Dendroidal sets as models for connective spectra

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Abstract

Dendroidal sets have been introduced as a combinatorial model for homotopy coherent operads. We introduce the notion of fully Kan dendroidal sets and show that there is a model structure on the category of dendroidal sets with fibrant objects given by fully Kan dendroidal sets. Moreover we show that the resulting homotopy theory is equivalent to the homotopy theory of connective spectra.

1 Introduction

The notion of a dendroidal set is an extension of the notion of a simplicial set, introduced to serve as a combinatorial model for ∞-operads [MW07]. The homotopy theory of ∞-operads is defined as an extension of Joyal's homotopy theory of ∞-categories to the category of dendroidal sets. More precisely there is a class of dendroidal sets called inner Kan dendroidal sets (or simply ∞-operads) which are defined analogously to inner Kan complexes (also known as ∞-categories) by lifting conditions [MW09]. These objects form fibrant objects in a model structure on the category of dendroidal sets, which is Quillen equivalent to topological operads as shown in a series of papers by Cisinski and Moerdijk [CM10, CM11a, CM11b].

The category of dendroidal sets behaves in many aspects similar to that of simplicial sets. One instance of this analogy is the model structure described above generalizing the Joyal model structure. Another instance is the fact that there is a nerve functor from (coloured) operads into dendroidal sets generalizing the nerve functor from categories into simplicial sets. But there are two important aspects of the theory of simplicial sets that have not yet a counterpart in the theory of dendroidal sets:

1. Kan complexes and the Kan-Quillen model structure on simplicial sets[1].
2. The geometric realization of simplicial sets.

The two aspects are closely related since the geometric realization $|-| : \mathrm{sSet} \to \mathrm{Top}$ is a left Quillen equivalence with respect to the Kan-Quillen model structure on simplicial sets. With respect to the Joyal model structure it is still a Quillen adjunction (but not an equivalence), as follows from the fact that the Kan-Quillen model structure is a left Bousfield localization

[1]In fact there is a model structure constructed by Heuts [Heu11a] that could be seen as a counterpart. We comment on this model structure later.

of the Joyal model structure. The problem of finding counterparts for these structures in the theory of dendroidal sets has been raised almost with the introduction of dendroidal sets, see e.g. [Wei11, Section 5].

In the present paper we construct analogues of 1 and 2 for the category of dendroidal sets. More precisely we introduce the notion of a fully Kan dendroidal set which (in analogy to a Kan complex in simplicial sets) has fillers for all horns of dendroidal sets and not just for inner horns (as for inner Kan dendroidal sets), see Definition 3.1. As a first result we show that a certain subclass of fully Kan dendroidal sets, called strictly fully Kan dendroidal sets, spans a category equivalent to the category of Picard groupoids, Corollary 3.4. This already provides a hint as to what the geometric realization might be since it is well known that Picard groupoids model all connective spectra with vanishing π_n for $n \geq 2$, [May08, JO12].

In fact, fully Kan dendroidal sets model all connective spectra. This is the main result of this paper:

Theorem (Theorems 4.2,4.6 and 5.4)**:** *There is a model structure on dendroidal sets, called the* stable model structure*, with fibrant objects given by fully Kan dendroidal sets which is a left Bousfield localization of the Cisinski-Moerdijk model structure. Moreover the stable model structure on dendroidal sets is Quillen equivalent to connective spectra.*

The stable model structure has good formal properties, i.e. it is left proper, simplicial, tractable and combinatorial. Furthermore it allows an explicit characterization of weak equivalences. The Quillen equivalence factors through the category of group-like E_∞-spaces which is known to be equivalent to connective spectra.

The proof of our theorem is based on constructions of Heuts [Heu11a, Heu11b]. Heuts establishes a model structure on dendroidal sets, called the *covariant model structure*, which lies between the Cisinski-Moerdijk model structure and the stable model structure. Although we had at first obtained the stable model structure by different techniques, in this paper we construct it as a left Bousfield localization of the covariant model structure. This enables us to directly use another main result of Heuts: there is a Quillen equivalence between the covariant model structure and the model category of E_∞-spaces. Our Quillen equivalence (Theorem 5.4) can then be derived by showing that the stable localization on the side of dendroidal sets corresponds to the group-like localization of E_∞-spaces, see section 5. One disadvantage of this construction is that establishing the explicit description of fibrant objects is technically demanding, see sections 6 - 8.

Finally we want to mention that our results not only show that fully Kan dendroidal sets form a model for Picard ∞-groupoids but also that the ∞-category of Picard ∞-groupoids is a full reflective subcategory (in the sense of Lurie [Lur09, Remark 5.2.7.9]) of the ∞-category of ∞-operads. The functor associating a spectrum to a dendroidal set will be further investigated in [Nik] and related to the geometric realization of simplicial sets.

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2 Dendroidal sets and model structures

In the following we will review some facts from the theory of dendroidal sets without giving explicit reference and we refer the reader to the lecture notes [MT10] and the papers [MW07, MW09].

We briefly recall the definition of the category of dendroidal sets. It is based on the notion of trees. A (finite rooted) tree is a non-empty connected finite graph with no loops equipped with a distinguished outer vertex called the root and a (possibly empty) set of outer vertices not containing the root called leaves. By convention, the term vertex of a tree refers only to non-outer vertices. Each tree T generates a symmetric, coloured operad $\Omega(T)$ (in the category of sets) which has the edges of T as colours. Using this construction we can define the category Ω which has as objects finite rooted trees and a morphism $T \to T'$ is given by a morphism of operads $\Omega(T) \to \Omega(T')$. Similarly to simplicial sets we then define the category of dendroidal sets as the presheaf category on Ω:

$$\mathrm{dSet} := [\Omega^{op}, \mathrm{Set}].$$

The dendroidal set represented by a tree T is denoted by $\Omega[T]$. In particular for the tree with one colour (the graph with one edge) we set $\eta := \Omega[\,|\,]$. The inclusion of Ω into the category of coloured, symmetric operads induces a fully faithful functor $N_d : \mathrm{Oper} \to \mathrm{dSet}$ called the dendroidal nerve. We have $N_d(\Omega(T)) = \Omega[T]$.

There is a fully faithful embedding of the simplex category Δ into Ω by considering finite linear ordered sets as linear trees. This induces an adjunction

$$i_! : \mathrm{sSet} \rightleftarrows \mathrm{dSet} : i^*$$

where the left adjoint is fully faithful (there is also a further right adjoint i_* which does not play a role in this paper).

The theory of dendroidal sets behaves very much like the theory of simplicial sets. In particular, for each tree T there is a set of certain subobjects of $\Omega[T]$ in dSet called faces. There are two types of faces: inner faces which are labeled by inner edges of T and outer faces which are labeled by vertices of the tree T with exactly one inner edge attached to it. The boundary $\partial\Omega[T]$ of $\Omega[T]$ is by definition the union of all faces of T. A horn is defined as the union of all but one faces, see [MW09] or [MT10]. We talk of inner or outer horns and we write $\Lambda^a[T]$ where a is an inner edge or an appropriate vertex of T.

Definition 2.1. Let T be a tree with at least 2 vertices. We call a horn $\Lambda^a[T] \subset \Omega[T]$ a *root horn*, if a is the unique vertex attached to the root edge.

The corolla C_n is the tree with one vertex and n leaves. There are $n+1$ faces of a corolla C_n, one for each colour (edge). The horns are the unions of all but one colours, denoted by $\Lambda^a[C_n]$ for a the omitted colour. We call this horn a *leaf horn* if a is the root colour (i.e. the leaf horn is the inclusion of the leaves) and a *root horn* otherwise.

The horns are used to define certain objects by the right lifting property against them.

Definition 2.2. A dendroidal set D is called *inner Kan* if D admits fillers for all inner horns, i.e. for any inner edge e of a tree T and a morphism $\Lambda^e[T] \to D$ there is a morphism

$\Omega[T] \to D$ that renders the diagram

$$\begin{array}{ccc} \Lambda^e[T] & \longrightarrow & D \\ \downarrow & \nearrow & \\ \Omega[T] & & \end{array}$$

commutative. A *dendroidal Kan complex* is a dendroidal set that admits fillers for all non-root horns.

The two classes of inner Kan dendroidal sets and of dendroidal Kan complex have been introduced and studied in [MW09, CM10] and [Heu11b]. The main results are

Theorem 2.3 (Cisinski-Moerdijk)**.** *There is a left proper, combinatorial model structure on dendroidal sets with cofibrations given by normal monomorphisms and fibrant objects given by inner Kan dendroidal sets. This model category is Quillen equivalent to topological operads.*

Theorem 2.4 (Heuts)**.** *There is a simplicial left proper, combinatorial model structure on dendroidal sets with cofibrations given by normal monomorphisms and fibrant objects given by dendroidal Kan complexes. This model structure is called the* covariant model structure *and is Quillen equivalent to E_∞-spaces.*

The slogan is that inner Kan dendroidal sets are a combinatorial model for topological operads and dendroidal Kan complexes are a model for E_∞-spaces. The weak equivalences are called operadic equivalences in the Cisinski-Moerdijk model structure and covariant equivalences in the Heuts model structure. Note in particular that the covariant model structure is simplicial in contrast to the Cisinski-Moerdijk model structure. The simplicial enrichment in question is induced by the Boardman-Vogt type tensor product on the category dSet.

3 Fully Kan dendroidal sets

We define similarly to inner Kan complexes as in Definition 2.2:

Definition 3.1. A dendroidal set D is called *fully Kan* if it has fillers for all horn inclusions. This means that for each morphism $\Lambda^a[T] \to D$ (where a is an inner edge or an outer vertex) there is a morphism $\Omega[T] \to D$ rendering the diagram

$$\begin{array}{ccc} \Lambda^a[T] & \longrightarrow & D \\ \downarrow & \nearrow & \\ \Omega[T] & & \end{array}$$

commutative. D is called *strictly fully Kan* if additionally all fillers for trees T with more than one vertex are unique.

Remark 3.2.
- Each fully Kan dendroidal set is also a dendroidal Kan complex and an inner Kan dendroidal set.

- The reader might wonder why we do not impose uniqueness for corolla fillers in the strictly fully Kan condition. The reason is that this forces the underlying simplicial set to be discrete as we will see in Proposition 3.5.

Let C be a (small) symmetric monoidal category. We can define a coloured operad as follows. The colours are the objects of C. The set of n-ary operations is defined as

$$\mathcal{O}(c_1, ..., c_n; c) := \mathrm{Hom}_C(c_1 \otimes ... \otimes c_n, c).$$

The Σ_n-action is induced by the symmetric structure on C and the composition is given by composition in C. Note that the expression $c_1 \otimes ... \otimes c_n$ is strictly speaking not well-defined in a symmetric monoidal category. One can either make a choice of order in which to tensor (e.g. from the left to the right) or work with unbiased symmetric monoidal categories. These are symmetric monoidal categories which have not only two-fold, but also n-fold chosen tensor products. For a discussion of these issues see [Lei04, Chapter 3.3].

We denote by Sym the category of symmetric monoidal categories together with lax monoidal functors. Recall that a lax monoidal functor $F : C \to D$ is a functor together with morphisms $F(c) \otimes F(c') \to F(c \otimes c')$ for each $c, c' \in C$ and $1 \to F(1)$ which have to satisfy certain coherence conditions but do not have to be isomorphisms. The construction described above gives a fully faithful functor

$$\mathrm{Sym} \to \mathrm{Oper}.$$

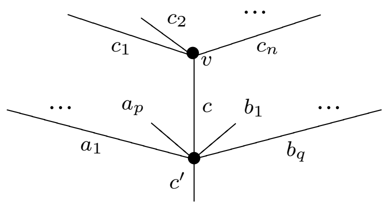
By composing with the dendroidal nerve $N_d : \mathrm{Oper} \to \mathrm{dSet}$ for each symmetric monoidal category C we obtain a dendroidal set which we denote by abuse of notation with $N_d(C)$.

In [MW09] it is shown that a dendroidal set is strictly inner Kan if and only if it is of the form $N_d(P)$ for a coloured operad P. An analogous statement is true for strictly fully Kan dendroidal sets. Therefore recall that a symmetric monoidal category is called a *Picard groupoid* if its underlying category is a groupoid and its set of isomorphism classes is a group, i.e. there are 'tensor inverses' for objects.

Proposition 3.3. *A dendroidal set D is strictly fully Kan if and only if there is a Picard groupoid C with $D \cong N_d(C)$.*

Proof. First assume that D is strictly fully Kan. Then in particular D is strictly inner Kan and [MW09, Theorem 6.1] shows that there is a coloured operad P with $N_d(P) \cong D$. Let C be the underlying category of P. Since the underlying simplicial set of $N_d(C)$ is a Kan complex we conclude that C is a groupoid.

By [Lei04, Theorem 3.3.4] an operad P comes from a unique symmetric monoidal category as described above if and only if for every sequence $c_1, ..., c_n$ of objects in P there is universal tensor product, that is an object c together with an operation $t \in P(c_1, ..., c_n; c)$ such that for all objects $a_1, ..., a_p, b_1, ..., b_q, c'$ and operations $t' \in P(a_1, ..., a_m, c_1, ..., c_n, b_1, ..., b_q; c')$ there is a unique element $s \in P(a_1, ..., a_m, c, b_1, ..., b_q; c')$ such that the partial composition of s and t in P is equal to t'. A sequence $c_1, ..., c_n$ of objects of P determines a map $\eta_{c_1} \sqcup \cdots \sqcup \eta_{c_n} \to N_d(P)$. Since $N_d(P)$ is fully Kan we can fill the horn $\eta_{c_1} \sqcup \cdots \sqcup \eta_{c_n} \to \Omega[C_n]$ and obtain a morphism $\Omega[C_n] \to N_d(P)$. The root colour of this morphism provides an object c in P and the corolla provides an operation $t \in P(c_1, ..., c_n; c)$. Assume we have another operation $t' \in P(a_1, ..., a_m, c_1, ..., c_n, b_1, ..., b_q; c')$. Then we consider the tree T which is given by



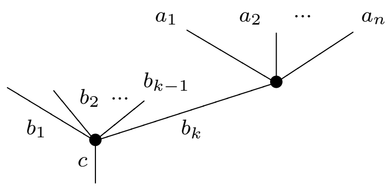
The operations t and t' provide a morphism $\Lambda^v[T] \to N_dP$, where $\Lambda^v[T]$ is the outer horn of $\Omega[T]$ at v. Since D is strictly fully Kan we obtain a unique filler $\Omega[T] \to N_d(P)$, i.e. a unique $s \in P(a_1, ..., a_m, c, b_1, ..., b_q; c')$ with the sought condition. This shows that c is the desired universal tensor product and that P comes from a symmetric monoidal category.

The last thing to show is that C is group-like. For a and b in C we obtain an object c together with a morphism $t \in P(a, c; b)$ by filling the horn $\eta_a \sqcup \eta_c \to \Omega[C_2]$. But this is the same as a morphism $a \otimes c \to b$ which is an isomorphism since C is a groupoid. If we let b be the unit in C then c is the necessary inverse for a.

Now assume conversely that C is a Picard groupoid. Then the associated dendroidal set $N_d(C)$ admits lifts for corolla horns since tensor products and inverses exist (the proof is essentially the same as above). It remains to show that all higher horns admit unique fillers. To see this let T be a tree with more than one vertex and $\Lambda^a[T]$ be any horn. A morphism $\Omega[T] \to N_d(C)$ is given by labeling the edges of T with objects of C and the vertices with operations in C of higher arity, i.e. morphisms out of the tensor product of the ingoing objects into the outgoing object of the vertex. The same applies for a morphism $\Lambda^a[T] \to N_d(C)$ where the faces in the horn are labeled in the same manner, but consistently.

The first observation is that for any labeling of the horn $\Lambda^a[T]$ already all edges of the tree T are labeled, since the horn contains all colours of T (for T with more than one vertex). If the horn is inner then also all vertices of T are already labeled if we label $\Lambda^a[T]$ and thus there is a unique filler. If a is an outer vertex and T has more then two vertices then the same applies as one easily checks. Thus the horn can be uniquely filled. Therefore we only have to deal with outer horns of trees with exactly two vertices. Such trees can all be obtained by grafting a n-corolla C_n for $n \geq 0$ on top of a k-corolla for $k \geq 1$. We call this tree $C_{n,k}$.

$C_{n,k} =$



A morphism from the non-root horn $\Lambda^v[C_{n,k}] \to N_d(C)$ is then given by a pair consisting of a morphism $f : a_1 \otimes ... \otimes a_n \longrightarrow b_k$ and a morphism $g : b_1 \otimes ... \otimes b_{k-1} \otimes a_1 \otimes ... \otimes a_n \longrightarrow c$ in C. Now we find a unique morphism $g \circ (id \otimes f^{-1}) : b_1 \otimes ... \otimes b_k \to c$ which renders the relevant diagram commutative, i.e. provides a filler $\Omega[C_{n,k}] \to N_d(C)$. A similar argument works for the case of the root horn of $C_{n,k}$. Together this finishes the proof. □

Corollary 3.4. *The functor N_d : Sym $\to$ dSet induces an equivalence between the full subcategory of Picard groupoids on the left and the full subcategory of strictly fully Kan dendroidal sets on the right.*

Proof. The functor N_d is fully faithful since both functors Sym $\to$ Oper and Oper $\to$ dSet are. The restriction is essentially surjective by the last proposition. □

One of the main results of this paper shows that this remains valid in a certain sense also for fully Kan dendroidal sets that are not strict. They form a model for Picard ∞-groupoids, as we will show in the next sections.

Finally we want to give a characterization of strictly fully Kan complexes, where also the corolla horns admit unique fillers. Let A be an abelian group, then we can associate to A a symmetric monoidal category A_{dis} which has A as objects and only identity morphisms. The tensor product is given by the group multiplication of A and is symmetric since A is abelian. This construction provides a fully faithful functor from the category AbGr of abelian groups to the category Sym. Composing with the functor Sym $\to$ dSet constructed above we obtain a fully faithful functor

$$i : \mathrm{AbGr} \to \mathrm{dSet}.$$

Now we can characterize the essential image of i.

Proposition 3.5. *For a dendroidal set D the following two statements are equivalent*

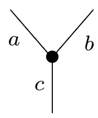
- *D is fully Kan with all fillers unique.*
- *$D \cong i(A)$ for an abelian group A.*

Proof. We already know by 3.3 that strictly fully Kan dendroidal sets are of the form $N_d(C)$ for C a Picard groupoid. We consider the underlying space $i^*D = NC$. This is now a strict Kan complex in the sense that all horn fillers are unique. In particular fillers for the horn $\Lambda^0[1] \to \Delta[1]$ are unique which shows that there are no non-degenerated 1-simplices in NC, hence no non-identity morphisms in C. Thus C is a discrete category. But a discrete category which is a Picard groupoid is clearly of the form A_{dis} for an abelian group A. This shows one direction of the claim. The other is easy and left to the reader. □

4 The stable model structure

In this section we want to describe another model structure on the category of dendroidal sets which we call the *stable* model structure. We construct it as a left Bousfield localization. We will further explore this model structure to give a simple characterization of fibrant objects and weak equivalences

The idea is to localize at a horn of the 2-corolla

$C_2 =$ 

The relevant horn is given by the inclusion of the colours a and c, i.e. by the map

$$s : \Lambda^b\Omega[C_2] = \eta_a \sqcup \eta_c \longrightarrow \Omega[C_2]. \tag{1}$$

Note that there is also the inclusion of the colours b and c but this is essentially the same map since we deal with symmetric operads.

Definition 4.1. The *stable* model structure on dendroidal sets is the left Bousfield localization of the covariant model structure at the map s. This means that stable cofibrations are normal maps of dendroidal sets. The stable fibrant objects are those dendroidal Kan complexes D for which the map

$$s^* : \underline{\mathrm{sHom}}(\Omega[C_2], D) \to \underline{\mathrm{sHom}}(\eta_a \sqcup \eta_c, D)$$

is a weak equivalence of simplicial sets.

The general theory of left Bousfield localization (see e.g. [Lur09, A.3]) yields the following:

Theorem 4.2. *1. The category of dendroidal sets together with the stable model structure is a left proper, combinatorial, simplicial model category.*

2. The adjoint pair

$$i_! : sSet \rightleftarrows dSet : i^*$$

is a Quillen adjunction (for the stable model structure on dendroidal sets and the Kan-Quillen model structure on simplicial sets).

3. The functor i^* is homotopy right conservative, that is a morphism $f : D \to D'$ between stably fibrant dendroidal sets D and D' is a stable equivalence if and only if the underlying map $i^*f : i^*D \to i^*D'$ is a homotopy equivalence of Kan complexes.

Proof. The first part follows from the general theory of Bousfield localizations. We now want to prove the fact about the adjunction. Therefore note that the corresponding fact for the covariant model structure is true. Since the stable model structure is a left Bousfield localization of the covariant model structure the claim follows by composition with the identity functor. The last assertion is true since a morphism between stably fibrant objects is a stable equivalence if and only if it is a covariant equivalence. And a covariant equivalence can be tested on the underlying spaces [Heu11b, Proposition 2.2.] □

Corollary 4.3. *Let $f : X \to Y$ be a map of dendroidal sets. Then f is a stable equivalence exactly if $i^*(f_K)$ is a weak equivalence where $f_K : X_K \to Y_K$ is the corresponding map between fully Kan (fibrant) replacements of X and Y.*

Remark 4.4. We could as well have localized at bigger collections of maps:

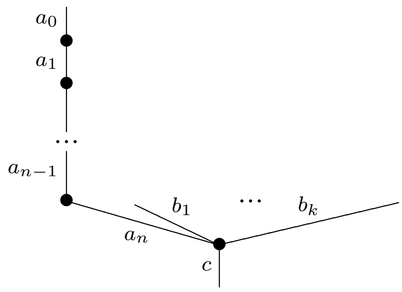
- all corolla root horns
- all outer horns

These localizations would yield the same model structure as we will see below. We decided to use only the 2-corolla in order to keep the localization (and the proofs) as simple as possible.

As a next step we want to identify the fibrant objects in the stable model structure as the fully Kan dendroidal sets. But first we need some terminology:

Definition 4.5. The *extended corolla* is the tree

$EC_{n,k} =$



In particular we have $EC_{0,k} = C_{k+1}$. The trees $EC_{n,1}$ are called *binary extended corollas*. The root horn of the extended corolla is the horn which is the union of all faces except the face obtained by chopping off the root corolla.

Theorem 4.6. *For a dendroidal set D are equivalent*

1. *D is fibrant in the stable model structure*
2. *D is dendroidal Kan and admits fillers for all root horns of extended corollas $EC_{n,1}$.*
3. *D is dendroidal Kan and admits fillers for all root horns of extended corollas $EC_{n,k}$.*
4. *D is fully Kan.*

We will prove Theorem 4.6 at the end of the paper. More precisely the equivalence of (1) and (2) is Proposition 6.2. The equivalence of (2) and (3) is Proposition 7.2 and the equivalence of (3) and (4) is in Proposition 8.2.

5 Equivalence to connective spectra

Let $\mathcal{E}_\infty \in \mathrm{dSet}$ be a cofibrant resolution of the terminal object in dSet which has the property that the underlying space $i^*\mathcal{E}_\infty$ is equal to the terminal object $\Delta[0] \in \mathrm{sSet}$. The existence of such an object can be easily seen, e.g. using the small object argument (note that the cofibrant objects are the same in all three model structures on dSet that we consider). In the following we denote $E_\infty := hc\tau_d(\mathcal{E}_\infty)$ which is an operad enriched over simplicial sets. Here

$$hc\tau_d : \mathrm{dSet} \to \mathrm{sOper}$$

is the left adjoint to the homotopy coherent nerve functor, see [CM11a]. The operad E_∞ is then cofibrant, has one colour and the property that each space of operations is contractible. Thus it is indeed an E_∞-operad in the classical terminology. Therefore for each E_∞-algebra X in sSet, the set $\pi_0(X)$ inherits the structure of an abelian monoid. Such an algebra X is called *group-like* if $\pi_0(X)$ is an abelian group, i.e. there exist inverses for each element.

Now denote by E_∞-spaces the category of E_∞-algebras in simplicial sets. Recall from [Heu11b, Section 3] that there is an adjoint pair $St : \mathrm{dSet}_{/\mathcal{E}_\infty} \rightleftarrows E_\infty\text{-spaces} : Un$ where $\mathrm{dSet}_{/\mathcal{E}_\infty}$ denotes the category of dendroidal sets over $\mathcal{E}_\infty$. We do not repeat the definition of St here since we need the formula only for a few particular simple cases and for these cases we give the result explicitly

Example 5.1.
- *The E_∞-algebra $St(\eta \to \mathcal{E}_\infty)$ is the free E_∞-algebra on one generator, which we denote by $Fr(a)$ where a is the generator.*

- *An object in $dSet_{/\mathcal{E}_\infty}$ of the form $p : \Omega[C_2] \to \mathcal{E}_\infty$ encodes a binary operation $- \cdot_p -$ in the operad E_∞. Then $St(p)$ is the free E_∞-algebras on two generators a,b and the square $\Delta[1] \times \Delta[1]$ subject to the relation that $a \cdot_p b \sim (1,1) \in \Delta[1] \times \Delta[1]$. We write this as*

$$St(\Omega[C_2] \to \mathcal{E}_\infty) = \frac{Fr(a, b, \Delta[1]^2)}{a \cdot_p b \sim (1,1)}.$$

- *The three inclusions $\eta \to \Omega[C_2]$ induce maps $St(\eta) \to St(p)$. As usual we let a, b be the leaves of C_2 and c the root. The first two maps are simply given by*

$$Fr(a) \to Fr(a, b, \Delta[1]^2)/_\sim \quad a \mapsto a \qquad \text{and} \qquad Fr(b) \to Fr(a, b, \Delta[1]^2)/_\sim \quad b \mapsto b$$

  The third map $Fr(c) \to Fr(a, b, \Delta[1]^2)/_\sim$ is given by sending c to $(0,0) \in \Delta[1]^2$. Note that this third map is obviously homotopic to the map sending c to $(1,1) = a \cdot_p b$.

The functor $P(D) := D \times \mathcal{E}_\infty$ induces a further adjoint pair $P : \mathrm{dSet} \rightleftarrows \mathrm{dSet}_{/\mathcal{E}_\infty} : \Gamma$. Composing the two pairs (St, Un) and (P, Γ) we obtain an adjunction

$$St_{\times\mathcal{E}_\infty} : \mathrm{dSet} \rightleftarrows E_\infty\text{-spaces} : Un_\Gamma \tag{2}$$

Moreover E_∞-spaces carries a left proper, simplicial model structure where weak equivalences and fibrations are just weak equivalences and fibrations of the underlying space of an E_∞-algebra, see [Spi01, Theorem 4.3. and Proposition 5.3] or [BM03]. For this model structure and the covariant model structure on dendroidal sets the above adjunction (2) is in fact a Quillen equivalence as shown by Heuts [Heu11b] [2].

Lemma 5.2. *Let X be a fibrant E_∞-space. Then X is group-like if and only if $Un_\Gamma(X) \in$ dSet is fully Kan.*

Proof. The condition that $Un_\Gamma(X)$ is fully Kan is by Theorem 4.6 equivalent to the map

$$s^* : \underline{\mathrm{sHom}}(\Omega[C_2], Un_\Gamma(X)) \to \underline{\mathrm{sHom}}(\eta_a \sqcup \eta_c, Un_\Gamma(X))$$

being a weak equivalence of simplicial sets. By the Quillen equivalence (2) and the fact that $\Omega[C_2]$ is cofibrant the space $\underline{\mathrm{sHom}}(\Omega[C_2], Un_\Gamma(X))$ is homotopy equivalent to the space $\underline{\mathrm{sHom}}(St(\Omega[C_2] \times \mathcal{E}_\infty \to \mathcal{E}_\infty), X)$. Next choose a morphism $p : \Omega[C_2] \to \mathcal{E}_\infty$. This choice exists and is essentially unique by the fact that $\Omega[C_2]$ is cofibrant and $\mathcal{E}_\infty \to *$ is a trivial fibration. In the covariant model structure on $\mathrm{dSet}_{/\mathcal{E}_\infty}$ (see [Heu11b, Section 2]) the objects $\Omega[C_2] \times \mathcal{E}_\infty \to \mathcal{E}_\infty$ and $\Omega[C_2] \to \mathcal{E}_\infty$ are cofibrant and equivalent. Cofibrancy is immediate and the fact that they are equivalent follows since the forgetful functor to dendroidal sets is a left Quillen equivalence and $\Omega[C_2] \simeq \Omega[C_2] \times \mathcal{E}_\infty$ in dSet. Therefore $St(\Omega[C_2] \times \mathcal{E}_\infty \to \mathcal{E}_\infty)$ is weakly equivalent to $St(\Omega[C_2] \to \mathcal{E}_\infty)$ in E_∞-spaces. Together we have the following weak equivalence of spaces

$$\underline{\mathrm{sHom}}(\Omega[C_2], Un_\Gamma(X)) \cong \underline{\mathrm{sHom}}(St(\Omega[C_2] \to \mathcal{E}_\infty), X).$$

[2] Note that Heuts in fact uses a slightly different variant where P is a right Quillen functor (instead of left Quillen). But if a right Quillen equivalence happens to be a left Quillen functor as well, then this left Quillen functor is also an equivalence. Thus Heuts' results immediately imply the claimed fact.

The same reasoning yields a weak equivalence $\underline{\text{sHom}}(\eta_a \sqcup \eta_c, Un_\Gamma(X)) \cong \underline{\text{sHom}}(St(\eta_a \sqcup \eta_c \to \mathcal{E}_\infty), X)$ such that the diagram

$$\begin{array}{ccc} \underline{\text{sHom}}(\Omega[C_2], Un_\Gamma(X)) & \xrightarrow{s^*} & \underline{\text{sHom}}(\eta_a \sqcup \eta_c, Un_\Gamma(X)) \\ \downarrow\sim & & \downarrow\sim \\ \underline{\text{sHom}}(St(\Omega[C_2] \to \mathcal{E}_\infty), X) & \xrightarrow{s^*} & \underline{\text{sHom}}(St(\eta_a \sqcup \eta_c \to \mathcal{E}_\infty), X) \end{array} \tag{3}$$

commutes.

Finally we use the fact that in the covariant model structure over $\mathcal{E}_\infty$ the leaf inclusion $i : \eta_a \sqcup \eta_b \to \Omega[C_2]$ is a weak equivalence. This implies that there is a further weak equivalence $St(\eta_a \sqcup \eta_b \to \mathcal{E}_\infty) \xrightarrow{\sim} St(\Omega[C_2] \to \mathcal{E}_\infty)$. As remarked above the straightening of $\eta \to \mathcal{E}_\infty$ is equal to $Fr(*)$, the free E_∞-algebra on one generator. Thus $St(\eta_a \sqcup \eta_b \to \mathcal{E}_\infty)$ is the coproduct of $Fr(a)$ and $Fr(b)$ which is isomorphic to $Fr(a, b)$ (here we used a and b instead of $*$ to label the generators). Then the above equivalence reads $Fr(a, b) \xrightarrow{\sim} St(\Omega[C_2] \to \mathcal{E}_\infty)$. The root inclusion $r : \eta_c \to \Omega[C_2]$ induces a further map $r^* : Fr(c) = St(\eta_c \to \mathcal{E}_\infty) \to St(\Omega[C_2] \to \mathcal{E}_\infty)$ and using the explicit description of $St(p)$ given above we see that there is a homotopy commutative diagram

$$\begin{array}{ccccc} St(\Omega[C_2] \to \mathcal{E}_\infty) & & \xleftarrow{St(r)} & & Fr(c) \\ & \nwarrow^{St(i)} & & \swarrow^{f} & \\ & & Fr(a, b) & & \end{array}$$

where f is defined as the map sending c to the product $a \cdot_p b$. Thus the horn $s : \eta_a \sqcup \eta_c \to C_2$ fits in a homotopy commutative diagram

$$\begin{array}{ccccc} St(\Omega[C_2] \to \mathcal{E}_\infty) & & \xleftarrow{St(s)} & & Fr(a, c) \\ & \nwarrow^{St(i)} & & \swarrow^{sh} & \\ & & Fr(a, b) & & \end{array}$$

with the map sh that sends c to the binary product of a and b and a to itself.

Putting the induced diagram together with diagram (3) we obtain a big diagram

$$\begin{array}{ccc} \underline{\text{sHom}}(\Omega[C_2], Un_\Gamma(X)) & \xrightarrow{s^*} & \underline{\text{sHom}}(\eta_a \sqcup \eta_c, Un_\Gamma(X)) \\ \downarrow\sim & & \downarrow\sim \\ \underline{\text{sHom}}(St(\Omega[C_2] \to \mathcal{E}_\infty), X) & \xrightarrow{s^*} & \underline{\text{sHom}}(Fr(a, c), X) \\ \downarrow\sim & \nearrow_{sh*} & \\ \underline{\text{sHom}}(Fr(a, b), X) & & \end{array} \tag{4}$$

in which all the vertical arrows are weak equivalences. This shows that $Un_\Gamma(X)$ is fully Kan if and only if $sh^* : \underline{\text{sHom}}(Fr(a, b), X) \to \underline{\text{sHom}}(Fr(a, c), X)$ is a weak equivalence. But we clearly have that domain and codomain of this map are given by $X \times X$. Thus the map in question is given by the shear map

$$Sh : X \times X \to X \times X \qquad (x, y) \mapsto (x, x \cdot_p y)$$

where $x \cdot_p y$ is the composition of x and y using the binary operation given by $hc\tau_d(p) : \Omega(C_2) \to E_\infty$

It remains to show that a fibrant E_∞-space X is group-like precisely when the shear map $Sh : X \times X \to X \times X$ is a weak homotopy equivalence. This is well known [Whi95, chapter III.4], but we include it for completeness. Assume first that the shear map is a weak equivalence. Then the induced shear map $\pi_0(X) \times \pi_0(X) \to \pi_0(X) \times \pi_0(X)$ is an isomorphism. This shows that $\pi_0(X)$ is a group, thus X is group-like. Assume conversely that X is group-like and $y \in X$ is a point in X. Then there is an inverse $y' \in X$ together with a path connecting $y' \cdot_p y$ to the point 1. This induces a homotopy inverse for the map $R_y : X \to X$ given by right multiplication with y (for the fixed binary operation). Now the shear map is a map of fibre bundles

$$\begin{array}{ccc} X \times X & \xrightarrow{Sh} & X \times X \\ & {}_{pr_1}\searrow \quad \swarrow_{pr_1} & \\ & X & \end{array}$$

Thus the fact that it is over each point $y \in X$ a weak equivalence as shown above already implies that the shear map is a weak equivalence. □

The last lemma shows that fully Kan dendroidal sets correspond to group-like E_∞-spaces. We want to turn this into a statement about model structures. Therefore we need a model structure on E_∞-spaces where the fibrant objects are precisely the group-like E_∞-spaces.

Proposition 5.3. *There is a left proper, combinatorial model structure on E_∞-spaces where the fibrant objects are precisely the fibrant, group-like E_∞-spaces and which is a left Bousfield localization of the standard model structure on E_∞-spaces. We call it the* group-completion *model structure.*

Proof. Since the model category of E_∞-spaces is left proper, simplicial and combinatorial the existence follows from general existence results provided that we can characterize the property of being group-like as a lifting property against a set of morphisms. It was already carried out how to do this in the last lemma, namely let the set consist of one map from the free E_∞-algebra on two generators to itself given by the shear map (actually there is one shear map for each binary operation in E_∞ but we simply pick one out). □

It is well known that group-like E_∞-spaces model all connective spectra by use of a delooping machine, see [May74]. More precisely the ∞-category of group-like E_∞-spaces obtained from the group-completion model structure is equivalent as an ∞-category to the ∞-category of connective spectra, see e.g. [Lur11, Remark 5.1.3.17].

Theorem 5.4. *The stable model structure on dendroidal sets is Quillen equivalent to the group-completion model structure on E_∞-spaces by the adjunction* (2)*. Thus the stable model structure on dendroidal sets is a model for connective spectra in the sense that there is an equivalence of ∞-categories.*

The theorem follows from Lemma 5.2 and the following more general statement about left Bousfield localization and Quillen equivalence. Therefore recall from [Bar07, Definition 1.3.] that a combinatorial model category is called *tractable* if it admits a set of generating cofibrations and generating trivial cofibrations with cofibrant domain and codomain. It turns out that it suffices to check this for generating cofibrations [Bar07, Corollary 1.12.]. Thus all model structures on dendroidal sets are clearly tractable.

Lemma 5.5. *Let C and D be simplicial model categories with C tractable and a (not necessarily simplicial) Quillen equivalence*

$$L : C \rightleftarrows D : R$$

Moreover let C' and D' be left Bousfield localizations of C and D. Assume R has the property that a fibrant object $d \in D$ is fibrant in D' if and only if $R(d)$ is fibrant in C'.

Then $(L \dashv R)$ is also a Quillen equivalence between C' and D'.

Proof. For simplicity we will refer to the model structures on C and D as the global model structures and to the model structures corresponding to C' and D' as the local model structures. First we have to show that the pair (L, R) induces a Quillen adjunction in the local model structures. We will show that L preserves local cofibrations and trivial cofibrations. Since local and global cofibrations are the same this is true for cofibrations. Thus we need to show it for trivial cofibrations and it follows by standard arguments if we can show it for generating trivial cofibrations. Thus let $i : a \to b$ be generating locally trivial cofibration in C. Now we can assume that a and b are cofibrant since C is tractable. Then the induced morphism $\underline{\mathrm{sHom}}(b, c) \to \underline{\mathrm{sHom}}(a, c)$ on mapping spaces is a weak equivalence for each locally fibrant object $c \in C$. In particular for $c = R(d)$ with $d \in D$ locally fibrant. Now we use that there are weak equivalences $\underline{\mathrm{sHom}}(b, R(d)) \cong \underline{\mathrm{sHom}}(Lb, d)$ and $\underline{\mathrm{sHom}}(a, R(d)) \cong \underline{\mathrm{sHom}}(La, d)$ of simplicial sets which stem from the fact that the pair (L, R) induces an adjunction of ∞-categories. Together this show that the induced morphism $\underline{\mathrm{sHom}}(Lb, d) \to \underline{\mathrm{sHom}}(La, d)$ is a weak equivalence for every locally fibrant objects $d \in D$. This shows that $La \to Lb$ is a local weak equivalence.

It remains to show that (L, R) is a Quillen equivalence in the local model structures. Therefore it suffice to show that the right derived functor

$$R' : Ho(D') \to Ho(C')$$

is an equivalence of categories. Not the fact that D' and C' are Bousfield localizations implies that $Ho(C')$ is a full reflective subcategory of $Ho(C)$ and correspondingly for D and D'. Moreover there is a commuting square

$$\begin{array}{ccc} Ho(D') & \xrightarrow{R'} & Ho(C') \\ \downarrow & & \downarrow \\ Ho(D) & \xrightarrow{R} & Ho(C) \end{array}$$

Since R is an equivalence if follows that R' is fully faithful. In order to show that R' is essentially surjective pick an element $c \in Ho(C')$ represented by a locally fibrant object of C. Since R is essentially surjective we find an element $d \in D$ which is globally fibrant such that $R(d)$ is equivalent to c in $Ho(C)$. But this implies that $R(d)$ is also locally fibrant (i.e. lies in $Ho(C')$) since this is a property that is invariant under weak equivalence in Bousfield localizations. Therefore we conclude that d is locally fibrant from the assumption on R. This shows that R' is essentially surjective, hence an equivalence of categories. □

The fact that the stable model structure is equivalent to connective spectra has the important consequence that a cofibre sequence in this model structure is also a fibre sequence, which is well-known for connective spectra (note that the converse is not true in connective spectra, but in spectra).

Corollary 5.6. *Let $X \to Y \to Z$ be a cofibre sequence of dendroidal sets in any of the considered model structures. Then*

$$i^*X_K \to i^*Y_K \to i^*Z_K$$

is a fibre sequence of simplicial sets. Here $(-)_K$ denotes a fully Kan (fibrant) replacement.

Proof. Since the stable model structure on dendroidal sets is a Bousfield localization of the other model structures we see that a cofibre sequence in any model structure is also a cofibre sequence in the stable model structure. But then it is also a fibre sequence as remarked above. The functor i^* is right Quillen, as shown in Theorem 4.2. Thus it sends fibre sequences in dSet to fibre sequences in sSet, which concludes the proof. □

6 Proof of Theorem 4.6, part I

Recall from Definition 4.5 the notion of binary extended corollas. Also recall from [Heu11a] that the weakly saturated class generated by non-root horns of arbitrary trees is called the class of left anodynes. The weakly saturated class generated by inner horn inclusions of arbitrary trees is called the class of inner anodynes. Analogously we set:

Definition 6.1. The weakly saturated class generated by non-root horns of all trees and root horns of binary extended corollas is called the class of *binary extended left anodynes.*

Proposition 6.2. *A dendroidal set D is stably fibrant if and only if D is a dendroidal Kan complex and it admits fillers for all root horns of extended corollas $EC_{n,1}$.*

Proof. Assume D is stably fibrant. Then by definition D is a dendroidal Kan complex and admits lifts against the maps

$$\Big(\Lambda^b[C_2] \otimes \Omega[L_n]\Big) \ \cup \ \Big(\Omega[C_2] \otimes \partial\Omega[L_n]\Big) \longrightarrow \Omega[C_2] \otimes \Omega[L_n]$$

for all $n \geq 0$. In Lemma 6.3 we will show that the root horn of $EC_{n,1}$ is a retract of this map. Thus D admits lifts against the root horn of $EC_{n,1}$.

Conversely assume D is dendroidal Kan and admits fillers against the root horn of $EC_{n,1}$. Then it clearly admits lifts against all binary extended left anodyne morphisms. In Lemma 6.4 we show that the inclusion

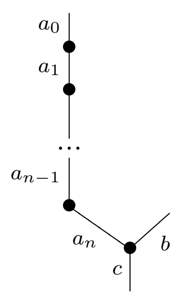
$$\Big(\Lambda^b[C_2] \otimes \Omega[L_n]\Big) \ \cup \ \Big(\Omega[C_2] \otimes \partial\Omega[L_n]\Big) \longrightarrow \Omega[C_2] \otimes \Omega[L_n])$$

is binary extended left anodyne. Thus D is stably fibrant. □

In the rest of the paper we prove technical lemmas and for this we fix some terminology. We denote:

- the edges of the binary extended corolla as in the following picture:

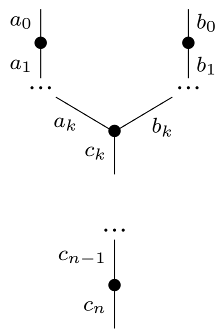
$EC_{n,1} =$



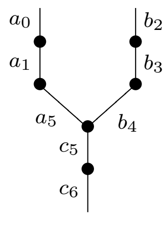
- the leaves of the corolla C_2 by a and b and its root by c;
- the edges of the linear tree L_n by $0, 1, ..., n$ with n being the root and 0 being the leaf;

$L_n =$ 

- the edges of the shuffles in the tensor product $\Omega[C_2] \otimes \Omega[L_n]$ by a_i, b_i, c_i instead of $(a, i), (b, i), (c, i)$;
- the shuffles of $\Omega[C_2] \otimes \Omega[L_n]$ by $T_k, k = 0, 1, ..., n$ where T_k is the unique shuffle that has the edges a_k, b_k and c_k;

$T_k =$ 

- the subtrees of a shuffle as sequences of its edges with indices in the ascending order (since there is no danger of ambiguity); for example we denote the following tree



(5)

  by $(a_0, a_1, a_5, b_2, b_3, b_4, c_5, c_6)$.
- by $D_i T_j$ the faces $\partial_{a_i}\partial_{b_i} T_j$ for $i < j$ and $\partial_{c_i} T_j$ for $i > j$;

Lemma 6.3. *The root horn of the binary extended corolla $EC_{n,1}$ is a retract of the map*

$$\Big(\Lambda^b[C_2] \otimes \Omega[L_n]\Big) \ \cup \ \Big(\Omega[C_2] \otimes \partial\Omega[L_n]\Big) \longrightarrow \Omega[C_2] \otimes \Omega[L_n] \tag{6}$$

Proof. As a first step we realize $\Omega[EC_{n,1}]$ as a retract of $\Omega[C_2] \otimes \Omega[L_n]$. Now we use the explicit form of the shuffles mentioned above. Clearly we can embed $EC_{n,1}$ into the shuffle T_n by sending a_i to a_i for $i = 0, ..., n$, b to b_n and c to c_n. This yields a morphism $\Omega[EC_{n,1}] \to$

$\Omega[C_2] \otimes \Omega[L_n]$. Conversely we define a map $\Omega[C_2] \otimes \Omega[L_n] \to \Omega[EC_{n,1}]$ which sends for each shuffle every edge b_i to b, every c_i to c and every a_i to the corresponding a_i.

Finally by definition it is clear that this retraction restricts to realize the root horn of $EC_{n,1}$ as a retract of $\Big(\Lambda^b[C_2] \otimes \Omega[L_n]\Big) \cup \Big(\Omega[C_2] \otimes \partial\Omega[L_n]\Big)$. □

Lemma 6.4. *The pushout product of the map $s : \Lambda^b[C_2] \to \Omega[C_2]$ with a simplex boundary inclusion*

$$\Big(\Lambda^b[C_2] \otimes \Omega[L_n]\Big) \cup \Big(\Omega[C_2] \otimes \partial\Omega[L_n]\Big) \longrightarrow \Omega[C_2] \otimes \Omega[L_n]$$

is a binary extended left anodyne map.

Proof. The case $n = 0$ is just the case of the inclusion $\Lambda^b[C_2] \to \Omega[C_2]$.

Fix $n \geq 1$. We set $A_0 := \Lambda^b[C_2] \otimes \Omega[L_n] \coprod_{\Lambda^b[C_2]\otimes\partial\Omega[L_n]} \Omega[C_2] \otimes \partial\Omega[L_n]$. Note that A_0 is the union of all $\Omega[D_iT_j]$ and of chains $\eta_a \otimes \Omega[L_n]$ and $\eta_c \otimes \Omega[L_n]$. We define dendroidal sets $A_k = A_{k-1} \cup \Omega[T_{k-1}]$ for $k = 1, ..., n+1$. So we have decomposed the map from the lemma into a composition of inclusions

$$A_0 \subset A_1 \subset ... \subset A_{n-1} \subset A_n \subset A_{n+1}.$$

We will show that $A_k \to A_{k+1}$ is inner anodyne for $k = 0, ..., n-1$ and binary extended left anodyne for $k = n$. Note that $A_{n+1} = \Omega[C_2] \otimes \Omega[L_n]$, so the inclusion $A_0 \to \Omega[C_2] \otimes \Omega[L_n]$ is binary extended left anodyne as a composition of such maps.

Case $k = 0$. The faces $\partial_{c_i}\Omega[T_0]$ are equal to $\Omega[D_iT_0]$ for all $i > 0$. The outer leaf face of T_0 is equal to $\eta_c \otimes \Omega[L_n]$. The remaining face $\partial_{c_0}\Omega[T_0]$ is in A_1, but not in A_0 so we have a pushout diagram

$$\begin{array}{ccc} \Lambda^{c_0}[T_0] & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ \Omega[T_0] & \longrightarrow & A_1 \end{array}.$$

Since inner anodyne extensions are closed under pushouts it follows that $A_0 \to A_1$ is inner anodyne.

Case $k < n$. We now construct a filtration

$$A_k = B_0^k \subset B_1^k \subset \cdots \subset B_{k+2}^k = A_{k+1}$$

as follows: informally speaking, we add representables of subtrees of T_k by the number of vertices starting from the minimal ones which are not contained in A_{k-1}. More precisely, set $B_0^k := A_k$ and for $l = 1, ..., k+2$ let B_l^k be the union of B_{l-1}^k and of all the representables of trees $(a_{j_1}, ..., a_{j_q}, b_{i_1}, ..., b_{i_p}, c_k, ..., c_n)$ with $q + p = l + k$ and $\{j_1, ..., j_q, i_1, ..., i_p\} = \{0, 1, ..., k\}$. We gave an example of such a tree for $k = 5, l = 1, p = q = 3$ and $n = 6$, c.f. tree (5).

If $p + q = k + 1$ and we fix a tree $U = (a_{j_1}, ..., a_{j_q}, b_{i_1}, ..., b_{i_p}, c_k, ..., c_n)$ we have an inclusion $\Lambda^{c_k}[U] \subset A_0 = B_0^k$ because $\partial_{c_i}\Omega[U] \subset \Omega[D_iT_k]$ for $i > k$, $\partial_{a_j}\Omega[U] \subset \Omega[D_jT_k]$ for $j \in \{j_1, ..., j_q\}$ and $\partial_{b_i}\Omega[U] \subset \Omega[D_iT_k]$ for $i \in \{i_1, ..., i_p\}$. Also note that $\partial_{c_k}\Omega[U]$ is not contained in A_0.

If $p + q = k + l, l \geq 2$ and we fix a tree $U = (a_{j_1}, ..., a_{j_q}, b_{i_1}, ..., b_{i_p}, c_k, ..., c_n)$ we have an inclusion $\Lambda^{c_k}[U] \subset B_{l-1}^k$. Indeed, for $j \in \{j_0, ..., j_q\}$, $\partial_{a_j}\Omega[U] \subset B_{l-1}^k$ by definition if $j \in \{i_0, ..., i_p\}$ and $\partial_{a_j}\Omega[U] \subset A_{k-1} \subset B_{l-1}^k$ if $j \notin \{i_0, ..., i_p\}$. Similarly, $\partial_{b_i}\Omega[U] \subset B_{l-1}^k$ for

$i \in \{i_0, ..., i_p\}$ and $\partial_{c_i}\Omega[U] \subset \Omega[D_i T_k] \subset A_0$ for $i > k$. The remaining face $\partial_{c_k}\Omega[U]$ is not contained B^k_{l-1}.

We conclude that for $l = 1, ..., k+2$ the map $B^k_{l-1} \to B^k_l$ is inner anodyne because it is a pushout of the inner anodyne map

$$\coprod_{q+p=k+l} \Lambda^{c_k}[U] \to \coprod_{q+p=k+l} \Omega[U]$$

where the coproduct is taken over all subtrees $U = (a_{j_0}, a_{j_1}, ..., a_{j_q}, b_{i_0}, ..., b_{i_p}, c_k, ..., c_n)$ of T_k such that $q + p = k + l$ and $\{j_1, ..., j_q, i_1, ..., i_p\} = \{0, 1, ..., k\}$.

Case $k = n$. Note that faces of the shuffle T_n are

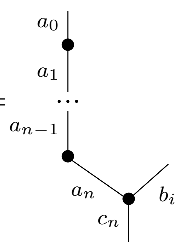
- $\partial_{b_i} T_n = (a_0, ..., a_n, b_0, ..., \widehat{b_i}, ..., b_n, c_n)$, $i = 0, ..., n$;
- $\partial_{a_j} T_n = (a_0, ..., \widehat{a_j}, ..., a_n, b_0, ..., b_n, c_n)$, $j = 0, ..., n$.

Our strategy is first to form the union of A_{n-1} with all $\partial_{b_i}\Omega[T_n], i = 0, ...n-1$, second to consider the union with all proper subsets of $\partial_{b_n}\Omega[T_n]$ that contain edges a_0 and a_n, third the union with $\partial_{a_j}\Omega[T_n], j = 1, ...n$ and then with $\partial_{a_0}\Omega[T_n]$. In the last step we use the horn inclusion $\Lambda^{b_n}[T_n] \subset \Omega[T_n]$. Thus we start with a filtration

$$A_n = P_0 \subset \cdots \subset P_{p-1} \subset P_p \subset \cdots \subset P_n,$$

where for $p = 1, ..., n-1$, P_p is the union of P_{p-1} with the representables of the trees of the form $(a_0, ..., a_n, b_{i_1}, ..., b_{i_p}, c_n)$. Also we define P_n as the union of P_{n-1} with $\partial_{b_i}\Omega[T_n], i \neq n$. We show that the maps $P_{p-1} \to P_p$ are left anodyne for $p = 1, 2, ..., n$.

- **Case** $p = 1$. For $i \in \{0, 1, ..., n\}$ and $V_i = (a_0, ..., a_n, b_i, c_n)$ all the faces of $\Omega[V_i]$, except $\partial_{a_i}\Omega[V_i]$, are in $P_0 = A_n$. The map $P_0 \to P_1$ is left anodyne as a pushout of the map $\coprod_{i=0}^n \Lambda^{a_i}[V_i] \to \coprod_{i=0}^n \Omega[V_i]$.

$V_i =$ 

- **Case** $p \leq n-1$. We give a further filtration

$$P_{p-1} = Q^p_0 \subset Q^p_1 \subset \cdots \subset Q^p_m \subset \cdots \subset Q^p_p = P_p$$

Let Q^p_m be the union of Q^p_{m-1} with $\Omega[U]$ for all the trees of the form

$$U = (a_{j_1}, ..., a_{j_q}, b_{i_1}, ..., b_{i_p}, c_n),\ q + p = n + m$$

such that there is a subset $I \subseteq \{i_1, ..., i_{p-1}\}$ with $\{j_1, ..., j_q\} = \{0, 1, ..., n\} \setminus I$. Note that $i_p \in \{j_1, ..., j_q\}$. We show that the inclusions $Q^p_{m-1} \to Q^p_m$ are left anodyne for all $m = 1, 2, ..., p-1$. For a fixed m and such a tree U the faces of $\Omega[U]$ are all in Q^p_{m-1}

except $\partial_{a_{i_p}}\Omega[U]$ (more precisely, the faces $\partial_{b_i}\Omega[U]$ are all in P_{p-1}, the faces $\partial_{a_j}\Omega[U]$ are in A_0 if $j \notin \{i_1, ..., i_p\}$ and in Q^p_{m-1} by definition if $j \in \{i_1, ..., i_p\}$).

We conclude that $Q^p_{m-1} \to Q^p_m$ is left anodyne as a pushout of the left anodyne maps $\coprod \Lambda^{a_{i_p}}[U] \to \coprod \Omega[U]$, where the coproduct is taken over trees U described above. We have $P_p = Q^p_p$, so $P_{p-1} \to P_p$ is also left anodyne.

- **Case** $p = n$. Here we do a slight modification of the previous argument. Let $Q^n_0 := P_{n-1}$ and for $m = 1, ..., n-1$ let Q^n_m be the union of Q^n_{m-1} with $\Omega[U_i]$ for the trees of the form

$$U_i = (a_{i_1}, ..., a_{i_m}, a_n, b_0, ..., \hat{b_i}, ..., b_n, c_n), i \neq n$$

or of the form

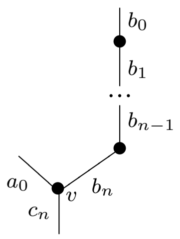
$$U_n = (a_0, a_{i_1}, ..., a_{i_{m-1}}, a_n, b_0, ..., b_{n-1}, c_n).$$

Let Q^n_n be the union of Q^n_{n-1} with $\partial_{b_i}\Omega[T_n], i \neq n$.

Similar argument (using horns $\Lambda^{a_n}[U_i], i \neq n$ and $\Lambda^{a_0}[U_n]$) shows that maps $Q^n_{m-1} \to Q^n_m$ are left anodyne for all $m = 1, ..., n$. Since $P_n = Q^n_n$ we have proven that $P_{n-1} \to P_n$ is left anodyne and hence $A_n \to P_n$ is left anodyne.

Next, we add $\partial_{a_i}\Omega[T_n]$ for $i = 1, 2, ..., n$ to the union. Let us denote the only binary vertex of the tree $W = (a_0, b_0, ..., b_n, c_n)$ by v. Let $P_{n+1} = P_n \cup \Omega[W]$. Then the map $P_n \to P_{n+1}$ is *binary extended left anodyne* because it is a pushout of the map $\Lambda^v[W] \to \Omega[W]$.

$W =$



For each $q = 2, ..., n$ we define P_{n+q} as the union of P_{n+q-1} and the representables of the trees of the form $Z_q = (a_0, a_{i_1}, ..., a_{i_q}, b_0, ..., b_n, c_n)$. The map $P_{n+q-1} \to P_{n+q}$ is left anodyne as the pushout of $\coprod \Lambda^{a_0}[Z_q] \to \coprod \Omega[Z_q]$.

The dendroidal set P_{2n} contains $\partial_{a_i}\Omega[T_n], i = 1, ..., n$. Furthermore, the faces of $\partial_{a_0}\Omega[T_n]$ except $\partial_{b_n}\partial_{a_0}\Omega[T_n]$ are in P_{2n}. Let $P_{2n+1} = P_{2n} \cup \partial_{a_i}\Omega[T_n]$. Then $P_{2n} \to P_{2n+1}$ is inner anodyne as the pushout of $\Lambda^{b_n}\partial_{a_i}[T_n] \to \partial_{a_i}\Omega[T_n]$. From this we conclude that $A_n \to P_{2n+1}$ is binary extended left anodyne. All the faces of $\Omega[T_n]$ are in P_{2n+1} except $\partial_{b_n}\Omega[T_n]$, so $P_{2n+1} \to A_{n+1}$ is left anodyne as the pushout of the map $\Lambda^{b_n}[T_n] \to \Omega[T_n]$. Hence $A_n \to A_{n+1}$ is binary extended left anodyne, which finishes the proof.

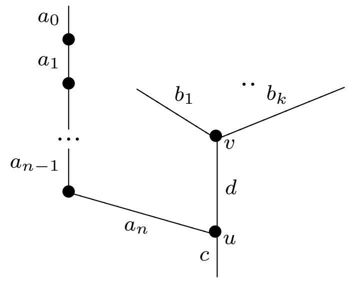
□

7 Proof of Theorem 4.6, part II

In order to compare lifts against binary extended corollas and all extended corollas we need the following lemma:

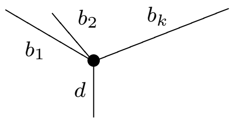
Lemma 7.1. *Consider the inclusion of the root horn of the extended corolla $\Lambda^u[EC_{n,k}] \to \Omega[EC_{n,k}]$. Then there is a tree T and a morphism $\Omega[EC_{n,k}] \to \Omega[T]$ such that the composition $\Lambda^u[EC_{n,k}] \to \Omega[T]$ is a binary extended left anodyne map.*

Proof. We use the labels for edges of the extended corolla $EC_{n,k}$ as given in its definition 4.5 and in addition we denote its root vertex by u. Now consider the tree T

$T =$ 

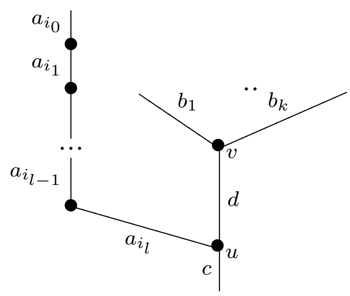
There is an obvious morphism $\Omega[EC_{n,k}] \to \Omega[T]$. We will show that the composition $\Lambda^u[EC_{n,k}] \to \Omega[T]$ is binary extended left anodyne.

We set $E_0 := \Lambda^u[EC_{n,k}]$. Let C_k be a corolla with root d and leaves $b_1, ..., b_k$

$C_k =$ 

Set $E_1 := E_0 \cup \Omega[C_k]$ which is a subobject of $\Omega[T]$. The map $E_0 \to E_1$ is a left anodyne because it is a pushout of the map $\coprod_{i=1}^{k} \eta_{b_i} \to \Omega[C_k]$, which is a left anodyne extension by definition.

As a next step consider subtrees of T which are of the form

$T_{i_0,...,i_l} =$ 

for $\{i_0, ...i_l\} \subset \{0, 1, ..., n\}$ and $l \leq n-1$. We define dendroidal sets E_{l+2} as the union of E_{l+1} and all representables $\Omega[T_{i_0,...,i_l}]$ for $\{i_0, ...i_l\} \subset \{0, 1, ..., n\}$ and $0 \leq l \leq n-1$. Thus we get a filtration

$$\Lambda^u[EC_{n,k}] = E_0 \subset E_1 \subset E_2 \subset ... \subset E_{n+1} \subset \Omega[T] \tag{7}$$

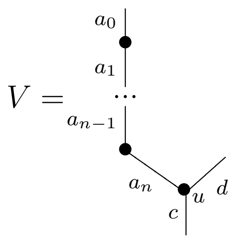
For a fixed $l \leq n-1$ and a subset $\{i_0, ..., i_l\}$ the inner face $\partial_d \Omega[T_{i_0,...,i_l}]$ is contained in E_0 and the faces $\partial_{a_j} \Omega[T_{i_0,...,i_l}]$ are contained in E_{l+1} for every $j \in \{i_0, ..., i_n\}$ (and for $l = 0$ the face $\partial_u \Omega[T_{i_0}]$ is in E_1).

Since $\partial_v \Omega[T_{i_0,\ldots,i_l}]$ is not in E_{l+1} we have the following pushout diagram

$$\begin{array}{ccc} \coprod \Lambda^v[T_{i_0,\ldots,i_l}] & \longrightarrow & E_{l+1} \\ \downarrow & & \downarrow \\ \coprod \Omega[T_{i_0,\ldots,i_l}] & \longrightarrow & E_{l+2} \end{array}$$

where the coproduct varies over all possible $(i_0, \ldots, i_l)$. This shows that $E_{l+1} \to E_{l+2}$ is left anodyne. From this we conclude that all maps in the above filtration (7) except for the last inclusion are left anodyne and therefore also the map $E_0 \to E_{n+1}$ is left anodyne.

We proceed by observing that for the tree



all faces of $\Omega[V]$ are in E_{n+1} except $\partial_u \Omega[V]$. Notice that $E_{n+1} \cup \Omega[V] = \Lambda^d[T]$. The map $E_{n+1} \to \Lambda^d[T]$ is the pushout of the binary extended left anodyne map $\Lambda^u[V] \to \Omega[V]$, so it is binary extended left anodyne map. Finally, since $\partial_d \Omega[T] \to \Omega[T]$ is inner anodyne, we conclude that $E_0 \to \Omega[T]$ is binary extended left anodyne. □

Proposition 7.2. *Let $D \in dSet$ be a dendroidal Kan complex. Then D admits fillers for all root horns of binary extended corollas $EC_{n,1}$ if and only if D admits fillers for all root horns of arbitrary extended corollas $EC_{n,k}$.*

Proof. One direction is a special case and thus trivial. Hence assume D admits fillers for all root horns of extended corollas $EC_{n,1}$. Then D admits lifts against all binary extended left anodynes (see Definition 6.1). We need to show that D admits lifts against the root horn inclusion $\Lambda^u[EC_{n,k}] \to \Omega[EC_{n,k}]$. By Lemma 7.1 we find a tree T and a morphism $\Omega[EC_{n,k}] \to \Omega[T]$ such that the composition $\Lambda^u[EC_{n,k}] \to \Omega[T]$ is binary extended left anodyne. Thus given a morphism $\Lambda^u[EC_{n,k}] \to D$ we find a filler $\Omega[T] \to D$. But the composition $\Omega[EC_{n,k}] \to \Omega[T] \to D$ is then the desired lift. □

8 Proof of Theorem 4.6, part III

Now we define similarly to Definition 6.1

Definition 8.1. The weakly saturated class generated by non-root horns of all trees and root horns of extended corollas is called the class of *extended left anodynes*. The weakly saturated class generated by all horn inclusions of trees is called the class of *outer anodynes*.

It would be more logical to call outer anodynes simply anodynes since it also includes the inner anodynes. But in order to distinguish it more clearly we call it outer anodynes here. By definition we have inclusions

$$\{\text{inner anodynes}\} \subset \{\text{left anodynes}\} \subset \{\text{binary ext. left anodynes}\}$$
$$\subset \{\text{ext. left anodynes}\} \subset \{\text{outer anodynes}\}$$

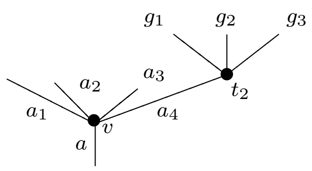
All of these inclusions are proper, except for the last, which we will prove now:

Proposition 8.2. *The class of extended left anodynes and the class of outer anodynes coincide.*

In particular a dendroidal set D admits lifts against all non-root horns and root horns of extended corollas if and only if it is fully Kan.

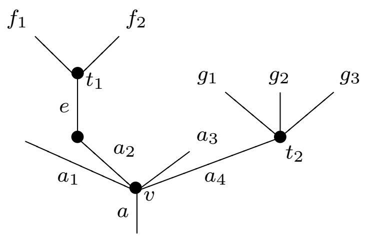
Proof. By the above inclusion of saturated classes it suffices to show that the root horn inclusions for arbitrary trees are contained in the class of extended left anodynes. But a root horn for a tree only exists if this tree is given by another tree T grafted on a corolla. For those trees the proof will be given in Lemma 8.6. □

Before we can prove the crucial lemma we need to introduce some terminology. Recall from [MW09] that *a top face map* is an outer face map with respect to a top vertex and *an initial segment* of a tree is a face obtained by composition of top face maps. For example, the tree



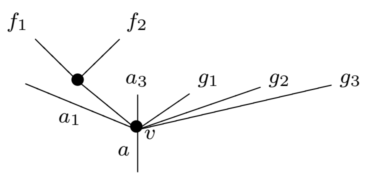
is an initial segment of the tree

$T =$



Definition 8.3. For a face that is a composition of an initial segment and inner face maps we say it is an *initial subtree of codimension k* if there are exactly k inner face maps in this composition.

By definition, every initial segment is an initial subtree of codimension zero. An example of subtree of codimension 2 of the tree T above is



Lemma 8.4 (Codimension argument)**.** *Let T be a tree and v a vertex of T. Let V be the maximal initial segment of T for which the input edges of v are leaves. Let moreover X_T be the subobject of $\Omega[T]$ defined as the union of the following dendroidal sets*

- *the representable $\Omega[V]$,*
- *the inner faces $\partial_e\Omega[T]$ where e is an inner edge of V,*
- *the outer faces $\partial_u\Omega[T]$ for vertices u of V, $u \neq v$.*

Then the inclusion $X_T \to \Omega[T]$ is inner anodyne.

Proof. Let $|V|$ and $|T|$ be the number of vertices of V and T. If S is an initial subtree of T of codimension k containing V with exactly $|V| - 1 + n$ vertices we will say that S is a (n,k)*-subtree*. Note that V is a $(1,0)$-subtree. Let $N = |T| - |V| + 1$. Note that T has $|T| - |V| = N - 1$ inner edges which are not inner edges of V (for each vertex of T which is not in V its output edge). Denote $X_{1,0} := X_T$. The strategy is to form an inner anodyne filtration by considering unions of $X_{1,0}$ with (n,k)-subtrees of T inductively on $n, 1 \leq n \leq N$ and $k, 0 \leq k \leq N-1$. For each $n \geq 1, k \geq 1$ and for each $(n+1,k-1)$-subtree S we choose one of its inner faces which is a (n,k)-subtree. The set of these chosen subtrees is denoted $\mathcal{F}_{n,k}$. Note that we can make this choice by ordering the inner edges of T and each time choose the inner face corresponding to the edge in S (and not in V) with the maximal label. Note that for $n = 2, k \geq 1$ such a subtree S has exactly $|V|+1$ vertices and only one inner face which is not an inner face of V, and that face belongs to $\mathcal{F}_{1,k}$. Hence $X_1 = X_{1,0} = X$. For $1 \leq n \leq N$, we define dendroidal sets X_n as the union $\bigcup_{k=0}^{N-1} X_{n,k}$. We define $X_{n,0}$ as the union of X_{n-1} and the representables of all $(n,0)$-subtrees. We define $X_{n,k}$ for $k \geq 1$ as the union of $X_{n,k-1}$ and the representables of all (n,k)-subtrees that are not in $\mathcal{F}_{n,k}$. The inclusions $X_{n-1} \to X_{n,0}$ are all inner anodyne because each of them is a pushout of the coproduct of the inner horn inclusions. More precisely, each $(n,0)$-subtree S has faces which are in X by definition, outer faces that are $(n-1,0)$-subtrees and hence are all in X_{n-1}, inner faces which are $(n-1,1)$-subtrees and by definition exactly one of them was chosen to be in $F_{n-1,1}$, so is not in X_{n-1}. Denote this inner face be $\partial_s S$. We have the pushout diagram (where the coproduct is taken over all $(n,0)$-subtrees)

$$\begin{array}{ccc} \coprod \Lambda^s[S] & \longrightarrow & X_{n-1} \; . \\ \downarrow & & \downarrow \\ \coprod \Omega[S] & \longrightarrow & X_{n,0} \end{array}$$

Note that the union of representables of $(n+1,k-1)$-subtrees and $X_{n+1,k-1}$ will also contain the representables of elements of $\mathcal{F}_{n,k}$ (since the elements of $\mathcal{F}_{n,k}$ will be faces of the $(n+1,k-1)$-subtrees). So $X_{n+1,k-1}$ will contain representables of all (n,k)-subtrees. The inclusions $X_{n+1,k-1} \to X_{n+1,k}$ are similarly shown to be inner anodyne. Faces of an $(n+1,k)$-subtree are in X or (n,k)-subtrees (and hence all in $X_{n+1,k-1}$ by the previous sentence) or $(n,k+1)$-subtrees (and hence all but one in $X_{n,k+1} \subset X_n \subset X_{n+1,k-1}$ by construction). We again have a horn inclusion with respect to the excluded face, and $X_{n+1,k-1} \to X_{n+1,k}$ is the pushout of the coproduct of these horn inclusions. Finally, we have shown that the inclusion $X_T = X_1 \subset X_2 \subset ... \subset X_N = \Omega[T]$ is left anodyne. □

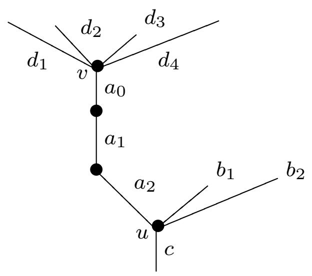
Definition 8.5. For a tree T the maximal subtree having non-unary root is unique and we call it *the tree top* of T. Exceptionally, if T is a linear tree (so it has no non-unary vertices), we say that its tree top is $L_0 = \,|\,$. The maximal initial segment of T which is a linear tree is also unique and we call it *the stem* of T. Note that T is obtained by grafting the tree top of T to the stem of T (conversely the tree top can be obtained from T by chopping of the stem).

For a fixed tree T with a root r we define the tree $U = U_{T,q}$ obtained by grafting T to the $(q+1)$-corolla with leaves $r, b_1, ..., b_q$, the root c and the root vertex u. Let T' be the tree that has one edge more than T such that this edge, called a', is the leaf of the stem of T' (and root of the tree top of T'). Let $W = W_{T,q}$ be the tree obtained by grafting T' to the $(q+1)$-corolla with leaves $r, b_1, ..., b_q$, the root c and the root vertex u.

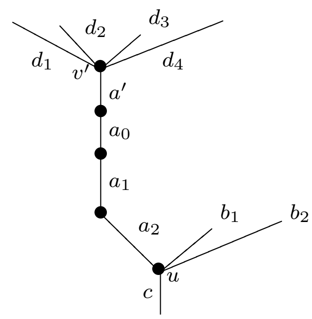
We will usually denote by v the root vertex of the tree top of T and the input edges of v by $d_1, ..., d_p$. We will denote by v' the vertex in W having the output a'. The edges of the stem of T will be denoted $a_0, ..., a_l$ with a_i and a_{i+1} being the input and the output of the same vertex for all $i = 0, ..., l-1$ (so a_l is the root). A tree whose tree top is a corolla C_p and whose stem has l vertices is denoted by $C_{p,l}$.

For example for the tree $C_{4,2}$ we have

$U_{C_{4,2},2} =$



$W_{C_{4,2},2} =$



For a subset $J \subset \{0, 1, ..., l\}$ we denote by

- U^0_J the unique subtree of W containing the edges $d_1, ..., d_p, a', b_1, ..., b_q, c$ and $a_j, j \in J$.
- U'_J the maximal subtree of W not containing the edges $a_j, j \in \{0, 1, ..., l\} \setminus J$. We denote $U'_j := U'_{[l]\setminus\{a_j\}} = \partial_{a_j} W$.
- T^0_J and T'_J the root face of U^0_J and U'_J, respectively.

Note that T'_J contains the whole tree top of T, while T^0_J only the non-unary root vertex of the tree top of T. Also, note that $T' = T'_{[l]}$.

Lemma 8.6. *For every tree T and any natural number $q \geq 0$ set $U := U_{T,q}$ as above. Then the inclusion $\Lambda^u[U] \to \Omega[U]$ is extended left anodyne.*

Proof. Let N be the number of vertices of its tree top S and let l be the number of vertices of its stem. We show the claim by induction on N.

For $N = 0$, i.e. if T is a linear tree with l vertices, the claim holds by definition of extended left anodynes.

Fix a tree top S with N vertices, $N \geq 1$, and assume that the claim holds for all trees with tree tops that have less than N vertices. We will prove that for fixed S and for every l, the inclusion $\Lambda^u[U] \to \Omega[U]$ is extended left anodyne Since $\Lambda^u[U] \to \Omega[U]$ is a retract of $\Lambda^u[U] \to \Omega[W]$, it is enough to show that $\Lambda^u[U] \to \Omega[W]$ is extended left anodyne. We divide the proof in three parts.

Step 1. We show that $\Lambda^u[U] \to \cup_{j=0}^{l} \partial_{a_j} W$ is left anodyne.

We denote $A_0 := \Lambda^u[U] =: B_0$. Inductively, for all $1 \leq k \leq l$, we define

$$A'_{k-1} := B_{k-1} \cup \bigcup_{|J|=k-1} \Omega[T^0_J], \quad A_k := A'_{k-1} \cup \bigcup_{|J|=k-1} \Omega[T'_J],$$
$$B'_{k-1} := A_k \cup \bigcup_{|J|=k-1} \Omega[U^0_J], \quad B_k := B'_{k-1} \cup \bigcup_{|J|=k-1} \Omega[U'_J].$$

Since $A'_0 = A_0 \cup \Omega[T^0_\emptyset]$ and $T^0_\emptyset$ is the p-corolla with inputs $d_1, ..., d_p$ and root a', the inclusion $A_0 \to A'_0$ is left anodyne because it is the pushout of $\eta_{d_1} \cup ... \cup \eta_{d_p} \to \Omega[T^0_\emptyset]$.
Let k be such that $1 \leq k \leq l$. The inclusion $A_k \to B'_{k-1}$ is left anodyne because it is the pushout of the coproduct of leaf horn inclusions $\coprod_{|J|=k} \Lambda^{v'}[U^0_J] \to \coprod_{|J|=k} \Omega[U^0_J]$.
The inclusion $B_k \to A'_k$ is left anodyne because it is the pushout of the coproduct of leaf horn inclusions $\coprod_{|J|=k} \Lambda^{v'}[T^0_J] \to \coprod_{|J|=k} \Omega[T^0_J]$.
For all trees $T'_J, |J| = k-1$ and vertex v' the codimension argument gives an inner anodyne $X_{T'_J} \to \Omega[T'_J]$. Since each $X_{T'_J}$ is a dendroidal subset of A'_{k-1}, the inclusion $A'_{k-1} \to A_k$ is inner anodyne as the pushout of the coproduct $\bigcup_{|J|=k-1} X_{T'_J} \to \bigcup_{|J|=k-1} \Omega[T'_J]$.
For all trees $U'_J, |J| = k-1$ and vertex v' the codimension argument gives an inner anodyne $X_{U'_J} \to \Omega[U'_J]$. Since each $X_{U'_J}$ is a dendroidal subset of B'_{k-1}, the inclusion $B'_{k-1} \to B_k$ is inner anodyne as the pushout of the coproduct $\bigcup_{|J|=k-1} X_{U'_J} \to \bigcup_{|J|=k-1} \Omega[U'_J]$.
Note that $B_l = \bigcup_{j=0}^{l} \partial_{a_j} W$, so we have proven that $\Lambda^u[U] \to \bigcup_{j=0}^{l} \partial_{a_j} W$ is left anodyne.

Step 2. Let V_0 be the initial segment of W for which a' is a leaf. Let $D_0 := \bigcup_{j=0}^{l} \partial_{a_j} W \cup \Omega[V_0]$. Define dendroidal sets $D_n, 1 \leq n \leq N-1$ as union of D_{n-1} and all the initial segments of W with exactly $n+l+2$ vertices. Note that all such subtrees must contain $a', a_0, ..., a_l, b_1, ..., b_q, c$ since they are initial and such subtrees have exactly n vertices more than V_0.

We have defined a filtration

$$\Lambda^u[U] \subset \bigcup_{j=0}^{l} \partial_{a_j} W \subset D_0 \subset D_1 \subset ... \subset D_{N-1}$$

In the previous step we have shown that the first inclusion is left anodyne. The map $\bigcup_{j=0}^{l} \partial_{a_j} W \to D_0$ is extended left anodyne because it is a pushout of outer root inclusion of $\Omega[V_0]$ (which is extended left anodyne by definition). For any $n, 1 \leq n \leq N-1$ and any initial segment V of W with $n+l+2$ vertices the outer root horn inclusion for V is extended left anodyne by the inductive hypothesis for the claim. For each such initial segment V, the horn $\Lambda^u[V]$ is a dendroidal subset of D_{n-1} because its faces $\partial_{a_j} \Omega[V], j = 0, 1, ..., l$ are in $B_l \subset D_0$ by the previous arguments, the face $\partial_{a'} V$ is in A_0, and other inner and outer leaf faces are in D_{n-1} by definition. We conclude that the inclusion $D_{n-1} \to D_n$ is also extended left anodyne because it is the pushout of the coproduct of root horn inclusions $\Lambda^u(V) \to \Omega[V]$ for all initial segments V with $n+l+2$ vertices.

Note that D_{N-1} contains all the faces of W except the outer root face T' and $\partial_{a'}W = U$. We have so far proven that $\Lambda^u[U] \to D_{N-1}$ is extended left anodyne.

Step 3. We show that $D_{N-1} \to \Omega[W]$ is inner anodyne.

The faces of the root face T' of W are all included in D_{N-1} except the inner face $\partial_{a'}T' = T$:

- $\partial_{a_j}T'$ is already in A_l;
- $\partial_e T'$ for inner edges e of the tree top S are in D_{N-1} because D_{N-1} contains $\partial_e W$;
- $\partial_t T'$ for top vertices t of the tree top S are in D_{N-1} because D_{N-1} contains $\partial_t W$.

So the map $D_{N-1} \to D_{N-1} \cup \Omega[T'] = \Lambda^{a'}[W]$ is inner anodyne because it is a pushout of an inner horn inclusion. Finally, $\Lambda^{a'}[W] \to \Omega[W]$ is inner anodyne and we have shown that the inclusion $A_0 \to \Omega[W]$ is extended left anodyne. □

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