{TP}(\mathcal{C}) \otimes_{\mathrm{TP}(k)} \mathrm{TP}(\mathcal{D})$ are automatically $p$-complete already. To prove the Künneth formula, it thus suffices to base-change along $\mathrm{TP}(k) \to Hk^{tS^1}$. But one already has a Künneth formula in $\mathrm{HP}(\cdot/k)$,[6] so one concludes. □

[6] For instance, this follows in a similar (but easier) fashion as dualizable objects in $\mathrm{Fun}(BS^1, \mathrm{Mod}_k)$ are perfect in view of the Postnikov filtration.

One also has a variant of Bökstedt's calculation in mixed characteristic. Let $A$ be the localization of the ring of integers in a number field at a prime ideal lying over $p$. Fix a uniformizer $\pi \in A$. Let $S^0[q]$ denote the $\mathbb{E}_\infty$-ring $\Sigma^\infty_+ \mathbb{Z}_{\geqslant 0}$. We consider the map of $\mathbb{E}_\infty$-rings $S^0[q] \to HA$ sending $q \mapsto \pi$. We denote the topological Hochschild homology of $A$ relative to $S^0[q]$ (i.e., in the $\infty$-category of $S^0[q]$-modules rather than spectra) as $\mathrm{THH}(A/S^0[q])$. The following result will appear in the work of Bhatt-Morrow-Scholze [BMS]. We are grateful to Lars Hesselholt for explaining it to us. For the reader's convenience, we include a proof.

**Theorem 3.5.** *Let $A$ be the localization of the ring of integers in a number field at a prime ideal $(\pi)$ lying over $p$ and consider $HA$ as an $S^0[q]$-algebra as above. There is an isomorphism of graded rings $\pi_*\mathrm{THH}(A/S^0[q]) \simeq A[\sigma]$ where $|\sigma| = 2$.*

*Proof sketch.* Note first that the Hochschild homology groups $\mathrm{HH}_*(A/\mathbb{Z}[q])$ are finitely generated $A$-modules in each degree as $A$ is a finitely generated module over $A \otimes_{\mathbb{Z}[q]} A$, which in turn is a noetherian ring. Since $\mathrm{THH}(A/S^0[q]) \otimes_{\mathrm{THH}(\mathbb{Z}[q]/S^0[q])} H\mathbb{Z}[q] \simeq \mathrm{HH}(A/\mathbb{Z}[q])$, it follows easily that the homotopy groups of $\mathrm{THH}(A/S^0[q])$ are finitely generated $A$-modules. After rationalization, we have $\pi_*\mathrm{THH}(A_\mathbb{Q}/\mathbb{Q}[q]) \simeq (A_\mathbb{Q})[\sigma]$ by a Hochschild-Kostant-Rosenberg argument since $\mathbb{Q}[q] \to A_\mathbb{Q}$ is an lci morphism. Since $\mathrm{THH}(A/S^0[q]) \otimes_{S^0[q]} S^0 \simeq \mathrm{THH}(A/(\pi))$, Theorem 3.5 thus follows from Bökstedt's calculation in Theorem 3.1 by the base-change $S^0[q] \to S^0$ setting $q \mapsto 0$. □

**Remark 3.6.** In [BMS], Theorem 3.5 (together with a base-change argument along $S^0[q] \to S^0_{\mathrm{A_{inf}}}$, where $S^0_{\mathrm{A_{inf}}}$ denotes a spectral lift of Fontaine's ring $\mathrm{A_{inf}}$) is also used to recover Hesselholt's calculation [Hes06] of the $p$-completion of $\mathrm{THH}(\mathcal{O}_C)$. Here $C$ is the completed algebraic closure of a local field. The argument in [BMS] extends Hesselholt's result to replace $\mathcal{O}_C$ by any integral perfectoid $\mathbb{Z}_p$-algebra.

One can use Theorem 3.5 to calculate $\mathrm{TP}(A/S^0[q])$ as in the case of perfect fields of characteristic $p > 0$. For example, $\pi_*\mathrm{TP}(\mathbb{Z}_{(p)}/S^0[q]) \simeq \widehat{\mathbb{Z}_{(p)}[q]}_{(q-p)}[x^{\pm 1}]$ for $|x| = 2$. Using regularity as before, one obtains the following result from Theorem 2.15.

**Corollary 3.7.** *Let $A$ be the localization of the ring of integers in a number field at a prime ideal lying over $p$ and consider $HA$ as an $S^0[q]$-algebra as above. In the symmetric monoidal $\infty$-category $\mathrm{Mod}_{\mathrm{THH}(A/S^0[q])}(\mathrm{Sp}^{BS^1})$, any dualizable object is perfect.*

Let $\mathrm{Cat}^{\mathrm{perf}}_{\infty,A}$ denote the $\infty$-category of small, $A$-linear stable $\infty$-categories. One has a symmetric monoidal functor

$$\mathrm{THH}(\cdot/S^0[q])\colon \mathrm{Cat}^{\mathrm{perf}}_{\infty,A} \to \mathrm{Mod}_{\mathrm{THH}(A/S^0[q])}(\mathrm{Sp}^{BS^1})$$

and one may define $\mathrm{TP}(\cdot/S^0[q]) \overset{\mathrm{def}}{=} \mathrm{THH}(\cdot/S^0[q])^{tS^1}$. Using arguments as above, one obtains the following.

**Corollary 3.8.** *Let $A$ be the localization of the ring of integers in a number field at a prime ideal lying over $p$ and consider $HA$ as an $S^0[q]$-algebra as above. Let $\mathcal{C}, \mathcal{D}$ be $A$-linear stable $\infty$-categories and suppose $\mathcal{C}$ is smooth and proper. Then $\mathrm{TP}(\mathcal{C}/S^0[q])$ is a perfect $\mathrm{TP}(A/S^0[q])$-module and the natural map*

$$\mathrm{TP}(\mathcal{C}/S^0[q]) \otimes_{\mathrm{TP}(A/S^0[q])} \mathrm{TP}(\mathcal{D}/S^0[q]) \to \mathrm{TP}(\mathcal{C} \otimes_A \mathcal{D}/S^0[q])$$

*is an equivalence.*

If $\pi \in A$ is a uniformizer, one also has a map

$$S^0[q^{\pm 1}] \to HA, \quad q \mapsto 1 + \pi.$$

One can carry out a slight variant of the above calculations for $\mathrm{THH}(\cdot/S^0[q^{\pm 1}])$ and $\mathrm{TP}(\cdot/S^0[q^{\pm 1}])$, and replace the base-change $q \mapsto 0$ with $q \mapsto 1$. One obtains

$$\pi_* \mathrm{THH}(A/S^0[q^{\pm 1}]) \simeq A[\sigma], \quad |\sigma| = 2.$$

Suppose now that the base ring $A$ is given by $\mathbb{Z}_{(p)}[\zeta_p]$ and $\pi = \zeta_p - 1$. In this case, there is an isomorphism

$$\pi_* \mathrm{TP}(\mathbb{Z}_{(p)}[\zeta_p]/S^0[q^{\pm 1}]) \simeq \widehat{\mathbb{Z}_{(p)}[q^{\pm 1}]}_{\Phi_p(q)}[x^{\pm 1}].$$

Here $\Phi_p(q)$ is the $p$th cyclotomic polynomial. Moreover, one obtains a functor

$$\mathrm{Cat}^{\mathrm{perf}}_{\infty, \mathbb{Z}_{(p)}[\zeta_p]} \to \mathrm{Mod}_{\mathrm{TP}(\mathbb{Z}_{(p)}[\zeta_p]/S^0[q^{\pm 1}])},$$

which analogs of our arguments show satisfies a Künneth formula for smooth and proper dg categories over $\mathbb{Z}_{(p)}[\zeta_p]$. Just as TP is analogous to 2-periodic crystalline cohomology, $\mathrm{TP}(\cdot/S^0[q^{\pm 1}])$ in this case is roughly analogous to a 2-periodic version of the $q$-de Rham cohomology proposed by Scholze [Sch]. We refer to [BMS] for details.

# 4 A finiteness result

We next apply our version of the Künneth theorem to prove the following finiteness result for the topological cyclic homology $\mathrm{TC}(\mathcal{C})$ of a smooth and proper dg category $\mathcal{C}$ over a *finite* field.

**Theorem 4.1.** *Suppose $k$ is a finite field of characteristic $p$. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\infty,k}$ be smooth and proper. Then $\mathrm{TC}(\mathcal{C})$ is a perfect $H\mathbb{Z}_p$-module.*

Of course, the above result fails for $\mathrm{THH}(\mathcal{C})^{hS^1}$, already for $\mathcal{C} = \mathrm{Perf}(k)$. The finiteness for TC follows from the Nikolaus–Scholze formula for TC in terms of $\mathrm{THH}^{hS^1}$ and TP and in particular the interactions with the cyclotomic Frobenius. After gathering the necessary facts about TC below and giving a lemma on natural transformations of symmetric monoidal functors, we prove Theorem 4.1 at the end of the section.

**Example 4.2.** Suppose $\mathcal{C} = \mathrm{Perf}(X)$ where $X$ is a smooth and proper variety over a finite field $k$. In this case, there is a stronger, more refined result of Geisser and Hesselholt [GH99]. They show [GH99, Prop. 5.1.1] that $\pi_i \mathrm{TC}(X)$ is finite for $i \neq 0, -1$. (It also follows from their descent spectral sequence and [GS88, Prop. 4.18] that $\pi_i \mathrm{TC}(X)$ is a finitely generated $W(k)$-module for $i = 0, -1$.) Earlier computations of Hesselholt [Hes96] imply that the homotopy groups of $\mathrm{TC}(X)$ vanish in degrees $> \dim X$, and for descent-theoretic reasons [GH99, Sec. 3] they vanish in degrees $< -\dim X - 1$. In addition, they show that in this case $\mathrm{TC}(X)$ is identified with the $p$-adic étale $K$-theory of $X$.

**Definition 4.3.** In this section, we write $\mathrm{TC}^-(\mathcal{C})$ for $\mathrm{THH}(\mathcal{C})^{hS^1}$.

We will need a number of preliminaries. In particular, we will use the Nikolaus–Scholze [NS17] description of the $\infty$-category CycSp of cyclotomic spectra. We will restrict to the $p$-local case.

**Definition 4.4** ([NS17, Def. II.1.6])**.** The homotopy theory CycSp of cyclotomic spectra is the presentably symmetric monoidal stable $\infty$-category of pairs

$$(X \in \mathrm{Sp}^{BS^1}, \{\varphi_p \colon X \to X^{tC_p}\}_{p \text{ prime}})$$

where, for each prime $p$, the map $\varphi_p$ is $S^1$-equivariant for the natural $S^1/C_p$-action on $X^{tC_p}$ and the natural identification $S^1 \simeq S^1/C_p$. For $X \in \mathrm{CycSp}$, the *topological cyclic homology* $\mathrm{TC}(X)$ is defined as the mapping spectrum $\mathrm{Hom}_{\mathrm{CycSp}}(\mathbb{1}, X)$ for $\mathbb{1} \in \mathrm{CycSp}$ the unit.

By [NS17, Sec. II.6], the above definition agrees with earlier definitions of cyclotomic spectra (e.g., those in terms of genuine equivariant stable homotopy theory) when the underlying spectrum $X$ is bounded below. In the rest of the paper, $X$ will always be local at a fixed prime $p$. In this case, the Tate constructions $X^{tC_q}$ for $q \neq p$ vanish, so that the only relevant map is $\varphi \stackrel{\mathrm{def}}{=} \varphi_p$.

Given an object $X \in \mathrm{CycSp}$ such that $X$ is bounded below and in addition $p$-complete, then $X^{hS^1}$ and $X^{tS^1}$ are also $p$-complete[7] and one obtains a Frobenius map $\varphi \colon X^{hS^1} \to X^{tS^1}$, in addition to the canonical map $\mathrm{can} \colon X^{hS^1} \to X^{tS^1}$. We will use the fundamental formula [NS17, Prop. II.1.9, Rmk. II.4.3] (valid for bounded below $p$-complete $X \in \mathrm{CycSp}$)

$$\mathrm{TC}(X) \simeq \mathrm{eq}\left(\mathrm{can}, \varphi \colon X^{hS^1} \rightrightarrows X^{tS^1}\right). \tag{3}$$

For a stable $\infty$-category $\mathcal{C}$, the topological Hochschild homology construction $\mathrm{THH}(\mathcal{C})$ naturally admits the structure of a cyclotomic spectrum (compare [AMGR17]). For $k$-linear dg categories one has a symmetric monoidal functor

$$\mathrm{THH} \colon \mathrm{Cat}^{\mathrm{perf}}_{\infty,k} \to \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp}).$$

There is a related discussion of the symmetric monoidality in [BGT14, Section 6] and we believe this is known to other authors as well. For example, one can use the symmetric monoidality of the classical point-set constructions and map that to the new $\infty$-category of cyclotomic spectra to get the desired functor.

Suppose $\mathrm{THH}(\mathcal{C})$ is bounded below, e.g., if $\mathcal{C}$ is smooth and proper over $k$ or if $\mathcal{C} = \mathrm{Perf}(A)$ for $A$ a connective $\mathbb{E}_1$-algebra in $k$-modules. Then, in this language, the *topological cyclic homology* $\mathrm{TC}(\mathcal{C})$ can be defined as

$$\mathrm{TC}(\mathcal{C}) = \mathrm{TC}(\mathrm{THH}(\mathcal{C})) = \mathrm{eq}\left(\mathrm{can}, \varphi \colon \mathrm{TC}^-(\mathcal{C}) \rightrightarrows \mathrm{TP}(\mathcal{C})\right). \tag{4}$$

**Example 4.5.** We need two basic properties of the two maps can and $\varphi$ (cf. [NS17, Sec. IV.4]). Here we use that $\pi_*\mathrm{TC}^-(k) \simeq W(k)[x,\sigma]/(x\sigma = p)$ for $x \in \pi_{-2}$ and $\sigma \in \pi_2$.

1. The map can carries $x \in \pi_{-2}\mathrm{TC}^-(k)$ to an invertible element in $\mathrm{TP}(k)$. For any $\mathcal{C}$, the map $\mathrm{TC}^-(\mathcal{C})[1/x] \to \mathrm{TP}(\mathcal{C})$ is an equivalence.

2. The map $\varphi$ carries $\sigma \in \pi_2\mathrm{TC}^-(k)$ to an invertible element in $\mathrm{TP}(k)$. The map $\varphi$ induces an equivalence $\mathrm{TC}^-(k)[1/\sigma] \to \mathrm{TP}(k)$. For any $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\infty,k}$ such that $\mathrm{THH}(\mathcal{C})$ is bounded below, one has a map $\mathrm{TC}^-(\mathcal{C})[1/\sigma] \to \mathrm{TP}(\mathcal{C})$ which is $\varphi$-semilinear. Alternatively, one has a map of $\mathrm{TP}(k)$-modules $\mathrm{TC}^-(\mathcal{C}) \otimes_{\mathrm{TC}^-(k)} {}_{\varphi}\mathrm{TP}(k) \to \mathrm{TP}(\mathcal{C})$, where the map $\mathrm{TC}^-(k) \to \mathrm{TP}(k)$ is $\varphi$.

We now prove some facts about these invariants for smooth and proper dg categories. First we need a preliminary proposition about dualizability.

[7] This follows by induction up the Postnikov tower.

**Proposition 4.6.** *1. Let $\mathcal{T}$ be a symmetric monoidal $\infty$-category and let $\mathrm{Fun}(\Delta^1, \mathcal{T})$ denote the $\infty$-category of arrows in $\mathcal{T}$, with the pointwise symmetric monoidal structure. Then any dualizable object $f\colon X_1 \to X_2$ of $\mathrm{Fun}(\Delta^1, \mathcal{T})$ has the property that the map $f$ is an equivalence.*

*2. Let $\mathcal{T}, \mathcal{T}'$ be a symmetric monoidal $\infty$-categories. Let $F_1, F_2\colon \mathcal{T} \to \mathcal{T}'$ be symmetric monoidal functors and let $t\colon F_1 \to F_2$ be a symmetric monoidal natural transformation. Suppose every object of $\mathcal{T}$ is dualizable. Then $t$ is an equivalence.*

*Proof.* The first assertion implies the second, so we focus on the first. Let $\vee$ denote duality on $\mathcal{T}$. If $f\colon X \to Y$ has a dual, then the source and target have to be the duals of $X$ and $Y$ since the evaluation functors are symmetric monoidal. Thus, the dual of $f\colon X \to Y$ is an arrow $\bar{f}\colon X^\vee \to Y^\vee$. We claim that $\bar{f}^\vee$ is the inverse of $f$.

To see this, we draw some diagrams. We write $\mathrm{ev}, \mathrm{coev}$ for evaluation and coevaluation maps, respectively. Since $\bar{f}$ is the dual of $f$, one has a commutative triangle

$$\begin{array}{ccc} \mathbb{1} & & \\ \downarrow{\scriptstyle \mathrm{coev}_X} & \searrow^{\mathrm{coev}_Y} & \\ X \otimes X^\vee & \xrightarrow{f\otimes \bar{f}} & Y \otimes Y^\vee. \end{array}$$

As a result, it follows that the diagram

$$\begin{array}{ccc} Y^\vee & \xrightarrow{\mathrm{id}} & Y^\vee \\ \downarrow{\scriptstyle \mathrm{id}\otimes \mathrm{coev}_X} & & \downarrow{\scriptstyle \mathrm{id}\otimes \mathrm{coev}_Y} \\ Y^\vee \otimes X \otimes X^\vee & \searrow^{\mathrm{id}\otimes f \otimes \bar{f}} & \\ \downarrow{\scriptstyle \mathrm{id}\otimes f \otimes \mathrm{id}} & & \\ Y^\vee \otimes Y \otimes X^\vee & \xrightarrow[\mathrm{id}\otimes\mathrm{id}\otimes \bar{f}]{} & Y^\vee \otimes Y \otimes Y^\vee \\ \downarrow{\scriptstyle \mathrm{ev}_Y \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{ev}_Y \otimes \mathrm{id}} \\ X^\vee & \xrightarrow{\bar{f}} & Y^\vee \end{array}$$

commutes. Chasing both ways around the diagram, one finds that $\mathrm{id}_{Y^\vee} \simeq \bar{f} \circ f^\vee$. Dualizing again, we get that $f \circ \bar{f}^\vee$ is equivalent to the identity of $Y$. In particular, $f$ admits a section. To see that $f$ is actually an equivalence, assume without loss of generality that all objects are dualizable. Now apply the symmetric monoidal equivalence $\vee\colon \mathrm{Fun}(\Delta^1, \mathcal{T}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{T})^{op}$. It follows that $f^\vee$ admits a section too. Therefore, $f$ is an equivalence. $\square$

**Proposition 4.7.** *Let $k$ be a perfect field of characteristic $p > 0$. For $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\infty,k}$ smooth and proper, the $\varphi$-semilinear map $\varphi\colon \mathrm{TC}^-(\mathcal{C})[1/\sigma] \to \mathrm{TP}(\mathcal{C})$ is an equivalence. Equivalently, one has an equivalence of $\mathrm{TP}(k)$-modules*

$$\mathrm{TC}^-(\mathcal{C}) \otimes_{\mathrm{TC}^-(k),\varphi} \mathrm{TP}(k) \simeq \mathrm{TP}(\mathcal{C}).$$

*Proof.* Let $\mathcal{T}$ denote the $\infty$-category of smooth and proper objects in $\mathrm{Cat}^{\mathrm{perf}}_{\infty,k}$. Then both sides of the above displayed map yield symmetric monoidal functors to $\mathrm{Mod}_{\mathrm{TP}(k)}$ in view of Theorem 3.3; for the right-hand-side this is the Blumberg–Mandell theorem. The natural map is a symmetric monoidal natural transformation, so the result follows from Proposition 4.6. $\square$

**Proposition 4.8.** *Let $k$ be a perfect field of characteristic $p > 0$. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\infty,k}$ be smooth and proper.*

1. *For $i \gg 0$, the map $\sigma\colon \pi_i\mathrm{TC}^-(\mathcal{C}) \to \pi_{i+2}\mathrm{TC}^-(\mathcal{C})$ is an isomorphism.*
2. *For $i \ll 0$, the map $x\colon \pi_i\mathrm{TC}^-(\mathcal{C}) \to \pi_{i-2}\mathrm{TC}^-(\mathcal{C})$ is an isomorphism.*
3. *For $i \gg 0$, the map $\varphi\colon \pi_i\mathrm{TC}^-(\mathcal{C}) \to \pi_i\mathrm{TP}(\mathcal{C})$ is an isomorphism.*
4. *For $i \ll 0$, the map $\mathrm{can}\colon \pi_i\mathrm{TC}^-(\mathcal{C}) \to \pi_i\mathrm{TP}(\mathcal{C})$ is an isomorphism.*

*Proof.* Assertions 1 and 2 follow from the fact that $\mathrm{TC}^-(\mathcal{C})$ is a perfect $\mathrm{TC}^-(k)$-module by Theorem 3.3 and the fact that they are true for $\mathrm{TC}^-(k)$ (recalling Example 4.5). Assertion 3 now follows from Assertion 1 and Proposition 4.7. Assertion 4 follows from Assertion 2 and the fact that $\mathrm{TP}(\mathcal{C}) \simeq \mathrm{TC}^-(\mathcal{C})[1/x]$ under can. □

*Proof of Theorem 4.1.* If $k$ is a finite field, then the $\mathbb{Z}_p$-modules $\pi_i\mathrm{TC}^-(\mathcal{C}), \pi_i\mathrm{TP}(\mathcal{C})$ are finitely generated by Theorem 3.3. Moreover, for $i \gg 0$, one finds that $\varphi$ is an isomorphism while can is divisible by $p$ (as $\mathrm{can}(\sigma)$ is divisible by $p$ in $\mathrm{TP}(k)$), while for $i \ll 0$, can is an isomorphism while $\varphi$ is divisible by $p$. The equalizer formula (4) for TC yields the assertion. □

# 5 A counterexample for cyclotomic spectra

Let $k$ be a perfect field of characteristic $p > 0$ and let $\mathcal{C}$ be a smooth and proper dg category over $k$. In view of the main result of the previous section and the fact that $\mathrm{THH}(\mathcal{C})$ is dualizable in $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$, one may now ask if $\mathrm{THH}(\mathcal{C})$ is perfect in $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$.

We show that this fails. For a supersingular K3 surface $X$ over $k$, we show that $\mathrm{THH}(X) = \mathrm{THH}(\mathrm{Perf}(X))$ is not perfect in CycSp. In fact, we show that $\mathrm{TF}(X)$ (see below) is not compact as a $\mathrm{TF}(k)$-module spectrum.

**Definition 5.1.** Recall that if $Y \in \mathrm{CycSp}$, then $Y$ in particular defines a genuine $C_{p^n}$-spectrum for each $n$.[8] As a result, one can form the fixed points $\mathrm{TR}^n(Y) = Y^{C_{p^n}}$. The spectrum $\mathrm{TF}(Y)$ can be defined as the homotopy limit $\mathrm{TF}(Y) = \lim_n \mathrm{TR}^n(Y)$ where the transition maps are the natural maps $F\colon \mathrm{TR}^{n+1}(Y) \to \mathrm{TR}^n(Y)$ (inclusions of fixed points). Thus TF defines an exact, lax symmetric monoidal functor

$$\mathrm{TF}\colon \mathrm{CycSp} \to \mathrm{Sp}.$$

For a scheme $X$, we will write $\mathrm{TF}(X) = \mathrm{TF}(\mathrm{Perf}(X))$.

We refer to [Hes96] for the basic structure theorems for $\mathrm{TR}^n$ and TF of smooth schemes over a perfect field $k$ and to [HM97] for the calculations over $k$ itself. In particular, the results of [Hes96] show that the homotopy groups of TF are closely related to the cohomology groups of the de Rham–Witt complex [Ill79]. For a smooth algebra $A/k$, we let $W_n\Omega^\bullet_A$ denote the level $n$ de Rham-Witt complex of $A$ and $W\Omega^\bullet_A$ denote the de Rham–Witt complex of $A$, so that $W\Omega^\bullet_A = \lim_R W_n\Omega^\bullet_A$ where the maps in the diagram are the restriction maps $R$. Recall also that the tower of graded abelian groups $\{W_n\Omega^\bullet_A\}$ is equipped with Frobenius maps $F\colon W_{n+1}\Omega^\bullet_A \to W_n\Omega^\bullet_A$.

**Theorem 5.2** (Hesselholt–Madsen [HM97, Theorems 3.3 and 5.5])**.** *Let $k$ be a perfect field of characteristic $p > 0$.*

[8] This was the basis of classical definitions of cyclotomic spectra. If one follows instead Definition 4.4, we refer to the proof of Theorem II.4.10 in [NS17] for a definition of the fixed points and to the proof of Theorem II.6.3 in loc. cit. for a construction of the genuine spectrum.

1. *One has* $\pi_*\mathrm{THH}(k)^{C_{p^n}} \simeq W_{n+1}(k)[\sigma_n]$ *for* $|\sigma_n| = 2$. *The map* $F\colon \pi_*\mathrm{THH}(k)^{C_{p^{n+1}}} \to \pi_*\mathrm{THH}(k)^{C_{p^n}}$ *is the Witt vector Frobenius on* $W_{n+1}(k)$ *and sends* $\sigma_{n+1}$ *to* $\sigma_n$.
2. *One has* $\mathrm{TF}_*(k) \simeq W(k)[\sigma]$ *for* $|\sigma| = 2$.

In particular, if $\mathrm{THH}(X)$ is perfect in $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$, then $\mathrm{TF}_i(X)$ is a finitely generated $W(k)$-module for all $i$. We show that if $X$ is a supersingular K3 surface, then $\mathrm{TF}_{-1}(X)$ is not finitely generated over $W(k)$. Here we use the following fundamental result.

**Theorem 5.3** (Hesselholt [Hes96, Thm. B])**.** *Let $k$ be a perfect field of characteristic $p > 0$. If $A$ is a smooth $k$-algebra, then* $\pi_*\mathrm{THH}(A)^{C_{p^n}} \simeq W_{n+1}\Omega^\bullet_A \otimes_{W_{n+1}(k)} W_{n+1}(k)[\sigma_n]$ *with* $|\sigma_n| = 2$. *The map* $F\colon \pi_*\mathrm{THH}(A)^{C_{p^{n+1}}} \to \pi_*\mathrm{THH}(A)^{C_{p^n}}$ *is the tensor product of the map* $F\colon W_{n+1}\Omega^\bullet_A \to W_n\Omega^\bullet_A$ *and the map* $W_{n+1}(k)[\sigma_{n+1}] \to W_n(k)[\sigma_n]$ *acting as the Witt vector Frobenius* $F\colon W_{n+1}(k) \to W_n(k)$ *and sending* $\sigma_{n+1}$ *to* $\sigma_n$.

Now let $X$ be a smooth and proper scheme over $k$. Note that the construction $U \mapsto \mathrm{TR}^n(U)$ is a sheaf of spectra $\underline{\mathrm{TR}^n}$ on the small étale site of $X$ (cf. [GH99, Sec. 3]). By Theorem 5.3, there is a strongly convergent descent spectral sequence

$$\mathrm{E}^2_{s,t} = \mathrm{H}^{-s}(X, \pi_t\underline{\mathrm{TR}^n}) \simeq \mathrm{H}^{-s}(X, \bigoplus_{j=0}^{\infty} W_{n+1}\Omega_X^{t-2j}) \Rightarrow \mathrm{TR}^n_{s+t}(X),$$

and taking the limit over $n$ we obtain a strongly convergent spectral sequence

$$\mathrm{E}^2_{s,t} = \lim_{n,F} \mathrm{H}^{-s}(X, \bigoplus_{j=0}^{\infty} W_n\Omega_X^{t-2j}) \Rightarrow \mathrm{TF}_{s+t}(X).$$

Note that since $X$ is proper, all of the terms at each stage of the inverse limit are finite length, so no $\lim^1$ terms appear. Note also that this spectral sequence comes from filtering the étale sheaf $\underline{\mathrm{TF}}$ given by $U \mapsto \mathrm{TF}(U)$ via the tower obtained by taking the inverse limit of the Postnikov towers of the $\underline{\mathrm{TR}^n}$ (and not the Postnikov tower of $\underline{\mathrm{TF}}$ itself). We will show that this spectral sequence forces $\mathrm{TF}_{-1}(X)$ to be non-finitely generated for $X$ a supersingular K3 surface.

We compare with crystalline cohomology. Recall that crystalline cohomology is computed by the hypercohomology of the de Rham–Witt complex $W\Omega^\bullet_X$. As in [IR83], there are two spectral sequences converging to crystalline cohomology: the Hodge spectral sequence, from the naive filtration of the de Rham–Witt complex, and the conjugate spectral sequence, which arises from the inverse limit of the Postnikov filtrations of the $W_n\Omega^\bullet_X$. The relevance to our setting is that the $\mathrm{E}_2$-terms of the conjugate spectral sequence appear in the above spectral sequence for $\mathrm{TF}(X)$. This follows from the higher Cartier isomorphisms [IR83, III, 1.4] $W_n\Omega^i_U \simeq \mathrm{H}^i(W_n\Omega^\bullet_U)$ for an affine $U$ étale over $X$, which implies together with Theorem 5.3 that the $\mathrm{E}_2^{s,t}$-term of the conjugate spectral sequence is equivalent to $\lim_{n,F}\mathrm{H}^s(X, W_n\Omega^t_X)$. Compare also [Hes16, Section 5].

**Proposition 5.4.** *The term* $\lim_{n,F}\mathrm{H}^2(X, W_n\Omega^1_X)$ *is not finitely generated if $X$ is a supersingular K3 surface.*

*Proof.* Recall from Figure 1 the Hodge–Witt cohomology groups $\mathrm{H}^i(X, W\Omega^j_X)$ for $X$ a supersingular K3 surface taken from [Ill79, Section 7.2]. By work of Illusie–Raynaud, this has consequences in the *conjugate* spectral sequence. Namely, in the conjugate spectral sequence, one has:

- The terms of total degree $0, 1, 4$ must be finitely generated $W(k)$-modules. In fact, they are torsion-free by comparing with the terms in the Hodge spectral sequence thanks to [IR83, IV.4.6.1] and because the quotient by torsion is always finitely generated [IR83, Introduction, 2.2].

| | $W\Omega^0_X$ | $W\Omega^1_X$ | $W\Omega^2_X$ |
|---|---|---|---|
| $\mathrm{H}^2$ | $k[[x]]$ | $k[[y]]$ | $W(k)$ |
| $\mathrm{H}^1$ | 0 | $W(k)^{22}$ | 0 |
| $\mathrm{H}^0$ | $W(k)$ | 0 | 0 |

Figure 1: The Hodge–Witt cohomology of a supersingular K3 surface.

- Some of the terms in the *conjugate* spectral sequence of total degrees 2 and 3 *must* be non-finitely generated. Compare [IR83, Introduction, 2.3].

We need to identify exactly where the failure of finite generation occurs. The conjugate spectral sequence degenerates after the $\mathrm{E}_2$-page for dimension reasons. Thus, the only possibly non-zero differentials are

$$\lim_{n,F} \mathrm{H}^0(X, W_n\Omega^2_X) \to \lim_{n,F} \mathrm{H}^2(X, W_n\Omega^1_X)$$

and

$$\lim_{n,F} \mathrm{H}^0(X, W_n\Omega^1_X) \to \lim_{n,F} \mathrm{H}^2(X, W_n\Omega^0_X).$$

The term $\lim_{n,F} \mathrm{H}^0(X, W_n\Omega^1_X)$ is in total degree 1 and hence is torsion-free and finitely generated. It follows that $\lim_{n,F} \mathrm{H}^2(X, W_n\Omega^0_X)$ is finitely generated as well since otherwise it would contribute something infinitely generated in crystalline cohomology. Therefore, both

$$\lim_{n,F} \mathrm{H}^0(X, W_n\Omega^2_X), \quad \lim_{n,F} \mathrm{H}^2(X, W_n\Omega^1_X)$$

are non-finite over $W(k)$. (The kernel and cokernel of the differential are, however, finite.) We see in particular that $\lim_{n,F} \mathrm{H}^2(X, W_n\Omega^1_X)$ is non-finite over $W(k)$. □

We can now state and prove the main conclusion of this section.

**Corollary 5.5.** *Let $k$ be a perfect field of characteristic $p > 0$. If $X$ is a supersingular K3 surface over $k$, then $\mathrm{TF}_{-1}(X)$ is not finitely generated as a $W(k)$-module. In particular, $\mathrm{THH}(X) \in \mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$ is not perfect.*

*Proof.* Returning to the local-global spectral sequence for $\mathrm{TF}(X)$, Figure 2 displays the region of the spectral sequence of interest to us. All terms and differentials contributing to $\mathrm{TF}_{s+t}(X)$ for $s+t \leqslant 1$ are shown. However, $\lim_{n,F} \mathrm{H}^0(X, W_n\Omega^0_X)$ is torsion-free and finite over $W(k)$ (since it has degree 0 in the conjugate spectral sequence for crystalline cohomology). Thus, the $d^2$-differential hitting $\lim_{n,F} \mathrm{H}^2(X, W_n\Omega^1)$ cannot possibly annihilate enough for the resulting term $\mathrm{E}^3_{-2,1} \cong \mathrm{E}^\infty_{-2,1}$ to be a finite $W(k)$-module. Of course, this implies that $\mathrm{TF}_{-1}(X)$ is not finitely generated, which is what we wanted to show. □

The reader might worry that this argument shows too much and can be used to contradict the finiteness of $\mathrm{TP}(X)$ over $\mathrm{TP}(k)$ given the spectral sequence of [Hes16, Theorem 6.8]. However, it is the periodicity of $\mathrm{TP}(k)$ that saves the day. When we periodicize the spectral sequence, more terms appear so that the differential hitting $\mathrm{E}^2_{-2,1}$ is the same as that hitting $\mathrm{E}^2_{-4,3}$. This fixes the non-finiteness.

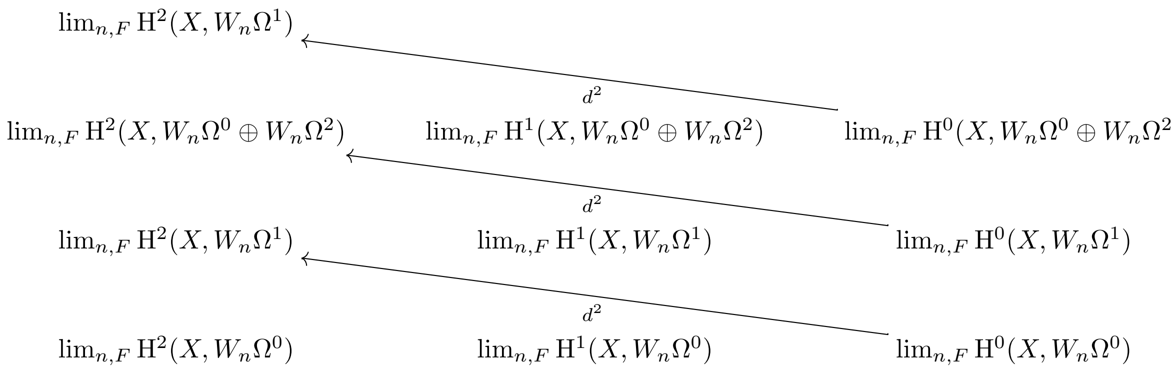


Figure 2: A part of the local-global spectral sequence for TF of a surface. To fix coordinates, the bottom left term displayed is $\mathrm{E}^2_{-2,0}$.

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