

Prismatic cohomology relative to δ -rings

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Abstract

We develop prismatic and syntomic cohomology relative to a δ -ring. This simultaneously generalizes Bhatt and Scholze’s absolute and relative prismatic cohomology and shows that the latter, which was defined relative to a prism, is in fact independent of the prism structure and only depends on the underlying δ -ring. We give several possible definitions of our new version of prismatic cohomology: a site theoretic definition, one using prismatic crystals, and a stack theoretic definition. These are equivalent under mild syntomicity hypotheses. As an application, we note how the theory of prismatic cohomology of filtered rings arises naturally in this context.

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1 Introduction

The p -adic syntomic complexes $\mathbf{Z}_p(i)(R)$ for $i \in \mathbf{Z}_{\geq 0}$ and a quasisyntomic ring R are objects of the p -complete derived ∞ -category $\mathbf{D}(\mathbf{Z}_p)_p^\wedge$ introduced by Bhatt, Morrow, and Scholze [9] in order to study topological cyclic homology and algebraic K-theory. They are a form of étale-sheafified motivic cohomology at the ‘bad’ prime [13] and were studied in a different guise by Fontaine and Messing [12]; see [6] for a comparison. Our goal in this paper is to introduce and study a relative form of the p -adic syntomic complexes $\mathbf{Z}_p(i)(R/A)$ when A is a δ -ring and R is a commutative A -algebra.¹

Syntomic cohomology is built from the theory of prismatic cohomology due to Bhatt, Morrow, and Scholze [9, 10] and we will similarly generalize prismatic cohomology to a relative setting. We assume that the reader either knows what a prism is or is willing to take this notion as a black box. For now we only need that (among other things) a prism is a commutative ring A with an endomorphism φ_A and an invertible ideal $I \subseteq A$. For a prism (A, I) and a commutative \overline{A} -algebra R , where $\overline{A} = A/I$, Bhatt and Scholze naturally associate the following objects:

- (a) an \mathbf{E}_∞ - A -algebra $\Delta_{R/A}$, the (derived) prismatic cohomology of R relative to A ;²
- (b) $\Delta_{R/A}$ -modules $\Delta_{R/A}\{i\}$ for every integer $i \in \mathbf{Z}$ called the Breuil–Kisin twists (with $\Delta_{R/A}\{0\} = \Delta_{R/A}$) which are graded multiplicative with respect to i ;
- (c) complete Hodge–Tate towers $\Delta_{R/A}^{[\star]}\{i\}$

$$\cdots \rightarrow \Delta_{R/A}^{[n+1]}\{i\} \rightarrow \Delta_{R/A}^{[n]}\{i\} \rightarrow \Delta_{R/A}^{[n-1]}\{i\} \rightarrow \cdots$$

for $i, n \in \mathbf{Z}$, multiplicative in i and n , with weight $n = 0$ part $\Delta_{R/A}\{i\}$, underlying object $\Delta_{R/A}\{i\}[1/I]$, and associated graded pieces $\mathrm{gr}^n \Delta_{R/A}^{[\star]}\{i\} \simeq \overline{\Delta}_{R/A}\{i+n\}$, the $(i+n)$ -Breuil–Kisin twisted Hodge–Tate cohomology of R over A ;³

- (d) complete \mathbf{Z} -filtered A -modules

$$\cdots \rightarrow N^{\geq n+1} \widehat{\Delta}_{R/A}^{(1)}\{i\} \rightarrow N^{\geq n} \widehat{\Delta}_{R/A}^{(1)}\{i\} \rightarrow N^{\geq n-1} \widehat{\Delta}_{R/A}^{(1)}\{i\} \rightarrow \cdots$$

for $i, n \in \mathbf{Z}$, multiplicative in i and n , called the Nygaard filtration on the Nygaard-completed, Frobenius-twisted, Breuil–Kisin twisted prismatic cohomology $\widehat{\Delta}_{R/A}^{(1)}\{i\}$, which have property that $N^{\geq \star} \widehat{\Delta}_{R/A}^{(1)}\{i\}$ is constant for $\star \leq 0$;

¹Bhatt and Lurie introduce p -adic syntomic complexes $\mathbf{Z}_p(i)(R)$ for a broad class of non- p -complete rings in [7]. When R is quasisyntomic (and in particular p -complete), these agree with the complexes defined in [9], but for example for $\mathbf{Z}_{(p)}$, the difference between $\mathbf{Z}_p(i)(\mathbf{Z}_{(p)})$ and $\mathbf{Z}_p(i)(\mathbf{Z}_p)$ is the same as the difference between the p -adic (pro-)étale cohomology in weight i of $\mathrm{Spec} \mathbf{Q}$ and of $\mathrm{Spec} \mathbf{Q}_p$. On the other hand, our relative syntomic complexes $\mathbf{Z}_p(i)(R/A)$ depend only on the derived p -completions: $\mathbf{Z}_p(i)(R/A) \simeq \mathbf{Z}_p(i)(\widehat{R}/\widehat{A})$. Thus, $\mathbf{Z}_p(i)(\mathbf{Z}_{(p)}/\mathbf{Z}_{(p)}) \simeq \mathbf{Z}_p(i)(\mathbf{Z}_p/\mathbf{Z}_p) \simeq \mathbf{Z}_p(i)(\mathbf{Z}_p)$, and not $\mathbf{Z}_p(i)(\mathbf{Z}_{(p)})$.

²When we want to distinguish this object carefully from our prismatic cohomology relative to δ -rings, we will write it as $\Delta_{R/A}^{\mathrm{rel}}$. As we will show in Theorem 1.2(2), the two agree when both are defined.

³In fact, this tower is simply given by the I -adic filtration on $\Delta_{R/A}\{i\}$, i.e. $\Delta_{R/A}^{[n]}\{i\} = \Delta_{R/A}\{i\} \otimes_A I^n$. We want to highlight the structure in this abstract way to foreshadow the generalization that is to come.

- (e) a φ_A -semilinear morphism $c: \Delta_{R/A}\{i\} \rightarrow \widehat{\Delta}_{R/A}^{(1)}\{i\}$ for every $i \in \mathbf{Z}$;
- (f) a filtered relative Frobenius map $N^{\geq \star} \widehat{\Delta}_{R/A}^{(1)}\{i\} \xrightarrow{\varphi/A} \Delta_{R/A}^{[\star-i]}\{i\}$ for every $i \in \mathbf{Z}$ (the variable \star is the filtration).

We will not give a detailed review of the definition of these objects here and just note that the functor $\Delta_{R/A}$ can be constructed by taking the cohomology of the prismatic site of R relative to A when R is p -adically formally smooth over \overline{A} , and then left Kan extending in the ∞ -category of (p, I) -complete \mathbf{E}_∞ - A -algebras from formally smooth \overline{A} -algebras.

Remark 1.1. (1) Let $\widehat{\Delta}_{R/A}^{(1)} = \widehat{\Delta}_{R/A}^{(1)}\{0\}$. In this case, the morphisms c and φ/A are maps of \mathbf{E}_∞ -rings and the composition

$$\Delta_{R/A} \xrightarrow{c} \widehat{\Delta}_{R/A}^{(1)} \xrightarrow{\varphi/A} \Delta_{R/A}$$

induces a φ_A -semilinear endomorphism φ that we call the (absolute) Frobenius.

- (2) We want to explain in which sense $\widehat{\Delta}_{R/A}^{(1)}$ is a completion to explain the terminology: using the map c one can ‘pull-back’ the Nygaard filtration to get a non-complete filtration on the Frobenius twisted prismatic cohomology $\Delta_{R/A}^{(1)}\{i\} := \Delta_{R/A}\{i\} \otimes_{A, \varphi_A} A$. Concretely this new filtration is given by the formula

$$N^{\geq \star} \Delta_{R/A}^{(1)}\{i\} := N^{\geq \star} \widehat{\Delta}_{R/A}^{(1)}\{i\} \times_{\widehat{\Delta}_{R/A}^{(1)}\{i, \tilde{c}\}} \Delta_{R/A}^{(1)}\{i\}$$

where \tilde{c} is the A -linear map $\Delta_{R/A}^{(1)}\{i\} \rightarrow \widehat{\Delta}_{R/A}^{(1)}\{i\}$ induced from c . By construction the completion of $\Delta_{R/A}^{(1)}\{i\}$ with respect to $N^{\geq \star} \Delta_{R/A}^{(1)}\{i\}$ is Nygaard-completed prismatic cohomology $\widehat{\Delta}_{R/A}^{(1)}\{i\}$.⁴

- (3) The objects (a) – (f) are naturally objects of the ∞ -category \mathcal{C}_A consisting of quadruples (H, N, c, φ) where

- H is a complete filtered, graded \mathbf{E}_∞ -algebra over A , i.e., a lax symmetric monoidal functor

$$H: (\mathbf{Z}, \geq) \times \mathbf{Z} \rightarrow D(A),$$

where (\mathbf{Z}, \geq) denotes the poset with the additive symmetric monoidal structure and \mathbf{Z} denotes the set with the additive symmetric monoidal structure. Completeness means that for fixed grading the inverse limit in the poset direction vanishes;

- N is another complete filtered, graded \mathbf{E}_∞ -algebra over A which is constant in non-positive filtration weights;
- c is a φ_A -semilinear map of graded A algebras $H^{\geq 0} \rightarrow N^{\geq 0}$;⁵
- φ is a map of graded filtered algebras $N \rightarrow \text{sh}(H)$ over A where sh is the shearing of H defined as changing the filtration degree by subtracting the grading degree.

There is also an absolute form of the theory also introduced in [10] and studied in [7], yielding for commutative rings R the following objects:

- (i) an \mathbf{E}_∞ -algebra Δ_R over \mathbf{Z}_p ;
- (ii) Breuil–Kisin twists $\Delta_R\{i\}$ for every integer $i \in \mathbf{Z}$;

⁴We could also consider the filtration $N^{\geq \star} \Delta_{R/A}^{(1)}\{i\}$ to be the more fundamental object, as is usually done. But we will see later that the Nygaard-completed filtration $N^{\geq \star} \widehat{\Delta}_{R/A}^{(1)}\{i\}$ has better formal properties, specifically with respect to descent in R . Also the syntomic cohomology is the same in both cases (Proposition 7.12)

⁵In fact, c carries slightly more structure which we will suppress for simplicity here.

- (iii) complete Hodge–Tate towers $\Delta_R^{[\star]} \{i\}$ with weight 0 part $\Delta_R^{[0]} \{i\} \simeq \Delta_R \{i\}$ and associated graded pieces $\mathrm{gr}^n \Delta_R^{[\star]} \{i\} \simeq \bar{\Delta}_R \{i+n\}$, the $(i+n)$ -Breuil–Kisin twisted absolute Hodge–Tate cohomology of R ;
- (iv) Nygaard filtrations $N^{\geq \star} \widehat{\Delta}_R \{i\}$;
- (v) maps $c: \Delta_R \{i\} \rightarrow \widehat{\Delta}_R \{i\}$;
- (vi) filtered Frobenius maps $N^{\geq \star} \widehat{\Delta}_R \{i\} \xrightarrow{\varphi} \Delta_R^{[\star-i]} \{i\}$.

These objects are naturally objects of the ∞ -category $\mathcal{C}_{\mathbf{Z}_p}$ of the previous remark, where $\varphi_{\mathbf{Z}_p} = \mathrm{id}$. The p -adic syntomic complexes are defined for $i \geq 0$ as

$$(vii) \quad \mathbf{Z}_p(i)(R) = \mathrm{fib} \left(N^{\geq i} \widehat{\Delta}_R \{i\} \xrightarrow{\mathrm{can}-c\varphi} \widehat{\Delta}_R \{i\} \right).$$

For example, $\mathbf{Z}_p(0) \simeq \lim \mathbf{Z}/p^m$ as a flat sheaf and $\mathbf{Z}_p(1)(R) \simeq \mathrm{T}_p \mathbf{G}_m$, the p -adic Tate module of \mathbf{G}_m ; see [9, Props. 6.16 and 6.17].

1.1 Results

To generalize syntomic cohomology to the case of a commutative algebra R with bounded p -primary torsion over a δ -ring A with bounded p -primary torsion (called a bounded δ -pair (A, R)), we extend prismatic cohomology and the structures (a)-(f) above to the setting of commutative R -algebras over general δ -rings A . The main result of this paper is the following:

Theorem 1.2. *There is an extension of prismatic cohomology to bounded δ -pairs. More precisely we construct objects*

$$\underline{\Delta}_{R/A} = \left(\Delta_{R/A}^{[\star]} \{\star\}, N^{\geq \star} \widehat{\Delta}_{R/A}^{(1)} \{\star\}, c, \varphi \right) \in \mathcal{C}_A$$

depending functorially on (R, A) which possess the following properties.

- (1) Insensitivity to localization and completion (Definition 3.3, Corollary 3.13, and Corollary 6.10): $\underline{\Delta}_{R/A}$ depends only on the S -localization of A where S is any set of elements of A that become invertible in R ; similarly, $\underline{\Delta}_{R/A}$ depends only on the derived (p, K) -adic completion of A and the derived p -completion of R where K is any finitely generated ideal in the kernel of $A \rightarrow R$.
- (2) Relative prismatic comparison (Propositions 4.10, 5.1, 5.17 and 6.12, [7, Thm. 7.17]): if A is a prism and R is an \bar{A} -algebra, then $\underline{\Delta}_{R/A}$ agrees naturally with the derived prismatic cohomology of Bhatt and Scholze [10].
- (3) Absolute prismatic comparison (Example 3.21 and Remark 6.8): if $A = \mathbf{Z}_p$ or if $A = W(k)$ for a perfect \mathbf{F}_p -algebra k and if R is an A -algebra, then $\underline{\Delta}_{R/A}$ agrees naturally with the absolute prismatic cohomology of Bhatt and Lurie [7]; note in this case that the Frobenius twist is equivalent to $\Delta_{R/A}$.
- (4) Preservation of sifted colimits (Corollary 3.17 and 6.10): for varying A and R , $\underline{\Delta}_{R/A}$ is left Kan extended from finitely presented free δ -pairs (as functors to $\mathcal{C}_{\mathbf{Z}}$, i.e., we get colimits in complete filtrations).⁶
- (5) Quasisyntomic descent (Proposition 3.15, Definition 9.1, and Corollary 6.10): for fixed A , $\underline{\Delta}_{R/A}$ satisfies p -completely quasisyntomic descent in relatively quasisyntomic δ -pairs (A, R) .
- (6) Quasismooth descent in A (Proposition 3.15 and Corollary 6.10): for fixed R , $\underline{\Delta}_{R/A}$ satisfies p -completely quasismooth descent in relatively quasisyntomic δ -pairs (A, R) .

⁶A δ -pair (A, R) is finitely presented free if A is a finitely generated free δ -ring and R is a finite polynomial ring over A .

- (7) Base change (Corollary 3.17 and Corollary 6.10): $\underline{\Delta}_{R/A}$ satisfies base change in A : for any map $A \rightarrow A'$ of δ -rings the natural map $\underline{\Delta}_{R/A} \otimes_A A' \rightarrow \underline{\Delta}_{R \otimes_A A'/A'}$ is an equivalence after completing the left-hand side with respect to the Hodge–Tate filtration $\underline{\Delta}_{R/A}^{[\star]} \otimes_A A'$ and similarly for the Nygaard-completed term.
- (8) Restriction along relatively perfect maps (Proposition 3.12 and Corollary 6.10): if $A \rightarrow A'$ is a p -adically relatively perfect map of bounded δ -rings, then the map $\underline{\Delta}_{R/A} \rightarrow \underline{\Delta}_{R/A'}$ is an equivalence for any bounded commutative A' -algebra R .

The functor $(A, R) \mapsto \underline{\Delta}_{R/A}$ is uniquely determined by these properties (Proposition 9.15).

Note that the previous theorem also implies that Bhatt–Scholze’s relative prismatic cohomology only depends on the δ -ring A and not the prism structure, which is slightly surprising. As a consequence of Theorem 1.2, there is an extension of the relative p -adic syntomic complexes $\mathbf{Z}_p(i)(R/A)$ to δ -pairs. If $A \rightarrow R$ is a δ -pair, we let

$$\mathbf{Z}_p(i)(R/A) = \text{fib} \left(\mathbb{N}^{\geq i} \widehat{\Delta}_{R/A}^{(1)} \{i\} \xrightarrow{\text{can}-c\varphi} \widehat{\Delta}_{R/A}^{(1)} \{i\} \right)$$

for $i \in \mathbf{Z}$, the i th relative syntomic complex of R over A .

It turns out that Nygaard-completed Frobenius twisted cohomology $\mathbb{N}^{\geq \star} \widehat{\Delta}_{R/A}$ satisfies a stronger form of descent for morphisms of δ -pairs $(A, R) \rightarrow (B, S)$, requiring only that $R \rightarrow S$ be of universal descent, meaning that for every map $R \rightarrow T$ of animated commutative rings the limit of the Čech complex of $T \rightarrow T \otimes_R S$ computes T . However, note that this generality requires prismatic cohomology relative to animated δ -rings [3]. To avoid this, in Corollary 1.3, one can assume that $(A, R) \rightarrow (B, S)$ be a map of bounded δ -pairs where $A \rightarrow B$ is flat and $R \rightarrow S$ is faithfully flat.

Corollary 1.3 (Descent for syntomic cohomology (Corollary 7.10)). *For each $i \in \mathbf{Z}$, the relative p -adic syntomic complexes $\mathbf{Z}_p(i)(R/A)$ satisfy descent for maps of pairs $(A, R) \rightarrow (B, S)$ such that $R \rightarrow S$ is a universal descent morphism (with no condition on $A \rightarrow B$).*

Adapting an argument from [9, Lem. 7.22] (see also [6, Thm. 5.1(2)]), we also find that the relative p -adic syntomic complexes $\mathbf{Z}_p(i)(R/A)$ preserve sifted colimits and hence are left Kan extended from their values on finitely presented free δ -pairs.

Corollary 1.4 (Left Kan extension of syntomic cohomology (Corollary 8.9)). *For each $i \in \mathbf{Z}$, the relative p -adic syntomic complexes $\mathbf{Z}_p(i)(R/A)$ are left Kan extended from finitely presented free δ -pairs (as a functor to the p -complete derived category $\mathbf{D}(\mathbf{Z})_p^\wedge$).*

Example 1.5. If $A = \mathbf{Z}$, the initial δ -ring, then by Theorem 1.2(3), $\underline{\Delta}_{R/\mathbf{Z}}$ recovers absolute prismatic cohomology $\underline{\Delta}_R$ and $\mathbb{N}^{\geq \star} \widehat{\Delta}_{R/\mathbf{Z}}^{(1)}$ agrees with $\mathbb{N}^{\geq \star} \widehat{\Delta}_R$. It follows that $\mathbf{Z}_p(i)(R/\mathbf{Z})$ agrees with the p -adic syntomic complexes defined in [9]. The Frobenius on \mathbf{Z} is the identity, which explains why no Frobenius twists appear for either the Nygaard filtration or the p -adic syntomic complexes in the absolute case.

1.2 Construction

One can use the descent properties of Theorem 1.2 to reduce to the derived prismatic cohomology of [10]. However, this makes it very hard to verify all the properties stated in Theorem 1.2 and also makes it hard to compare it to different definitions. Therefore we use the theory of prismatic crystals to give a direct definition. Prismatic crystals are by definition quasi-coherent sheaves on the Cartier–Witt stack \mathbf{WCart} of Drinfeld [11] and Bhatt–Lurie [7]. For every bounded δ -pair (R, A) we introduce prismatic crystals $\mathcal{H}_\Delta^{[n]}(R/A)\{i\}$ and set

$$\underline{\Delta}_{R/A}^{[n]}\{i\} = \mathbf{R}\Gamma(\mathbf{WCart}, \mathcal{H}_\Delta^{[n]}(R/A)\{i\}).$$

Concretely the value of $\mathcal{H}_\Delta^{[n]}(R/A)\{i\}$ on a transversal prism (B, J) is given by derived relative prismatic cohomology $\Delta_{R\widehat{\otimes}B/A\widehat{\otimes}B}^{\text{rel}, [n]}\{i\}$ as constructed in [10] (with notation from [7]). Here $A\widehat{\otimes}B$ denotes the prism obtained by (p, J) -adically completing the δ -ring $A \otimes B$ and giving it the prism structure induced from J , while $R\widehat{\otimes}B$ is the p -completion of $R \otimes B = R \otimes B/J$.

Using this description many of the properties of relative prismatic cohomology, such as the following Hodge–Tate comparison, follow immediately from properties of prismatic cohomology relative to prisms.

Proposition 1.6. Hodge–Tate comparison (Proposition 3.9): *There is a natural exhaustive increasing filtration $F_{\leq \star}^{\delta\text{conj}}\overline{\Delta}_{R/A}\{i\}$ on $\overline{\Delta}_{R/A}\{i\}$, called the δ -conjugate filtration, whose graded pieces are given as*

$$\text{gr}_u^{\delta\text{conj}}\overline{\Delta}_{R/A}\{i\} \simeq L\Omega_{R/A}^u \otimes \text{fib}(\mathbf{Z}_p \xrightarrow{i-u} \mathbf{Z}_p)[-u]$$

for all $u \in \mathbf{Z}$.⁷

The construction of the Nygaard-completed prismatic cohomology and the Nygaard filtration will also be given using the language of prismatic crystals as is done in [7, Sec. 5.5].

Finally we note that one can also give other descriptions of relative prismatic cohomology that are often more accessible or easier to describe.

Theorem 1.7. (1) Site-theoretic comparison (Theorem 5.6): *when $L_{R/A}$ has p -complete Tor-amplitude in $[0, 1]$, $\Delta_{R/A}$ has a site theoretic description, that is it agrees with the cohomology of the relative prismatic site $(R/A)_\Delta$ as defined in Section 2.*

(2) Comparison to prismaticization (Proposition 5.1): *for each bounded δ -pair (R/A) , there is a formal stack $\text{WCart}_{R/A}$ over WCart . The pushforward of the structure sheaf $\mathcal{O}_{\text{WCart}_{R/A}}$ along $\text{WCart}_{R/A} \rightarrow \text{WCart}$ agrees with $\mathcal{H}_\Delta(R/A)$ when $L_{R/A}$ has p -complete Tor-amplitude in $[0, 1]$. In particular in this situation the global sections of the structure sheaf of $\text{WCart}_{R/A}$ is equivalent to $\Delta_{R/A}$.*

1.3 Applications

The idea that there should be relative p -adic syntomic complexes in the generality of this paper is motivated by topological considerations such as applying TC to $\text{THH}(R/\mathbf{S}[z])$ where z maps to some element of R . In this case, the complexes could have been defined in [9, Sec. 11]. The idea that the relative syntomic complexes should satisfy descent results as in Corollary 1.3 is heavily influenced by the work of Liu and Wang [19] on $\text{TC}_*(\mathcal{O}_K; \mathbf{F}_p)$, in which the authors recover Hesselholt and Madsen’s verification [14] of the Quillen–Lichtenbaum conjecture for local fields, and by [18] on THH of (quotients of) discrete valuation rings. These papers both take a topological approach to calculations using descent, so it was natural to look for a purely prismatic approach. The explicit connection between the two approaches will be studied in [5] in the context of the modern approach to cyclotomic spectra developed in [26].

In [4] (see [2] for a survey), we use Corollary 1.3(d) for $R = \mathcal{O}_K/\varpi^n$ when K is a finite extension of \mathbf{Q}_p with uniformizer ϖ and residue field k to show that the absolute p -adic syntomic complexes $\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n)$ can be computed from descent along $W(k) \rightarrow W(k)[[z]]$ using the relative p -adic syntomic complexes. The point is that $\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n/W(k)[[z]])$ admits a purely algebraic description in terms of the prismatic envelopes introduced in [10].

In order to make this approach amenable to computer calculations, we use the ϖ -adic filtration and argue that one has to compute $\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n)$ only up to finite filtration level. To make this precise, we introduce filtered prismatic cohomology below, following a suggestion of Bhatt. It turns out that this is most naturally viewed as a specific form of prismatic cohomology relative to a δ -stack, $\widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m$, so it fits naturally into the context of the present paper. We use filtered prismatic cohomology to prove the following result.

⁷This filtration does not agree with the conjugate filtration constructed in [10] in the prismatic case; see Warning 3.22. Rather it is an analog of the conjugate filtration on absolute prismatic cohomology studied in [7].

Theorem 1.8 (Crystalline degeneration). *If $F^{\geq \star}R$ is a filtered p -complete commutative ring with $F^0R = R$ and which is constant for $\star \leq 0$, then the syntomic complexes of R admit natural filtrations $F^{\geq \star}\mathbf{Z}_p(i)(R)$ and there are natural identifications $\mathrm{gr}^*\mathbf{Z}_p(i)(R) \simeq \mathrm{gr}^*\mathbf{Z}_p(i)(\mathrm{gr}^*R)$.*

We call this crystalline degeneration as in the cases of interest to us in [4], we have $p \in F^{\geq 1}R$ so that the associated graded is an \mathbf{F}_p -algebra.

Outline. We give three different constructions of prismatic cohomology relative to a δ -ring A in Sections 2, 3, and 4: one is site theoretic and extends the prismatic site of [10]; one is by constructing prismatic crystals $\mathcal{H}_{\Delta}(R/A)$ on the stack WCart of Bhatt–Lurie [7] and Drinfeld [11]. The final method is via a prismaticization and gives relative Cartier–Witt stacks $\mathrm{WCart}_{R/A}$, extending the definition of the relative prismaticization from [8]. We compare these approaches under mild hypotheses in Section 5 and we introduce the Frobenius twisted variants and the Nygaard filtration in Section 6. We discuss syntomic cohomology in Section 7 and we wrap it all up and discuss the prismatic package in Section 8. We discuss the use of relative quasisyntomic descent to compute prismatic cohomology relative to δ -rings in Section 9 and prove the uniqueness statement of Theorem 1.2 there. In Section 10, we explain how to work over $\mathbf{A}^1/\mathbf{G}_m$ to construct filtered variants. Finally, we explain how the theory of prismatic cohomology relative to δ -rings gives a natural way to extend the theory of prismatic cohomology to filtered and graded rings by working over the stack $\mathrm{WCart}_{\mathbf{A}^1/\mathbf{G}_m}$. In Appendix A, we give background on the theory of quasi-coherent sheaves on formal stacks and prove some results on base change for quasi-coherent cohomology needed for the computation of prismatic crystals.

Background. We will freely use the theory of prismatic cohomology as developed by Bhatt, Morrow, and Scholze [9], Bhatt and Scholze [10], and Bhatt and Lurie [7, 8]. In particular, we use the notions of quasisyntomic rings from [9], δ -rings from [10] (and going back to [16, 17]), animated δ -rings and animated prisms from [8], generalized Cartier–Witt divisors from [7, 8], and the prismatic site from [10] as well as its animated analogue from [8]. We also use animated commutative and derived commutative rings (as defined by Mathew), see [27] for the latter.

Notation. Throughout this paper, we fix a prime number p . All δ -ring theoretic notions are taken with respect to p and quasisyntomic will be shorthand for quasisyntomic with respect to the prime p . Thus, for example, a map $A \rightarrow A'$ of commutative rings is quasisyntomic if A' is p -completely flat over A and $L_{A'/A}$ has p -complete Tor-amplitude in $[0, 1]$ (where we index throughout using the homological convention).⁸ A commutative ring is bounded if it has bounded p -power torsion. A map $A \rightarrow A'$ of commutative rings is p -completely quasismooth if $L_{A'/A}$ has p -complete Tor-amplitude in $[0, 0]$. A map $A \rightarrow A'$ of commutative rings is p -completely quasi-étale if $L_{A'/A}$ vanishes p -adically.

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⁸Note that our definition of quasisyntomic is not the same as that found in [9], which requires the rings themselves to be derived p -complete, but agrees with the notion of p -quasisyntomic morphisms found in [7, Def. C.9].

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2 The prismatic site

In this first section we define a relative prismatic site. The cohomology of this with respect to the structure sheaf \mathcal{O}_Δ is a version of relative prismatic cohomology which we call the *site theoretic (relative) prismatic cohomology* $\Delta_{R/A}^{\text{site}}$. We will see later that the site theoretic prismatic cohomology agrees on a large class of rings with the ‘correct’ prismatic cohomology. We begin this paper with the site theoretic variant since it is the easiest one to understand conceptually.

Definition 2.1 (δ -pairs). Let Pairs^δ be the category consisting of δ -pairs (A, R) (or $A \rightarrow R$), meaning a δ -ring A and a commutative A -algebra R . The morphisms $(A, R) \rightarrow (A', R')$ consist of commutative diagrams

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ R & \longrightarrow & R', \end{array}$$

where $A \rightarrow A'$ is a δ -ring map. In particular, we emphasize that R is not equipped with a δ -ring structure, in fact it typically does not even admit a (natural) δ -ring structure. In general, we say a δ -pair is bounded, (derived) p -complete, or quasisyntomic if A and R are bounded, (derived) p -complete, or quasisyntomic as commutative rings.

Definition 2.2 (Pre-prismatic δ -pairs). Say that a δ -pair (A, R) is *pre-prismatic* if the kernel of $A \rightarrow R$ contains a Cartier divisor I such that Zariski locally on $\text{Spec } A$ any generator d of I has the property that $\delta(d)$ maps to a unit in R_p^\wedge . A pre-prismatic δ -pair is prismatic if (A, I) is a prism.

Remark 2.3. Note that if (A, R) is a pre-prismatic bounded δ -pair, as exhibited by a Cartier divisor I , then $A[\delta(I)^{-1}]_{(I,p)}^\wedge$ inherits a δ -ring structure by [10, Rem. 2.16, Lem. 2.18] and becomes a prism with respect to the completion of I . By property (1) of Theorem 1.2, the prismatic cohomology will not see the difference between (A, R) and $(A[\delta(I)^{-1}]_{(I,p)}^\wedge, R_p^\wedge)$.

Definition 2.4 (The prismatic site). Let $A \rightarrow R$ be a δ -pair. The prismatic site $(R/A)_\Delta$ of R relative to the δ -ring A is the opposite of the category of commutative squares

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ R & \longrightarrow & \overline{B} \end{array} \tag{1}$$

where $B \rightarrow \overline{B} = B/I$ is a bounded prism and $A \rightarrow B$ is a map of δ -rings; we equip this category with the (p, I) -completely faithfully flat topology in B . The proof that $(R/A)_\Delta$ is indeed a site is the same as the first paragraph of the proof of [10, Cor. 3.12].

Remark 2.5 (Comparison to other prismatic sites). (1) If A is itself a bounded prism and the map $A \rightarrow R$ factors through \bar{A} , then $(R/A)_\Delta$ agrees by definition with the relative prismatic site of R over A defined in [10, Def. 4.1]. This follows since $A \rightarrow B$ has to take the prismatic ideal in A to ideal $I = \ker(B \rightarrow \bar{B})$ by commutativity of (1).
 (2) If $A = \mathbf{Z}_p$, then $(R/\mathbf{Z}_p)_\Delta$ agrees with the absolute prismatic site $(R)_\Delta$ of R defined in [7, Def. 4.4.27] since \mathbf{Z}_p is initial as a p -complete δ -ring. In particular, in the special case of $(\mathbf{Z}_p/\mathbf{Z}_p)_\Delta$, we recover the site of all bounded prisms.

Definition 2.6 (Breuil–Kisin twists). Each prism B admits a natural line bundle $B\{1\}$ constructed in [7, Sec. 2] called the first Breuil–Kisin twist of B . It is the home for the prismatic logarithm map and is the prismatic analogue of the Tate twist $\mathbf{Z}_p(1)$ in the étale setting. It is also compatible with base change: if $B \rightarrow B'$ is a map of prisms, then there is a natural isomorphism $B' \otimes_B B\{1\} \cong B'\{1\}$.⁹ The tensor power $B\{i\} = B\{1\}^{\otimes_B i}$ is the i th Breuil–Kisin twist. These define presheaves $\mathcal{O}_\Delta\{i\}$ on $(R/A)_\Delta$ by sending a square (1) to $B\{i\}$. The presheaf $\mathcal{O}_\Delta = \mathcal{O}_\Delta\{0\}$ is a sheaf of δ -rings and is called the prismatic structure sheaf, terminology which will be justified below. Given any sheaf of \mathcal{O}_Δ -modules \mathcal{M} on $(R/A)_\Delta$, we denote by $\mathcal{M}\{i\}$ the tensor product $\mathcal{M} \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta\{i\}$ in sheaves of abelian groups on $(R/A)_\Delta$.

Definition 2.7 (The Hodge–Tate tower). The assignment to (1) of the prismatic ideal I defines an invertible sheaf of ideals \mathcal{J}_Δ in \mathcal{O}_Δ called the Hodge–Tate ideal. We let $\mathcal{O}_\Delta^{[n]}\{i\} = \mathcal{J}_\Delta^{\otimes n} \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta\{i\}$ for $n \in \mathbf{Z}$. These assemble into a tower

$$\dots \rightarrow \mathcal{O}_\Delta^{[2]}\{i\} \rightarrow \mathcal{O}_\Delta^{[1]}\{i\} \rightarrow \mathcal{O}_\Delta^{[0]}\{i\} \rightarrow \mathcal{O}_\Delta^{[-1]}\{i\} \rightarrow \dots$$

with weight 0 term the i th Breuil–Kisin twist. We also have $\mathcal{O}_{\bar{\Delta}} = \mathcal{O}_\Delta/\mathcal{O}_\Delta^{[1]}$, which takes $(B, I) \in (R/A)_\Delta$ to $\bar{B} = B/I$ and defines a sheaf of commutative R -algebras.

Note that the Hodge–Tate tower for $\mathcal{O}_\Delta\{i\}$ is tensored up from the Hodge–Tate tower for \mathcal{O}_Δ : we have $\mathcal{O}_\Delta^{[*]}\{i\} \cong \mathcal{O}_\Delta\{i\} \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta^{[*]}$.

Lemma 2.8. *For each $n, i \in \mathbf{Z}$, $\mathcal{O}_\Delta^{[n]}\{i\}$ and $\mathcal{O}_{\bar{\Delta}}\{i\}$ are sheaves on $(R/A)_\Delta$ with vanishing higher cohomology for any object $(B, I) \in (R/A)_\Delta$.*

Proof. For \mathcal{O}_Δ and $\mathcal{O}_{\bar{\Delta}}$, this follows from [10, Cor. 3.12], which says that these define sheaves with vanishing higher cohomology in the case of the site of all bounded prisms, i.e., $(\mathbf{Z}_p/\mathbf{Z}_p)_\Delta$. We use that the value of \mathcal{O}_Δ and $\mathcal{O}_{\bar{\Delta}}$ on $(B, I) \in (R/A)_\Delta$ with respect to $(R/A)_\Delta$ can be computed by considering (B, I) as an object in $(\mathbf{Z}_p/\mathbf{Z}_p)_\Delta$, which follows by observing that the Čech complexes associated to $(B, I) \rightarrow (C, IC)$ in $(R/A)_\Delta$ and $(\mathbf{Z}_p/\mathbf{Z}_p)_\Delta$ agree.

The result for $\mathcal{J}_\Delta = \mathcal{O}_\Delta^{[1]} = \ker(\mathcal{O}_\Delta \rightarrow \mathcal{O}_{\bar{\Delta}})$ follows because sheaves are closed under kernels and because $\mathcal{O}_\Delta \rightarrow \mathcal{O}_{\bar{\Delta}}$ is surjective pointwise. The fact that the Breuil–Kisin twists $\mathcal{O}_\Delta\{i\}$ are sheaves with vanishing higher cohomology on $(B, I) \in (R/A)_\Delta$ follows from the sheaf property for \mathcal{O}_Δ and the crystal property for the Breuil–Kisin twists. Given a (p, I) -completely faithfully flat map of prisms $(B, I) \rightarrow (C, IC)$, let (C^\bullet, IC^\bullet) be the Čech nerve. Then, $C^\bullet\{i\}$ is equivalent as a cosimplicial object to $(C^\bullet) \otimes_B B\{i\}$ by the base change property of the Breuil–Kisin twists. So, the sheaf property for $\mathcal{O}_\Delta\{i\}$ reduces to (p, I) -completely faithfully flat descent for the invertible B -module $B\{i\}$. The result for the Hodge–Tate towers is proven in the same way. \square

Remark 2.9. There is a natural equivalence $\mathcal{J}_\Delta/\mathcal{J}_\Delta^2 \cong \mathcal{O}_\Delta^{[1]}/\mathcal{O}_\Delta^{[2]} \cong \mathcal{O}_{\bar{\Delta}}\{1\}$. It follows in general that

$$\mathcal{O}_\Delta^{[n]}\{i\}/\mathcal{O}_\Delta^{[n+1]}\{i\} \cong \mathcal{O}_{\bar{\Delta}}\{i+n\}.$$

See [7, Rem. 2.5.7].

⁹In the language of [7], the assignment $B \mapsto B\{1\}$ defines a prismatic crystal, or a quasi-coherent sheaf on the Cartier–Witt stack WCart .

Definition 2.10 (Site-theoretic prismatic cohomology). We define

$$\Delta_{R/A}^{\text{site},[n]}\{i\} = \text{R}\Gamma((R/A)_\Delta, \mathcal{O}_\Delta^{[n]}\{i\}).$$

The superscript $(-)^{\text{site}}$ will be used to distinguish this form of prismatic cohomology from other forms discussed below, before we have established our comparison theorems under quasisyntomicity assumptions in Section 5.

In the case of $n = i = 0$, the resulting object of the p -complete derived ∞ -category $\text{D}(\mathbf{Z}_p)_p^\wedge$ is denoted by $\Delta_{R/A}^{\text{site}}$, the site-theoretic prismatic cohomology of R relative to A . It naturally has the structure of an \mathbf{E}_∞ -algebra over A with an endomorphism φ called Frobenius induced by the Frobenius endomorphism of the sheaf \mathcal{O}_Δ . The objects $\Delta_{R/A}^{\text{site},[n]}\{i\}$ assemble into Hodge–Tate towers

$$\Delta_{R/A}^{\text{site},[*]}\{i\}: \cdots \rightarrow \Delta_{R/A}^{\text{site},[n+1]}\{i\} \rightarrow \Delta_{R/A}^{\text{site},[n]}\{i\} \rightarrow \Delta_{R/A}^{\text{site},[n-1]}\{i\} \rightarrow \cdots$$

of $\Delta_{R/A}^{\text{site}}$ -modules; these towers are complete filtrations on the colimit $\Delta_{R/A}^{\text{site}}\{i\}[1/I]$ with weight 0 part $\Delta_{R/A}^{\text{site}}\{i\}$.

Warning 2.11 (No Breuil–Kisin twists for δ -rings). When the δ -pair (A, R) is prismatic, $\Delta_{R/A}^{\text{site}}\{i\} = \text{R}\Gamma((R/A)_\Delta, \mathcal{O}_\Delta\{i\})$ is equivalent to $\Delta_{R/A}^{\text{site}} \otimes_A A\{i\}$, but when A is a general δ -ring, this need not be the case. In fact, the construction $A \mapsto A\{i\}$ does not extend from prisms to δ -rings since it depends on the ideal I defining the prism structure.

Moreover, even in the special case $A = R = \mathbf{Z}_p$ the Breuil–Kisin twisted absolute prismatic cohomologies $\Delta_{\mathbf{Z}_p/\mathbf{Z}_p}\{i\}$ are, for $i \neq 0$, not invertible over $\Delta_{\mathbf{Z}_p/\mathbf{Z}_p}$ and are not generally tensor powers of $\Delta_{\mathbf{Z}_p/\mathbf{Z}_p}\{1\}$.

Definition 2.12 (Site-theoretic Hodge–Tate cohomology). The Breuil–Kisin twisted Hodge–Tate cohomology of R relative to A is $\overline{\Delta}_{R/A}^{\text{site}}\{i\} = \text{R}\Gamma((R/A)_\Delta, \mathcal{O}_\Delta\{i\})$. If $i = 0$, we call $\overline{\Delta}_{R/A}^{\text{site}}$ the Hodge–Tate cohomology of R over A , which is naturally a p -complete \mathbf{E}_∞ - R -algebra. More generally, for any i , we have the Breuil–Kisin twisted Hodge–Tate cohomology

$$\overline{\Delta}_{R/A}^{\text{site}}\{i\} = \text{R}\Gamma((R/A)_\Delta, \mathcal{O}_\Delta),$$

which is naturally a $\overline{\Delta}_{R/A}^{\text{site}}$ -module. For each $i, n \in \mathbf{Z}$, there are fiber sequences

$$\Delta_{R/A}^{\text{site},[n+1]}\{i\} \rightarrow \Delta_{R/A}^{\text{site},[n]}\{i\} \rightarrow \overline{\Delta}_{R/A}^{\text{site}}\{i+n\}$$

by Remark 2.9.

Remark 2.13 (Insensitivity to p -completion). Suppose that A and R are bounded so that their derived p -completions are discrete and agree with the classical p -completions. Since all prisms B and their Hodge–Tate quotients $\overline{B} = B/I$ are derived p -complete, the prismatic site $(R/A)_\Delta$ is naturally equivalent to $(\widehat{R}/\widehat{A})_\Delta$, the relative prismatic site of the derived p -completion \widehat{R} over the derived p -completion \widehat{A} , where the δ -ring structure on A extends to \widehat{A} by [10, Lem. 2.18]. It follows that $\Delta_{R/A}^{\text{site}}$ depends only on the derived p -completion, and similarly for the Breuil–Kisin twists and Hodge–Tate towers. The boundedness hypothesis can also be replaced at the cost of using the animated prismatic site of Variant 2.16.

Remark 2.14 (Relative prismatic cohomology comparison). If (A, R) is a bounded prismatic δ -pair (so that A is a bounded prism and R is a bounded commutative \widehat{A} -algebra), then by Remark 2.5 our site-theoretic prismatic cohomology agrees with the cohomology of the prismatic site of [10, Def. 4.1] and [7, Def. 4.1.1].¹⁰

¹⁰Note that the definition in [7] should include the hypothesis that the prisms $(B, IB) \in (R/A)_\Delta$ be bounded.

Remark 2.15 (Absolute prismatic cohomology comparison). It follows by definition from Remark 2.5 that $\Delta_{R/A}^{\text{site},[n]}\{i\}$ agrees with the cohomology of $\mathcal{O}_{\Delta}^{[n]}\{i\}$ on the absolute prismatic site of $\text{Spf } R$ studied in [7, Def. 4.4.27] and denoted there by $\text{R}\Gamma_{\Delta}^{\text{site},[n]}(\text{Spf } R)\{i\}$.

Variation 2.16 (The animated prismatic site). Suppose more generally that A is an animated δ -ring in the sense of [8, App. A] and that R is an animated commutative A -algebra. We let $\text{Pairs}^{\text{an}\delta}$ be the ∞ -category of such animated δ -pairs. Recall from [8] that an animated prism is an animated δ -ring B equipped with a generalized Cartier divisor $\alpha: I \rightarrow B$ such that B is (p, I) -complete and for any perfect \mathbf{F}_p -algebra k and any animated δ -ring map $B \rightarrow W(k)$ the extension of scalars $I \otimes_B W(k) \rightarrow W(k)$ is equivalent to the inclusion of the ideal $(p) \subseteq W(k)$. Every ordinary prism is an animated prism.

The ∞ -category $(R/A)_{\Delta}^{\text{an}}$ is defined to be the opposite of the ∞ -category of commutative squares (1) where $B \rightarrow \overline{B} = B/I = \text{cofib}(\alpha)$ is an animated prism, $A \rightarrow B$ is a map of animated δ -rings and $R \rightarrow \overline{B}$ is a map of animated commutative rings. We equip $(R/A)_{\Delta}^{\text{an}}$ with the structure of a site by declaring the covers to be (p, I) -completely faithfully flat maps (of animated δ -rings) in B . The objects $\mathcal{O}_{\Delta}^{[n]}\{i\}$ naturally extend to sheaves on $(R/A)_{\Delta}^{\text{an}}$ with values in $\text{D}(\mathbf{Z}_p)_{\wedge}^{\wedge}$. We obtain upon taking global sections

$$\Delta_{R/A}^{\text{ansite},[n]}\{i\} = \text{R}\Gamma((R/A)_{\Delta}^{\text{an}}, \mathcal{O}_{\Delta}^{\text{an},[n]}\{i\}),$$

which are all modules over animated site-theoretic prismatic cohomology $\Delta_{R/A}^{\text{ansite}}$. If (A, R) is a δ -Pair, there is an inclusion of the (underived) prismatic site $(R/A)_{\Delta}$ into the animated prismatic site $(R/A)_{\Delta}^{\text{an}}$. Upon taking global sections, this inclusion induces a natural map

$$\Delta_{R/A}^{\text{ansite}} \simeq \lim_{(B, \overline{B}) \in (R/A)_{\Delta}^{\text{an}}} B \rightarrow \lim_{(B, \overline{B}) \in (R/A)_{\Delta}} B \simeq \Delta_{R/A}^{\text{site}}.$$

In the prismatic case, if A is a bounded prism, R is bounded, and $L_{R/\overline{A}}$ has p -complete Tor-amplitude in $[0, 1]$, then the natural map

$$\Delta_{R/A}^{\text{ansite}} \rightarrow \Delta_{R/A}^{\text{site}}$$

is an equivalence; see [8, Rem. 7.14]. Similar results hold for the Breuil–Kisin twists, Hodge–Tate cohomology, and so forth.

3 The prismatic crystal

In this section we give the general definition of relative prismatic cohomology using the theory of prismatic crystals.

Notation 3.1. If A is a commutative ring and $M, N \in \text{D}(A)$, we let $M \widehat{\otimes}_A N$ be the derived p -completion of the derived tensor product over A ; this makes the p -complete derived ∞ -category $\text{D}(A)_{\wedge}^{\wedge}$ into a symmetric monoidal stable ∞ -category. In particular, without further decoration, $M \widehat{\otimes} N$ denotes the tensor product in $\text{D}(\mathbf{Z}_p)_{\wedge}^{\wedge} \simeq \text{D}(\mathbf{Z}_p)_{\wedge}^{\wedge}$. If (B, I) is a prism and $M, N \in \text{D}(B)$, we write $(M \otimes_B N)_{\wedge}^{\wedge}_{(p, I)}$, or $M \widehat{\otimes}_B N$ if the prism structure is clear from context, for the derived (p, I) -completed derived tensor product; again, this endows the (p, I) -complete derived ∞ -category $\text{D}(B)_{\wedge}^{\wedge}_{(p, I)}$ with a symmetric monoidal structure.

Definition 3.2 (Prismatic crystals). A prismatic crystal is a functorial assignment to each bounded prism (B, I) of a derived (p, I) -complete object $\mathcal{F}(B) \in \text{D}(B)_{\wedge}^{\wedge}_{(p, I)}$ where this assignment satisfies base change in the sense that if $(B, I) \rightarrow (C, IC)$ is a map of bounded prisms, then the natural map $\mathcal{F}(B) \widehat{\otimes}_B C \rightarrow \mathcal{F}(C)$ is an

equivalence.¹¹ The stable ∞ -category of prismatic crystals is equivalent to $D(\mathrm{WCart})$, the stable ∞ -category of quasi-coherent sheaves on the formal stack WCart by [7, Prop. 3.3.5]. Moreover, to define a prismatic crystal, it is enough to make such a functorial assignment on transversal prisms, by [7, Lem. 3.3.10].

We introduce a prismatic crystal $\mathcal{H}_\Delta(R/A)$ associated to any δ -pair where A is bounded. The prismatic crystal definition of relative prismatic cohomology will be based upon $\mathrm{R}\Gamma(\mathrm{WCart}, \mathcal{H}_\Delta(R/A))$.

Our construction is based on derived relative prismatic cohomology, which is introduced in [10, Sec. 7.2] (see also [7, Sec. 4.1]). If k is a commutative ring, let $\widehat{\mathrm{CAlg}}_k^{\mathrm{an}}$ denote the ∞ -category of derived p -complete animated k -algebras. If (B, I) is a bounded prism, then $\Delta_{-/B}: \widehat{\mathrm{CAlg}}_B^{\mathrm{an}} \rightarrow D(B)_{(p,I)}^\wedge$ is defined to be the unique functor which preserves sifted colimits and agrees with the site-theoretic relative prismatic cohomology of [10] on p -complete finitely presented polynomial \widehat{B} -algebras.

Definition 3.3 (Relative prismatic cohomology crystals). Let $A \rightarrow R$ be a δ -pair and assume that A is bounded; we can in fact allow R to be a p -complete animated commutative A -algebra. The prismatic crystal $\mathcal{H}_\Delta(R/A)$ associated to the pair (R/A) is the assignment, for each transversal prism (B, I) , of the derived prismatic cohomology $\mathcal{H}_\Delta(R/A)(B) = \Delta_{R\widehat{\otimes}B/A\widehat{\otimes}B}^{\mathrm{rel}}$, where $A\widehat{\otimes}B = (A \otimes B)_{(p,I)}^\wedge$. As $\Delta_{R\widehat{\otimes}B/A\widehat{\otimes}B} \widehat{\otimes}_B C \simeq \Delta_{R\widehat{\otimes}C/A\widehat{\otimes}C}$ for a map of bounded prisms $B \rightarrow C$ by [7, Rem. 4.1.5], this functor defines a prismatic crystal, i.e., an object of $D(\mathrm{WCart})$.

Remark 3.4. The boundedness hypothesis on A guarantees that if B is a transversal prism, then $A\widehat{\otimes}B$ is a bounded prism. However, as we do not impose any assumptions on R , the ring $R\widehat{\otimes}B$ is in general an animated commutative ring, which is why we require the generality of derived relative prismatic cohomology. We could drop the boundedness condition on A and describe the value $\mathcal{H}_\Delta(R/A)(B)$ for a non-transversal prism B at the cost of using animated prisms as in [8]. Alternatively, note that every prism B admits a map of prisms $C \rightarrow B$ where C is transversal by [7, Prop. 2.4.1]. Thus, the value of $\mathcal{H}_\Delta(R/A)$ at B is computed as $\Delta_{R\widehat{\otimes}C/A\widehat{\otimes}C} \widehat{\otimes}_C B$.

Definition 3.5 (Hodge–Tate crystals). If $A \rightarrow R$ is a δ -pair where A is bounded, let $\mathcal{H}_\Delta^-(R/A)$ be the crystal which assigns to a transversal bounded prism (B, I) the derived Hodge–Tate complex

$$\mathcal{H}_\Delta^-(R/A)(B) = \overline{\Delta}_{R\widehat{\otimes}B/A\widehat{\otimes}B}^{\mathrm{rel}}.$$

The Hodge–Tate crystal $\mathcal{H}_\Delta^-(R/A)$ is equivalent to $\iota_* \iota^* \mathcal{H}_\Delta(R/A)$ where $\iota: \mathrm{WCart}^{\mathrm{HT}} \hookrightarrow \mathrm{WCart}$ is the inclusion of the Hodge–Tate locus.

VARIANT 3.6 (Breuil–Kisin twisted prismatic crystals). Besides $\mathcal{H}_\Delta(R/A)$, there are the Breuil–Kisin twists and these admit associated Hodge–Tate towers. In general, let $\mathcal{H}_\Delta^{[n]}(R/A)\{i\}$ be the prismatic crystal $\mathcal{J}^n \otimes_{\mathcal{O}_{\mathrm{WCart}}} \mathcal{H}_\Delta(R/A) \otimes_{\mathcal{O}_{\mathrm{WCart}}} \mathcal{O}_{\mathrm{WCart}}\{i\}$, where \mathcal{J} is the ideal of the Hodge–Tate divisor $\iota: \mathrm{WCart}^{\mathrm{HT}} \hookrightarrow \mathrm{WCart}$ and $\mathcal{O}_{\mathrm{WCart}}\{i\}$ is the i th tensor power of the Breuil–Kisin line bundle $\mathcal{O}_{\mathrm{WCart}}\{1\}$ of [7, Ex. 3.3.8]. In other words,

$$\mathcal{H}_\Delta^{[n]}(R/A)\{i\}(B) = \Delta_{R\widehat{\otimes}B/A\widehat{\otimes}B}^{\mathrm{rel}, [n]} \{i\}$$

for transverse prisms B in the notation of [7, Const. 4.4.10]. There are fiber sequences

$$\mathcal{H}_\Delta^{[n+1]}(R/A)\{i\} \rightarrow \mathcal{H}_\Delta^{[n]}(R/A)\{i\} \rightarrow \mathcal{H}_\Delta^-(R/A)\{i+n\}$$

in $D(\mathrm{WCart})$ as in [7, Rem. 4.5.7] thanks to the equivalence $\mathcal{J}/\mathcal{J}^2 \cong \iota_* \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{1\}$. These assemble into Hodge–Tate towers $\mathcal{H}_\Delta^{[*]}(R/A)\{i\}$ of prismatic crystals.

¹¹More rigorously, a prismatic crystal is a cocartesian section of the cocartesian fibration whose classifying functor associates to a prism (B, I) the stable ∞ -category $D(B)_{(p,I)}^\wedge$ and to a morphism of prisms the derived completed base change.

Definition 3.7 (Cohomology of prismatic crystals). Suppose that $A \rightarrow R$ is a δ -pair where A is bounded. Let the relative prismatic cohomology be $\Delta_{R/A} = \mathrm{R}\Gamma(\mathrm{WCart}, \mathcal{H}_\Delta(R/A))$, the cohomology of the prismatic crystal $\mathcal{H}_\Delta(R/A)$. Similarly, let $\overline{\Delta}_{R/A} = \mathrm{R}\Gamma(\mathrm{WCart}^{\mathrm{HT}}, \iota^* \mathcal{H}_\Delta(R/A)) \simeq \mathrm{R}\Gamma(\mathrm{WCart}, \mathcal{H}_{\overline{\Delta}}(R/A))$, the cohomology of the Hodge–Tate crystal. More generally, we have

$$\Delta_{R/A}^{[n]} \{i\} = \mathrm{R}\Gamma(\mathrm{WCart}, \mathcal{H}_\Delta^{[n]}(R/A)\{i\}),$$

which assemble into towers $\Delta_{R/A}^{[\star]} \{i\}$ with associated graded pieces computed by fiber sequences

$$\Delta_{R/A}^{[n+1]} \{i\} \rightarrow \Delta_{R/A}^{[n]} \{i\} \rightarrow \overline{\Delta}_{R/A} \{i+n\}.$$

Remark 3.8 (Cohomology as a limit over prismatic points). Given a transversal prism (B, I) there is a canonical map $\rho_B: \mathrm{Spf} B \rightarrow \mathrm{WCart}$. It is shown in [7] that WCart is the colimit of these ρ maps over the opposite of the category of transversal prisms. It follows that the cohomology of a prismatic crystal such as $\mathcal{H}_\Delta(R/A)$ is given as a limit

$$\Delta_{R/A} = \mathrm{R}\Gamma(\mathrm{WCart}, \mathcal{H}_\Delta(R/A)) = \lim_{\text{transversal } B} \mathcal{H}_\Delta(R/A)(B) = \lim_{\text{transversal } B} \Delta_{R \widehat{\otimes}_{\overline{B}}/A \widehat{\otimes} B}^{\mathrm{rel}}.$$

A similar remark applies to the Breuil–Kisin twists and the Hodge–Tate cohomology.

The Hodge–Tate comparison theorem applies out of the box.

Proposition 3.9 (The conjugate filtration on Hodge–Tate crystals). *If $A \rightarrow R$ is a δ -pair where A is bounded, then the Hodge–Tate crystal $\mathcal{H}_{\overline{\Delta}}(R/A)$, viewed as a quasi-coherent sheaf on $\mathrm{WCart}^{\mathrm{HT}}$, admits an increasing exhaustive multiplicative conjugate filtration $F_{\leq \star}^{\delta \text{conj}} \mathcal{H}_{\overline{\Delta}}(R/A)$ with graded pieces given by*

$$\mathrm{gr}_{\star}^{\delta \text{conj}} \mathcal{H}_{\overline{\Delta}}(R/A) \simeq \widehat{\mathrm{L}}\Omega_{R/A}^{\star} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}} \{-\star\}[-\star],$$

where $\widehat{\mathrm{L}}\Omega_{R/A}^{\star}$ denotes the p -complete derived differential forms of R over A .

Proof. Recall that, as for the Cartier–Witt stack, the ∞ -category $\mathrm{D}(\mathrm{WCart}^{\mathrm{HT}})$ of quasi-coherent sheaves on the Hodge–Tate locus is given by the limit over $\mathrm{D}(\overline{B})_p^{\wedge}$ as (B, I) ranges over all bounded prisms. For a fixed bounded prism (B, I) and a p -complete animated \overline{B} -algebra S , there is a natural increasing multiplicative conjugate filtration $F_{\leq \star}^{\mathrm{conj} \overline{\mathrm{rel}}} \overline{\Delta}_{S/B}$ with graded pieces given by

$$\mathrm{gr}_{\star}^{\mathrm{conj} \overline{\mathrm{rel}}} \overline{\Delta}_{S/B} \simeq \widehat{\mathrm{L}}\Omega_{S/\overline{B}}^{\star} \{-\star\}[-\star],$$

where $\widehat{\mathrm{L}}\Omega_{S/\overline{B}}^{\star}$ denotes the p -complete derived differential forms on S relative to \overline{B} ; see [10, Thm. 6.3] or [7, Rem. 4.1.7]. The conjugate filtration is functorial and satisfies base change: if $(B, I) \rightarrow (C, IC)$ is a map of bounded prisms, then the natural map $\overline{\Delta}_{S/B}^{\overline{\mathrm{rel}}} \rightarrow \overline{\Delta}_{S \widehat{\otimes}_{\overline{B}} \overline{C}/C}^{\overline{\mathrm{rel}}}$ preserves the conjugate filtration and the induced map

$$(F_{\leq \star}^{\mathrm{conj} \overline{\Delta}_{S/B}}) \widehat{\otimes}_{\overline{B}} \overline{C} \rightarrow F_{\leq \star}^{\mathrm{conj} \overline{\Delta}_{S \widehat{\otimes}_{\overline{B}} \overline{C}/C}}$$

is a filtered equivalence. From this, it follows that the conjugate filtration descends to give a filtration $F_{\leq \star}^{\delta \text{conj}} \mathcal{H}_{\overline{\Delta}}(R/A)$ on the Hodge–Tate crystal $\mathcal{H}_{\overline{\Delta}}(R/A)$. As $L_{R \widehat{\otimes}_{\overline{B}}/A \widehat{\otimes} B} \simeq L_{R/A} \widehat{\otimes}_{\overline{B}}$, the graded pieces of the conjugate filtration are the crystals

$$(B, I) \mapsto \mathrm{gr}_u^{\delta \text{conj}} \mathcal{H}_\Delta(R/A)(B) \simeq \mathrm{gr}_u^{\mathrm{conj} \overline{\Delta}_{R \widehat{\otimes}_{\overline{B}}/A \widehat{\otimes} B}} \simeq \widehat{\mathrm{L}}\Omega_{R/A}^u \widehat{\otimes}_{\overline{B}} \widehat{\otimes}_{\overline{B}} \overline{B} \{-u\}[-u],$$

where $\overline{B}\{-u\} \simeq (I/I^2)^{\otimes -u}$. In other words,

$$\mathrm{gr}_u^{\delta\mathrm{conj}} \mathcal{H}_{\overline{\Delta}}(R/A) \simeq \widehat{\mathrm{L}\Omega}_{R/A}^u \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{-u\}[-u].$$

The exhaustiveness follows from the fact that $\mathrm{D}(\mathrm{WCart}^{\mathrm{HT}}) \simeq \lim_B \mathrm{D}(\overline{B})_p^\wedge$, where the limit is over all bounded prisms (B, I) . In particular, $\mathrm{colim}_u \mathrm{F}_{\leq u}^{\delta\mathrm{conj}} \mathcal{H}_{\overline{\Delta}}(R/A) \rightarrow \mathcal{H}_{\overline{\Delta}}(R/A)$ is an equivalence since it evaluates to an equivalence in each $\mathrm{D}(\overline{B})_p^\wedge$. \square

Variante 3.10 (Conjugate filtration on Breuil–Kisin twists). There is a conjugate filtration on the i th Breuil–Kisin twist of the Hodge–Tate crystal obtained by tensoring $\mathrm{F}_{\leq \star}^{\delta\mathrm{conj}} \mathcal{H}_{\overline{\Delta}}(R)$ over $\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}$ with $\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{i\}$. The associated graded pieces are

$$\mathrm{gr}_u^{\delta\mathrm{conj}} \mathcal{H}_{\overline{\Delta}}(R/A)\{i\} \simeq \widehat{\mathrm{L}\Omega}_{R/A}^\star \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{i-u\}[-u].$$

Construction 3.11. Taking global sections yields conjugate filtrations

$$\mathrm{F}_{\leq \star}^{\delta\mathrm{conj}} \overline{\Delta}_{R/A}\{i\}$$

for any $i \in \mathbf{Z}$. The associated graded pieces are computed using the Sen operator:

$$\mathrm{gr}_u^{\delta\mathrm{conj}} \overline{\Delta}_{R/A}\{i\} \simeq \widehat{\mathrm{L}\Omega}_{R/A}^u \widehat{\otimes} \mathrm{R}\Gamma(\mathrm{WCart}^{\mathrm{HT}}, \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{i-u\}[-u]) \simeq \widehat{\mathrm{L}\Omega}_{R/A}^u \widehat{\otimes} \mathrm{fib}(\mathbf{Z}_p \xrightarrow{i-u} \mathbf{Z}_p)[-u]$$

by [7, Cor. 3.5.14]. These filtrations are exhaustive because $\mathrm{R}\Gamma(\mathrm{WCart}^{\mathrm{HT}}, -)$ commutes with colimits by [7, Cor. 3.5.13].

Proposition 3.12 (Invariance under quasi-étale extensions for prismatic crystals). *If $(A, R) \rightarrow (A', R)$ is a map of bounded δ -pairs where $\mathrm{L}_{A'/A}$ vanishes p -adically (we call such a map $A \rightarrow A'$ p -adically quasi-étale), then the map*

$$\mathcal{H}_{\overline{\Delta}}^{[n]}(R/A)\{i\} \rightarrow \mathcal{H}_{\overline{\Delta}}^{[n]}(R/A')\{i\}$$

is an equivalence for each n and i .

Proof. This follows directly from the properties of the conjugate filtration, since $\mathrm{L}\Omega_{R/A}^i \simeq \mathrm{L}\Omega_{R/A'}^i$ if $\mathrm{L}_{A'/A} = 0$. \square

Corollary 3.13 (Invariance under localizations and completions).

- (i) *For any bounded δ -pair (A, R) , if $K \subseteq A$ is a finitely generated ideal contained in the kernel of $A \rightarrow R$ and if the derived (p, K) -completion of A is discrete, then the natural map $\mathcal{H}_{\overline{\Delta}}^{[n]}(R/A)\{i\} \rightarrow \mathcal{H}_{\overline{\Delta}}^{[n]}(R_p^\wedge/A_{(p, K)}^\wedge)\{i\}$ is an equivalence for all $i, n \in \mathbf{Z}$.*
- (ii) *For any bounded δ -pair (A, R) , if $S \subseteq A$ is a set of elements which map to units in R , then the natural map $\mathcal{H}_{\overline{\Delta}}^{[n]}(R/A)\{i\} \rightarrow \mathcal{H}_{\overline{\Delta}}^{[n]}(R_p^\wedge, A[S^{-1}]_p^\wedge)\{i\}$ is an equivalence for all $i, n \in \mathbf{Z}$.*

Proof. For (i), let A' be the derived (p, K) -adic completion of A . The δ -ring structure on A extends uniquely to A' by [10, Lem. 2.18]. It follows from the δ -conjugate filtration that it is enough to show that the natural map $\mathrm{L}_{R/A} \rightarrow \mathrm{L}_{R/A'}$ is a p -adic equivalence. This occurs if and only if $R \otimes_{A'} \mathrm{L}_{A'/A}$ vanishes p -adically. As R is K -complete, this happens if and only if $\mathrm{L}_{A'/A}$ vanishes (p, K) -completely, which holds because $A \rightarrow A'$ is a (p, K) -adic equivalence, by definition. From (i), we can assume that A and R are derived p -complete. In particular, it follows that the δ -ring structure on A extends uniquely across $A[S^{-1}]$ by [10, Rem. 2.16]. Now, part (ii) follows from Proposition 3.12. \square

The following is a slightly stronger version of [9, Thm. 3.1], namely descent for differential forms where we also vary the base ring.

Lemma 3.14 (Descent for differential forms). *Let $(R/A) \rightarrow (R^0/A^0)$ be a map of pairs of animated rings where $R \rightarrow R^0$ is p -completely an effective descent morphism and let (R^\bullet, A^\bullet) denote the Čech nerve. Then, the natural map*

$$\widehat{L}\Omega_{R/A}^k \rightarrow \text{Tot } \widehat{L}\Omega_{R^\bullet/A^\bullet}^k$$

is an equivalence.

Proof. We observe that in the diagram of animated graded-commutative rings

$$\begin{array}{ccc} \widehat{L}\Omega_{R/A}^* & \longrightarrow & \widehat{L}\Omega_{R^0/A^0}^* \\ \downarrow & & \downarrow \\ R & \longrightarrow & R^0 \end{array}$$

the left vertical map is an effective descent morphism since the fiber is positively graded, the bottom horizontal map is an effective descent morphism by assumption, and hence it follows that the top map is an effective descent morphism (compare [20, Lem. 3.1.2]). In particular, this implies that

$$\widehat{L}\Omega_{R/A}^* \xrightarrow{\cong} \text{Tot } \widehat{L}\Omega_{R^\bullet/A^\bullet}^*,$$

which degreewise (with respect to the grading given by $*$) implies the claim. \square

Proposition 3.15 (Descent for prismatic crystals). *Let $(A, R) \rightarrow (A^0, R^0)$ be a map of bounded δ -pairs with Čech nerve (A^\bullet, R^\bullet) , and assume that $R \rightarrow R^0$ is faithfully flat, $A \rightarrow A^0$ is flat, and all L_{R^\bullet/A^\bullet} have p -complete Tor-amplitude in $[0, 1]$. Then,*

$$\mathcal{H}_\Delta^{[n]}(R/A)\{i\} \rightarrow \text{Tot } \mathcal{H}_\Delta^{[n]}(R^\bullet/A^\bullet)\{i\}$$

is an equivalence for each n and i .

Proof. Since the $\mathcal{H}_\Delta^{[n]}(R/A)\{i\}$ for varying $n \geq 0$ form a complete filtration on $\mathcal{H}_\Delta(R/A)$, it suffices to check the claim for the associated graded $\mathcal{H}_{\overline{\Delta}}(R/A)\{i\}$. We have that

$$\widehat{L}\Omega_{R/A}^k \rightarrow \text{Tot } \widehat{L}\Omega_{R^\bullet/A^\bullet}^k$$

is an equivalence, by Lemma 3.14. This implies that

$$F_{\leq k}^{\delta\text{conj}} \mathcal{H}_{\overline{\Delta}}(R/A)\{i\} \rightarrow \text{Tot } F_{\leq k}^{\delta\text{conj}} \mathcal{H}_{\overline{\Delta}}(R^\bullet/A^\bullet)\{i\}$$

is an equivalence for each k . Since the Tor-amplitude condition ensures that all of the terms on the right are coconnective, Tot commutes with the colimit over k , leading to the desired statement. \square

Lemma 3.16. *For each u , the functor $L\Omega_{-/-}^u : \text{Pairs}^\delta \rightarrow D(\mathbf{Z})$ commutes with sifted colimits.*

Proof. The forgetful functor $\text{Pairs}^\delta \rightarrow \text{Pairs}$ commutes with all colimits. Thus, it is enough to prove that $L\Omega_{-/-}^u : \text{Pairs} \rightarrow D(\mathbf{Z})$ commutes with sifted colimits. To see this let $(A_i, R_i)_{i \in I}$ be a sifted diagram of pairs

with colimit (A, R) . As $I \rightarrow I^{\Delta^1}$ is cofinal and we have

$$\begin{aligned} \operatorname{colim}_I \mathrm{L}\Omega_{R_i/A_i}^u &\simeq \operatorname{colim}_{I^{\Delta^1}} \mathrm{L}\Omega_{R_i/A_i}^u \otimes_{A_i} A_j \\ &\simeq \operatorname{colim}_I \mathrm{L}\Omega_{R_i/A_i}^u \otimes_{A_i} A \\ &\simeq \operatorname{colim}_I \mathrm{L}\Omega_{R_i \otimes_{A_i} A/A}^u \\ &\simeq \mathrm{L}\Omega_{\operatorname{colim} R_i \otimes_{A_i} A/A}^u \\ &\simeq \mathrm{L}\Omega_{R/A}^u. \end{aligned} \quad \square$$

Corollary 3.17.

(a) *The functors of bounded δ -pairs to $\mathrm{D}(\mathrm{WCart}^{\mathrm{HT}})$ and $\mathrm{D}(\mathbf{Z}_p)_{\widehat{p}}$, respectively, given by*

$$(A, R) \mapsto \mathcal{H}_{\overline{\Delta}}(R/A)\{i\} \quad \text{and} \quad (A, R) \mapsto \overline{\Delta}_{R/A}\{i\}$$

preserves sifted colimits for all $i \in \mathbf{Z}$ and hence are left Kan extended from their values on finitely presented free δ -pairs.

(b) *The functors $\Delta_{-/-}^{[\star]}\{i\}$ from bounded δ -pairs to $\widehat{\mathrm{FD}}(\mathrm{D}(\mathbf{Z}_p)_{\widehat{p}}^{[\star]})$, the ∞ -category of complete filtrations with values in $\mathrm{D}(\mathbf{Z}_p)_{\widehat{p}}$, preserves sifted colimits for all $j \in \mathbf{Z}$.*

Proof. The preservation of sifted colimits in Part (a) follows from the facts that the conjugate filtration on $\mathcal{H}_{\overline{\Delta}}(R/A)\{i\}$ is complete and exhaustive and that the associated graded pieces are left Kan extended by Variant 3.10 and Lemma 3.16. For the fact that every δ -pair is a sifted colimit of free δ -pairs we note that the free δ -pairs are compact, projective objects and that they generate, since mapping out of them is conservative, which is easily seen. Part (b) follows from part (a) since the Hodge–Tate tower on $\Delta_{R/A}\{i\}$ is complete by construction. \square

Remark 3.18 (Hodge–Tate filtration). When restricted to $\star \geq 0$, we view $\Delta_{R/A}^{[\star]}\{i\}$ as a complete decreasing filtration on $\Delta_{R/A}\{i\}$ and refer to this as the Hodge–Tate filtration.

Warning 3.19. It is not the case that $\Delta_{-/-}$ is itself left Kan extended. For example, if A is a prism, then the prismatic comparison of Proposition 5.17 implies that

$$\Delta_{\overline{A}[t_1^{1/p^\infty}, \dots, t_s^{1/p^\infty}]_{\widehat{p}}/A} \simeq A[t_1^{1/p^\infty}, \dots, t_s^{1/p^\infty}]_{(p, I)}^{\widehat{p}}$$

Taking the colimit as $s \rightarrow \infty$ results in a ring whose p -completion is not typically I -complete and hence is not the prismatic cohomology of $\overline{A}[t_1^{1/p^\infty}, \dots, t_s^{1/p^\infty}, \dots]_{\widehat{p}}$ relative to A .

Warning 3.20. While relative Hodge–Tate cohomology is left Kan extended, it does not typically preserve all colimits, although this is the case in the prismatic setting over a fixed prism. For example, absolute Hodge–Tate cohomology $\overline{\Delta}_{-/\mathbf{Z}_p}$ does not preserve pushouts as a functor to p -complete \mathbf{E}_∞ -rings. Indeed, we have $\overline{\Delta}_{\mathbf{Z}_p\langle x \rangle/\mathbf{Z}_p} \simeq \mathbf{Z}_p\langle x \rangle \widehat{\otimes} \overline{\Delta}_{\mathbf{Z}_p/\mathbf{Z}_p}$ as one sees using the conjugate filtration. On the other hand, $\overline{\Delta}_{\mathbf{Z}_p\langle x_1, \dots, x_p \rangle/\mathbf{Z}_p}$ is not equivalent to $\mathbf{Z}_p\langle x_1, \dots, x_p \rangle \widehat{\otimes} \overline{\Delta}_{\mathbf{Z}_p/\mathbf{Z}_p}$, using that $\mathrm{gr}_p^{\delta \mathrm{conj}}$ is non-zero.

Example 3.21 (Absolute prismatic comparison). (a) If $A = \mathbf{Z}_p$, then $\mathcal{H}_{\Delta}(R/\mathbf{Z}_p) \simeq \mathcal{H}_{\Delta}(R)$, the prismatic crystal introduced in [7, Const. 4.4.1], and the conjugate filtration $\mathrm{F}_{\leq \star}^{\delta \mathrm{conj}} \mathcal{H}_{\overline{\Delta}}(R/\mathbf{Z}_p)$ agrees with the conjugate filtration on the absolute Hodge–Tate crystal of [7, Const. 4.5.1].

- (b) More generally, if A is a perfect δ -ring, then L_{A/\mathbf{Z}_p} vanishes after p -completion and hence $\widehat{L\Omega}_{R/A}^* \simeq \widehat{L\Omega}_{R/\mathbf{Z}_p}^*$. It follows that for each $i \in \mathbf{Z}$ the natural map $\mathcal{H}_{\overline{\Delta}}(R)\{i\} \rightarrow \mathcal{H}_{\overline{\Delta}}(R/A)\{i\}$ is an equivalence of quasi-coherent sheaves on $\mathrm{WCart}^{\mathrm{HT}}$ and thus, by \mathcal{J} -completeness, $\mathcal{H}_{\overline{\Delta}}^{[n]}(R)\{i\} \rightarrow \mathcal{H}_{\overline{\Delta}}^{[n]}(R/A)\{i\}$ is an equivalence of quasi-coherent sheaves on WCart for all $n, i \in \mathbf{Z}$. In particular, $\overline{\Delta}_R \simeq \overline{\Delta}_{R/A}$ and $\Delta_R \simeq \Delta_{R/A}$, where Δ_R denotes the absolute prismatic cohomology of [7, Const. 4.4.10], defined as the global sections of the absolute prismatic crystal $\mathcal{H}_{\overline{\Delta}}(R)$. This is a generalization to δ -rings of the relative-to-absolute prismatic cohomology comparison theorem [7, Prop. 4.4.12] for perfect prisms.
- (c) If $A = R$ is derived p -complete, then the conjugate filtration implies that $\mathcal{H}_{\overline{\Delta}}(A/A)\{i\} \simeq A \otimes_{\mathrm{WCart}^{\mathrm{HT}}} \{i\}$ for all $i \in \mathbf{Z}$. Hence, since $\mathrm{R}\Gamma(\mathrm{WCart}^{\mathrm{HT}}, -)$ preserves colimits as a functor to $\widehat{D}(\mathbf{Z}_p)$ by [7, Cor. 3.5.13],

$$\overline{\Delta}_{A/A} \simeq A \widehat{\otimes}_{\mathbf{Z}_p} \overline{\Delta} \simeq A \oplus A[-1].$$

By \mathcal{J} -completeness, the natural map $A \widehat{\otimes}_{\mathrm{WCart}} \overline{\Delta} \rightarrow \mathcal{H}_{\overline{\Delta}}(A/A)$ is an equivalence. Thus, the natural map $A \widehat{\otimes}_{\mathbf{Z}_p} \overline{\Delta} \rightarrow \overline{\Delta}_{A/A}$ is an equivalence after completing the tensor product with respect to the Hodge–Tate filtration on $\overline{\Delta}_{\mathbf{Z}_p}$.

Warning 3.22 (Two conjugate filtrations). As indicated by the notation, the conjugate filtration differs in general from that considered in [10] in the prismatic case. Let $F_{\leq \star}^{\delta \mathrm{conj}} \overline{\Delta}_{R/A}$ denote the filtration defined by using that A is a δ -ring, and let $F_{\leq \star}^{\Delta \mathrm{conj}} \overline{\Delta}_{R/A}^{\mathrm{rel}}$ denote the prismatic conjugate filtration on derived prismatic cohomology, as studied in [10] when A is a prism and R is a p -complete animated commutative \overline{A} -algebra. As proven in Proposition 5.17, $\overline{\Delta}_{R/A}^{\mathrm{rel}} \simeq \overline{\Delta}_{R/A}$, however the conjugate filtrations do not, for example when $A = \mathbf{Z}_p$ and $R = \mathbf{F}_p$. In this case, the δ -ring variant agrees with the absolute conjugate filtration, which has

$$\mathrm{gr}_u^{\delta \mathrm{conj}} \overline{\Delta}_{\mathbf{F}_p/\mathbf{Z}_p} \simeq \mathrm{fib}(\mathrm{L}\Omega_{\mathbf{F}_p/\mathbf{Z}_p}^u \xrightarrow{-u} \mathrm{L}\Omega_{\mathbf{F}_p/\mathbf{Z}_p}^u)[-u] \simeq \mathrm{fib}(\mathbf{F}_p \xrightarrow{-u} \mathbf{F}_p),$$

while

$$\mathrm{gr}_u^{\Delta \mathrm{conj}} \overline{\Delta}_{\mathbf{F}_p/\mathbf{Z}_p}^{\mathrm{rel}} \simeq \mathrm{L}\Omega_{\mathbf{F}_p/\mathbf{F}_p}^u \{-u\}[-u] \simeq \begin{cases} \mathbf{F}_p & \text{if } u = 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The prismatic conjugate spectral sequence degenerates at the E_1 -page, while the δ -ring theoretic conjugate spectral sequence supports d_p -differentials and degenerates at the E_{p+1} -page.

We end this section by giving a different perspective on the definition of $\Delta_{R/A}$ that avoids the theory of prismatic crystals. To this end we let Pairs^{Δ} be the category whose objects are given by pairs consisting of a bounded prism (A, I) together with a map of commutative rings $A/I \rightarrow R$. Note that this is not a full subcategory of Pairs^{δ} since we require the choice and existence of a prismatic ideal I , in contrast to the definition of prismatic δ -pairs (Definition 2.2) where we only required the mere existence. Still there is a forgetful functor

$$\mathrm{Pairs}^{\Delta} \rightarrow \mathrm{Pairs}^{\delta}.$$

Moreover Bhatt–Scholze’s derived prismatic cohomology [10, Sec. 7.2] defines a functor

$$\Delta_{R/A}^{\mathrm{rel}}: \mathrm{Pairs}^{\Delta} \rightarrow \mathrm{D}(\mathbf{Z})_p^{\wedge}.$$

Proposition 3.23. *Prismatic cohomology $\Delta_{R/A}: \mathrm{Pairs}^{\Delta} \rightarrow \mathrm{D}(\mathbf{Z})_p^{\wedge}$ is the right Kan extension of $\Delta_{R/A}^{\mathrm{rel}}$ along $\mathrm{Pairs}^{\Delta} \rightarrow \mathrm{Pairs}^{\delta}$.*

Note that since $\mathrm{Pairs}^{\Delta} \rightarrow \mathrm{Pairs}^{\delta}$ is not fully faithful, it is not a priori clear that the right Kan extension indeed leaves the value on a prismatic δ -pairs unchanged. However, we will show that it does in Section 5.3.

Proof. Let $(A, R) \in \text{Pairs}^\delta$ be a δ -pair. We would like to compute the value of the right Kan extension on (A, R) . This is given by the limit of $\Delta_{S/C}^{\text{rel}}$ where C and S vary over the category $\text{Pairs}_{(A,R)'}^\Delta$, i.e. C has a prism structure $K \subseteq C$ a map of rings $C/K \rightarrow S$ and is equipped with a map of δ -pairs $(A, R) \rightarrow (C, S)$. We now consider the functor

$$\text{TransPrisms} \rightarrow \text{Pairs}_{(A,R)'}^\Delta$$

from the category of transversal prisms given by

$$(B, J) \mapsto (A \widehat{\otimes} B, R \widehat{\otimes} \overline{B}),$$

as in the definition of the prismatic crystal (Definition 3.3). Now to prove the proposition it suffices to show that this functor $\text{TransPrisms} \rightarrow \text{Pairs}_{(A,R)'}^\Delta$ is initial (in the sense that taking limits induces an equivalence, i.e., the opposite of the notion of cofinality from [21, Sec. 4.1.1]), as the limit over the composite of this functor with $\Delta_{R/A}^{\text{rel}}$ is the definition of $\Delta_{R/A}$ (see Remark 3.8). By Quillen's Theorem A (see [21, Thm. 4.1.3.1]), this initiality comes down to checking that for each object

$$(C, S) \in \text{Pairs}_{(A,R)'}^\Delta$$

the category of transversal prisms (B, J) with a map of prism pairs

$$(A \widehat{\otimes} B, R \widehat{\otimes} \overline{B}) \rightarrow (C, S)$$

under (A, R) is weakly contractible. Such a map however is uniquely determined on the part of the tensor product given by A and R so that it is equivalently given by a map of prismatic δ -pairs

$$(B, \overline{B}) \rightarrow (C, S)$$

or equivalently a map of prisms $(B, J) \rightarrow (C, K)$ since the map $\overline{B} \rightarrow S$ is uniquely determined. Thus, the category is equivalent to the category of transversal prisms over (C, S) which is weakly contractible since it is sifted [7, Cor. 2.4.7]. \square

We note that the proof actually shows something stronger, namely that one can even right Kan extend from transversal prisms instead of all prisms, but we shall not need this here.

4 Prismaticization

The goal of prismaticization as developed by Drinfeld [11] and Bhatt–Lurie [7, 8] is to construct for each p -adic formal scheme X a formal stack (in groupoids on the site of p -nilpotent affine schemes with the flat topology), called WCart_X , such that (1) quasi-coherent sheaves on X provide a good formalism for coefficients for prismatic cohomology and (2) $\text{R}\Gamma(\text{WCart}_X, \mathcal{O}_{\text{WCart}_X})$ computes Δ_X , the absolute (derived) prismatic cohomology of X . There is also a relative version, denoted by $\text{WCart}_{X/A}$ when A is a prism, introduced in [8]. For more on formal (higher) stacks, see Appendix A.

Definition 4.1 (Cartier–Witt divisors). Recall from [7, Def. 3.1.4] that if S is a p -nilpotent commutative ring, then a Cartier–Witt divisor of S is a generalized Cartier divisor $\alpha: I \rightarrow W(S)$ such that the image of $I \xrightarrow{\alpha} W(S) \rightarrow S$ is nilpotent and the image of $I \xrightarrow{\alpha} W(S) \xrightarrow{\delta} W(S)$ generates the unit ideal of $W(S)$.

Definition 4.2 (Absolute Cartier–Witt stacks). Let R be a commutative ring with bounded p -power torsion. Define WCart_R to be the formal stack which assigns to a p -nilpotent ring S the groupoid of pairs $(I \xrightarrow{\alpha} W(S), R \rightarrow \overline{W(S)})$, where α is a Cartier–Witt divisor of S and $\overline{W(S)} = W(S)//I$ is the animated commutative ring with underlying chain complex given by the cofiber of α ; see [8, Sec. 3].

Morally one should think of an S -point of WCart_R as making $W(S)$ an object of the animated version of the absolute prismatic site of R using [8, Remark 3.1.5]. The stack $\mathrm{WCart}_{R/A}$ that we define below similarly has S points given by ways of making $W(S)$ an object of the animated version of the relative prismatic site of R relative to A .

Remark 4.3 (Insensitivity to derived p -completion). As $\pi_0 \overline{W(S)}$ is a p -nilpotent ring by [8, Lem. 3.3], WCart_R depends only on the derived p -completion of R .

Remark 4.4 (Extension to stacks). We will need the natural extension of WCart_- to formal stacks. If \mathcal{F} is any presheaf of spaces on p -nilpotent affine schemes, we can define a presheaf $\mathrm{WCart}_{\mathcal{F}}$ as follows: for any p -nilpotent ring S , we let $\mathrm{WCart}_{\mathcal{F}}(S)$ be the space of pairs of a Cartier–Witt divisor $\alpha: I \rightarrow W(S)$ and a $\overline{W(S)}$ -point of \mathcal{F} . With this notation, $\mathrm{WCart}_R = \mathrm{WCart}_{\mathrm{Spf} R}$ if R is bounded. The natural map $\mathrm{colim}_{\mathrm{Spf} R \rightarrow \mathcal{F}} \mathrm{WCart}_R \rightarrow \mathrm{WCart}_{\mathcal{F}}$ is an equivalence of presheaves of spaces, where the colimit ranges over maps from p -nilpotent affine schemes.

Notation 4.5. If (A, I) is a bounded prism, we distinguish between the p -adic formal scheme $\mathrm{Spf} A$ and the object $\mathrm{Spf} A$. If S is a p -nilpotent commutative ring, then $(\mathrm{Spf} A)(S) = \mathrm{colim}_m \mathrm{Map}_{\mathrm{CAlg}}(A/p^m, S)$, while $(\mathrm{Spf} A)(S) = \mathrm{colim}_{m,n} \mathrm{Map}_{\mathrm{CAlg}}(A/(p^m, I^n), S)$. Alternatively, $\mathrm{Spf} A \subseteq \mathrm{Spf} A$ is the formal completion at I , corresponding to maps $A \rightarrow S$ where I is sent to a nilpotent ideal.

Definition 4.6 (Relative Cartier–Witt stacks). If A is a bounded prism, then there is a natural map $\mathrm{Spf} A \rightarrow \mathrm{WCart}_{\overline{A}}$. If S is a p -nilpotent commutative ring and $g: A \rightarrow S$ is a point of $\mathrm{Spf} A$, so that the image of I in S is nilpotent, then we can take the adjoint δ -ring map $g^\#: A \rightarrow W(S)$ and take the Cartier–Witt divisor $I \otimes_A W(S) \rightarrow W(S)$ obtained by extension of scalars. If R is a commutative \overline{A} -algebra with bounded p -power torsion, then Bhatt and Lurie define the relative Cartier–Witt stack $\mathrm{WCart}_{R/A}$ in [8, Variant 5.1] as the pullback

$$\begin{array}{ccc} \mathrm{WCart}_{R/A} & \longrightarrow & \mathrm{WCart}_R \\ \downarrow & & \downarrow \\ \mathrm{Spf} A & \longrightarrow & \mathrm{WCart}_{\overline{A}}. \end{array}$$

Example 4.7. If B is a bounded prism, then the natural map

$$\mathrm{WCart}_{\overline{B}/B} \rightarrow \mathrm{Spf} B$$

is an equivalence.

Construction 4.8 ([8, Const. 3.11]). If A is a bounded δ -ring, there is a functor $\mathrm{Spf} A \times \mathrm{WCart} \rightarrow \mathrm{WCart}_A$ obtained as follows. Given a p -nilpotent commutative ring R and an R -point of $\mathrm{Spf} A \times \mathrm{WCart}$, which consists of a pair consisting of a map $A \rightarrow R$ and a Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$ over R , one uses the universal property of the Witt vectors to obtain a δ -ring map $A \rightarrow W(R)$. The composition $A \rightarrow W(R) \rightarrow W(R)//I$ together with (I, α) defines an R -point of WCart_A .

Definition 4.9 (Prismaticization of δ -pairs). Suppose that $A \rightarrow R$ is a bounded δ -pair. We define the relative Cartier–Witt stack of R over A as the pullback

$$\begin{array}{ccc} \mathrm{WCart}_{R/A} & \longrightarrow & \mathrm{WCart}_R \\ \downarrow & & \downarrow \\ \mathrm{Spf} A \times \mathrm{WCart} & \longrightarrow & \mathrm{WCart}_A. \end{array}$$

In other words, if S is a p -nilpotent commutative ring, $\mathrm{WCart}_{R/A}(S)$ is the space consisting of quadruples of a Cartier–Witt divisor $I \xrightarrow{\alpha} W(S)$, a map $R \rightarrow W(S)//I$ of animated commutative rings, a map $A \rightarrow S$ of commutative rings, and an equivalence between the compositions $A \rightarrow W(S) \rightarrow W(S)//I$ and $A \rightarrow R \rightarrow W(S)//I$. More generally, one can make an analogous definition when X is a bounded p -adic formal scheme over $\mathrm{Spf} A$ to obtain $\mathrm{WCart}_{X/A}$.

This definition agrees with the definition given in [8] when both make sense.

Proposition 4.10 (Agreement with prismaticization in the prismatic case). *If A is a bounded prism and R is a bounded \bar{A} -algebra, then there is a natural commutative diagram*

$$\begin{array}{ccc} \mathrm{WCart}_{R/A} & \longrightarrow & \mathrm{WCart}_R \\ \downarrow & & \downarrow \\ \mathrm{Spf} A & \xrightarrow{\rho_A} & \mathrm{WCart}_{\bar{A}} \\ \mathrm{id} \times \rho_A \downarrow & & \downarrow \\ \mathrm{Spf} A \times \mathrm{WCart} & \longrightarrow & \mathrm{WCart}_A \end{array}$$

of pullback squares where the outer square agrees with the defining square of Definition 4.9.

In particular, the definition given in Definition 4.9 of $\mathrm{WCart}_{R/A}$ agrees with that of [8, Const. 7.1] for prismatic δ -pairs.

Proof. The pullback of the top square defines the relative Cartier–Witt stack in the prismatic case by definition [8, Var. 5.1, Const. 7.1]. It is enough to identify the bottom pullback as $\mathrm{Spf} A$; write P for this pullback. Evaluated at a p -nilpotent commutative ring S , $P(S)$ is the space consisting of

- (i) a Cartier–Witt divisor $I \xrightarrow{\alpha} W(S)$ equipped with a map $f: \bar{A} \rightarrow W(S)//I$,
- (ii) a map $g: A \rightarrow S$ and a Cartier–Witt divisor $J \xrightarrow{\beta} W(S)$, and
- (iii) an equivalence $(I, \alpha) \simeq (J, \beta)$ of Cartier–Witt divisors on S and an equivalence of maps between $A \xrightarrow{g^\#} W(S) \rightarrow W(S)//J$ and $A \rightarrow \bar{A} \xrightarrow{f} W(S)//I$, where $g^\#$ is the map of δ -rings adjoint to g .

This data determines and is determined by a map $A \rightarrow S$ which sends I to a nilpotent ideal of S by rigidity for maps between prisms [10, Lem. 3.59]. So, the pullback P is equivalent to $\mathrm{Spf} A$, as desired. \square

Lemma 4.11 (Insensitivity to I -adic completion). *Suppose that $A \rightarrow R$ is a bounded p -complete δ -pair and that the kernel contains an ideal I which is locally generated by a distinguished element which is a non-zero divisor. Let A_I^\wedge denote the I -adic completion of A , which is a prism. Then, the natural map*

$$\mathrm{WCart}_{R/A_I^\wedge} \rightarrow \mathrm{WCart}_{R/A}$$

is an equivalence.

Proof. If S is a p -nilpotent commutative ring and x is a point of $\mathrm{WCart}_{R/A}$, corresponding to the data of a map $A \xrightarrow{g} S$, a Cartier–Witt divisor $J \xrightarrow{\alpha} W(S)$, a map $R \rightarrow W(S)/J$, and a specified equivalence between $A \xrightarrow{g^\#} W(S) \rightarrow W(S)/J$ and $A \rightarrow R \rightarrow W(S)/J$, then it follows from commutativity that $g^\#$ sends the ideal $I \subseteq A$ into J . On the other hand, J maps to a nilpotent ideal of S by definition of Cartier–Witt divisors. It follows that g and $g^\#$ factor uniquely through the I -adic completion of A and hence that the map on Cartier–Witt stacks is an equivalence. \square

Remark 4.12. In fact, in Lemma 4.11, we do not need I to be locally generated by a distinguished element or for A_I^\wedge to be a prism. We need only for I to be some ideal such that the derived I -adic completion of A is discrete. However, we will only apply the lemma in the prismatic situation.

Variante 4.13 (Derived (animated) Cartier–Witt stacks). Suppose that R is an animated commutative ring. The derived Cartier–Witt stack $\mathrm{WCart}_R^{\mathrm{an}}$ of R is defined to be the functor on p -nilpotent animated commutative rings S with value the space consisting of pairs of an animated prism structure $I \xrightarrow{\alpha} W(S)$ such that the image of $\pi_0(I) \rightarrow \pi_0(W(S)) \rightarrow \pi_0(S)$ is nilpotent together with a map $R \rightarrow W(S) = \mathrm{cofib}(\alpha)$.

Let $A \rightarrow R$ be an animated δ -pair. There is a natural extension of Construction 4.8 to a map $\mathrm{Spf} A \times \mathrm{WCart}^{\mathrm{an}} \rightarrow \mathrm{WCart}_A^{\mathrm{an}}$ by using the universal property [8, Prop. A.23] of the Witt vectors for animated commutative rings as the cofree animated δ -ring functor. Define $\mathrm{WCart}_{R/A}^{\mathrm{an}}$ as the pullback (in presheaves of spaces on derived schemes) of $\mathrm{Spf} A \times \mathrm{WCart}^{\mathrm{an}} \rightarrow \mathrm{WCart}_A^{\mathrm{an}}$ along $\mathrm{WCart}_R^{\mathrm{an}} \rightarrow \mathrm{WCart}_A^{\mathrm{an}}$.

If A is a bounded prism and R is an animated commutative \bar{A} -algebra, then the proof of Proposition 4.10 can be extended to the animated situation to give an equivalence between $\mathrm{WCart}_{R/A}^{\mathrm{an}}$, as defined above, and the definition given in [8, Const. 7.1].

Lemma 4.14 (The Cartier–Witt stacks are sheaves). *If $A \rightarrow R$ is a bounded δ -pair, then $\mathrm{WCart}_{R/A}$ is a sheaf for the flat topology on p -nilpotent commutative rings.*

Proof. Compare the proof of [8, Lem. 7.3]. By definition of $\mathrm{WCart}_{R/A}$ as a pullback, it is enough to show that WCart is a flat sheaf and that WCart_A is a flat sheaf for any commutative ring A with bounded p -torsion. The functor WCart is a flat sheaf by flat descent for generalized Cartier divisors, which can be seen as follows (see also Remark 3.1.7 in [7]): the construction $S \mapsto W(S)$ satisfies flat descent. The category of invertible modules satisfies flat descent, the conditions of being nilpotent satisfies flat descent (since for every exponent it can be written as the vanishing of a certain module map), and finally the prism condition can be checked after base change along a faithfully flat map (as one sees by a direct verification).

Suppose that $G \rightarrow F$ is a map of presheaves of spaces on some site. If F is a sheaf of spaces and if the fibers of $G \rightarrow F$, computed as presheaves, are in fact sheaves, then G is a sheaf. Thus, it suffices to show that the fibers of $\mathrm{WCart}_A \rightarrow \mathrm{WCart}$ are flat sheaves.

Let $S \rightarrow S^0$ be a faithfully flat map of p -nilpotent commutative rings with Čech nerve S^\bullet . If $I \xrightarrow{\alpha} W(S) \in \mathrm{WCart}(S)$ is fixed, then

$$\overline{W(S)} \rightarrow \mathrm{Tot} \overline{W(S^\bullet)}$$

is an equivalence since $W(S) \rightarrow \mathrm{Tot} W(S^\bullet)$ is an equivalence and I is a perfect $W(S)$ -module. But, $\mathrm{Map}(A, \overline{W(S)}) \simeq \mathrm{Tot} \mathrm{Map}(A, \overline{W(S^\bullet)})$. It follows that the fiber over α of $\mathrm{WCart}_A \rightarrow \mathrm{WCart}$ is a flat sheaf, as desired. \square

Definition 4.15 (Prismatic cohomology via Cartier–Witt stacks). For a bounded δ -pair (A, R) , let

$$\Delta_{R/A}^{\mathrm{wc}} = \mathrm{R}\Gamma(\mathrm{WCart}_{R/A}, \mathcal{O}_{\mathrm{WCart}_{R/A}}).$$

There are Breuil–Kisin twists $\mathcal{O}_{\mathrm{WCart}_{R/A}}\{i\}$ on $\mathrm{WCart}_{R/A}$ obtained by pullback from WCart ; there is also a quasi-coherent sheaf of ideals \mathcal{J} obtained by pullback from the Hodge–Tate ideal on WCart . Thus, we can also define $\Delta_{R/A}^{\mathrm{wc},[n]}\{i\} = \mathrm{R}\Gamma(\mathrm{WCart}_{R/A}, \mathcal{J}^{\otimes n} \otimes_{\mathcal{O}_{\mathrm{WCart}_{R/A}}} \mathcal{O}_{\mathrm{WCart}_{R/A}}\{i\})$ and the Hodge–Tate tower $\Delta_{R/A}^{\mathrm{wc},[*]}\{i\}$. The associated graded pieces of the Hodge–Tate tower fit into fiber sequences

$$\Delta_{R/A}^{\mathrm{wc},[n+1]}\{i\} \rightarrow \Delta_{R/A}^{\mathrm{wc},[n]}\{i\} \rightarrow \overline{\Delta}_{R/A}^{\mathrm{wc}}\{i+n\}.$$

Definition 4.16 (Relative Hodge–Tate stacks). Pulling back $\mathrm{WCart}^{\mathrm{HT}} \hookrightarrow \mathrm{WCart}$ along $\mathrm{WCart}_{R/A} \rightarrow \mathrm{WCart}$ produces a relative Hodge–Tate stack $\mathrm{WCart}_{R/A}^{\mathrm{HT}}$. By construction,

$$\mathrm{R}\Gamma(\mathrm{WCart}_{R/A}^{\mathrm{HT}}, \mathcal{O}_{\mathrm{WCart}_{R/A}^{\mathrm{HT}}}\{i\}) \simeq \overline{\Delta}_{R/A}^{\mathrm{wc}}\{i\} = \mathrm{cofib} \left(\Delta_{R/A}^{\mathrm{wc},[1]}\{i\} \rightarrow \Delta_{R/A}^{\mathrm{wc},[0]}\{i\} \right).$$

Example 4.17. In the situation of Proposition 4.10, when A is a bounded prism and $R = \overline{A}$, one has a natural equivalence $\mathrm{WCart}_{\overline{A}/A}^{\mathrm{HT}} \simeq \mathrm{Spf} \overline{A}$.

Variante 4.18. If $A \rightarrow R$ is an animated δ -pair, we can define $\Delta_{R/A}^{\mathrm{anwc}} = \mathrm{R}\Gamma(\mathrm{WCart}_{R/A}^{\mathrm{an}}, \mathcal{O}_{\mathrm{WCart}_{R/A}^{\mathrm{an}}})$ as well as Breuil–Kisin twists and so on. If R/A is a bounded δ -pair, then the classical locus of $\mathrm{WCart}_{R/A}^{\mathrm{an}}$ is $\mathrm{WCart}_{R/A}$ by definition, so there is a natural map $\Delta_{R/A}^{\mathrm{anwc}} \rightarrow \Delta_{R/A}^{\mathrm{wc}}$.

Before comparing different definitions of prismatic cohomology, we develop some of the theory of the prismaticization of δ -pairs.

Lemma 4.19 (Compatibility with limits). *Suppose that $\{X_i \rightarrow \mathrm{Spf} A_i\}_{i \in I}$ is a natural transformation of I -indexed diagrams of bounded p -adic formal schemes. Assume that the A_i are equipped with δ -ring structures and the maps in the diagram are δ -ring maps. If the diagrams $\{X_i\}_{i \in I}$ and $\{\mathrm{Spf} A_i\}_{i \in I}$ are Tor-independent, meaning that their limit in p -adic formal schemes agrees with their limit in derived p -adic formal schemes, then letting $X = \lim X_i$ and $\mathrm{Spf} A = \lim \mathrm{Spf} A_i$ the natural map*

$$\mathrm{WCart}_{X/A} \rightarrow \lim_I \mathrm{WCart}_{X_i/A_i}$$

is an equivalence.

Proof. Indeed, each of the other vertices in the definition of WCart_{X_*/A_*} in Definition 4.6 is compatible with limits by [8, Rem. 3.5]. \square

The following lemma is similar to, but easier than, [8, Lem. 6.3].

Lemma 4.20. *Suppose that $A \rightarrow A^0$ is a map of bounded δ -rings and R is a bounded commutative A^0 -algebra. If $A \rightarrow A^0$ is p -completely faithfully flat, then the natural map*

$$\mathrm{WCart}_{R/A^0} \rightarrow \mathrm{WCart}_{R/A}$$

is surjective in the flat topology.

Proof. Let $R^0 = A^0 \widehat{\otimes}_A R$. Then, because R is already a commutative A^0 -algebra, there are maps of δ -pairs $(A, R) \rightarrow (A^0, R) \rightarrow (A^0, R^0)$. To prove that $\mathrm{WCart}_{R/A^0} \rightarrow \mathrm{WCart}_{R/A}$ is surjective, it is thus enough to prove that $\mathrm{WCart}_{R^0/A^0} \rightarrow \mathrm{WCart}_{R/A}$ is surjective. However, this morphism is the top arrow of a pullback diagram

$$\begin{array}{ccc} \mathrm{WCart}_{R^0/A^0} & \longrightarrow & \mathrm{WCart}_{R/A} \\ \downarrow & & \downarrow \\ \mathrm{Spf} A^0 \times \mathrm{WCart} & \longrightarrow & \mathrm{Spf} A \times \mathrm{WCart}. \end{array}$$

The bottom arrow is surjective by hypothesis, and hence the top one is too. \square

Corollary 4.21 (Prismaticization preserves quasisyntomic covers). *Suppose that $(A, R) \rightarrow (A^0, R^0)$ is a p -completely faithful flat map of bounded δ -pairs, meaning that $A \rightarrow A^0$ and $R \rightarrow R^0$ are p -completely faithfully flat. If $L_{R^0/R}$ has p -complete Tor-amplitude in $[0, 1]$, then the natural forgetful map*

$$\mathrm{WCart}_{R^0/A^0} \rightarrow \mathrm{WCart}_{R/A}$$

is surjective in the flat topology.

Proof. The map in question factors as $\mathrm{WCart}_{R^0/A^0} \rightarrow \mathrm{WCart}_{R^0/A} \rightarrow \mathrm{WCart}_{R/A}$. The first map is surjective by Lemma 4.20. The second map is the left vertical map in the cartesian diagram

$$\begin{array}{ccc} \mathrm{WCart}_{R^0/A} & \longrightarrow & \mathrm{WCart}_{R^0} \\ \downarrow & & \downarrow \\ \mathrm{WCart}_{R/A} & \longrightarrow & \mathrm{WCart}_R. \end{array}$$

The right vertical map is surjective by [8, Lem. 6.3], which applies by our assumption on $L_{R^0/R}$. Thus, the left vertical map is surjective. \square

5 Comparison theorems

If $A \rightarrow R$ is a bounded δ -pair, then there is a natural commutative diagram of maps between the various forms of prismatic cohomology of R relative to A constructed in the previous sections:

$$\begin{array}{ccccc} \Delta_R & \xleftarrow[\text{Ex. 3.21}]{A=\mathbf{Z}_p} & \Delta_{R/A} & \xrightarrow[\text{Prop. 5.17}]{\text{prismatic}} & \Delta_{R/A}^{\text{rel}} \\ & & \downarrow \text{Cor. 5.5} & & \\ \Delta_{R/A}^{\text{anwc}} & \xrightarrow{\text{Lem. 5.16}} & \Delta_{R/A}^{\text{wc}} & & \\ \downarrow & & \downarrow \text{Thm. 5.6} & & \\ \Delta_{R/A}^{\text{ansite}} & \xrightarrow{\text{Var. 5.15}} & \Delta_{R/A}^{\text{site}} & & \end{array}$$

The lower horizontal arrows are constructed using the inclusion of discrete commutative rings into animated commutative rings; see Variants 2.16 and 4.18. The vertical arrows as well as the upper horizontal arrow are constructed below. The top right arrow exists in the case of a bounded prismatic δ -pair and the top left arrow exists, and is an equivalence, by Example 3.21.

We prove that these maps, as well as the corresponding maps on Breuil–Kisin twists and Hodge–Tate towers, are equivalences for a large class of δ -pairs satisfying quasisyntomicity conditions.

5.1 Stack and crystal

We use the base change results of the appendix to identify the pushforward of the structure sheaf along $\mathrm{WCart}_{R/A} \rightarrow \mathrm{WCart}$.

Proposition 5.1 (Crystal comparison). *Given a bounded δ -pair (A, R) where $L_{R/A}$ has p -complete Tor-amplitude in $[0, 1]$, the pushforward of the Breuil–Kisin twisted structure sheaf $\mathcal{J}_{\mathrm{WCart}_{R/A}}^{\otimes n} \otimes \mathcal{O}_{\mathrm{WCart}_{R/A}}\{i\}$ along the composition $\mathrm{WCart}_{R/A} \rightarrow \mathrm{Spf} A \times \mathrm{WCart} \xrightarrow{\mathrm{pr}_2} \mathrm{WCart}$ is naturally equivalent to $\mathcal{H}_{\Delta}^{[n]}(R/A)\{i\}$ for each $n, i \in \mathbf{Z}$.*

Proof. First, we handle the case where $n = i = 0$. Consider the pullback diagram

$$\begin{array}{ccc} \mathrm{WCart}_{R \widehat{\otimes} B / A \widehat{\otimes} B} & \longrightarrow & \mathrm{WCart}_{R/A} \\ \downarrow q' & & \downarrow q \\ \mathrm{Spf} B & \xrightarrow{\rho_B} & \mathrm{WCart}, \end{array}$$

where (B, I) is a transversal prism and where the upper-left vertex is identified using Tor-independence of $(B, \bar{B}) \leftarrow (\mathbf{Z}_p, \mathbf{Z}_p) \rightarrow (A, R)$ and Lemma 4.19 to obtain identifications

$$\mathrm{Spf} B \times_{\mathrm{WCart}} \mathrm{WCart}_{R/A} \simeq \mathrm{WCart}_{\bar{B}/B} \times_{\mathrm{WCart}_{\mathbf{Z}_p/\mathbf{Z}_p}} \mathrm{WCart}_{R/A} \simeq \mathrm{WCart}_{R \widehat{\otimes} \bar{B}/A \widehat{\otimes} B}.$$

Note that the p -completed tensor product $A \widehat{\otimes} B$ is possibly not a prism so that $\mathrm{WCart}_{R \widehat{\otimes} \bar{B}/A \widehat{\otimes} B}$ is not by definition the relative Cartier–Witt stack studied in [8]. However, the natural map $\mathrm{WCart}_{R \widehat{\otimes} \bar{B}/(A \widehat{\otimes} B)^\wedge} \rightarrow \mathrm{WCart}_{R \widehat{\otimes} \bar{B}/A \widehat{\otimes} B}$ is an equivalence by Lemma 4.11. It follows that

$$q_* \mathcal{O}_{\mathrm{WCart}_{R/A}}(B) = \rho_B^* q_* \mathcal{O}_{\mathrm{WCart}_{R/A}} \simeq q'_* \mathcal{O}_{\mathrm{WCart}_{R \widehat{\otimes} \bar{B}/(A \widehat{\otimes} B)^\wedge}} \simeq \mathrm{R}\Gamma(\mathrm{WCart}_{R \widehat{\otimes} \bar{B}/(A \widehat{\otimes} B)^\wedge}, \mathcal{O}_{\mathrm{WCart}_{R \widehat{\otimes} \bar{B}/(A \widehat{\otimes} B)^\wedge}})$$

by base change for bounded above quasi-coherent cohomology; see Corollary A.41. The latter is equivalent to $\Delta_{R \widehat{\otimes} \bar{B}/A \widehat{\otimes} B}^{\mathrm{rel}}$ under our p -complete Tor-amplitude hypothesis by [8, Thm. 6.4, Rem. 7.23]. This identification is natural in B , so the lemma follows for $i = 0$.

For general n, i , we use the fact that $\mathcal{O}_{\mathrm{WCart}_{R/A}}\{i\} \simeq q^* \mathcal{O}_{\mathrm{WCart}}\{i\}$ and $\mathcal{J}_{\mathrm{WCart}_{R/A}}^{\otimes n} \simeq q^* \mathcal{J}_{\mathrm{WCart}}^{\otimes n}$, so the projection formula (see Proposition A.37) implies that $q_*(\mathcal{J}^{\otimes n} \otimes \mathcal{O}_{\mathrm{WCart}_{R/A}}\{i\}) \simeq \mathcal{J}^{\otimes n} \otimes q_*(\mathcal{O}_{\mathrm{WCart}_{R/A}}\{i\}) \simeq \mathcal{H}_\Delta^{[n]}(R/A)\{i\}$. \square

Remark 5.2. As $\mathcal{H}_\Delta(R/A)$ is naturally an A -module in $\mathrm{D}(\mathrm{WCart})$, it can be considered as a quasi-coherent sheaf on $\mathrm{Spf} A \times \mathrm{WCart}$. The proposition shows in fact that it agrees with the pushforward of $\mathcal{O}_{\mathrm{WCart}_{R/A}}$ along the canonical map $\mathrm{WCart}_{R/A} \rightarrow \mathrm{Spf} A \times \mathrm{WCart}$.

Variation 5.3. If (A, R) is a δ -pair where A is bounded and R is an animated commutative A -algebra such that $\Omega_{\pi_0(R/p)/A/p}^1$ is finitely generated over $\pi_0(R)/p$ and if (B, I) is a transversal prism, then Bhatt and Lurie show in [8, Thm. 7.20] that $\Delta_{R \widehat{\otimes} \bar{B}/(A \widehat{\otimes} B)^\wedge} \simeq \mathrm{R}\Gamma(\mathrm{WCart}_{R \widehat{\otimes} \bar{B}/(A \widehat{\otimes} B)^\wedge}^{\mathrm{an}}, \mathcal{O}_{\mathrm{WCart}_{R \widehat{\otimes} \bar{B}/(A \widehat{\otimes} B)^\wedge}^{\mathrm{an}}})$. Thus, it follows from the argument in Proposition 5.1, and an appeal to Variant 4.13, that $q_* \mathcal{O}_{\mathrm{WCart}_{R/A}^{\mathrm{an}}} \simeq \mathcal{H}_\Delta(R/A)$.

Corollary 5.4. *Given a bounded δ -pair (A, R) where $L_{R/A}$ has p -complete Tor-amplitude in $[0, 1]$, the pushforward $q_* \mathcal{O}_{\mathrm{WCart}_{R/A}^{\mathrm{HT}}}\{i\}$ along $q: \mathrm{WCart}_{R/A}^{\mathrm{HT}} \rightarrow \mathrm{WCart}^{\mathrm{HT}}$ is naturally equivalent to $\iota^* \mathcal{H}_\Delta(R/A)\{i\}$ for each i , where $\iota: \mathrm{WCart}^{\mathrm{HT}} \hookrightarrow \mathrm{WCart}$.*

Proof. We have to show that the canonical map

$$\iota^* \mathcal{H}_\Delta(R/A)\{i\} \rightarrow q_* \mathcal{O}_{\mathrm{WCart}_{R/A}^{\mathrm{HT}}}\{i\}$$

is an equivalence. This can be verified after applying ι_* in which case both sides can be identified with the cofiber of the map $\mathcal{H}_\Delta^{[1]}(R/A)\{i\} \rightarrow \mathcal{H}_\Delta(R/A)\{i\}$ by Proposition 5.1. \square

Corollary 5.5. *If (A, R) is a bounded δ -pair where $L_{R/A}$ has p -complete Tor-amplitude in $[0, 1]$, then there are natural equivalences $\Delta_{R/A}^{[n]}\{i\} \simeq \Delta_{R/A}^{\mathrm{wc}, [n]}\{i\}$ and $\bar{\Delta}_{R/A}\{i\} \simeq \bar{\Delta}_{R/A}^{\mathrm{wc}}\{i\}$ for each $i, n \in \mathbf{Z}$.*

Proof. Take global sections in Proposition 5.1 and Corollary 5.4. \square

5.2 Stack and site

If (A, R) is a bounded δ -pair and $(B, I) \in (R/A)_\Delta$, then by functoriality of the relative Cartier–Witt stacks there is a canonical morphism $\mathrm{WCart}_{\bar{B}/B} \rightarrow \mathrm{WCart}_{R/A}$, which induces a map $\Delta_{R/A}^{\mathrm{wc}} \rightarrow \Delta_{\bar{B}/B}^{\mathrm{wc}} \simeq B$. These assemble into a natural morphism

$$\Delta_{R/A}^{\mathrm{wc}} \rightarrow \Delta_{R/A}^{\mathrm{site}}$$

by definition of site-theoretic relative prismatic cohomology. This natural map extends to a natural transformation on Hodge–Tate towers and Breuil–Kisin twists.

Theorem 5.6 (Stack-site comparison). *Let (A, R) be a bounded δ -pair. If $L_{R/A}$ has p -complete Tor-amplitude in $[0, 1]$, then the natural map*

$$\Delta_{R/A}^{[n], \text{wc}} \{i\} \rightarrow \Delta_{R/A}^{[n], \text{site}} \{i\}$$

is an equivalence for all $n, i \in \mathbf{Z}$.

We will need a few preliminary results. Our strategy is to show that both sides satisfy a restricted form of quasisyntomic descent, which lets us reduce to the case of pairs (A, R) where A is a prism and R is a commutative \overline{A} -algebra, i.e., the situation studied in [10]. In that case, the comparison theorem will follow from [8].

Lemma 5.7 (Rezk). *Given δ -rings A, A', B , and maps $A \rightarrow A' \rightarrow B$ such that*

- (1) *the map $A \rightarrow A'$ is p -completely formally étale and a δ -ring map and*
- (2) *the composite $A \rightarrow B$ is a δ -ring map,*

the map $A' \rightarrow B$ is also a δ -ring map.

Proof. Consider the diagram

$$\begin{array}{ccccc} A & \longrightarrow & A' & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ W_2(A) & \longrightarrow & W_2(A') & \longrightarrow & W_2(B) \\ & & \downarrow & & \downarrow \\ & & A' & \longrightarrow & B \end{array}$$

where the upper vertical maps are the maps encoding the δ -structure, and the lower vertical maps are the canonical projections. The lower square commutes trivially, the upper left square commutes by the assumption that $A \rightarrow A'$ is a δ -ring map, the upper outer square commutes by the assumption that $A \rightarrow B$ is a δ -ring map, and the right outer square commutes trivially. To show that $A' \rightarrow B$ is a δ -ring map, we need to show that the upper right square commutes, i.e. that the two composites $A' \rightarrow W_2(B)$ agree. By assumption, they fit into a commutative square

$$\begin{array}{ccc} A & \longrightarrow & W_2(B) \\ \downarrow & \nearrow & \downarrow \\ A' & \longrightarrow & B \end{array},$$

and by the uniqueness part of lifting formally étale extensions along square-zero maps, they agree. \square

Lemma 5.8 (Insensitivity to formally étale extensions I). *Given a sequence $A \rightarrow A' \rightarrow R$, where $A \rightarrow A'$ is a map of δ -rings which is p -completely formally étale, the natural map $\text{WCart}_{R/A'} \rightarrow \text{WCart}_{R/A}$ is an equivalence. In particular, the natural maps*

$$\Delta_{R/A}^{[n], \text{wc}} \{i\} \rightarrow \Delta_{R/A'}^{[n], \text{wc}} \{i\}$$

are equivalences for all $n, i \in \mathbf{Z}$.

Proof. Consider the diagram

$$\begin{array}{ccc}
\mathrm{WCart}_{R/A'} & \longrightarrow & \mathrm{WCart}_R \\
\downarrow & & \downarrow \\
\mathrm{Spf} A' \times \mathrm{WCart} & \longrightarrow & \mathrm{WCart}_{A'} \\
\downarrow & & \downarrow \\
\mathrm{Spf} A \times \mathrm{WCart} & \longrightarrow & \mathrm{WCart}_A.
\end{array}$$

The upper diagram is a pullback square by definition. If we show that the lower square is a pullback diagram, it follows that the outer diagram is a pullback, and thus $\mathrm{WCart}_{R/A} \simeq \mathrm{WCart}_{R/A'}$.

Let P denote the pullback of the lower square; there is a canonical map $\mathrm{Spf} A' \times \mathrm{WCart} \rightarrow P$. An S -point in P is given by a map $A \rightarrow S$, a generalized Cartier-Witt divisor $I \rightarrow W(S)$, and a factorization of the composite map $A \rightarrow W(S)/I$ through A' .

Since $A \rightarrow A'$ is p -completely formally étale, this factorization lifts uniquely to a map $A' \rightarrow W(S)$ agreeing with the δ -ring map $A \rightarrow W(S)$ on A . This map is automatically a δ -ring map by Lemma 5.7. Since $W(S)$ is the cofree δ -ring on S , this is the same datum as a map $A' \rightarrow S$ extending the original map $A \rightarrow S$. So a point in the pullback is the same datum as a pair of map $A' \rightarrow S$ and generalized Cartier-Witt divisor $I \rightarrow W(S)$, which is to say a point of $\mathrm{Spf} A' \times \mathrm{WCart}$. \square

Remark 5.9. The consequence for prismatic cohomology in Lemma 5.8 can be proved using the conjugate filtration on Hodge–Tate cohomology (Proposition 3.9) and Corollary 5.5 under the stronger hypothesis that $L_{A'/A}$ vanishes p -adically.

Lemma 5.10 (Insensitivity to formally étale extensions II). *Given $A \rightarrow A' \rightarrow R$ with $A \rightarrow A'$ a p -completely formally étale map of δ -rings, the sites $(R/A)_\Delta$ and $(R/A')_\Delta$ agree. In particular,*

$$\Delta_{R/A}^{[n], \mathrm{site}} \{i\} \simeq \Delta_{R/A'}^{[n], \mathrm{site}} \{i\}$$

for all $n, i \in \mathbf{Z}$.

Proof. Given an object B of $(R/A)_\Delta$, we get a unique dashed lift making the diagram

$$\begin{array}{ccccc}
A & \xrightarrow{\quad} & A' & \dashrightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
R & \xrightarrow{\mathrm{id}} & R & \longrightarrow & B/J
\end{array}$$

of commutative rings commute. The map $A' \rightarrow B$ is automatically a δ -ring map by Lemma 5.7, so this defines an equivalence $(R/A)_\Delta \simeq (R/A')_\Delta$. \square

Lemma 5.11 (Comparison for prismatic δ -pairs). *Given a δ -ring A and an A -algebra R with bounded p -power torsion, if the kernel of the map $A \rightarrow R$ contains a Cartier divisor I which makes the pair (A, I) into a bounded prism and if $L_{R/\bar{A}}$ has p -complete Tor-amplitude in $[0, 1]$, then the natural map*

$$\Delta_{R/A}^{[n], \mathrm{wc}} \{i\} \rightarrow \Delta_{R/A}^{[n], \mathrm{site}} \{i\}$$

is an equivalence for all $n, i \in \mathbf{Z}$.

Proof. By Proposition 4.10, our $\mathrm{WCart}_{R/A}$ agrees in this case with the relative prismaticization as studied in [8, Sec. 5]. It follows from [8, Thm. 6.4] (and [8, Rem. 7.23]) that the cohomology of $\mathrm{WCart}_{R/A}$ with coefficients in $\mathcal{J}_{\mathrm{WCart}_{R/A}}^{\otimes n}\{i\}$ computes the cohomology of the corresponding sheaf on the relative prismatic site. \square

Corollary 5.12. *Fix a δ -pair $A \rightarrow R$. If $A \rightarrow R$ factors as $A \rightarrow A' \rightarrow R$ where $A \rightarrow A'$ is a p -completely formally étale map of δ -rings, the kernel of $A' \rightarrow R$ contains a Cartier divisor I making A' into a prism, and $L_{R/\overline{A}}$ has p -complete Tor-amplitude in $[0, 1]$, then the canonical map*

$$\Delta_{R/A}^{[n], \mathrm{wc}}\{i\} \rightarrow \Delta_{R/A}^{[n], \mathrm{site}}\{i\}$$

is an equivalence for all $n, i \in \mathbf{Z}$.

Proof. Combine Lemmas 5.8, 5.10, and 5.11. \square

Lemma 5.13. *As a functor of commutative A -algebras, $\Delta_{-/A}^{[n], \mathrm{site}}\{i\}$ has descent with respect to quasisyntomic covers of the form*

$$R \rightarrow R[z^{1/p^\infty}]^\wedge / (z - p)$$

for all $n, i \in \mathbf{Z}$.

Proof. For fixed A and R we have a canonical ‘forgetful’ functor $(R/A)_\Delta \rightarrow (\mathbf{Z}_p/\mathbf{Z}_p)_\Delta$ that induces an adjunction

$$f_! : \mathrm{Shv}((R/A)_\Delta) \rightleftarrows \mathrm{Shv}((\mathbf{Z}_p/\mathbf{Z}_p)_\Delta) : f^*$$

between ∞ -topoi whose right adjoint f^* is the restriction functor. The left adjoint $f_!$ sends the terminal object to the object

$$\underline{R/A} : (B, B/J) \mapsto \mathrm{Map}_{\delta\text{-pair}}((A, R), (B, B/J)) .$$

Moreover the right adjoint f^* preserves the structure sheaves $\mathcal{O}_\Delta^{[n]}\{i\}$. As a result we can write $\Delta_{R/A}^{[n], \mathrm{site}}\{i\}$, which is defined as the maps out of the terminal object in $\mathrm{Shv}((R/A)_\Delta)$ to $\mathcal{O}_\Delta^{[n]}\{i\}$, as maps in $\mathrm{Shv}((\mathbf{Z}_p/\mathbf{Z}_p)_\Delta)$ from $\underline{R/A}$ to $\mathcal{O}_\Delta^{[n]}\{i\}$.

For a general morphism $R \rightarrow R'$ of rings under A this procedure translates the Čech complex

$$R \longrightarrow R' \rightrightarrows R' \otimes_R R' \rightrightarrows R' \otimes_R R' \otimes_R R' \rightrightarrows \cdots$$

into the Čech complex of the map $\underline{R'/A} \rightarrow \underline{R/A}$ (considered as an augmented simplicial object) in the ∞ -topos $\mathrm{Shv}((\mathbf{Z}_p/\mathbf{Z}_p)_\Delta)$, as one immediately verifies. Thus to verify descent for a map $R \rightarrow R'$, that is the fact that the diagram

$$\Delta_{R/A}^{[n], \mathrm{site}}\{i\} \longrightarrow \Delta_{R'/A}^{[n], \mathrm{site}}\{i\} \rightrightarrows \Delta_{R' \otimes_R R'/A}^{[n], \mathrm{site}}\{i\} \rightrightarrows \Delta_{R' \otimes_R R' \otimes_R R'/A}^{[n], \mathrm{site}}\{i\} \rightrightarrows \cdots$$

is a limit diagram, it then suffices to show that the morphism $\underline{R'/A} \rightarrow \underline{R/A}$ is an effective epimorphism in the topos $\mathrm{Shv}((\mathbf{Z}_p/\mathbf{Z}_p)_\Delta)$.

Concretely, we need to show that for every object B in $(R/A)_\Delta$, there exists a $B' \in (R'/A)_\Delta$ whose restriction covers B , i.e. for each B we find a B' in the following diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & B' \\ \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & B/J & \longrightarrow & B'/J \\ \downarrow & & \nearrow & & \\ R' & & & & \end{array}$$

such that $B \rightarrow B'$ is faithfully flat.

Now we let $R' = R[z^{1/p^\infty}]^\wedge/(z-p)$. and simply take B' to be the prismatic envelope

$$B[z^{1/p^\infty}] \left\{ \frac{z-p}{J} \right\}_{(p,J)}^\wedge,$$

with $\delta(z) = 0$. This is p -completely faithfully flat over B by [10, Prop. 3.13]. \square

Proof of Theorem 5.6. Both sides of the comparison map have descent with respect to an extension of the form $R \rightarrow R[z^{1/p^\infty}]^\wedge/(z-p)$. For the right-hand side, this is Lemma 5.13. For the left hand side, it follows from Proposition 3.15 and Corollary 5.5.

The n th term in the Čech-nerve of the map $R \rightarrow R^0 = R[z^{1/p^\infty}]^\wedge/(z-p)$ is given by

$$R^n = R^{n-1}[z^{1/p^\infty}]^\wedge/(z-p)$$

and $L_{R^{n-1}/A}$ has p -complete Tor-amplitude in $[0, 1]$. Therefore, it suffices to see that the comparison map is an equivalence for $A \rightarrow R[z^{1/p^\infty}]^\wedge/(z-p) =: R'$. This map factors through $A \rightarrow A'$, where $A' = A[z^{1/p^\infty}]^\wedge_{(p,z-p)}$ (with $\delta(z) = 0$). As the kernel of $A' \rightarrow R[z^{1/p^\infty}]^\wedge/(z-p)$ contains a distinguished non-zero divisor, $z-p$, making A' into a prism, Corollary 5.12 applies to give the desired comparison equivalence since

$$L_{R'/(A'/(z-p))} \simeq L_{R/A} \otimes_A (A'/(z-p))$$

has p -complete Tor-amplitude in $[0, 1]$ since $(A'/(z-p))$ is p -completely free as an A -module. \square

Remark 5.14. Note that for a prism A and an A/I -algebra R , the condition that $L_{R/A}$ has p -complete Tor-amplitude in $[0, 1]$ is slightly weaker than that $L_{R/(A/I)}$ having p -complete Tor-amplitude in $[0, 1]$. So Theorem 5.6 is slightly more general than the comparison result obtained in [8].

Variant 5.15. Using a similar argument as the proof of Theorem 5.6, one can prove that the animated site-theoretic prismatic cohomology $\Delta_{R/A}^{\text{ansite}}$ satisfies sufficiently fine descent in R in order to reduce to the prismatic case, where it follows from [8, Rem. 7.14] that it agrees with $\Delta_{R/A}^{\text{site}}$. Thus, one finds that if (A, R) is a bounded δ -pair and $L_{R/A}$ has p -complete Tor-amplitude in $[0, 1]$, then the natural maps $\Delta_{R/A}^{[n], \text{ansite}} \{i\} \rightarrow \Delta_{R/A}^{[n], \text{site}} \{i\}$ are equivalences for all $n, i \in \mathbf{Z}$.

To conclude the section, we give a generalization of the classicality result [8, Cor. 8.13] to our relative Cartier–Witt stacks.

Lemma 5.16 (Classicality of relative Cartier–Witt stacks). *If A is a quasisyntomic δ -ring and $R \in \mathcal{Q}\text{Syn}_A$, then $\text{WCart}_{R/A}^{\text{an}}$ is classical and*

$$\Delta_{R/A}^{[n], \text{anwc}} \{i\} \rightarrow \Delta_{R/A}^{[n], \text{wc}} \{i\}$$

is an equivalence for each $n, i \in \mathbf{Z}$.

Proof. It suffices to prove that $\mathrm{WCart}_{R/A}^{\mathrm{an}}$ is a colimit (in derived stacks) of objects represented by discrete commutative rings. By quasisyntomic descent, Corollary 4.21, and Lemma 4.19, we can assume that $R \in \mathrm{QRSPerfd}_A$ and in particular that $L_{R/A}$ has p -complete Tor-amplitude in $[1, 1]$. Consider the map of δ -pairs $(A, R) \rightarrow (A[z^{1/p^\infty}], R[z^{1/p^\infty}]/(z-p)) =: (A^0, R^0)$ and let (A^\bullet, R^\bullet) be the associated Čech complex in δ -pairs. By Corollary 4.21 (or rather its derived variant), the induced map $\mathrm{WCart}_{R^0/A^0}^{\mathrm{an}} \rightarrow \mathrm{WCart}_{R/A}^{\mathrm{an}}$ is a surjective map of stacks in the flat topology on p -nilpotent commutative rings. By the derived version of Lemma 4.19, the Čech complex of this map is $\mathrm{WCart}_{R^\bullet/A^\bullet}^{\mathrm{an}}$. However, for each $s \geq 0$, the δ -pair (A^s, R^s) is in fact prismatic and the prismatic cohomology Δ_{R^s/A^s} is discrete since L_{R^s/A^s} has p -complete Tor-amplitude in $[1, 1]$, so that the Cartier–Witt stack $\mathrm{WCart}_{R^s/A^s}^{\mathrm{an}}$ is affine and classical by [8, Cor. 7.18]. The lemma now follows by taking the geometric realization and global sections. \square

5.3 Comparison to [10]

Let (A, I) be a bounded prism and let R be an animated commutative \bar{A} -algebra. There are now two definitions of $\Delta_{R/A}$, namely as the global sections of the prismatic crystal $\mathcal{H}_\Delta(R/A)$ and as the derived relative prismatic cohomology of [10, Sec. 7.2]. For the remainder of this section, we write $\Delta_{R/A}^{\mathrm{rel}}$ for the latter. By combining Propositions 5.1 and 4.10 with [8, Rem. 7.23], we find that $\Delta_{R/A} \simeq \Delta_{R/A}^{\mathrm{rel}}$ when R is discrete and $L_{R/\bar{A}}$ has p -complete Tor-amplitude in $[0, 1]$. The next proposition shows that in fact no condition on R is required.¹²

Proposition 5.17. *If (A, I) is a bounded prism and R is an animated commutative \bar{A} -algebra, then there is a natural equivalence $\gamma_{R/A}: \Delta_{R/A}^{[\star]} \xrightarrow{\simeq} \Delta_{R/A}^{\mathrm{rel}, [\star]}$.*

Proof. To construct the natural transformation γ , let \mathcal{T}_A be the category of transversal prisms (B, J) equipped with a map of prisms $B \xrightarrow{f} A$. For each $(B, f) \in \mathcal{T}_A$, let $\widehat{A \otimes B}$ denote the (p, J) -adic completion of $A \otimes B$. The map f and multiplication induce a map of prisms $\widehat{A \otimes B} \rightarrow A$. We can evaluate the crystal $\mathcal{H}_\Delta(R/A)$ at B to obtain a natural composition

$$\Delta_{R/A} \rightarrow \mathcal{H}_\Delta(R/A)(B) \simeq \Delta_{R \widehat{\otimes} B / A \widehat{\otimes} B}^{\mathrm{rel}} \rightarrow \Delta_{R \widehat{\otimes} A \bar{A} / A}^{\mathrm{rel}} \rightarrow \Delta_{R/A}^{\mathrm{rel}},$$

where the final map uses the \bar{A} -linear multiplication map $R \widehat{\otimes} A \bar{A} \rightarrow R$.

The construction of the map is natural in B : for another choice B' with a map $B \rightarrow B'$ in \mathcal{T}_A we get a natural homotopy between the two induced maps $\Delta_{R/A} \rightarrow \Delta_{R/A}^{\mathrm{rel}}$ which is induced from a ‘hammock’ between the constructions for these maps:

$$\begin{array}{ccccc} & & \Delta_{R \widehat{\otimes} B / A \widehat{\otimes} B}^{\mathrm{rel}} & \longrightarrow & \Delta_{R \widehat{\otimes} A \bar{A} / A}^{\mathrm{rel}} & & \\ & \nearrow & \downarrow & & \downarrow & \searrow & \\ \Delta_{R/A} & & & & & & \Delta_{R/A}^{\mathrm{rel}} \\ & \searrow & \downarrow & & \downarrow & \nearrow & \\ & & \Delta_{R \widehat{\otimes} B' / A \widehat{\otimes} B'}^{\mathrm{rel}} & \longrightarrow & \Delta_{R \widehat{\otimes} A \bar{A} / A}^{\mathrm{rel}} & & \end{array}$$

As \mathcal{T}_A is sifted by [7, Cor. 2.4.7], this implies that the map $\Delta_{R/A} \rightarrow \Delta_{R/A}^{\mathrm{rel}}$ does not depend on the transversal prism B chosen (for example by taking the colimit over all these maps in the arrow category).

¹²One could also prove this by a Kan extension argument from the case just explained, but we would like to give an explicit argument.

This specifies the natural transformation $\gamma_{R/A}$ when A is fixed. A similar argument can be made with the help of an intermediary transversal prism $B \in \mathcal{T}_A$ to construct commutative diagrams

$$\begin{array}{ccc} \Delta_{R/A} & \longrightarrow & \Delta_{R/A}^{\text{rel}} \\ \downarrow & & \downarrow \\ \Delta_{R'/A'} & \longrightarrow & \Delta_{R'/A'}^{\text{rel}} \end{array}$$

for maps of bounded prismatic δ -pairs $(A, R) \rightarrow (A', R')$. Siftedness of \mathcal{T}_A again implies that this diagram does not depend on the choice of B . Continuing in this way, we obtain a natural transformation as desired.¹³

Both sides commute with colimits in the variable R as functors to $\mathbf{D}(A)_{(p,I)}^\wedge$. Thus, we can immediately reduce to the case of $R = \overline{A}[x]$. Now, we can use base change in A and Zariski descent to reduce to the case of the universal oriented prism A^0 . Then, we can test whether γ_{R/A^0} is an equivalence by base change along the faithfully flat map to the perfection A_{perf}^0 of A^0 . But, then, we have a commutative square

$$\begin{array}{ccc} \Delta_{R/\mathbf{Z}} & \longrightarrow & \Delta_R \\ \downarrow & & \downarrow \\ \Delta_{R/A_{\text{perf}}^0} & \xrightarrow{\gamma_{R/A_{\text{perf}}^0}} & \Delta_{R/A_{\text{perf}}^0}^{\text{rel}} \end{array}$$

The vertical arrows are equivalences because A_{perf}^0 is perfect, using the conjugate filtrations on the Hodge–Tate cohomology in both cases, and the top arrow is an equivalence by Example 3.21. \square

Remark 5.18. From now on, we drop the distinction between $\Delta_{R/A}^{\text{rel}}$ and $\Delta_{R/A}$ in the case of prismatic δ -pairs.

6 The Nygaard filtration

In this section, we introduce the Nygaard filtration and Nygaard-completed Frobenius-twisted prismatic cohomology relative to δ -pairs and prove the related statements of Theorem 1.2.

Definition 6.1. Let (A, R) be a bounded δ -pair and let $\mathcal{H}_\Delta(R/A)\{i\}$ be the i th Breuil–Kisin twisted prismatic crystal associated to (A, R) . This object is an A -module in $\mathbf{D}(\text{WCart})$, so we can form its Frobenius twist

$$\mathcal{H}_\Delta^{(1)}(R/A)\{i\} = \mathcal{H}_\Delta(R/A)\{i\} \otimes_{A, \varphi_A} A.$$

If (B, I) is a transversal prism, then the value of this crystal on B is naturally equivalent to the (p, I) -completion of

$$\Delta_{R \widehat{\otimes} B / (A \widehat{\otimes} B)_I^\wedge} \{i\} \otimes_{A, \varphi} A.$$

Note that this is not typically equivalent to the usual Frobenius twist $\Delta_{R \widehat{\otimes} B / (A \widehat{\otimes} B)_I^\wedge}^{(1)} \{i\}$ of [10] because we use only the Frobenius of A and not of $(A \widehat{\otimes} B)_I^\wedge$.

¹³To make this construction rigorous, consider the ∞ -category $\text{Pairs}^{\delta\Delta\text{an}}$ of prismatic δ -pairs (A, R) where A is bounded and R is an animated commutative \overline{A} -algebra. Let $q: \mathcal{T} \rightarrow \text{Pairs}^{\delta\Delta\text{an}}$ denote the coCartesian fibration with fiber over (A, R) given by \mathcal{T}_A . On $\text{Pairs}^{\delta\Delta\text{an}}$ are defined sections $\Delta_{(-)/(-)}$ and $\Delta_{(-)/(-)}^{\text{rel}}$ of the coCartesian fibration $\widehat{\text{DAlg}} \rightarrow \text{Pairs}^{\delta\Delta\text{an}}$ with fiber over (A, R) given by the ∞ -category of (p, I) -complete derived commutative A -algebras. The construction above induces a natural transformation $\tilde{\gamma}: \Delta_{(-)/(-)} \circ q \rightarrow \Delta_{(-)/(-)}^{\text{rel}} \circ q$ where naturality can be seen similar to the case above. Taking a left Kan extension along q produces a natural transformation from $\Delta_{(-)/(-)} \rightarrow \Delta_{(-)/(-)}^{\text{rel}}$ since the fibers of q are contractible.

In this section, we use an approach analogous to the one from [7] to construct the Nygaard filtration. Our goal will first be to produce a filtration on $\mathrm{R}\Gamma(\mathrm{WCart}, \mathcal{H}_\Delta^{(1)}(R/A)\{i\})$.

Let F denote the Frobenius endomorphism of WCart . Recall from [7, Theorem 3.6.7] that the diagram

$$\begin{array}{ccc} \mathrm{WCart}^{\mathrm{HT}} & \longrightarrow & \mathrm{WCart} \\ \downarrow & & \downarrow F \\ \mathrm{Spf} \mathbf{Z}_p & \xrightarrow{\rho_{\mathrm{dR}}} & \mathrm{WCart} \end{array}$$

yields pullback diagrams on global sections for any quasi-coherent sheaf on WCart , where ρ_{dR} is the point of WCart corresponding to the crystalline prism $(\mathbf{Z}_p, (p))$. Commutativity of the diagram yields a canonical equivalence

$$F^* \mathcal{E}|_{\mathrm{WCart}^{\mathrm{HT}}} \simeq \mathcal{E}_{\mathrm{dR}} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}} \quad (2)$$

for any prismatic crystal \mathcal{E} , where $\mathcal{E}_{\mathrm{dR}}$ denotes the p -complete complex $\rho_{\mathrm{dR}}^* \mathcal{E}$. We will apply this for $\mathcal{E} = \mathcal{H}_\Delta^{(1)}(R/A)\{i\}$.

Remark 6.2 (Prismatic cohomology relative to animated crystalline prisms). Since $(\mathbf{Z}_p, (p))$ is not transversal, our definition of $\mathcal{H}_\Delta(R/A)$ when (A, R) is a bounded δ -pair does not allow us to *a priori* compute $\rho_{\mathrm{dR}}^* \mathcal{H}_\Delta(R/A) = \mathcal{H}_\Delta(R/A)_{\mathrm{dR}}$ if A has p -torsion. Rather, we view $\mathcal{H}_\Delta(R/A)_{\mathrm{dR}}$ as the definition of $\Delta_{R \otimes_{\mathbf{F}_p} A}$ where we view A as an animated crystalline prism.¹⁴ Nevertheless, this notation is unambiguous as it agrees with the definition of $\Delta_{R \otimes_{\mathbf{F}_p} A}^{\mathrm{anwc}}$ which uses that A is only a δ -ring. A similar comment applies to the Frobenius twist $\mathcal{H}_\Delta^{(1)}(R/A)_{\mathrm{dR}}$.

Remark 6.3 (Prismatic cohomology relative to animated prisms). More generally, one can use the formalism of prismatic cohomology relative to animated δ -pairs to give a definition of prismatic cohomology $\Delta_{R/A}$ when (A, \bar{A}) is an animated prism and R is an animated commutative \bar{A} -algebra, for example as $\mathrm{R}\Gamma(\mathrm{WCart}_{R/A}^{\mathrm{an}}, \mathcal{O})$; see Variant 4.13.

Lemma 6.4 (de Rham comparison for prismatic cohomology relative to δ -rings). *Let (A, R) be a bounded δ -pair. There is a canonical multiplicative equivalence $\mathcal{H}_\Delta^{(1)}(R/A)_{\mathrm{dR}} \simeq \widehat{\mathrm{dR}}_{R/A} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}$, where the target denotes p -completed derived de Rham cohomology of R over A . It fits into a commutative diagram*

$$\begin{array}{ccc} F^* \mathcal{H}_\Delta^{(1)}(R/A)|_{\mathrm{WCart}^{\mathrm{HT}}} & \longrightarrow & \widehat{\mathrm{dR}}_{R/A} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}} \\ \downarrow & \nearrow & \\ \mathcal{H}_\Delta^{(1)}(R/A)_{\mathrm{dR}} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}} & & \end{array}$$

of equivalences, where the vertical map comes from (2) and the horizontal map from relative de Rham comparison.

Proof. We follow the argument from [7, Section 5.4]. For a prism B , we have a canonical pullback diagram

$$\begin{array}{ccc} \mathrm{Spf} \bar{B} & \longrightarrow & \mathrm{WCart}^{\mathrm{HT}} \\ \downarrow & & \downarrow \\ \mathrm{Spf} B & \longrightarrow & \mathrm{WCart}. \end{array}$$

¹⁴Note that while animated prisms are introduced in [8], they are used there only to define derived Cartier–Witt stacks and not as possible bases for relative prismatic cohomology.

The quasi-coherent sheaf $F^*\mathcal{H}_\Delta^{(1)}(R/A)$ pulls back to

$$\Delta_{R\widehat{\otimes}B/A\widehat{\otimes}B} \otimes_{A\widehat{\otimes}B,\varphi} A\widehat{\otimes}B$$

on $\mathrm{Spf} B$, and hence to

$$\Delta_{R\widehat{\otimes}B/A\widehat{\otimes}B} \otimes_{A\widehat{\otimes}B,\varphi} \overline{A\widehat{\otimes}B}$$

on $\mathrm{Spf} \overline{B}$. By the relative de Rham comparison [7, Prop. 5.2.5], this is naturally equivalent to

$$\widehat{\mathrm{dR}}_{R\widehat{\otimes}B/A\widehat{\otimes}B} \simeq \widehat{\mathrm{dR}}_{R/A}\widehat{\otimes}\overline{B}.$$

Hence we have a natural equivalence

$$F^*\mathcal{H}_\Delta^{(1)}(R/A)|_{\mathrm{WCart}^{\mathrm{HT}}} \simeq \widehat{\mathrm{dR}}_{R/A} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}},$$

providing the horizontal map in the commutative diagram. Together with the equivalence (2), this determines a unique multiplicative equivalence

$$\mathcal{H}_\Delta^{(1)}(R/A)_{\mathrm{dR}} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}} \simeq \widehat{\mathrm{dR}}_{R/A} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}.$$

of quasi-coherent sheaves on $\mathrm{WCart}^{\mathrm{HT}}$ making the diagram commute. To finish the proof, we need to check that this is induced from a natural multiplicative equivalence

$$\mathcal{H}_\Delta^{(1)}(R/A)_{\mathrm{dR}} \simeq \widehat{\mathrm{dR}}_{R/A}$$

in $D(\mathbf{Z}_p)$. To see this, it suffices to check that the map

$$\mathcal{H}_\Delta^{(1)}(R/A)_{\mathrm{dR}} \rightarrow \mathrm{R}\Gamma(\mathrm{WCart}^{\mathrm{HT}}, \widehat{\mathrm{dR}}_{R/A} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}})$$

factors uniquely and multiplicatively through the canonical map

$$\widehat{\mathrm{dR}}_{R/A} \rightarrow \mathrm{R}\Gamma(\mathrm{WCart}^{\mathrm{HT}}, \widehat{\mathrm{dR}}_{R/A} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}).$$

Since the $\mathcal{H}_\Delta^{(1)}(R/A)_{\mathrm{dR}}$ is left Kan extended from δ -pairs (A, R) with R quasisyntomic over A we may reduce to that situation. Then, since $\widehat{\mathrm{dR}}_{R/A}$ satisfies quasisyntomic descent, we may reduce to the case where $L_{R/A}$ has Tor-amplitude in $[1, 1]$. But then $\widehat{\mathrm{dR}}_{R/A}$ is discrete, and agrees with the connective cover of $\mathrm{R}\Gamma(\mathrm{WCart}^{\mathrm{HT}}, \widehat{\mathrm{dR}}_{R/A} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}})$, $\mathcal{H}_\Delta^{(1)}(R/A)_{\mathrm{dR}}$ is connective, and the desired lift is uniquely determined. \square

Corollary 6.5. *If (A, R) is a bounded δ -pair, then there is a natural pullback diagram*

$$\begin{array}{ccc} \mathrm{R}\Gamma(\mathrm{WCart}, \mathcal{H}_\Delta^{(1)}(R/A)\{i\}) & \longrightarrow & \widehat{\mathrm{dR}}_{R/A} \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma(\mathrm{WCart}, F^*\mathcal{H}_\Delta^{(1)}(R/A)\{i\}) & \longrightarrow & \widehat{\mathrm{dR}}_{R/A} \otimes \mathrm{R}\Gamma(\mathrm{WCart}^{\mathrm{HT}}, \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}) \end{array}$$

for each $i \in \mathbf{Z}$.

Proof. We combine the pullback diagram obtained from (6) with the equivalence $\mathcal{H}_\Delta^{(1)}(R/A)\{i\}_{\mathrm{dR}} \simeq \widehat{\mathrm{dR}}_{R/A}$ obtained by multiplying the equivalence from Lemma 6.4 with the canonical trivialization of the Breuil–Kisin twist $\mathbf{Z}_p\{i\}$ (compare [7, Variant 5.4.13]). \square

Definition 6.6 (Nygaard-filtered Frobenius-twisted prismatic cohomology). Let (A, R) be a bounded δ -pair and fix $i \in \mathbf{Z}$. We define the Nygaard filtration on $\mathrm{R}\Gamma(\mathrm{WCart}, \mathcal{H}_\Delta^{(1)}(R/A)\{i\})$ as the pullback of filtered spectra

$$\begin{array}{ccc} \mathrm{N}^{\geq *}\mathrm{R}\Gamma(\mathrm{WCart}, \mathcal{H}_\Delta^{(1)}(R/A)\{i\}) & \longrightarrow & \mathrm{F}_H^{\geq *}\widehat{\mathrm{dR}}_{R/A} \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma(\mathrm{WCart}, \mathrm{N}^{\geq *}\mathrm{F}^*\mathcal{H}_\Delta^{(1)}(R/A)\{i\}) & \longrightarrow & \mathrm{F}_H^{\geq *}\widehat{\mathrm{dR}}_{R/A} \otimes \mathrm{R}\Gamma(\mathrm{WCart}^{\mathrm{HT}}, \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}), \end{array}$$

where the lower left term uses the pointwise Nygaard filtration on the prismatic crystal

$$B \mapsto \Delta_{R \widehat{\otimes} \overline{B}/A \widehat{\otimes} B}\{i\} \otimes_{A \widehat{\otimes} B, \varphi} A \widehat{\otimes} B$$

corresponding to $\mathrm{F}^*\mathcal{H}_\Delta^{(1)}(R/A)$, and the bottom map uses that the composite

$$\mathrm{F}^*\mathcal{H}_\Delta^{(1)}(R/A)\{i\} \rightarrow \mathcal{H}_\Delta^{(1)}(R/A)\{i\}_{\mathrm{dR}} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}} \simeq \widehat{\mathrm{dR}} \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}$$

is a combination of the canonical trivialisation of $\mathrm{F}^*\mathcal{O}_{\mathrm{WCart}}\{i\}$ on the Hodge–Tate locus, and the de Rham comparison map

$$\Delta_{R \widehat{\otimes} \overline{B}/A \widehat{\otimes} B} \otimes_{A \widehat{\otimes} B, \varphi} A \widehat{\otimes} B \rightarrow \Delta_{R \widehat{\otimes} \overline{B}/A \widehat{\otimes} B} \otimes_{A \widehat{\otimes} B, \varphi} \overline{A \widehat{\otimes} B} \simeq \widehat{\mathrm{dR}}_{R/A} \widehat{\otimes} \overline{B}$$

which takes the Nygaard filtration into the Hodge filtration (see [7, Const. 5.5.3]).

Definition 6.7 (Nygaard-completed Frobenius-twisted prismatic cohomology). If (A, R) is a bounded δ -pair, we define Nygaard-completed (Frobenius-twisted, Breuil–Kisin twisted) prismatic cohomology of R over A as the completion

$$\mathrm{N}^{\geq *}\widehat{\Delta}_{R/A}^{(1)}\{i\} := (\mathrm{N}^{\geq *}\mathrm{R}\Gamma(\mathrm{WCart}, \mathcal{H}_\Delta^{(1)}(R/A)\{i\}))^\wedge.$$

Remark 6.8 (Absolute Nygaard comparison). Since the Frobenius on the δ -ring \mathbf{Z}_p is the identity, $\mathrm{N}^{\geq *}\widehat{\Delta}_{R/\mathbf{Z}_p}^{(1)}\{i\} \simeq \mathrm{N}^{\geq *}\widehat{\Delta}_R$, where the latter denotes the Nygaard complete absolute prismatic cohomology of [9, 7]. Indeed, in this case $\mathcal{H}_\Delta^{(1)}(R/\mathbf{Z}_p) \simeq \mathcal{H}_\Delta(R)$, in the notation of [7], and our construction of the Nygaard filtration and completion agrees with the one in [7, Sec. 5.5].

We now recall the Hodge–Tate point $\eta: \mathrm{Spf}(\mathbf{Z}_p) \rightarrow \mathrm{WCart}$ which assigns to every p -nilpotent ring R the Cartier–Witt divisor given by $V(1): W(R) \rightarrow W(R)$. In analogy to [7], we write $\Omega_{R/A}^{\mathcal{D}}$ for $\eta^*\mathcal{H}_\Delta(R/A)$. This inherits a conjugate filtration from the δ -conjugate filtration on the relative Hodge–Tate crystal by Proposition 3.9, and, since

$$\mathrm{gr}_i^{\delta\text{-conj}}\mathcal{H}_\Delta(R/A) \simeq \widehat{\mathrm{L}}\Omega_{R/A}^i \otimes \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{-i\}[-i],$$

we have $\mathrm{gr}_i^{\text{conj}}\Omega_{R/A}^{\mathcal{D}} \simeq \widehat{\mathrm{L}}\Omega_{R/A}^i[-i]$.

Lemma 6.9. *If (A, R) is a bounded δ -pair, then there are natural maps of fiber sequences*

$$\begin{array}{ccccc} \mathrm{gr}_N^i\widehat{\Delta}_{R/A}^{(1)}\{j\} & \longrightarrow & \mathrm{F}_{\leq i}^{\text{conj}}\Omega_{R/A}^{\mathcal{D}} & \xrightarrow{\Theta+i} & \mathrm{F}_{\leq i-1}^{\text{conj}}\Omega_{R/A}^{\mathcal{D}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{R}\Gamma(\mathrm{WCart}, \mathrm{gr}_N^i\mathrm{F}^*\mathcal{H}_\Delta^{(1)}(R/A)\{j\}) & \longrightarrow & \mathrm{F}_{\leq i}^{\text{conj}}\Omega_{R/A}^{\mathcal{D}} & \xrightarrow{\Theta+i} & \mathrm{F}_{\leq i}^{\text{conj}}\Omega_{R/A}^{\mathcal{D}} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\Delta}_{R/A}\{i\} & \longrightarrow & \Omega_{R/A}^{\mathcal{D}} & \xrightarrow{\Theta+i} & \Omega_{R/A}^{\mathcal{D}} \end{array}$$

in $D(A)_p^\wedge$, where Θ is the Sen operator on $F_{\leq i}^{\text{conj}} \Omega_{R/A}^{\mathcal{D}}$ induced from the Hodge–Tate crystal $\mathcal{H}_{\Delta}^-(R/A)$ under the description of quas-coherent sheaves on WCart^{HT} of [7, Sec. 3.5].

Proof. The Hodge–Tate comparison provides that for every transversal prism B the relative Frobenius map

$$\text{gr}_N^i(\Delta_{R \otimes \overline{B}/A \otimes B} \{j\} \otimes_{A \otimes B, \varphi} A \widehat{\otimes} B) \rightarrow \overline{\Delta}_{R \otimes \overline{B}/A \otimes B} \{i\}$$

factors through an equivalence to $F_{\leq i}^{\text{conj}} \overline{\Delta}_{R \otimes \overline{B}/A \otimes B} \{i\}$. In particular, $\text{gr}_N^i F^* \mathcal{H}_{\Delta}^{(1)}(R/A)$ is supported on the Hodge–Tate locus, and global sections can be expressed through the Sen operator, leading to the bottom two rows.

For the top row (and the map between the top two rows), we first observe that the pullback diagram defining $N^{\geq *}\widehat{\Delta}_{R/A}^{(1)}$ provides a pullback diagram

$$\begin{array}{ccc} \text{gr}_N^i \widehat{\Delta}_{R/A}^{(1)} \{j\} & \longrightarrow & \widehat{\text{L}}\Omega_{R/A}^i[-i] \\ \downarrow & & \downarrow \\ \text{RF}(\text{WCart}, \text{gr}_N^i F^* \mathcal{H}_{\Delta}^{(1)}(R/A) \{j\}) & \longrightarrow & \widehat{\text{L}}\Omega_{R/A}^i[-i] \otimes \text{RF}(\text{WCart}^{\text{HT}}, \mathcal{O}_{\text{WCart}^{\text{HT}}}). \end{array}$$

Relative de Rham comparison on the level of prismatic crystals identifies $\text{gr}_N^i F^* \mathcal{H}_{\Delta}^{(1)}(R/A) \{j\}$ with the prismatic crystal

$$B \mapsto F_{\leq i}^{\text{conj}} \overline{\Delta}_{R \otimes \overline{B}/A \otimes B},$$

and the bottom map is obtained from the canonical map

$$F_{\leq i}^{\text{conj}} \overline{\Delta}_{R \otimes \overline{B}/A \otimes B} \rightarrow \text{gr}_i^{\text{conj}} \overline{\Delta}_{R \otimes \overline{B}/A \otimes B} \simeq \widehat{\text{L}}\Omega_{R/A}^i[-i].$$

The bottom two terms can now be expressed as fibers of the respective Sen operators $\Theta + i$, and the right vertical map is the canonical map

$$\widehat{\text{L}}\Omega_{R/A}^i[-i] \rightarrow \text{fib}(\widehat{\text{L}}\Omega_{R/A}^i[-i] \xrightarrow{\Theta+i} \widehat{\text{L}}\Omega_{R/A}^i[-i]) \simeq \widehat{\text{L}}\Omega_{R/A}^i[-i] \oplus \widehat{\text{L}}\Omega_{R/A}^i[-i-1].$$

This yields the claimed first row (and the map to the second row). \square

Corollary 6.10. *For each $i \in \mathbf{Z}$, $\text{gr}_N^i \widehat{\Delta}_{R/A}^{(1)} \{j\}$ is sifted-colimit preserving, has faithfully flat descent as a functor on bounded δ -pairs, satisfies base change in the base δ -ring, and is invariant under quasi-étale extensions, finitely generated completions, and localizations in the base δ -ring (as in Proposition 3.12 and Corollary 3.13). Moreover, if we consider $N^{\geq *}\widehat{\Delta}_{R/A}^{(1)} \{j\}$ as a functor to complete filtered objects then it is sifted-colimit preserving, has faithfully flat descent as a functor on bounded δ -pairs, satisfies base change in the base δ -ring, and is invariant under quasi-étale extensions in the base δ -ring.*

Proof. The second claim follows from the first. The first claim follows from the fact that $\text{gr}_N^i \widehat{\Delta}_{R/A}^{(1)}$ is built out of differential forms by Lemma 6.9. Differential forms have the necessary descent by Lemma 3.14, the necessary invariance under quasi-étale extensions, finitely generated completions, and localizations, and by definition preserve sifted colimits and satisfy base change. \square

In the case of a prismatic δ -pair (A, R) , i.e. one where A admits the structure of a prism, and R the structure of an \overline{A} -algebra, $N^{\geq *}\widehat{\Delta}_{R/A}^{(1)} \{i\}$ now potentially has two meanings: The one defined above, and the Nygaard filtration from relative prismatic cohomology. Our goal is now to identify the two. Temporarily, we will write $N_{\text{rel}}^{\geq *}\widehat{\Delta}_{R/A}^{(1)} \{i\}$ for the Nygaard filtration from relative prismatic cohomology.

Definition 6.11. I updated this to use the argument from 5.22 to handle the non-transverse case. We define a comparison map $N^{\geqslant*} \widehat{\Delta}_{R/A}^{(1)} \{i\} \rightarrow N_{\text{rel}}^{\geqslant*} \widehat{\Delta}_{R/A}^{(1)} \{i\}$ as in the proof of Proposition 5.17: before completion, given a transversal prism B with a map of prisms to A , we have the composite map of filtered spectra

$$N^{\geqslant*} R\Gamma(\text{WCart}, \mathcal{H}_{\Delta}^{(1)}(R/A)\{i\}) \rightarrow R\Gamma(\text{WCart}, N^{\geqslant*} F^* \mathcal{H}_{\Delta}^{(1)}(R/A)\{i\}) \rightarrow N_{\text{rel}}^{\geqslant*} \Delta_{R \otimes \overline{B}/A \otimes \overline{B}}^{(1)} \{i\} \rightarrow N_{\text{rel}}^{\geqslant*} \Delta_{R/A}^{(1)} \{i\},$$

where the first map comes from the definition, the second is evaluation at $\rho_B: \text{Spf } B \rightarrow \text{WCart}$, and the third uses the multiplication map. After completion, this gives rise to a map

$$N^{\geqslant*} \widehat{\Delta}_{R/A}^{(1)} \{i\} \rightarrow N_{\text{rel}}^{\geqslant*} \widehat{\Delta}_{R/A}^{(1)} \{i\},$$

which one shows as in Proposition 5.17 does not depend on B and which is natural in prismatic δ -pairs (A, R) .

Proposition 6.12 (Relative prismatic Nygaard comparison). *The map from Definition 6.11 is an equivalence for all prismatic δ -pairs (A, R) .*

Proof. As functors to complete filtered spectra, both sides preserve sifted colimits. For the left hand side this is part of Corollary 6.10, for the right hand side this is seen analogously using $\text{gr}_{N_{\text{rel}}^{\geqslant*}}^i \widehat{\Delta}_{R/A} \{j\} \simeq F_{\leqslant i}^{\Delta \text{conj}} \overline{\Delta}_{R/A} \{j\}$ and the fact that the associated graded pieces of the Δ -conjugate filtration are also built out of differential forms. This reduces us to the case $R = \overline{A}[x_1, \dots, x_n]$. Also, both sides satisfy base change as functors to complete filtered spectra, so we may assume that A is the free oriented prism $\mathbf{Z}\{d\}[\delta(d)^{-1}]_{p,d}^{\wedge}$. For this A , the map $A \rightarrow A^{\text{perf}}$ to its colimit perfection is (p, d) -completely faithfully flat, so by again appealing to base change, we may reduce to the case of perfect prisms A . In that case, we have a commutative diagram

$$\begin{array}{ccc} N^{\geqslant*} \widehat{\Delta}_{R/\mathbf{Z}_p}^{(1)} \{j\} & & \\ \downarrow \simeq & \searrow & \\ N^{\geqslant*} \widehat{\Delta}_{R/A}^{(1)} \{j\} & \longrightarrow & N_{\text{rel}}^{\geqslant*} \widehat{\Delta}_{R/A}^{(1)} \{j\}. \end{array}$$

The left vertical map is an equivalence, which follows from the fact that $\widehat{\text{L}\Omega}_{R/\mathbf{Z}_p}^i \rightarrow \widehat{\text{L}\Omega}_{R/A}^i$ is an equivalence, since $(\text{L}_{A/\mathbf{Z}_p})_p^{\wedge} \simeq 0$ as A is perfect. The diagonal composite agrees with the completion of the composite

$$N^{\geqslant*} R\Gamma(\text{WCart}, \mathcal{H}_{\Delta}(R/\mathbf{Z}_p)^{(1)} \{j\}) \rightarrow R\Gamma(\text{WCart}, N^{\geqslant*} F^* \mathcal{H}_{\Delta}(R/\mathbf{Z}_p)^{(1)} \{j\}) \rightarrow N_{\text{rel}}^{\geqslant*} \Delta_{R \otimes \overline{A}/\mathbf{Z}_p \otimes \overline{A}}^{(1)} \{j\} \rightarrow N_{\text{rel}}^{\geqslant*} \Delta_{R/A}^{(1)} \{j\},$$

which is the map from [7, Construction 5.6.1]. According to [7, Theorem 5.6.2], it is an equivalence. \square

From now on we drop the notation $N_{\text{rel}}^{\geqslant*}$.

Corollary 6.13. *If (A, R) is a bounded δ -pair and $A \rightarrow R$ is relatively quasiregular semiperfectoid, then $N^{\geqslant*} \widehat{\Delta}_{R/A}^{(1)} \{j\}$ is discrete with discrete associated graded, and the filtration on it agrees with the underived pullback of the Hodge–Tate filtration along $\varphi: \widehat{\Delta}_{R/A}^{(1)} \{j\} \rightarrow \Delta_{R/A}^{[-j]} \{j\}$.*

Proof. Let $A \rightarrow A' \rightarrow R$ be a factorization of $A \rightarrow R$ such that $A \rightarrow A'$ is relatively perfect, (A', R) is a prismatic δ -pair, and $\text{L}_{R/A'}$ has p -complete Tor-amplitude in $[1, 1]$. In the diagram

$$\begin{array}{ccc} \widehat{\Delta}_{R/A}^{(1)} \{j\} & \longrightarrow & \Delta_{R/A}^{[-j]} \{j\} \\ \downarrow & & \downarrow \\ \widehat{\Delta}_{R/A'}^{(1)} \{j\} & \longrightarrow & \Delta_{R/A'}^{[-j]} \{j\} \end{array}$$

the vertical maps are all isomorphisms, so it suffices to prove the claim in the case of a prismatic δ -pair. For that, it suffices to check that the associated graded terms are discrete, and that the map on associated graded pieces is injective. In the prismatic case, we have a factorization

$$\mathrm{gr}_N^i \widehat{\Delta}_{R/A'}^{(1)}\{j\} \simeq \mathrm{F}_{\leq i}^{\mathrm{conj}} \overline{\Delta}_{R/A'}\{i\} \rightarrow \overline{\Delta}_{R/A'}\{i\},$$

where the latter map is an injective map between discrete objects since all associated graded terms of the conjugate filtration $\widehat{\mathrm{L}}\Omega_{R/A}^i[-i]$ are discrete. \square

Remark 6.14. Since we can cover every R/A with R quasisyntomic over A by relatively quasiregular semiperfectoids, Corollary 6.13 yields another characterization of the Nygaard filtration through descent and left Kan extension by Lemma 9.14.

Remark 6.15. If R is smooth over A (which never happens in the prismatic case), then the conjugate filtration on $\Omega_{R/A}^{\mathbb{P}}$ is the Postnikov filtration, and the fiber sequences from Lemma 6.9 exhibit $\mathrm{gr}_N^i \widehat{\Delta}_{R/A}^{(1)}\{j\}$ as the $(-i)$ -connective cover of $\overline{\Delta}_{R/A}\{i\}$. This identifies $\mathrm{N}^{\geq \star} \widehat{\Delta}_{R/A}^{(1)}\{j\}$ as the Beilinson-connective cover of the Hodge–Tate filtered $\Delta_{R/A}\{j\}$, providing yet another characterization. The analogue of this fact curiously *also* holds in the prismatic case if R is smooth over \overline{A} . The relationship between these two facts can be explained as follows: if R is a smooth A -algebra, then the I -adic filtration on R has associated graded given by the smooth \overline{A} -algebra $\overline{R}[I]$, with $\overline{R} = R \otimes_A \overline{A}$. Filtered prismatic cohomology provides a filtration on $\mathrm{gr}_N^i \widehat{\Delta}_{R/A}^{(1)}$ with associated graded given by $\mathrm{gr}_N^i \widehat{\Delta}_{\overline{R}[I]/A}^{(1)}$, from which the connectivity statement from relative prismatic cohomology implies the one in prismatic cohomology relative δ -rings.

7 Relative syntomic cohomology

We write $\mathrm{can}: \mathrm{N}^{\geq i} \widehat{\Delta}_{R/A}^{(1)}\{i\} \rightarrow \widehat{\Delta}_{R/A}^{(1)}\{i\}$ for the canonical map. We define the divided Frobenius maps following [7, Sec. 5.7]. Finally, we define relative syntomic cohomology.

Definition 7.1 (The relative Frobenius). The Frobenius on the prismatic crystal $\mathcal{H}_{\Delta}(R/A)$ factors as

$$\mathcal{H}_{\Delta}(R/A) \xrightarrow{\mathcal{O}\text{-linear}} \mathcal{H}_{\Delta}^{(1)}(R/A) \xrightarrow{\varphi\text{-semilinear}} F^* \mathcal{H}_{\Delta}^{(1)}(R/A) \xrightarrow{\mathcal{O}\text{-linear}} \mathcal{H}_{\Delta}(R/A).$$

By definition, $\mathrm{N}^{\geq i} F^* \mathcal{H}_{\Delta}^{(1)}(R/A) \rightarrow \mathcal{H}_{\Delta}(R/A)$ factors canonically through $\mathcal{J}^i \otimes \mathcal{H}_{\Delta}(R/A)$. Tensoring with $\mathcal{O}_{\mathrm{WCart}}\{i\}$ and using the equivalence $F^*(\mathcal{O}_{\mathrm{WCart}}\{i\}) \simeq \mathcal{J}^{-i}\{i\}$ we obtain a map

$$\mathrm{N}^{\geq i} F^* \mathcal{H}_{\Delta}^{(1)}(R/A)\{i\} \rightarrow \mathcal{H}_{\Delta}(R/A)\{i\},$$

which is moreover compatible with the Nygaard filtration on the left-hand side (in weights at least i) and the \mathcal{J} -adic filtration on the right-hand-side. Hence, after taking global sections, there are maps

$$\mathrm{N}^{\geq i} \mathrm{R}\Gamma(\mathrm{WCart}, \mathcal{H}_{\Delta}^{(1)}(R/A)\{i\}) \rightarrow \mathrm{R}\Gamma(\mathrm{WCart}, \mathrm{N}^{\geq i} F^* \mathcal{H}_{\Delta}^{(1)}(R/A)\{i\}) \rightarrow \Delta_{R/A}\{i\}.$$

Since the \mathcal{J} -adic filtration on $\mathcal{H}_{\Delta}(R/A)\{i\}$ is complete, inducing the complete Hodge–Tate filtration on $\Delta_{R/A}\{i\}$, this map factors through the Nygaard completion, inducing a filtered map

$$\mathrm{N}^{\geq \star} \widehat{\Delta}_{R/A}^{(1)}\{i\} \rightarrow \Delta_{R/A}^{[\star-i]}\{i\}.$$

Definition 7.2 (The map c_0). The $\mathcal{O}_{\mathrm{WCart}}$ -linear map of crystals $\mathcal{H}_{\Delta}(R/A)\{i\} \rightarrow \mathcal{H}_{\Delta}^{(1)}(R/A)\{i\}$ induces a map $\Delta_{R/A}\{i\} \rightarrow \mathrm{R}\Gamma(\mathrm{WCart}, \mathcal{H}_{\Delta}^{(1)}(R/A)\{i\})$ on global sections, which we may compose with the map to the Nygaard completion $\widehat{\Delta}_{R/A}^{(1)}\{i\}$ to obtain a map

$$c_0: \Delta_{R/A}\{i\} \rightarrow \widehat{\Delta}_{R/A}^{(1)}\{i\}.$$

Construction 7.3. For every filtration $N^{\geq \star}$ indexed on (\mathbf{N}, \geq) , we form a new filtration indexed on (\mathbf{N}, \geq) given as

$$N^{\geq p \star} \otimes p^{\star} \mathbf{Z}_p,$$

the Day convolution product of the re-scaled filtration $N^{\geq p \star}$ and the p -adic filtration on \mathbf{Z}_p . Concretely, this filtration has the form

$$\dots \rightarrow N^{\geq 2p} \oplus_{N^{\geq p, p}} N^{\geq p} \oplus_{N^{\geq 0, p}} N^{\geq 0} \rightarrow N^{\geq p} \oplus_{N^{\geq 0, p}} N^{\geq 0} \xrightarrow{(\mathrm{can}, p)} N^{\geq 0},$$

where the pushout maps to the left are the canonical maps and the maps to the right are the multiplication by p maps. For example, if $N^{\geq \star}$ is the (d) -adic filtration on an oriented transversal prism, then this construction produces the (d^p, p) -adic filtration.

Remark 7.4. If $N^{\geq \star}$ is complete and $N^{\geq 0}$ is p -complete, then the filtration is $N^{\geq p \star} \otimes p^{\star} \mathbf{Z}_p$ complete too.

Lemma 7.5 (The map c). *There exists a natural φ -semilinear, \mathbf{N} -filtered multiplicative map*

$$c: \Delta_{R/A}^{[\star]}\{\star\} \rightarrow (N^{\geq p \star} \widehat{\Delta}_{R/A}^{(1)}\{\star\} \otimes p^{\star} \mathbf{Z}_p).$$

Proof. By Remark 7.4, both sides are complete filtrations, and they are left Kan extended as such from finitely presented free δ -pairs by Corollaries 3.17 and 6.10. Thus to construct the map we can assume that (A, R) is a finitely presented free δ -pair. By relative quasisyntomic descent in R (Proposition 3.15 and Corollary 6.10) we further reduce to the prismatic case where (A, I) is a prism and R is an animated commutative \overline{A} -algebra and assume additionally that $L_{R/\overline{A}}$ has p -complete Tor-amplitude in $[1, 1]$. In this case, all objects in question are discrete. In this case, the map $c_0: \Delta_{R/A}\{i\} \rightarrow \widehat{\Delta}_{R/A}^{(1)}\{i\}$ is the natural map obtained by (p, I) -completed extension of scalars along φ_A by Proposition 6.12. It is enough to prove the result when $i = 0$ and when A is orientable and transversal, with $I = (d)$, in which case $c: \Delta_{R/A} \rightarrow \widehat{\Delta}_{R/A}^{(1)}$ is a φ_A -semilinear map of commutative rings, which sends d to $\varphi_A(d) = d^p + p\delta(d)$. As $d \in N^{\geq 1} \widehat{\Delta}_{R/A}^{(1)}$, it follows that $\varphi_A(d)$ is in weight ≥ 1 with respect to the Day convolution filtration $(N^{\geq p \star} \widehat{\Delta}_{R/A}^{(1)} \otimes p^{\star} \mathbf{Z}_p)$. By multiplicativity, $\varphi_A(d^n)$ is in weight $\geq n$. This shows that c_0 refines to a filtered map as desired for such pairs (A, R) . Descending and left Kan extending produces the map c . \square

Remark 7.6. We expect that by forgetting the filtered structure on c one recovers the map c_0 , but we do not explore that here.

Definition 7.7 (The divided Frobenius). We let $c\varphi$ denote the resulting composition

$$N^{\geq i} \widehat{\Delta}_{R/A}^{(1)}\{i\} \rightarrow \Delta_{R/A}\{i\} \xrightarrow{c} \widehat{\Delta}_{R/A}^{(1)}\{i\}.$$

Definition 7.8 (Relative syntomic complexes). The relative p -adic syntomic complexes $\mathbf{Z}_p(i)(R/A)$ are defined as

$$\mathbf{Z}_p(i)(R/A) = \mathrm{fib} \left(N^{\geq i} \widehat{\Delta}_{R/A}^{(1)}\{i\} \xrightarrow{\mathrm{can} - c\varphi} \widehat{\Delta}_{R/A}^{(1)}\{i\} \right).$$

Remark 7.9. When $A = \mathbf{Z}_p$, we have that $\mathbf{Z}_p(i)(R/\mathbf{Z}_p)$ agrees with the syntomic complexes of [9, 7] by Remark 6.8.

The following result establishes Corollary 1.3 from the introduction.

Corollary 7.10. *For each $i \in \mathbf{Z}$, the relative p -adic syntomic complexes $\mathbf{Z}_p(i)(R/A)$ satisfy descent for maps of pairs $(A, R) \rightarrow (B, S)$ such that $R \rightarrow S$ is a universal descent morphism (with no condition on $A \rightarrow B$).*

Proof. The corollary follows immediately from Corollary 6.10. \square

We close this section by noting that one can compute syntomic cohomology also by a non-completed version of Frobenius-twisted prismatic cohomology. Specifically we set $\Delta_{R/A}^{(1)}\{i\} = \Delta_{R/A}\{i\} \widehat{\otimes}_{A, \varphi} A$ and denote by \tilde{c} the map $\Delta_{R/A}^{(1)}\{i\} \rightarrow \widehat{\Delta}_{R/A}^{(1)}\{i\}$ induced from c .

Construction 7.11. We can define the Nygaard filtration on $\Delta_{R/A}^{(1)}\{i\}$ as the pullback

$$\begin{array}{ccc} N^{\geq *}\Delta_{R/A}^{(1)}\{i\} & \longrightarrow & N^{\geq *}\widehat{\Delta}_{R/A}^{(1)}\{i\} \\ \downarrow & \lrcorner & \downarrow \\ \Delta_{R/A}^{(1)}\{i\} & \xrightarrow{\tilde{c}} & \widehat{\Delta}_{R/A}^{(1)}\{i\}. \end{array}$$

The completion of $N^{\geq *}\Delta_{R/A}^{(1)}\{i\}$ recovers $\widehat{\Delta}_{R/A}^{(1)}\{i\}$ by construction. Since the map $c: \Delta_{R/A}\{i\} \rightarrow \widehat{\Delta}_{R/A}\{i\}$ factors through \tilde{c} by definition, there are decompleted divided Frobenius maps $c\varphi: N^{\geq i}\Delta_{R/A}^{(1)}\{i\} \rightarrow \Delta_{R/A}^{(1)}\{i\}$ compatible with the completed ones. We can then define the equalizer

$$\mathbf{Z}_p(i)(R/A)^{\text{nc}} = \text{fib}\left(N^{\geq i}\Delta_{R/A}^{(1)}\{i\} \xrightarrow{\text{can}-c\varphi} \Delta_{R/A}^{(1)}\{i\}\right).$$

The clumsy notation is only used temporarily, since this constructions turn out to be nothing new:

Proposition 7.12. *The canonical map*

$$\mathbf{Z}_p(i)(R/A)^{\text{nc}} \rightarrow \mathbf{Z}_p(i)(R/A)$$

is an equivalence for any δ -pair where A is bounded and R is an animated commutative A -algebra.

Proof. We treat the second map, the first works completely analogous. We can write any abstract equalizer $\text{fib}(A \xrightarrow{f-g} B)$ as a pullback of the diagram

$$A \xrightarrow{(f,g)} B \times B \xleftarrow{\Delta} B.$$

Now consider the following commutative diagram

$$\begin{array}{ccccc} N^{\geq i}\Delta_{R/A}^{(1)}\{i\} & \xrightarrow{(\text{can}, c\varphi)} & \Delta_{R/A}^{(1)}\{i\} \times \Delta_{R/A}^{(1)}\{i\} & \xleftarrow{\Delta} & \Delta_{R/A}^{(1)}\{i\} \\ \downarrow & \lrcorner & \downarrow & & \parallel \\ N^{\geq i}\widehat{\Delta}_{R/A}^{(1)}\{i\} & \xrightarrow{(\text{can}, c\varphi)} & \widehat{\Delta}_{R/A}^{(1)}\{i\} \times \Delta_{R/A}^{(1)}\{i\} & \xleftarrow{\Delta} & \Delta_{R/A}^{(1)}\{i\} \\ \parallel & & \downarrow & \lrcorner & \downarrow \\ N^{\geq i}\widehat{\Delta}_{R/A}^{(1)}\{i\} & \xrightarrow{(\text{can}, c\varphi)} & \widehat{\Delta}_{R/A}^{(1)}\{i\} \times \widehat{\Delta}_{R/A}^{(1)}\{i\} & \xleftarrow{\Delta} & \widehat{\Delta}_{R/A}^{(1)}\{i\}. \end{array}$$

The pullback of the top and bottom horizontal row are $\mathbf{Z}_p(i)(R/A)^{\text{nc}}$ and $\mathbf{Z}_p(i)(R/A)$, respectively. We claim that in the diagram the maps between the pullback of all rows are in fact equivalences. For the map from the pullback of the first row to the second this follows by the fact that the upper left square is a pullback and the right upper vertical map is an equivalence. For the map from the pullback of the middle row to the pullback of the last row this follows similarly since the left lower vertical map is an equivalence and the right lower square a pullback. \square

8 The prismatic package

Recall from the introduction the definition of the ∞ -category \mathcal{C}_A which is the natural home of the prismatic package (we will review this soon). The main goal of this section is to conclude from the results of the previous sections that our objects

$$\underline{\Delta}_{R/A} = \left(\Delta_{R/A}^{[\star]}, \mathbf{N}^{\geq \star} \widehat{\Delta}_{R/A}^{(1)} \{\star\}, c, \varphi \right)$$

indeed form objects of this category and then use this to define and study syntomic cohomology.

First we would like to rigorously define \mathcal{C}_A . It will be the category of \mathbf{E}_∞ -algebras in a symmetric monoidal category \mathcal{G}_A so that we would like to define \mathcal{G}_A .

By a filtered, graded A -module we mean a functor

$$H: (\mathbf{Z}, \geq) \times \mathbf{Z} \rightarrow \mathbf{D}(A),$$

where (\mathbf{Z}, \geq) denotes the poset of integers and \mathbf{Z} denotes the discrete set of integers. We shall write $H^{\geq i}\{j\}$ for the evaluation at i and j . The category of filtered, graded A -module admits a symmetric monoidal structure given by Day convolution with respect to the sum operation on both \mathbf{Z} 's. We note that this tensor product does not involve any signs or so shifts (as sometimes used on graded/filtered things). Such a filtered graded A -module is called complete if for fixed j all the filtered objects $H^{\geq i}\{j\}$ are complete, i.e. $\varprojlim_i H^{\geq i}\{j\} = 0$. The tensor product descends to a tensor product on complete things.

There are two operations that we can perform on a filtered, graded object. The first is the *shearing*, which changes the filtration degree by subtracting the grading degree:

$$(\text{sh}H)^{\geq i}\{j\} = H^{\geq i-j}\{j\}$$

This is indeed a lax symmetric monoidal endofunctor of the ∞ -category of filtered, graded objects since it is induced from a symmetric monoidal endofunctor of $(\mathbf{Z}, \geq) \times \mathbf{Z}$.

The second construction sends a filtration N to a new filtration indexed on $(\mathbf{N}, \geq) \times \mathbf{Z}$ given as

$$\text{scp}(N) = (N^{\geq p^\star}\{j\} \otimes p^\star\mathbf{Z})$$

explained in Construction 7.3 above. Again this is a lax symmetric monoidal operation since $p^\star\mathbf{Z}$ is an algebra. The filtration $\text{scp}(N)$ has the property that if we take the levelwise modulo p -reduction we get the convolution of the scaled filtration $N^{\geq p^\star}/p$ with the filtration on \mathbf{F}_p given by $\dots \rightarrow \mathbf{F}_p \xrightarrow{0} \mathbf{F}_p \xrightarrow{0} \mathbf{F}_p$. This filtration has a multiplicative map to the filtration $\dots \rightarrow 0 \rightarrow 0 \xrightarrow{0} \mathbf{F}_p$ which is the Day convolution tensor unit over \mathbf{F}_p . Thus we conclude that there is a canonical map of filtrations

$$\text{scp}(N)/p \rightarrow N^{\geq p^\star}/p.$$

Also note that if N is complete and p -complete, then so is $\text{scp}(N)$.

Definition 8.1. We define the ∞ -category \mathcal{G}_A to be the ∞ -category consisting of quadruples (H, N, c, φ) where H and N are p -complete and complete filtered graded A -modules, the filtration on N is constant in non-positive degrees (equivalently it is $(\mathbf{N}, \geq) \times \mathbf{Z}$ filtered), the map $c: H \rightarrow \text{scp}(N)$ is a φ_A -semilinear map of $(\mathbf{N}, \geq) \times \mathbf{Z}$ -filtered, graded A -modules and φ is a map of graded filtered objects $N \rightarrow \text{sh}(H)$ over A .

As an ∞ -operad, the category \mathcal{G}_A is given as the pullback

$$\begin{array}{ccc} & & \widehat{\text{Fun}}((\mathbf{N}, \geq) \times \mathbf{Z}, \text{D}(A))^{\Delta^1 \amalg \Delta^1} \\ & & \downarrow \text{ev} \\ \widehat{\text{Fun}}((\mathbf{Z}, \geq) \times \mathbf{Z}, \text{D}(A)) \times \widehat{\text{Fun}}((\mathbf{N}, \geq) \times \mathbf{Z}, \text{D}(A)) & \xrightarrow{(\pi_1, \text{sh} \circ \pi_2, \pi_2, \varphi^* \circ \text{oscp} \circ \pi_1)} & \widehat{\text{Fun}}((\mathbf{N}, \geq) \times \mathbf{Z}, \text{D}(A))^4, \end{array}$$

where each vertex is viewed as an ∞ -operad using the Day convolution symmetric monoidal structures on Fun and the pointwise monoidal structures on $(-)^{\Delta^1 \amalg \Delta^1}$ and $(-)^4$, where the notion $\widehat{\text{Fun}}$ indicates that we consider complete and p -complete filtrations, and where the map ev is evaluation at the four vertices of $\Delta^1 \amalg \Delta^1$.

The ∞ -category \mathcal{C}_A is defined as the category of \mathbf{E}_∞ -algebras in \mathcal{G}_A .

Note that since \mathcal{C}_A is algebras in a pullback, this ∞ -category is also the pullback of the respective categories of algebras, which are given by categories of lax symmetric functors [resp. arrows in lax symmetric monoidal functors. monoidal functors.](#)

Remark 8.2.

- (i) The category \mathcal{G}_A is a variant of the category of prismatic F -gauges of Bhatt–Lurie, except that it is not a category of quasi-coherent sheaves on a variant of WCart but rather the global sections of such sheaves. We view this category \mathcal{C}_A simply as a tool to formalize the structure present on prismatic cohomology and streamline some of the coming proofs.
- (ii) For any δ -ring A we have an object $\underline{A} \in \mathcal{G}_\mathbf{Z}$ given by setting

$$N = H = \left(\cdots \rightarrow 0 \rightarrow 0 \rightarrow A \xrightarrow{\text{id}} A \xrightarrow{\text{id}} \cdots \right)$$

and the map $c = \varphi$ and $\varphi = \text{id}$. (The reason for the surprising interchange is that we think of the N as the Frobenius twist of H , but that identifies with A .) Then an A -module internal to $\mathcal{G}_\mathbf{Z}$ is the same as an object in \mathcal{G}_A . Thus we might as well work with the absolute category.

- (iii) The assignment $A \mapsto \mathcal{C}_A$ defines an fpqc stack (in stable ∞ -categories) on the opposite of the category of δ -rings and δ -ring maps. Thanks to the pullback definition of \mathcal{C}_A this boils down to checking that each vertex in the pullback is an fpqc stack which in turn follows from the fact that $A \mapsto \widehat{\text{Fun}}((\mathbf{Z}, \geq), \text{D}(A))$ defines an fpqc stack as $\widehat{\text{Fun}}((\mathbf{Z}, \geq), \text{D}(A)) \simeq \text{Mod}_{\mathbf{D}_-}(\text{GrD}(A))$ where \mathbf{D}_- is the graded ring $\mathbf{Z} \oplus \mathbf{Z}[-1](1)$. See for example [27, Thm. 3.2.14].

Proposition 8.3. *The ∞ -categories \mathcal{C}_A and \mathcal{G}_A have all colimits and taking the associated graded of H and N preserves colimits.*

Proof. All functors on the pullback description of \mathcal{G}_A preserve colimits. Thus colimits in \mathcal{G}_A are computed underlying in $\widehat{\text{Fun}}((\mathbf{Z}, \geq) \times \mathbf{Z}, \text{D}(A)) \times \widehat{\text{Fun}}((\mathbf{N}, \geq) \times \mathbf{Z}, \text{D}(A))$. In the category of complete filtrations colimits are computed by taking the colimit in filtrations and then completing. These induce colimits on associated graded pieces and can in fact be determined there.

The case of algebras either follows the same way (it is also a pullback of the categories of algebras) or simply using that algebras have colimits if the underlying category has operadic colimits. \square

Definition 8.4. The syntomic complex $\mathbf{Z}_p(i)(H, N, c, \varphi)$ for an object of our category (H, N, c, φ) is defined as

$$\mathrm{fib} \left(N^{\geq i} \{i\} \xrightarrow{\mathrm{can}-c\varphi} N^{\geq 0} \{i\} \right) \in \mathrm{D}(\mathbf{Z})_p^\wedge$$

Similarly, we define $\mathbf{F}_p(i)$ as the mod p reduction $\mathbf{Z}_p(i)/p$.

Lemma 8.5. Fix an integer i . If j satisfies $(p-1)j > pi$, then for every object of \mathcal{C}_A the morphism

$$N^{\geq j} \{i\}/p \xrightarrow{\mathrm{can}-c\varphi} N^{\geq j-i} \{i\}/p$$

lifts canonically and functorially as

$$N^{\geq j} \{i\}/p \xrightarrow{\simeq} N^{\geq j} \{i\}/p \xrightarrow{\mathrm{can}} N^{\geq j-i} \{i\}/p.$$

Proof. We use that mod p we have a morphism $c: H^{\geq *}/p \rightarrow N^{\geq p^*}/p$ of filtrations. Thus in particular a map

$$l: H^{\geq j-i} \xrightarrow{c} N^{\geq p(j-i)} \xrightarrow{\mathrm{can}} N^{\geq j}$$

since $p(j-i) > j$ by assumption. Thus the map $c\phi: N^{\geq j} \rightarrow N^{\geq j-i}$ in fact lifts through a map $N^{\geq j} \rightarrow N^{\geq j}$ and thus the difference $\mathrm{can} - c\phi$ lifts through the map $\mathrm{id} - l$. Now, we claim that $\mathrm{id} - l$ is in fact an equivalence. This follows since l lifts one filtration bit further and both id and l are filtered maps (applying the construction for all j). Since the filtration on $N^{\geq j}$ is complete we can test equivalences on graded pieces, where id is obviously an isomorphism and l is zero, since it lifts further. \square

Proposition 8.6. Fix an integer i . If j satisfies $(p-1)j \geq i$, then we have an equivalence

$$\mathbf{F}_p(i)(H, N, \varphi, c) \simeq \mathrm{fib} \left(N^{\geq i} \{i\}/N^{\geq j} \{i\}/p \xrightarrow{\mathrm{can}-c\varphi} N^{\geq 0} \{i\}/N^{\geq j} \{i\}/p \right).$$

In other words: we may quotient out a high enough Nygaard filtered pieces to compute syntomic cohomology.

Proof. Consider the diagram

$$\begin{array}{ccc} N^{\geq j} \{i\}/p & \xrightarrow{\simeq} & N^{\geq j-i} \{i\}/p \\ \downarrow \mathrm{can} & & \downarrow \mathrm{can} \\ N^{\geq i} \{i\}/p & \xrightarrow{\mathrm{can}-c\varphi} & N^{\geq 0} \{i\}/p \end{array}$$

which exist by the previous lemma. We take the vertical cofiber. Now the claim is that this doesn't change the horizontal fibers, which is obvious since the upper horizontal fiber is zero. \square

Corollary 8.7. The functors $\mathbf{Z}_p(i), \mathbf{F}_p(i): \mathcal{G}_A \rightarrow \mathrm{D}(\mathbf{Z})_p^\wedge$ preserve all colimits and all limits.

Proof. For a fixed i , fix a j as in the statement of Proposition 8.6. Then, $\mathbf{F}_p(i)$ is a fiber of two functors which evidently preserve all colimits since they only depend on finitely many graded pieces. Thus, $\mathbf{F}_p(i)$ preserves all finite colimits. To check the statement for $\mathbf{Z}_p(i)$ we can reduce mod p since we land in p -complete complexes and thus to the statement about $\mathbf{F}_p(i)$. The preservation of limits is clear. \square

Theorem 8.8. The object

$$\underline{\Delta}_{R/A} = \left(\underline{\Delta}_{R/A}^{[*]}, N^{\geq *} \widehat{\underline{\Delta}}_{R/A}^{(1)} \{*\}, c, \varphi \right)$$

is an object of \mathcal{C}_A . In fact it refines to a functor $\mathrm{Pairs}^\delta \rightarrow \mathcal{C}_{\mathbf{Z}}$, which is an algebra over the functor $(A, R) \mapsto \underline{A}$ (see Remark 8.2). As such it is left Kan extended from free δ -pairs.

Proof. The object $\Delta_{R/A}^{[\star]}$ is constructed in Section 3 as a functor to complete filtered objects. It is shown in Corollary 3.17 that it is Kan extended as such (note that this Kan extension only involves filtered colimits, so we do not have to worry about the distinction whether we Kan extend with or without algebra structure). The fact that it is a graded, filtered \mathbf{E}_∞ -algebra follows from the combination of two facts. First the facts that the crystal $\mathcal{H}_\Delta^{[\star]}(R/A)\{\star\}$ is a graded filtered algebra, which follows from the fact that the filtration is multiplicative (since it is I -adic on the crystal) and that we tensor with a tensor algebra over a line bundle, which is also multiplicative. Secondly the functor taking global sections of prismatic crystals is a lax symmetric monoidal functor.

The object $N^{\geq \star} \widehat{\Delta}_{R/A}^{(1)}\{\star\}$ is produced in Definition 6.7 and it is multiplicative for the same reason as $\Delta_{R/A}^{[\star]}\{\star\}$ above combined with the global pullback against another multiplicative filtrations. One simply has to check that the comparison functors in the defining pullback are also multiplicative, which is clear by construction, see Lemma 6.4. This term is Kan extended by Corollary 6.10.

The maps φ and c are produced at the beginning of Section 7 and are clearly multiplicative (the filtration by definition). Finally we note that the description of colimits in \mathcal{C}_Z then implies the Kan extension statement and we are done. \square

Now, we can prove Corollary 1.4.

Corollary 8.9. *For each $i \in \mathbf{Z}$, syntomic cohomology as a functor*

$$\mathbf{Z}_p(i)(-/-): \text{Pairs}^\delta \rightarrow D(\mathbf{Z})_p^\wedge$$

is left Kan extended from finitely presented free δ -pairs.

Proof. We consider the prismatic package

$$\underline{\Delta}_{R/A} = \left(\Delta_{R/A}^{[\star]}\{\star\}, N^{\geq \star} \widehat{\Delta}_{R/A}^{(1)}\{\star\}, c, \varphi \right) \in \mathcal{C}_A$$

as an object of \mathcal{C}_Z by forgetting the A -algebra structure. As such a functor, it is left Kan extended by Theorem 8.8, so the result follows by Corollary 8.7 and the fact that syntomic cohomology on \mathcal{C}_A factors through the forgetful functor $\mathcal{C}_A \rightarrow \mathcal{C}_Z$. \square

Corollary 8.10. *For $i < 0$, $\mathbf{Z}_p(i)(R/A) \simeq 0$.*

Proof. By Corollary 8.9, it is enough to prove the corollary when (A, R) is a finitely presented free δ -pair and then we can use Corollary 7.10 to reduce to the bounded prismatic case and even to the bounded oriented prismatic case, so assume that $(A, (d))$ is an oriented bounded prism. Using Proposition 6.12, we know our construction agrees with the Nygaard filtration on relative prismatic cohomology introduced in [10]. For $i < 0$, the Frobenius $c\varphi: \widehat{\Delta}_{R/A}^{(1)}\{i\} \rightarrow \widehat{\Delta}_{R/A}^{(1)}\{i\}$ is divisible by d^{-i} . It follows that, for $i < 0$, $c\varphi$ is topologically nilpotent, so that $1 - c\varphi$ is an equivalence. \square

9 Computation using descent

Recall that $\Delta_{R/A}$ has descent in A and R (Proposition 3.15) and is invariant under relatively perfect maps and completion in A (Proposition 3.12 and Corollary 3.17). Knowing this, we can give a descent style description of $\Delta_{R/A}$ using a relative version of the quasisyntomic site, that we will describe now. In fact one could also use this to give an independent definition of the prismatic package, specifically the Nygaard filtration. This is not only a theoretical tool, but will be applied for practical calculations. After this discussion, we prove the uniqueness claim from Theorem 1.2.

Definition 9.1 (Relatively quasisyntomic rings). A bounded δ -pair (A, R) is relatively quasisyntomic if R is derived p -complete and $L_{R/A}$ has p -complete Tor-amplitude in $[0, 1]$. If A is a bounded δ -ring, let \mathcal{QSyn}_A be the category of relatively quasisyntomic δ -pairs (A, R) .¹⁵

Definition 9.2 (Relatively quasiregular semiperfectoids). Say that a bounded δ -pair (A, R) is relatively quasiregular semiperfectoid (or relatively qrsp) if R is p -complete, there is a factorization $A \rightarrow A' \rightarrow R$ where $A \rightarrow A'$ is a p -completely relatively perfect map of δ -rings, the δ -pair (A', R) is pre-prismatic as exhibited by an ideal $I \subseteq A'$, the map $A' \rightarrow R$ is surjective, and $L_{R/A'}$ has p -complete Tor-amplitude in $[1, 1]$. Call such a factorization a pre-prismatic factorization. Note that a relatively quasiregular semiperfectoid is relatively quasisyntomic in the sense of Definition 9.1. Let $\mathcal{RQRSPerfd} \subseteq \mathbf{Pairs}^\delta$ be the full subcategory consisting of the relatively quasiregular semiperfectoid δ -pairs.

Note that we can replace A' by $A'[\delta(I)^{-1}]_{(p,I)}^\wedge$, so that we can assume in fact that A' is a prism, with $\Delta_{R/A} = \Delta_{R/A'}$.

Example 9.3. Let (A, I) be a bounded δ -ring with a Cartier divisor I . If r_1, r_2, \dots forms a regular sequence in A/I and if $A \rightarrow R = A/(I, r_1, r_2, \dots)$ is pre-prismatic, then it is relatively quasiregular semiperfectoid with $A' = A$.

Example 9.4. Assume that R is a semiperfectoid ring in the sense of [9, Definition 4.20]. Then we claim that $\mathbf{Z}_p \rightarrow R$ is relatively quasiregular semiperfectoid in the sense of Definition 9.2. To see this we pick a perfectoid ring R' such that $R = R'/I$ and $L_{R/R'}$ has Tor amplitude in $[1, 1]$, see [9, Lemma 4.25]. Then $A' = A_{\text{inf}}(R)$ exhibits that $\mathbf{Z}_p \rightarrow R$ is relatively qrsp. Assume conversely that $\mathbf{Z}_p \rightarrow R$ is relatively quasiregular semiperfectoid in the sense of Definition 9.2. Then, the prism in the definition has to be perfect so that A'/I is perfectoid and R a quotient thereof, which implies that R is qrsp in the sense of [9, Definition 4.20].

Example 9.5. Assume that (A, I) is a bounded prism. Recall from [10, Definition 15.1] that a A/I -algebra S is called large if it is flat, the cotangent complex has Tor-Amplitude in $[1, 1]$ and there is a surjection

$$A/I[X_i^{1/p^\infty}]_p^\wedge \rightarrow S$$

for some set $i \in I$. In this case we can consider $A' = A[X_i^{1/p^\infty}]_{(p,I)}^\wedge$ which is relatively perfect over A . Moreover we have an induced surjective map $A' \rightarrow S$, for which $L_{S/A'}$ has Tor-Amplitude in $[1, 1]$.

The significance of relatively qrsp pairs is that prismatic cohomology can be understood very explicitly.

Theorem 9.6 (Bhatt–Scholze). *Let $A \rightarrow R$ be relatively qrsp. Then the prismatic cohomologies $\Delta_{R/A}^{[\star]} \{\star\}$ and $N^{\geq \star} \widehat{\Delta}_{R/A}^{(1)} \{\star\}$ including all the filtered pieces and all the graded pieces are concentrated in degree 0 and the relative Frobenius $N^{\geq \star} \widehat{\Delta}_{R/A}^{(1)} \{\star\} \rightarrow \Delta_{R/A}^{[\star]} \{\star\}$ is injective.*

If more specifically $R = A'/(I, r_1, r_2, \dots)$ for some prism (A', I) and a regular sequence $r_1, r_2, \dots \in A'/I$ then we have

$$\Delta_{R/A} \simeq A' \left\{ \frac{r_1}{I}, \frac{r_2}{I}, \dots \right\}_{(p,I)}^\wedge \quad \text{and} \quad \widehat{\Delta}_{R/A}^{(1)} \simeq A' \left\{ \frac{\varphi(r_1)}{\varphi(I)}, \frac{\varphi(r_2)}{\varphi(I)}, \dots \right\}_{(p,N)}^\wedge$$

¹⁵The notation \mathcal{QSyn}_A diverges from that of [6], which would require also that A and R be quasisyntomic in the sense of [9]. However, $\mathcal{QSyn}_{\mathbf{Z}_p}$ does agree with the category by the same name in [6] and also with \mathcal{QSyn} from [9].

with the Hodge-Tate filtration on $\Delta_{R/A}$ given by the I -adic filtration and the Nygaard filtration on $\widehat{\Delta}_{R/A}^{(1)}$ given by the intersection¹⁶ of the I -adic filtration with the obvious inclusion

$$A' \left\{ \frac{\varphi(r_1)}{\varphi(I)}, \frac{\varphi(r_2)}{\varphi(I)}, \dots \right\}_{(p,N)}^\wedge \subseteq A' \left\{ \frac{r_1}{I}, \frac{r_2}{I}, \dots \right\}_{(p,I)}^\wedge$$

The map $c: \Delta_{R/A} \rightarrow \widehat{\Delta}_{R/A}^{(1)}$ is given by applying φ and the relative Frobenius $\widehat{\Delta}_{R/A}^{(1)} \rightarrow \Delta_{R/A}$ is given by the inclusion. The Breuil–Kisin twisted versions have the analogous form with A' replaced by $A'\{i\}$.

Proof. Using that $\underline{\Delta}_{R/A} \simeq \underline{\Delta}_{R/A'}$ this reduces to the work of Bhatt–Scholze [10] where it is shown in Sections 12 and 15. \square

Definition 9.7 (Quasismooth maps). Say that a map of δ -rings $A \rightarrow A'$ is quasismooth if it is p -completely flat and $L_{A'/A}$ has p -complete Tor-amplitude in $[0, 0]$. Such a map is a quasismooth cover if it is additionally p -completely faithfully flat.

Remark 9.8. Note that if we have a factorization $A \rightarrow A' \rightarrow R$ with $A \rightarrow A'$ quasismooth and $R \in \mathcal{QSyn}_A$ then the transitivity triangle

$$L_{A'/A} \otimes_{A'} R \rightarrow L_{R/A} \rightarrow L_{R/A'}$$

implies that $R \in \mathcal{QSyn}_{A'}$.

Lemma 9.9. *Assume that (A, R) is a relatively quasisyntomic δ -pair. Then there is a factorization $A \rightarrow A' \rightarrow R$ with $A \rightarrow A'$ a quasismooth cover and (A', R) relatively qrsp. In fact we can choose A' in such a way that (A', R) is pre-prismatic, exhibited by an ideal I , such that $A'/I \rightarrow R$ is a surjection whose cotangent complex has Tor-Amplitude in $[1, 1]$.*

Proof. Choose a surjection $\kappa: A[\{z_i\}_{i \in I}]_p^\wedge \rightarrow R$ extending $A \rightarrow R$. Now set

$$A' = A[\{z_i\}_{i \in I}, z]_p^\wedge$$

with the δ -ring structure with $\delta(z_i) = \delta(z) = 0$. Moreover A' is a quasismooth A algebra. Now consider the map $A' \rightarrow R$ extending κ with $z \mapsto p$. Then, the kernel contains $I = (z - p)$ which exhibits the δ -pair (A', R) as pre-prismatic. Moreover the map $\overline{A'} = A'/I \rightarrow R$ is surjective and the transitivity triangle for $A \rightarrow \overline{A'} \rightarrow R$ takes the form

$$L_{\overline{A'}/A} \otimes_{\overline{A'}} R \rightarrow L_{R/A} \rightarrow L_{R/\overline{A'}}$$

The first term has Tor-Amplitude in degree 0. The second term has, by quasisyntomicity of (A, R) , Tor-Amplitude in $[0, 1]$. Moreover since $\overline{A'} \rightarrow R$ is surjective, the first map is surjective on π_0 . It follows that $L_{R/\overline{A'}}$ has Tor-Amplitude in $[1, 1]$. \square

We note that the converse also holds: whenever we have a bounded p -complete δ -pair (A, R) that admits a quasismooth cover by a relatively qrsp (A', R) , the pair (A, R) is relatively quasisyntomic.

Lemma 9.10. *For any factorization $A \rightarrow A^0 \rightarrow R$ as in the last lemma (i.e., $A \rightarrow A^0$ is a quasismooth cover and (A^0, R) is relatively qrsp), consider the Čech diagram A^\bullet for $A \rightarrow A^0$ with $A^n \simeq A^0 \otimes_A \dots \otimes_A A^0$. All the terms $A^n \rightarrow R$ are relatively qrsp.*

¹⁶In particular, (p, N) -convergent sequences are exactly the ones that converge (p, I) -adically in the ambient space of the inclusion $\widehat{\Delta}_{R/A}^{(1)} \subseteq \Delta_{R/A}$.

Proof. Let us first assume that (A^0, R) is already a prismatic δ -pair. Then all the terms A^n are also prisms using the ideals given as base changes along any of the flat maps $A^0 \rightarrow A^n$. Moreover consider the transitivity triangles

$$L_{A^n/A^0} \otimes_{A^n} R \rightarrow L_{R/A^0} \rightarrow L_{R/A^n} .$$

where the first term has Tor-Amplitude in $[0, 0]$ and the second in $[1, 1]$. It follows that L_{R/A^n} has Tor-Amplitude in $[1, 1]$.

In the general case, suppose that $A^0 \rightarrow B \rightarrow R$ is a factorization where $A^0 \rightarrow B$ is p -completely relatively perfect, $B \rightarrow R$ is a surjective pre-prismatic δ -pair, and $L_{R/B}$ has Tor-amplitude in $[1, 1]$. Then, $A^n \rightarrow R$ factors as $A^n \rightarrow B \otimes_{A^0} A^n \rightarrow R$, which is a factorization exhibiting R as relatively qrsp over A^n , as desired. \square

For a fixed bounded R , consider the category \mathcal{QSyn}_R of relatively quasisyntomic bounded δ -pairs (A, R) . We have the full subcategory

$$\mathcal{RQRSPerf}_R \subseteq \mathcal{QSyn}_R$$

and consider the opposite of both categories as sites with the quasismooth topologies. We leave it as an exercise to see that these are indeed sites: the key is to verify that for a quasismooth cover $A \rightarrow A'$ and a map $A \rightarrow B$ over R the map $A' \otimes_A B \rightarrow R$ is relatively quasisyntomic. This follows using that $B \rightarrow A' \otimes_A B$ is a quasismooth cover using Remark 9.8.

Proposition 9.11 (Unfolding in the base). *For each fixed R , the inclusion induces an equivalence*

$$\mathrm{Shv}_{\mathcal{C}}(\mathcal{QSyn}_R^{\mathrm{op}}) \simeq \mathrm{Shv}_{\mathcal{C}}(\mathcal{RQRSPerf}_R^{\mathrm{op}})$$

of ∞ -categories of \mathcal{C} -valued sheaves for any ∞ -category \mathcal{C} .

Proof. This follows from the previous lemmas by an argument similar to the proof of [9, Prop. 4.31]. \square

Definition 9.12 (Quasisyntomic maps). Say that a map $R \rightarrow R'$ of commutative rings is quasisyntomic if it is p -completely flat and $L_{R'/R}$ has p -complete Tor-amplitude in $[0, 1]$. Such a map is a quasisyntomic cover if it is additionally p -completely faithfully flat. Note that if A is a bounded δ -ring, $R \in \mathcal{QSyn}_A$, and $R \rightarrow R'$ is quasisyntomic, then $R' \in \mathcal{QSyn}_A$.

Lemma 9.13. *A bounded δ -pair (A, R) with R derived p -complete is relatively quasisyntomic if and only if there is a quasisyntomic cover $R \rightarrow R'$ with $R' \in \mathcal{RQRSPerf}_A$.*

Proof. Assume that there exists a quasisyntomic cover $R \rightarrow R'$ with $R' \in \mathcal{RQRSPerf}_A$. Then, there is a transitivity triangle

$$L_{R/A} \otimes_R R' \rightarrow L_{R'/A} \rightarrow L_{R'/R}$$

which shows that $L_{R/A} \otimes_R R'$ has Tor-Amplitude in $[-1, 1]$, which implies that $L_{R/A}$ has Tor-Amplitude in $[-1, 1]$ by faithful flatness. Since $L_{R/A}$ is connective it follows that it has Tor-Amplitude in $[0, 1]$, i.e. (A, R) is relatively quasisyntomic.

Assume conversely that (A, R) is relatively quasisyntomic. Choose a set of p -complete generators $\{r_i\}_{i \in I}$ of R as an A -algebra and consider the cover

$$R \rightarrow R[p^{1/p^\infty}, \{r_i^{1/p^\infty}\}_{i \in I}]_p^\wedge = R'$$

This cover is quasisyntomic. We claim moreover that (A, R') is relatively qrsp. To see this we consider

$$A' = A[\{z_i^{1/p^\infty}\}_{i \in I}, z^{1/p^\infty}]_p^\wedge$$

with the δ -ring structure with $\delta(z_i^{1/p^k}) = 0$ and $\delta(z^{1/p^k}) = 0$. The map $A \rightarrow A'$ of δ -rings is p -completely relatively perfect. Moreover, the map

$$A' \rightarrow R'$$

which sends z^{1/p^k} to p^{1/p^k} and z_i^{1/p^k} to r_i^{1/p^k} is surjective by construction and has the ideal $(z - p)$ in its kernel, so (A', R) is pre-prismatic. \square

Similar to the unfolding in the base we would like also see that one can use descent in the ring to compute prismatic cohomology. For a fixed δ -ring A let $\mathcal{RQRSPerfd}_A$ be the full subcategory of \mathcal{QSyn}_A consisting of the qrsp A -algebras. We claim that for fixed A the categories $\mathcal{QSyn}_A^{\text{op}}$ and $\mathcal{RQRSPerfd}_A^{\text{op}}$ form sites with respect to the quasisyntomic topologies. To see this, it is enough to see that if S' is the p -completed pushout of a diagram $R' \leftarrow R \rightarrow S$ of maps in \mathcal{QSyn}_A (resp. $\mathcal{RQRSPerfd}_A$) where $R \rightarrow R'$ is a quasisyntomic cover, then $S' \in \mathcal{QSyn}_A$ (resp. $\mathcal{RQRSPerfd}_A$). This is left to the reader as an exercise with the conormal sequence for the cotangent complex.

Proposition 9.14 (Unfolding in the ring). *The inclusion $\mathcal{RQRSPerfd}_A^{\text{op}} \hookrightarrow \mathcal{QSyn}_A^{\text{op}}$ induces an equivalence $\text{Shv}_{\mathcal{C}}(\mathcal{QSyn}_A^{\text{op}}) \simeq \text{Shv}_{\mathcal{C}}(\mathcal{RQRSPerfd}_A^{\text{op}})$ of ∞ -categories of \mathcal{C} -valued sheaves for any ∞ -category \mathcal{C} .*

Proof. If $R \in \mathcal{QSyn}_A$ and $f: R \rightarrow S$ is a quasisyntomic map with $S \in \mathcal{RQRSPerfd}_A$, then every S^n in the Čech complex of f is in $\mathcal{RQRSPerfd}_A$ as well. Indeed, $L_{S/A}$ has p -complete Tor-amplitude in $[1, 1]$, so $L_{S/R}$ has in fact p -complete Tor-amplitude in $[1, 1]$ as well. Thus, every $L_{S^n/S^{n-1}}$ (via any choice of maps in the cosimplicial diagram) has p -complete Tor-amplitude in $[1, 1]$, so $L_{S^n/A'}$ has p -complete Tor-amplitude in $[1, 1]$ for any choice of prismatic factorization of $S = S^0$. Now, follow the proof of [9, Prop. 4.31] using the previous lemma to show that every object is locally relatively quasiregular semiperfectoid. \square

Now we want to explain why and how Theorem 1.2 uniquely determines relative prismatic cohomology and how to compute it in practice. This will also show the advantage of the relative approach since we will use descent in A which would not be possible otherwise. We would like to argue that the assignment

$$(R, A) \mapsto \underline{\Delta}_{R/A} \in \mathcal{C}_A$$

is uniquely determined by properties (1), (2), (4), (6) and (8) (or alternatively by (1), (2), (4), (5) and (8)). In fact a much weaker axiom (§) replacing (1), (2) and (8) is sufficient to determine the whole theory in the presence of (4) and (5) or (6):

(§) If (A, R) is relatively qrsp, with prism (A', I) then $\underline{\Delta}_{R/A}$ agrees naturally with the derived prismatic cohomology $\underline{\Delta}_{R/A'}$ of Bhatt and Scholze [10], which in this case is discrete and the relative Frobenius is injective.

The advantage of this formulation is that it can be formulated without reference to Bhatt–Scholze’s theory, as long as one constructs the prismatic cohomology $\underline{\Delta}_{R/A'}$ explicitly, e.g. as an initial prism. Here naturality in (A', I) means that we have to make the choice of (A', I) and it depends on that, since the Bhatt–Scholze theory a priori does. More precisely there is a category $\mathcal{RQRSPerfd}'$ of relatively qrsp pairs (A, R) with a choice of a prism (A', I) and maps given by maps of δ -pairs that come with a compatible choice of a map between the respective prisms. Axiom (§) asserts the existence of a commutative diagram

$$\begin{array}{ccc} \mathcal{RQRSPerfd}' & \longrightarrow & \text{Pairs}^{\Delta} \\ \downarrow T & & \downarrow \underline{\Delta}^{\text{rel}} \\ \mathcal{RQRSPerfd} & \xrightarrow{\underline{\Delta}} & \mathcal{C}_{\mathbf{Z}}, \end{array} \quad (3)$$

where the first two functors are the forgetful functors.

Proposition 9.15. *The functor $\underline{\Delta}$ is uniquely¹⁷ characterized on Pairs^δ by (§), (4) and (6) or alternatively by (§), (4) and (5).*

Proof. Using (4), the functor is uniquely determined by its restriction to quasisyntomic δ -pairs since it is Kan extended from those. Now using (6) and Proposition 9.11 or (5) and Proposition 9.14 it is uniquely determined by its restriction to relatively qrsp pairs. Now we want to argue that diagram (3) already uniquely determines the right hand functor $\underline{\Delta}$, in other words there is at most one dashed functor

$$\begin{array}{ccc} \mathcal{RQRSPerfd}' & \xrightarrow{\underline{\Delta}^{\text{rel}}} & \mathcal{C}_{\mathbf{Z}} \\ \downarrow T & \dashrightarrow & \\ \mathcal{RQRSPerfd} & & \end{array}$$

As a first step we note that the upper horizontal functor $\underline{\Delta}^{\text{rel}}$ really takes values in the full subcategory $\mathcal{C}'_{\mathbf{Z}} \subseteq \mathcal{C}_{\mathbf{Z}}$ given by those quadruples (H, N, c, φ) where H and N are discrete, strict filtrations (i.e. all transition maps are injective), and φ is injective. This subcategory is actually a 1-category. Since the left vertical functor $T: \mathcal{RQRSPerfd}' \rightarrow \mathcal{RQRSPerfd}$ is by definition essentially surjective this also implies that any lift has to take values in $\mathcal{C}'_{\mathbf{Z}}$ and therefore we may replace $\mathcal{C}_{\mathbf{Z}}$ by $\mathcal{C}'_{\mathbf{Z}}$. Moreover, there is a forgetful functor

$$U: \mathcal{C}'_{\mathbf{Z}} \rightarrow \widehat{\text{Fil}}(\text{Ab}) \quad (H, N, c, \varphi) \mapsto H,$$

where $\widehat{\text{Fil}}(\text{Ab})$ is the category of derived complete strictly filtered abelian groups, i.e. functors $(\mathbf{Z}, \geq) \rightarrow \text{Ab}$ whose transition functions are injective and whose derived inverse limit vanishes. The functor U is faithful, which follows since maps between quadruples are determined on the filtration H since φ is injective, i.e. the map on N is determined by the map on H . Applying the next Lemma 9.16 to U and T shows that to prove the proposition it suffices to show that there is at most one dashed lift in the diagram

$$\begin{array}{ccc} \mathcal{RQRSPerfd}' & & \\ \downarrow T & \searrow \underline{\Delta}^{\text{rel}} & \\ \mathcal{RQRSPerfd} & \dashrightarrow & \widehat{\text{Fil}}(\text{Ab}). \end{array} \quad (4)$$

To see this we note that for a given object $A \rightarrow A' \rightarrow R \in \mathcal{RQRSPerfd}'$ we have that $(B, \overline{B}) = (\underline{\Delta}_{R/A}^{\text{rel}}, \overline{\underline{\Delta}}_{R/A'}^{\text{rel}})$ is a prism equipped with a natural map $\chi: A' \rightarrow B$ of prisms by [10, Lemma 7.7]. Moreover, this map induces an equivalence

$$\underline{\Delta}_{R/A}^{\text{rel}} \xrightarrow{\simeq} \underline{\Delta}_{\overline{B}/B}^{\text{rel}}.$$

We declare a class of morphisms in $\mathcal{RQRSPerfd}'$ by

$$W' = \left\{ (A \rightarrow A' \rightarrow R) \xrightarrow{\chi} (B \xrightarrow{\text{id}} B \rightarrow \overline{B}) \mid A \rightarrow A' \rightarrow R \in \mathcal{RQRSPerfd}' \right\}$$

and let W be the image of W' under the functor $T: \mathcal{RQRSPerfd}' \rightarrow \mathcal{RQRSPerfd}$. Now the functor $\underline{\Delta}^{\text{rel}}$ in diagram (4) sends W' to equivalences. Thus any factorization $\mathcal{RQRSPerfd} \rightarrow \widehat{\text{Fil}}(\text{Ab})$ has to send W' to equivalences as well. Therefore we can invert the classes W' and W in Diagram (4). But then the uniqueness of the dashed arrow follows from the assertion that the induced functor

$$\Psi: \mathcal{RQRSPerfd}'[(W')^{-1}] \rightarrow \mathcal{RQRSPerfd}[W^{-1}]$$

¹⁷Meaning up to a contractible space of choices.

is an equivalence. To see this, we consider the functor

$$\mathcal{RQRSPerfd} \rightarrow \mathcal{RQRSPerfd}' \quad (A \rightarrow R) \mapsto (\Delta_{R/A} \rightarrow \Delta_{R/A} \rightarrow \overline{\Delta}_{R/A}).$$

This functor depends on the choice of an extension Δ in (4) which we fix (if there were no extension, then there would be nothing to show). This functor sends W to equivalences since the prismatic cohomology agrees. Thus it induces a functor

$$\Omega: \mathcal{RQRSPerfd}[W^{-1}] \rightarrow \mathcal{RQRSPerfd}'[(W')^{-1}].$$

We have natural isomorphisms

$$(A \rightarrow R) \xrightarrow{\simeq \in W} (\Delta_{R/A} \rightarrow \overline{\Delta}_{R/A}) = \Psi\Omega(A \rightarrow R)$$

and

$$(A \rightarrow A' \rightarrow R) \xrightarrow{\chi} (B \rightarrow B \rightarrow \overline{B}) = \Omega\Psi(A \rightarrow A' \rightarrow R),$$

which finishes the proof. \square

Lemma 9.16. *Assume that we have a commutative¹⁸ square of 1-categories*

$$\begin{array}{ccc} \mathcal{R}_1 & \xrightarrow{f} & \mathcal{C}_1 \\ \downarrow T & \nearrow & \downarrow U \\ \mathcal{R}_2 & \xrightarrow{g} & \mathcal{C}_2 \end{array}$$

where T is essentially surjective and U is faithful. Then there exists either an empty or a contractible space of lifts $\mathcal{R}_2 \rightarrow \mathcal{C}_1$ as indicated.

Proof. Fix two lifts F_1, F_2 , with natural isomorphisms $\eta_i: U \circ F_i \rightarrow g$ and $\nu_i: F_i \circ T \rightarrow f$. (These are required to be compatible in the sense that their composites $g \circ T \rightarrow U \circ f$ have to agree with the natural isomorphism provided as part of the commutative square.) For $x \in \mathcal{R}_2$, we have the composite

$$U(F_1(x)) \xrightarrow{\eta_{1,x}} g(x) \xrightarrow{\eta_{2,x}^{-1}} U(F_2(x)).$$

By faithfulness of U , there exists at most one lift of this composite along U . If there exists one for each x , together the resulting maps $\varepsilon_x: F_1(x) \rightarrow F_2(x)$ form a natural transformation, and any natural transformation compatible with the diagram needs to be of this form. For existence, we pick \tilde{x} with $T(\tilde{x}) \cong x$, and compose the resulting isomorphism

$$U(F_i(x)) \cong U(F_i(T(\tilde{x}))) \cong U(f(\tilde{x}))$$

for $i = 1$ with the inverse of the corresponding isomorphism for $i = 2$. \square

10 Filtered prismatic cohomology

As a consequence of the base change property, prismatic cohomology of δ -pairs inherits gradings and filtrations, generalizing the observation from [?, Remark 3.8]. Indeed, recall that graded objects can be encoded as quasicoherent sheaves on \mathbf{BG}_m , i.e., as objects with an $\mathcal{O}_{\mathbf{G}_m} = \mathbf{Z}[x^{\pm 1}]$ -coaction. Since $\mathbf{Z}[x^{\pm 1}]$ carries a

¹⁸This means up to a chosen natural isomorphism, as usual in the higher categorical context.

canonical δ -ring structure with $\delta(x) = 0$, if (A, R) is a δ -pair, we may consider $(A \otimes \mathcal{O}_{\mathbf{G}_m}, R \otimes \mathcal{O}_{\mathbf{G}_m})$ as a δ -pair as well. Then, we may define graded δ -pairs as δ -pairs with an $\mathcal{O}_{\mathbf{G}_m}$ -coaction. By base change, the map

$$\Delta_{R/A} \rightarrow \Delta_{R \otimes \mathcal{O}_{\mathbf{G}_m} / A \otimes \mathcal{O}_{\mathbf{G}_m}} \simeq \Delta_{R/A} \widehat{\otimes} \mathcal{O}_{\mathbf{G}_m}$$

then induces a $\mathcal{O}_{\mathbf{G}_m}$ -coaction on $\Delta_{R/A}$ (as a complete filtered object), so $\Delta_{R/A}$ inherits a grading for a graded δ -pair (A, R) . Analogous statements hold for filtered δ -pairs. We now want to develop this theory in more detail.

Convention 10.1 (Nonnegative decreasing filtrations). Throughout this section, all filtered commutative rings F^*R and filtered δ -rings F^*A (to be defined below) will be decreasingly filtered commutative rings with $F^0R \rightarrow F^{-m}R$ is an equivalence for $m \geq 0$; we set $R = F^0R$.

Definition 10.2. A filtered abelian group F^*M is strict if $F^mM \rightarrow M$ is a monomorphism for each $m \in \mathbf{Z}$.

Definition 10.3 (Filtered δ -rings). A filtered δ -ring is a pair (F^*A, δ) consisting of a strict filtered commutative ring F^*A and a δ -ring structure on $A = F^0A$ with the property that $\delta(F^m A) \subseteq F^{pm} A$ for all $m \geq 0$.

Remark 10.4. Given a filtered δ -ring F^*A , the Frobenius φ on A also has the property that $\varphi(F^m A) \subseteq F^{pm} A$. Conversely, given a filtered commutative ring F^*A such that A and each graded piece $\mathrm{gr}^m A$ is p -torsion free, then any lift of Frobenius φ satisfying $\varphi(F^m A) \subseteq F^{pm} A$ arises from a unique filtered δ -ring structure in this way. This follows since $p\delta(x) \in F^{pm} A$ implies $\delta(x) \in F^{pm} A$ by the assumption on the graded pieces.

Remark 10.5. If F^*A is a filtered δ -ring, then there is an induced δ -ring structure on $\mathrm{gr}^0 A$. Note that this implies that no power of p can be in positive filtration weight in A .

Definition 10.6 (Filtered prisms). A filtered prism consists of a filtered δ -ring F^*A together with a prism structure (A, I) on $A = F^0A$ such that each $\mathrm{gr}^i A$ is $I \otimes_A \mathrm{gr}^0 A$ -torsion free and $I \otimes_A \mathrm{gr}^0 A$ -complete. Note that the assumption means that $(\mathrm{gr}^0 A, I \otimes_A \mathrm{gr}^0 A)$ is a prism as is the $(p, I \otimes_A \mathrm{gr}^0 A)$ -completion of $\bigoplus_{m \geq 0} \mathrm{gr}^m A$. A filtered prism is called complete if the filtered commutative ring F^*A is derived complete with respect to the filtration.

Example 10.7 (Filtered Breuil–Kisin prisms). Consider the ring $A = W(k)[[z]]$ equipped with the z -adic filtration and a prism structure given by $(E(z))$, where $E(z)$ is an Eisenstein polynomial. Then $(E(z)) \otimes_A \mathrm{gr}^0 A = (p) \subseteq W(k)$. As each graded piece $\mathrm{gr}^u A \cong W(k)$ is p -torsion free, this defines a (complete) filtered prism, which we call a filtered Breuil–Kisin prism.

Definition 10.8 (Flat filtrations). Given a filtered ring F^*A and an $A = F^0A$ -module M , the flat filtration on M is defined to be $F^*M = M \otimes_A F^*A$.

Remark 10.9. If F^*A is a filtered δ -ring and (A, I) is a prism structure, not assumed yet to make A into a filtered prism, then

$$0 \rightarrow F^*I \rightarrow F^*A \rightarrow F^*\overline{A} \rightarrow 0$$

is an exact sequence of filtered abelian groups, which follows because A is I -torsion free and hence so is each $F^i A$ by the strictness assumption in the definition of a filtered δ -ring. The filtration F^*I is automatically strict in this situation. The pair (F^*A, I) is a filtered prism if and only if the associated graded pieces of the flat filtration on \overline{A} are discrete or equivalently that the flat filtration on \overline{A} is strict.

Definition 10.10. A filtered δ -pair consists of filtered δ -ring F^*A and a map of filtered commutative rings $F^*A \rightarrow F^*R$. A δ -pair is strict if the filtration on R is strict (recalling that the filtration on the filtered δ -ring is strict by definition). We say that a filtered δ -pair is prismatic if the kernel of $F^*A \rightarrow F^*R$ contains a Cartier divisor $I \subseteq A$ making (F^*A, I) into a filtered prism.

We introduce the filtered prismatic site and filtered prismatic cohomology. This is not the main focus of our study, so we include it only for the sake of the curious reader. Below, we will give a more detailed study of filtered prismatic cohomology using prismatization and we compare the two theories in the case of a regular filtered quotient of a filtered prism (see Proposition 10.41).

Definition 10.11 (Filtered prismatic site). Fix a strict filtered δ -pair $F^*A \rightarrow F^*R$. The filtered prismatic site $(F^*R/F^*A)_\Delta$ of F^*R relative to the filtered δ -ring F^*A is the opposite of the category of commutative squares

$$\begin{array}{ccc} F^*A & \longrightarrow & F^*B \\ \downarrow & & \downarrow \\ F^*R & \longrightarrow & F^*\overline{B}, \end{array} \quad (5)$$

of filtered commutative ring maps where (F^*B, I) is a bounded filtered prism and $F^*A \rightarrow F^*B$ is a filtered δ -ring map. The topology is the filtered p -adically faithfully flat topology, meaning that $F^*B \rightarrow F^*C$ is a cover if $B \rightarrow C$ is (p, I) -adically faithfully flat and $\mathrm{gr}^0 B \rightarrow \mathrm{gr}^0 C$ is a $(p, \mathrm{gr}^0 I)$ -adically faithfully flat map of graded commutative rings.

Definition 10.12 (Filtered structure sheaves). The presheaf $F^*\mathcal{O}_\Delta$ which to any square (5) associates F^*B is a sheaf of strictly filtered commutative rings by arguing as in [10, Cor. 3.12]; see also Lemma 2.8. Similarly, the presheaf $\mathcal{O}_{\overline{\Delta}}$, which sends (5) to $F^*\overline{B}$ is a sheaf of strictly filtered commutative rings.

Definition 10.13 (Filtered Breuil–Kisin twists). The filtered Breuil–Kisin twists $F^*\mathcal{O}_\Delta\{i\}$ assign to each filtered prism B the flat filtration $B\{i\} \otimes_B F^*B$ on $B\{i\}$.

Definition 10.14 (Filtered prismatic cohomology). We let $F^*\Delta_{F^*R/F^*A}^{\mathrm{site}}$, or more briefly $F^*\Delta_{R/A}^{\mathrm{site}}$, denote $\mathrm{R}\Gamma((F^*R/F^*A)_\Delta, F^*\mathcal{O}_\Delta)$ and similarly for filtered Hodge–Tate cohomology $F^*\overline{\Delta}_{R/A}^{\mathrm{site}}$ and the Breuil–Kisin twists $F^*\overline{\Delta}_{R/A}^s\{i\}$, etc.

Example 10.15 (Filtered absolute prismatic cohomology). The case where $F^*A = \mathbf{Z}_p$ with the trivial filtration (so $F^i\mathbf{Z}_p = 0$ for $i > 0$) gives $F^*\Delta_{R/\mathbf{Z}_p}^{\mathrm{site}}$, a filtered version of absolute prismatic cohomology for any filtered commutative ring F^*R .¹⁹

Example 10.16. Suppose that the filtrations on F^*A and F^*R are trivial, meaning that $F^iA \simeq 0$ and $F^iR \simeq 0$ for $i > 0$. Then, any square (5) can be factored as

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & F^*B \\ \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & \overline{B} & \longrightarrow & F^*\overline{B}. \end{array}$$

Conversely, any object in the unfiltered site $(R/A)_\Delta$ can be equipped with the trivial filtration. It follows in this case that $F^*\Delta_{R/A}^{\mathrm{site}}$ is naturally equivalent to $\Delta_{R/A}^{\mathrm{site}}$ with the trivial filtration.

The goal of the remainder of this section is to outline the general theory of filtered prismatic cohomology from the stacky perspective of [8].

¹⁹Note again that $p^*\mathbf{Z}_p$ does not define a filtered δ -ring.

Construction 10.17 (Graded and filtered Spec). Given a filtered commutative ring F^*R we can associate to it the graded Rees algebra

$$\text{Rees}(F^*R) = \bigoplus_{m \in \mathbf{Z}} F^m R \cdot t^m,$$

where t has weight 1, whose spectrum $\text{Spec Rees}(F^*R)$ is a \mathbf{G}_m -equivariant affine scheme over \mathbf{A}^1 (on the coordinate t). Let $\text{FSpec } R = \text{FSpec } F^*R$ denote the quotient

$$(\text{Spec Rees}(F^*R)) / \mathbf{G}_m,$$

viewed as a stack over $\mathbf{A}^1 / \mathbf{G}_m$. We have a commutative diagram in which we can identify the fibers of $\text{FSpec } F^*R$ over $0, 1 \in \mathbf{A}^1 / \mathbf{G}_m$:

$$\begin{array}{ccccccc} \text{Spec } \bigoplus_{m \in \mathbf{Z}} \text{gr}^m R & \longrightarrow & \text{GrSpec } \text{gr}^* R & \longrightarrow & \text{FSpec } F^* R & \longleftarrow & \text{Spec } R \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Spec } \mathbf{Z} & \longrightarrow & \text{B}\mathbf{G}_m & \xrightarrow{0} & \mathbf{A}^1 / \mathbf{G}_m & \xleftarrow{1} & \text{Spec } \mathbf{Z}, \end{array}$$

where $\text{GrSpec } \text{gr}^* R$ denotes the quotient stack $(\text{Spec } \bigoplus_{m \in \mathbf{Z}} \text{gr}^m R) / \mathbf{G}_m$. Note that these constructions are examples of relative Spec for F^*R and $\text{gr}^* R$ viewed as commutative algebra objects in $\mathcal{D}(\mathbf{A}^1 / \mathbf{G}_m)$ and $\mathcal{D}(\text{B}\mathbf{G}_m)$, respectively.

As we are interested in functors on p -nilpotent commutative rings, we will in fact use the following p -adic formal version of the construction above. See Appendix A for more details on p -adic formal stacks, especially Warning A.3 which clarifies that $\widehat{\mathbf{A}}^1 = \text{Spf } \mathbf{Z}[t] \simeq \text{colim } \text{Spf } \mathbf{Z}[t]/p^n$, not $\text{colim } \text{Spf } \mathbf{Z}_p[t]/t^n$.

Construction 10.18 (Graded and filtered Spf). For a filtered commutative ring F^*R , we can restrict the functors above to p -nilpotent commutative rings to obtain a commutative diagram of pullback squares

$$\begin{array}{ccccccc} \text{Spf } \bigoplus_{m \in \mathbf{Z}} \text{gr}^m R & \longrightarrow & \text{GrSpf } \text{gr}^* R & \longrightarrow & \text{FSpf } F^* R & \longleftarrow & \text{Spf } R \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Spf } \mathbf{Z}_p & \longrightarrow & \text{B}\widehat{\mathbf{G}}_m & \xrightarrow{0} & \widehat{\mathbf{A}}^1 / \widehat{\mathbf{G}}_m & \xleftarrow{1} & \text{Spf } \mathbf{Z}_p \end{array}$$

of formal stacks.

Remark 10.19 (Functor of points for graded and filtered Spf). Given a filtered commutative ring F^*R , the filtered formal spectrum $\text{FSpf } R$ has the following universal property as a stack over $\widehat{\mathbf{A}}^1 / \widehat{\mathbf{G}}_m$. Given a point $\text{Spf } S \rightarrow \widehat{\mathbf{A}}^1 / \widehat{\mathbf{G}}_m$, described by the choice of a rank 1 projective module L over $\text{Spf } R$ and a section $s: L \rightarrow R$, the pullback

$$\begin{array}{ccc} P & \longrightarrow & \text{FSpf } R \\ \downarrow & & \downarrow \\ \text{Spf } S & \xrightarrow{(L,s)} & \widehat{\mathbf{A}}^1 / \widehat{\mathbf{G}}_m. \end{array}$$

giving the fiber over (L, s) is the space of maps of graded commutative rings $\text{Rees}(F^*R) \rightarrow S[L^{\pm 1}]$ taking $t^{-1} \in F^{-1}R$ to the element s^{-1} of $L^{\otimes -1}$ corresponding to the dual of s , where $S[L^{\pm 1}] = \bigoplus_{m \in \mathbf{Z}} L^{\otimes m}$ is the graded commutative ring with the natural multiplication of sections.

Similarly, if R^* is a graded commutative ring, then $\text{GrSpf } R$ is described as a stack over $\text{B}\widehat{\mathbf{G}}_m$ by saying that the fiber of $\text{GrSpf } R$ over a point $\text{Spf } S \rightarrow \text{B}\widehat{\mathbf{G}}_m$ corresponding to a rank 1 projective module P is the space of maps $R^* \rightarrow S[L^{\pm 1}]$ of graded commutative rings.

Now, we describe the Cartier–Witt stack controlling filtered prismatic cohomology.

Lemma 10.20. *Let F^*A be a filtered δ -ring. Let $\mathrm{FSpf} F^*A$, or $\mathrm{FSpf} A$ for short, denote the formal stack defined above. Then, $\mathrm{FSpf} A$ is naturally a δ -stack and the canonical map $\mathrm{FSpf} A \rightarrow \widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m$ is a map of δ -stacks.*

Proof. By definition, $\mathrm{FSpf} A = (\mathrm{Spf} \bigoplus F^m A \cdot t^m) / \widehat{\mathbf{G}}_m$, so it can be computed as the colimit of the simplicial object $(\mathrm{Spf} \bigoplus F^m A \cdot t^m) \times \widehat{\mathbf{G}}_m^{\times \bullet}$ corresponding to the one-sided bar construction. The filtered δ -ring structure on A gives $\mathrm{Spf} \bigoplus F^m A \cdot t^m$ the structure of a formal δ -scheme. Specifically, the δ -operation on $\bigoplus F^m A \cdot t^m$ sends $x \cdot t^m \in F^m A \cdot t^m$ to $\delta(x) \cdot t^{pm} \in F^{pm} A \cdot t^{pm}$. As $\varphi(x) = x^p + p\delta(x)$ is a lift of Frobenius on A , it follows that the formula $\varphi(x \cdot t^m) = (x^p + p\delta(x)) \cdot t^{pm}$ defines a lift of Frobenius on the direct sum. Now, by [10, Lems. 2.17 and 2.18], there is an induced δ -ring structure on the p -completion. Finally, the comultiplication $\nabla: \bigoplus F^m A \cdot t^m \rightarrow \bigoplus F^m A \cdot t^m \otimes_{\mathbf{Z}} \mathbf{Z}[t^{\pm 1}]$ encoding the $\widehat{\mathbf{G}}_m$ -action is defined for $x \in F^m A$ by $\nabla(x \cdot t^m) = (x \cdot t^m) \otimes t^m$. The comultiplication is thus a map of δ -rings as

$$\delta(\nabla(x \cdot t^m)) = \delta((x \cdot t^m) \otimes t^m) = \delta(x \cdot t^m) \otimes t^{pm} + (x^p \cdot t^{pm}) \otimes \delta(t^m) + p\delta(x \cdot t^m) \otimes \delta(t^m) = \delta(x \cdot t^m) \otimes t^{pm} = \nabla(\delta(x \cdot t^m))$$

and remains such upon p -completion. It follows that the simplicial object from the start of the proof is a simplicial object in formal affine δ -schemes. \square

Construction 10.21 (Filtered prismaticization). Let $F^*A \rightarrow F^*R$ be a filtered δ -pair. The filtered prismaticization $\mathrm{FWCart}_{F^*R/F^*A}$, or $\mathrm{FWCart}_{R/A}$ for short when the filtrations are clear from context, is the formal stack defined as the pullback

$$\begin{array}{ccc} \mathrm{FWCart}_{F^*R/F^*A} & \longrightarrow & \mathrm{WCart}_{\mathrm{FSpf} R} \\ \downarrow & & \downarrow \\ \mathrm{FSpf} A \times \mathrm{WCart} & \longrightarrow & \mathrm{WCart}_{\mathrm{FSpf} A}, \end{array}$$

where the bottom map is defined thanks to the naturality of [8, Const. 3.11]; see also Construction 4.8.

Construction 10.22 (Graded prismaticization). Given a graded δ -ring A^* and a graded A^* -algebra R^* , define $\mathrm{GrWCart}_{R^*/A^*}$ as the analogous pullback where $\mathrm{GrSpf} A$ replaces the filtered formal spectrum.

Remark 10.23 (Functor of points of filtered and graded prismaticizations). Using Remark 10.19, we can describe $\mathrm{FWCart}_{R/A}$ as a stack over $\widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m$ along the compositions $\mathrm{FWCart}_{R/A} \rightarrow \mathrm{FSpf} A \times \mathrm{WCart} \rightarrow \mathrm{FSpf} A \rightarrow \widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m$. If p is nilpotent in S , an S -point of $\mathrm{FSpf} A \times \mathrm{WCart}$ corresponds to a collection

$$(L, s, f: \mathrm{Rees}(F^*A) \rightarrow S[[L^{\pm 1}]], \alpha: I \rightarrow W(S)),$$

where α is a Cartier–Witt divisor and (L, s, f) describes an S point of $\mathrm{FSpf} A$. On the other hand, an S -point of $\mathrm{WCart}_{\mathrm{FSpf} A}$ corresponds to a Cartier–Witt divisor $(\alpha: I \rightarrow W(S))$ together with a $W(S)/I$ -point of $\mathrm{FSpf} A$, and similarly for $\mathrm{WCart}_{\mathrm{FSpf} R}$. The bottom natural transformation $\mathrm{FSpf} A \times \mathrm{WCart} \rightarrow \mathrm{WCart}_{\mathrm{FSpf} A}$ sends the collection (L, s, f, α) to the Cartier–Witt divisor $\alpha: I \rightarrow W(S)$ together with the Teichmüller lift $[L]$ of L and $[s]$ of s and the composition

$$\mathrm{Rees}(F^*A) \rightarrow W(S)[[L^{\pm 1}]] \rightarrow W(S)/I[[L^{\pm 1}]]$$

obtained from adjunction using the δ -stack structure of $\mathrm{FSpf} A$. Thus, an S -point of $\mathrm{FWCart}_{R/A}$ consists of (L, s, f, α) as above together with a Cartier–Witt divisor $\beta: J \rightarrow W(S)$, a rank 1 projective module M over $W(S)/I$, a section $u: M \rightarrow W(S)/I$, and a map $\mathrm{Rees}(F^*R) \rightarrow W(S)/I[[M^{\pm 1}]]$ sending t^{-1} to u^{-1} together with a fixed equivalence between α with β , a fixed equivalence between $[L]$ and M over $W(S)/I$ compatible with the equivalence between α and β , and a fixed equivalence between $\mathrm{Rees}(F^*A) \rightarrow W(S)/I[[L^{\pm 1}]]$ and $\mathrm{Rees}(F^*R) \rightarrow \mathrm{Rees}(F^*R) \rightarrow W(S)/J[[M^{\pm 1}]]$ compatible with the other identifications.

Lemma 10.24. *There are natural equivalences*

$$D(\widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m \times \mathrm{WCart}) \simeq \mathrm{FD}(\mathrm{WCart}) \quad \text{and} \quad D(\widehat{\mathbf{B}}\widehat{\mathbf{G}}_m \times \mathrm{WCart}) \simeq \mathrm{GrD}(\mathrm{WCart}).$$

Proof. See Appendix A.1. □

Construction 10.25 (Filtered and graded prismatic crystals). Define the filtered prismatic crystal $F^*\mathcal{H}_\Delta(R/A)$ to be the (derived) pushforward of the structure sheaf of $\mathrm{FWCart}_{R/A}$ along the maps $\mathrm{FWCart}_{R/A} \rightarrow \mathrm{FSpf} A \times \mathrm{WCart} \rightarrow \widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m \times \mathrm{WCart}$. Define the graded prismatic crystal $\mathrm{gr}^*\mathcal{H}_\Delta(R/A)$ in the analogous way as a sheaf on $\widehat{\mathbf{B}}\widehat{\mathbf{G}}_m \times \mathrm{WCart}$.

Warning 10.26. As in Proposition 5.1, we view this definition of the filtered prismatic crystals as correct only under filtered quasisyntomicity conditions.

Definition 10.27 (Filtered and graded prismatic cohomology). If $F^*A \rightarrow F^*R$ is a filtered δ -pair, let $F^*\Delta_{R/A} = \mathrm{R}\Gamma(\mathrm{FWCart}_{R/A}, \mathcal{O}_{\mathrm{FWCart}_{R/A}})$, the filtered prismatic cohomology of F^*R relative to F^*A . Similarly, if $A^* \rightarrow R^*$ is a graded δ -pair, let $\mathrm{gr}^*\Delta_{R/A} = \mathrm{R}\Gamma(\mathrm{GrWCart}_{R^*/A^*}, \mathcal{O}_{\mathrm{GrWCart}_{R^*/A^*}})$.

Remark 10.28. By Lemma 10.24, we can view $F^*\mathcal{H}_\Delta(R/A)$ as a filtered object in prismatic crystals, and similarly for the graded prismatic crystal. By pushing forward along $\widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m \times \mathrm{WCart} \rightarrow \widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m$, we can view $F^*\Delta_{R/A}$ as being a filtered object of $D(\mathbf{Z})_p^\wedge$.

Proposition 10.29. *The commutative diagram*

$$\begin{array}{ccccccccc} \mathrm{WCart}_{\widehat{\bigoplus_{\mathrm{gr}^m R}/\widehat{\bigoplus_{\mathrm{gr}^m A}}} & \longrightarrow & \mathrm{GrWCart}_{R/A} & \longrightarrow & \mathrm{FWCart}_{R/A} & \longleftarrow & \mathrm{WCart}_{\mathrm{Rees}(R)/\mathrm{Rees}(A)} & \longleftarrow & \mathrm{WCart}_{R/A} \\ \downarrow & & \downarrow \mathrm{gr}q & \text{(A)} & \downarrow \mathrm{F}q & & \downarrow & & \downarrow \\ \mathrm{WCart} & \longrightarrow & \widehat{\mathbf{B}}\widehat{\mathbf{G}}_m \times \mathrm{WCart} & \xrightarrow{0} & \widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m \times \mathrm{WCart} & \longleftarrow & \widehat{\mathbf{A}}^1 \times \mathrm{WCart} & \xleftarrow{1} & \mathrm{WCart} \end{array}$$

consists of pullback squares which satisfy base change for bounded above quasi-cohomology, where $\widehat{\bigoplus_{\mathrm{gr}^m R}}$ and $\widehat{\bigoplus_{\mathrm{gr}^m A}}$ denote the p -completed direct sums.

Proof. For simplicity, we will prove the result for square (A); the rest of the proof uses the same ideas. Square (A) fits into a commutative diagram

$$\begin{array}{ccccc} \mathrm{GrWCart}_{R/A} & \longrightarrow & \mathrm{FWCart}_{R/A} & \longrightarrow & \mathrm{WCart}_{\mathrm{FSpf} R} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{GrSpf} A \times \mathrm{WCart} & \longrightarrow & \mathrm{FSpf} A \times \mathrm{WCart} & \longrightarrow & \mathrm{WCart}_{\mathrm{FSpf} A} \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathbf{B}}\widehat{\mathbf{G}}_m \times \mathrm{WCart} & \longrightarrow & \widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m \times \mathrm{WCart} & \longrightarrow & \mathrm{WCart}_{\widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m}. \end{array} \tag{6}$$

The top outer square fits into another commutative diagram

$$\begin{array}{ccccc} \mathrm{GrWCart}_{R/A} & \longrightarrow & \mathrm{WCart}_{\mathrm{GrSpf} R} & \longrightarrow & \mathrm{WCart}_{\mathrm{FSpf} R} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{GrSpf} A \times \mathrm{WCart} & \longrightarrow & \mathrm{WCart}_{\mathrm{GrSpf} A} & \longrightarrow & \mathrm{WCart}_{\mathrm{FSpf} A}. \end{array}$$

In this diagram, the left square is a pullback by definition and the right square is a pullback as $\mathrm{WCart}_{(-)}$ commutes with limits. It follows that the outer square is a pullback which proves that the top outer square of (6) is a pullback. As the top right square of (6) is a pullback by definition, it follows that the top left square of (6) is a pullback, as desired (see for example [21, Lem. 4.4.2.1]). But, the bottom left square of (6) is a pullback square as it is the pullback of a pullback square to WCart . It follows that square (A) is a pullback square as it is the outer left square in (6).

To prove base change for bounded above quasi-coherent cohomology for square (A) in the proposition, it is enough to prove it for the top-left and bottom-left squares of (6). The bottom-left square is left to the reader: it is geometric and standard arguments work. For the top-left square, we can pull back everything along the Čech complex of $\widehat{\mathbf{A}}^1 \times \mathrm{WCart} \rightarrow \widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m \times \mathrm{WCart}$. This results in a cosimplicial commutative diagram

$$\begin{array}{ccc}
\mathrm{WCart}_{\mathrm{gr}R^\bullet/\mathrm{gr}A^\bullet} & \longrightarrow & \mathrm{WCart}_{R^\bullet/A^\bullet} \\
\downarrow & & \downarrow \\
\mathrm{Spf}(\bigoplus \mathrm{gr}^i A) \times \widehat{\mathbf{G}}_m^\bullet \times \mathrm{WCart} & \longrightarrow & \mathrm{Spf}(\mathrm{Rees}(A)) \times \widehat{\mathbf{G}}_m^\bullet \times \mathrm{WCart} \\
\downarrow & & \downarrow \\
\widehat{\mathbf{G}}_m^\bullet \times \mathrm{WCart} & \longrightarrow & \widehat{\mathbf{A}}^1 \times \widehat{\mathbf{G}}_m^\bullet \times \mathrm{WCart},
\end{array}$$

of pullback squares, where A^\bullet and R^\bullet are the global sections of $\mathrm{Spf}(\mathrm{Rees}(A)) \times \widehat{\mathbf{G}}_m^\bullet$ and $\mathrm{Spf}(\mathrm{Rees}(R)) \times \widehat{\mathbf{G}}_m^\bullet$, respectively, while $\mathrm{gr}A^\bullet$ and $\mathrm{gr}R^\bullet$ are shorthands for the global sections of $\mathrm{Spf}(\bigoplus \mathrm{gr}^i A) \times \widehat{\mathbf{G}}_m^\bullet$ and $\mathrm{Spf}(\bigoplus \mathrm{gr}^i R) \times \widehat{\mathbf{G}}_m^\bullet$, respectively. Again, the bottom square satisfies base change for quasi-coherent cohomology, which uses that the closed inclusions $\widehat{\mathbf{G}}_m^\bullet \hookrightarrow \widehat{\mathbf{A}}^1 \times \widehat{\mathbf{G}}_m^\bullet$ have finite flat dimension. The top square satisfies base change for bounded above quasi-coherent cohomology by Corollary A.42 in each cosimplicial degree. Arguing as in the proof of that corollary, one deduces base change for bounded above quasi-coherent cohomology for the colimit diagram. \square

Definition 10.30. Say that a strict filtered δ -pair $F^*A \rightarrow F^*R$ is filtered relatively quasisyntomic if $\mathrm{Rees}(F^*A)_p^\wedge \rightarrow \mathrm{Rees}(F^*R)_p^\wedge$ is relatively quasisyntomic.

Remark 10.31. A strict δ -pair $F^*A \rightarrow F^*R$ is filtered relatively quasisyntomic if and only if $A \rightarrow R$ and $(\bigoplus \mathrm{gr}^i A)_p^\wedge \rightarrow (\bigoplus \mathrm{gr}^i R)_p^\wedge$ are relatively quasisyntomic.

Corollary 10.32. For a relatively quasisyntomic filtered δ -pair $F^*A \rightarrow F^*R$,

- (i) $\mathrm{R}\Gamma(\mathrm{WCart}, \bigoplus_{m \in \mathbf{Z}} \mathrm{gr}^m \Delta_{\mathrm{gr}^*R/\mathrm{gr}^*A}) \simeq \Delta_{\widehat{\bigoplus_{m \in \mathbf{Z}} \mathrm{gr}^m R}/\widehat{\bigoplus_{m \in \mathbf{Z}} \mathrm{gr}^m A}}$,
- (ii) $\mathrm{gr}^* \Delta_{F^*R/F^*A} \simeq \mathrm{gr}^* \Delta_{\mathrm{gr}^*R/\mathrm{gr}^*A}$, and
- (iii) $\mathrm{Rees}(F^* \Delta_{F^*R/F^*A})_{\mathrm{HT}}^\wedge \simeq \Delta_{\mathrm{Rees}(F^*R)/\mathrm{Rees}(F^*A)}$, and
- (iv) $F^{-\infty} \Delta_{F^*R/F^*A} \simeq \Delta_{R/A}$,

where the direct sum in (i) is taken in $\mathrm{D}(\mathrm{WCart})$ and hence implicitly completed along p and the Hodge–Tate locus and the Rees algebra in (iii) is completed at the Hodge–Tate tower. In particular, the associated graded pieces of the filtration $F^* \Delta_{R/A}$ depend only on the graded δ -pair $\mathrm{gr}^*A \rightarrow \mathrm{gr}^*R$.

Proof. This follows from Prop 10.29 and Proposition 5.1. \square

Warning 10.33. The filtration $F^* \Delta_{R/A}$ is not typically complete, even when $F^*A \rightarrow F^*R$ is a complete prismatic δ -pair.

Remark 10.34. The right square in

$$\begin{array}{ccccc} \mathrm{WCart} & \xrightarrow{1} & \widehat{\mathbf{A}}^1 \times \mathrm{WCart} & \longrightarrow & \widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m \times \mathrm{WCart} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spf} \mathbf{Z}_p & \xrightarrow{1} & \widehat{\mathbf{A}}^1 & \longrightarrow & \widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m \end{array}$$

does not satisfy base change for quasi-coherent cohomology. The issue is that pullback along the bottom horizontal arrow is the p -completed Rees construction, which need not be complete with respect to the filtration induced by the Hodge–Tate locus. However, the outer square does satisfy base change for quasi-coherent cohomology on those filtered crystals $F^*\mathcal{H}$ which eventually stabilize at equivalences $\cdots \rightarrow F^m\mathcal{H} \simeq F^{m-1}\mathcal{H} \simeq F^{m-2}\mathcal{H} \simeq \cdots$. This explains the Hodge–Tate completion necessary in (iii) of Corollary 10.32 while there is no completion necessary in (iv).

Proposition 10.35 (Filtered and graded Hodge–Tate comparison). *Suppose that $F^*A \rightarrow F^*R$ is a prismatic filtered δ -pair, which is additionally formally smooth. Then, $\mathrm{FWCart}_{R/A}^{\mathrm{HT}} \rightarrow \mathrm{FSpf} R$ is naturally a $T_{F^*R/F^*A}\{1\}^\#$ -gerbe. Similarly in the graded case.*

Proof. This follows immediately from [8, Prop. 5.12] by pulling back over the Čech complex $\mathrm{Spf} \mathrm{Rees} \bar{A} \times \widehat{\mathbf{G}}_m^\bullet$ of $\mathrm{Spf} \mathrm{Rees} \bar{A} \rightarrow \mathrm{FSpf} \bar{A}$. The graded case is similar. \square

In order to gain understanding of filtered prismatic cohomology, especially in the case of prismatic filtered δ -pairs, we need to verify that a specific filtered prismatic envelope construction does produce a filtered prism.

Construction 10.36 (Filtered Koszul complexes). Recall that if $J = (x_1, \dots, x_c)$ is an ideal in A , then we will say that the sequence (x_1, \dots, x_c) is Koszul-regular if the Koszul complex $\mathrm{Kos}(x_1, \dots, x_c)$ is a resolution of A/J (so has no higher homology). If F^*A is a strictly filtered commutative ring and each x_j has weight $w(j)$, meaning that $x_j \in F^{w(j)}A$ but $x_j \notin F^{w(j)+1}$, then one can make the Koszul complex $\mathrm{Kos}(x_1, \dots, x_c)$ into a complex of filtered F^*A -modules, which we will call the filtered Koszul complex $F^*\mathrm{Kos}(x_1, \dots, x_c)$. For example, if $J = (x_1, x_2)$, then the filtered Koszul complex of the sequence (x_1, x_2) will be

$$\cdots \rightarrow 0 \rightarrow F^{*-w(1)-w(2)}A \xrightarrow{\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}} F^{*-w(1)}A \oplus F^{*-w(2)}A \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} F^*A \rightarrow 0 \rightarrow \cdots$$

We say that (x_1, \dots, x_c) is a filtered Koszul-regular sequence if the filtered Koszul complex $F^*\mathrm{Kos}(x_1, \dots, x_c)$ has vanishing positive homology and if $H_0(F^*\mathrm{Kos}(x_1, \dots, x_c))$ is a strict filtration, necessarily on A/J . This implies that (x_1, \dots, x_c) is a regular sequence and much more. Equivalently, $F^*\mathrm{Kos}(x_1, \dots, x_c)$ has vanishing positive homology (in the abelian category of (non-strict) filtered complexes) and the associated graded complex $\mathrm{gr}^*\mathrm{Kos}(x_1, \dots, x_c)$ also has vanishing positive homology. We let $F^*A/J = H_0(F^*\mathrm{Kos}(x_1, \dots, x_c))$ and $F^*J = \ker(F^*A \rightarrow F^*A/J)$ when (x_1, \dots, x_c) is a filtered Koszul-regular sequence generating J . In this case, $F^*A/F^*J \cong F^*A/J$; moreover the filtration on A/J is the image filtration obtained by letting $F^n A/J = \mathrm{im}(F^n A \rightarrow A/J)$.

Remark 10.37. There is a natural extension of the definition above to infinite sequences of elements; we will use this extension below.

We will be applying this to the case where F^*A is a filtered prism and $J = (d, x_1, \dots, x_c)$. In this case, note that d has weight 0.

Example 10.38 (Breuil–Kisin quotients). For example if k is a perfect field of characteristic p and $E(z_0) \in A^s = W(k)[[z_0, \dots, z_s]]$ is an Eisenstein polynomial corresponding to a uniformizer $\varpi \in \mathcal{O}_K = W(k)[[z_0]]/E(z_0)$, then we can view A^s as having the adic filtration with respect to the ideal (z_0, \dots, z_s) . The ideal $J = (E(z_0), z_0^n, z_1 - z_0, \dots, z_s - z_0)$ is filtered Koszul. To see this, use the decomposition

$$F^* \text{Kos}(E(z_0), z_0^n, z_1 - z_0, \dots, z_s - z_0) \simeq F^* \text{Kos}(E(z_0)) \otimes_{F^* A} F^* \text{Kos}(z_0^n) \otimes_{F^* A} F^* \text{Kos}(z_1 - z_0) \otimes_{F^* A} \cdots \otimes_{F^* A} F^* \text{Kos}(z_s - z_0)$$

to realize the filtered Koszul complex as quasi-isomorphic to $\varpi^* \mathcal{O}_K / \varpi^n$, which is a strict filtration on the quotient.

Example 10.39 (The stack of a filtered prism). Let $(F^* A, I)$ be a filtered prism. We compute the stack $\text{FWCart}_{\overline{A}/A}$. Recall that the unfiltered stack $\text{WCart}_{\overline{A}/A}$ is equivalent to $\text{Spf } A$. We show that the filtered stack is equivalent to $F\text{Spf } A = \text{Spf}(\text{Rees}(A)_{(p,I)}^\wedge) / \widehat{\mathbf{G}}_m$. For this, we can form the Čech complex of $\text{Spf}(\text{Rees}(A)) \times \text{WCart} \rightarrow F\text{Spf } A \times \text{WCart}$ and pull back $\text{FWCart}_{\overline{A}/A} \rightarrow F\text{Spf } A \times \text{WCart}$ to the Čech complex. This results in a commutative diagram

$$\begin{array}{ccc} \text{WCart}_{\text{Rees}(\overline{A})^\bullet / \text{Rees}(A)^\bullet} & \longrightarrow & \text{FWCart}_{\overline{A}/A} \\ \downarrow & & \downarrow \\ \text{Spf}(\text{Rees}(A)) \times \widehat{\mathbf{G}}_m & \longrightarrow & F\text{Spf } A \times \text{WCart} \end{array}$$

where the geometric realization of the left vertical arrow is the right vertical arrow and where $\text{Rees}(\overline{A})^\bullet$ and $\text{Rees}(A)^\bullet$ are the global sections of $\text{Spf}(\text{Rees}(\overline{A})) \times \widehat{\mathbf{G}}_m$ and $\text{Spf}(\text{Rees}(A)) \times \widehat{\mathbf{G}}_m$, respectively. Now, the fiber of $\text{Rees}(A) \rightarrow \text{Rees}(\overline{A})$ is $\text{Rees}(I)$ by the filtered prism condition. Zariski locally on $\text{Spf}(\text{Rees}(A))$, $\text{Rees}(I)$ is generated by a distinguished element d and hence $\text{Rees}(I) \simeq \text{Rees}(A)$, i.e., the δ -pair $\text{Rees}(A) \rightarrow \text{Rees}(\overline{A})$ determines a prism after $\text{Rees}(I)$ -adic completion (which is the same as I -adic completion using that $\text{Rees}(A)$ is an A -algebra). The same is true of $\text{Rees}(A)^\bullet$ in each cosimplicial degree. Thus, there is an identification of simplicial objects

$$\text{WCart}_{\text{Rees}(\overline{A})^\bullet / \text{Rees}(A)^\bullet} \simeq \text{Spf}(\text{Rees}(A)_{(p,I)}^{\bullet, \wedge})$$

by Lemma 4.11; by definition, the right-hand side has geometric realization $F\text{Spf } A$, as desired.

Construction 10.40 (Filtered prismatic envelopes). Let $(F^* A, (d))$ be a filtered prism and let $J = (d, x_1, \dots, x_c)$ be a filtered Koszul-regular ideal, so that the image sequence x_1, \dots, x_c defines a filtered Koszul-regular ideal in $F^* \overline{A}$. Let $F^* A\{\frac{x_1}{d}, \dots, \frac{x_c}{d}\}_{(p,d)}^\wedge$ denote the (p, d) -completed filtered (derived) tensor product

$$(F^* A\{a_1, \dots, a_c\}_\delta \otimes_{F^* A\{r_1, \dots, r_c\}_\delta} F^* A)_{(p,d)}^\wedge.$$

Here $F^* A\{r_1, \dots, r_c\}_\delta$ is the free filtered δ -ring over $F^* A$ on weight w_u generators r_u and similarly for $F^* A\{a_1, \dots, a_c\}$. The left map is the filtered δ -ring map sending r_u to $da_u - x_u$; the right map sends r_u to zero. Let $F^* R = F^* \overline{A}/(x_1, \dots, x_c)$. We will show in the next proposition that the filtered prismatic envelope computes the filtered prismatic cohomology of R relative to A .

Proposition 10.41. *If $(F^* A, (d))$ is an oriented filtered prism and $J = (d, x_1, \dots, x_c) \subseteq A$ is an ideal generated by a filtered Koszul-regular sequence (d, x_1, \dots, x_c) , then*

- $F^* A\{\frac{x_1}{d}, \dots, \frac{x_c}{d}\}_{(p,d)}^\wedge$ is a filtered prism,
- $F^* A\{\frac{x_1}{d}, \dots, \frac{x_c}{d}\}_{(p,d)}^\wedge$ defines a final object of the filtered relative prismatic site $(F^* R/F^* A)_\Delta$, and
- the natural maps $F\text{Spf } A\{\frac{x_1}{d}, \dots, \frac{x_c}{d}\}_{(p,d)}^\wedge \rightarrow \text{FWCart}_{R/A}$ and $F^* \Delta_{R/A} \rightarrow F^* \Delta_{R/A}^{\text{site}} \simeq F^* A\{\frac{x_1}{d}, \dots, \frac{x_c}{d}\}_{(p,d)}^\wedge$ are equivalences.

Proof. We give the case when $c = 1$ and write x for x_1 and w for $w(1)$. The general case is obtained by taking suitable (p, d) -completed tensor products. The Rees algebra $\text{Rees}(\mathbb{F}^*A\{\frac{x}{d}\}_{(p,d)}^\wedge)$ has (p, d) -completion the prismatic envelope $(\text{Rees}(\mathbb{F}^*A))\{\frac{x}{d}\}_{(p,d)}^\wedge$ by symmetric monoidality of the Rees construction. By our assumption on x , $(\text{Rees}(\mathbb{F}^*A))\{\frac{x}{d}\}_{(p,d)}^\wedge$ is discrete. Thus, the Rees algebra $\text{Rees}(\mathbb{F}^*A\{\frac{x}{d}\}_{(p,d)}^\wedge)$ is a connective A -module whose (p, d) -completion is discrete. Moreover, it is a countable coproduct of (p, d) -complete A -modules. This implies that $\text{Rees}(\mathbb{F}^*A\{\frac{x}{d}\}_{(p,d)}^\wedge)$ is already discrete, which shows that the filtration on the filtered prismatic envelope is filtered discrete. Moreover, the (p, d) -completion of $\text{Rees}(\mathbb{F}^*A\{\frac{x}{d}\}_{(p,d)}^\wedge)/t^{-1}$ is equivalent to $\text{Rees}(\mathbb{F}^*A\{\frac{x}{d}\}_{(p,d)}^\wedge)^\wedge_{(p,d)}/t^{-1}$. The latter is seen to be equivalent to the prismatic envelope $(\bigoplus_{m \in \mathbb{Z}} \text{gr}^m A)\{\frac{x}{d}\}_{(p,d)}^\wedge$ by symmetric monoidality of taking the cofiber of t^{-1} (the associated graded). This is again discrete by the result for prismatic envelopes in Bhatt–Scholze. So, the same decompletion argument shows that $\text{Rees}(\mathbb{F}^*A\{\frac{x}{d}\}_{(p,d)}^\wedge)/t^{-1}$, which is again a direct sum of connective (p, d) -complete A -module spectra, is discrete. This implies that the filtration on the filtered prismatic envelope $\mathbb{F}^*A\{\frac{x}{d}\}_{(p,d)}^\wedge$ is strict. Hence, the filtered prismatic envelope is a filtered δ -ring. To show it is a filtered prism, we have to show strictness of the induced filtration on the Hodge–Tate locus, which is to say that we want to show that $\text{Rees}(\mathbb{F}^*A\{\frac{x}{d}\}_p^\wedge)$ and $\text{Rees}(\mathbb{F}^*A\{\frac{x}{d}\}_p^\wedge)/t^{-1}$ are discrete. After p -completing $\text{Rees}(\mathbb{F}^*A\{\frac{x}{d}\}_p^\wedge)$, we obtain an object equivalent to $\text{Rees}(\mathbb{F}^*A)\{\frac{x}{d}\}_p^\wedge/d$, which is discrete by the d -torsion freeness of the prismatic envelope of [10]. Similarly, p -completed and mod t we obtain an object equivalent to $(\bigoplus_{m \in \mathbb{Z}} \text{gr}^m A)\{\frac{x}{d}\}_p^\wedge/d$ (using the graded regularity of the element x), which is discrete again. Thus, arguing as above and decompleting, we obtain discreteness, as desired. This proves (a).

By part (a), the filtered prismatic envelope defines an object of the filtered prismatic site. Given an object \mathbb{F}^*B of the filtered prismatic site, there is a unique induced map $A\{\frac{x}{d}\}_{(p,d)}^\wedge \rightarrow B$ of δ -rings. For this to be a map of filtered prisms is a property since the filtrations on filtered δ -rings is assumed to be strict. To see that this induced map is indeed filtered it is enough to note that the induced map $A\{a\}_\delta \rightarrow B$ respects the filtrations by construction and hence the factorization through the (p, d) -completed quotient $A\{\frac{x}{d}\}_{(p,d)}^\wedge$ is filtered. This proves part (b).

For part (c), we claim that $\mathbb{F}^*\Delta_{R/A}$ is a filtered prism, which follows from the filtered Hodge–Tate comparison theorem. Thus, it defines an object of the filtered prismatic site and hence there is a map $\mathbb{F}^*A\{\frac{x}{d}\}_{(p,d)}^\wedge \rightarrow \mathbb{F}^*\Delta_{R/A}$. The composite $\mathbb{F}^*\Delta_{R/A} \xrightarrow{f} A\{\frac{x}{d}\}_{(p,d)}^\wedge \xrightarrow{g} \mathbb{F}^*\Delta_{R/A}$ is an idempotent map of filtered δ -rings. By [7, Lem. 4.3.10], f and g are isomorphisms on \mathbb{F}^0 and hence the composite $g \circ f$ is the identity on \mathbb{F}^0 . The only way to make the identity map filtered is by letting it act as the identity on each filtered piece, so $g \circ f$ is the identity on each $\mathbb{F}^m\Delta_{R/A}$. It follows from strictness of the filtrations that f and g are isomorphisms on each \mathbb{F}^m , proving the second statement of part (c). The first statement is proved by a Rees-algebra argument. \square

Construction 10.42 (Filtered syntomic cohomology). Given a bounded δ -pair (A, R) , the prismatic package $\underline{\Delta}_{R/A}$ includes, for each integer i , the following information:

- (a) the Nygaard-filtered, Nygaard-complete, Frobenius-twisted, Breuil–Kisin cohomology $\mathbb{N}^{\geq i}\widehat{\Delta}_{R/A}^{(1)}\{i\}$, which is a complete filtered A -module;
- (b) an A -linear map $\text{can}: \mathbb{N}^{\geq i}\widehat{\Delta}_{R/A}^{(1)}\{i\} \rightarrow \widehat{\Delta}_{R/A}^{(1)}\{i\}$;
- (c) a φ_A -semilinear map $c\varphi: \mathbb{N}^{\geq i}\widehat{\Delta}_{R/A}^{(1)}\{i\} \rightarrow \widehat{\Delta}_{R/A}^{(1)}\{i\}$,

from which one builds

- (d) relative syntomic cohomology complexes

$$\mathbf{Z}_p(i)(R/A) = \text{fib} \left(\mathbb{N}^{\geq i}\widehat{\Delta}_{R/A}^{(1)}\{i\} \xrightarrow{\text{can}-c\varphi} \widehat{\Delta}_{R/A}^{(1)}\{i\} \right).$$

This data is functorial in maps of bounded δ -pairs.

Now, suppose that (F^*A, F^*R) is a filtered δ -pair. We can use descent along the formally smooth morphism $\mathrm{Spf} \mathrm{Rees}(F^*A) \rightarrow \mathrm{FSpf} A$ and the naturality of the constructions above to construct an ∞ -category $\mathcal{C}_{\mathrm{FSpf} A}$ by right Kan extension (thanks to Remark 8.2(iii)) and a filtered prismatic package $\underline{\Delta}_{F^*R/F^*A} \in \mathcal{C}_{\mathrm{FSpf} A} = (\underline{\Delta}_{F^*R/F^*A}^{[*]}, N^{\geq *}\widehat{\Delta}_{F^*R/F^*A}^{(1)}\{*\}, c, \varphi)$ using Theorem 1.2(6, 7). To see that the underlying prismatic part $\underline{\Delta}_{F^*R/F^*A}^{[*]}$ agrees with $F^*\underline{\Delta}_{R/A}^{[*]}$ of Definition 10.27, one uses that $\mathrm{Spf} \mathrm{Rees}(F^*A) \times \mathrm{WCart} \rightarrow \mathrm{FSpf} A \times \mathrm{WCart}$ is a formally smooth cover together with base change (after Hodge–Tate completion) for relative prismatic cohomology (Theorem 1.2(7)). The filtered prismatic package includes, for each integer i ,

- (a) $F^{\geq *}\mathbf{N}^{\geq *}\widehat{\Delta}_{R/A}^{(1)}\{i\} = N^{\geq *}\widehat{\Delta}_{F^*R/F^*A}^{(1)}\{i\}$, which is a filtered F^*A -module equipped with a second filtration (the Nygaard filtration) for which it is complete.;
- (b) a filtered $F^{\geq *}\mathbf{N}^{\geq *}\widehat{\Delta}_{R/A}^{(1)}\{i\} \rightarrow F^{\geq *}\widehat{\Delta}_{R/A}^{(1)}\{i\}$;
- (c) a p -filtered²⁰ φ_{F^*A} -semilinear map $c\varphi: F^{\geq *}\mathbf{N}^{\geq *}\widehat{\Delta}_{R/A}^{(1)}\{i\} \rightarrow F^{\geq *}\widehat{\Delta}_{R/A}^{(1)}\{i\}$.

From this, we can define, for $i \geq 0$,

- (d) filtered relative syntomic cohomology complexes

$$F^{\geq *}\mathbf{Z}_p(i)(F^*R/F^*A) = \mathrm{fib} \left(F^{\geq *}\mathbf{N}^{\geq *}\widehat{\Delta}_{R/A}^{(1)}\{i\} \xrightarrow{\mathrm{can}-c\varphi} F^{\geq *}\widehat{\Delta}_{R/A}^{(1)}\{i\} \right)$$

for $i \in \mathbf{Z}$.²¹

As above, we often write $F^*\mathbf{Z}_p(i)(R/A)$ for $F^*\mathbf{Z}_p(i)(F^*R/F^*A)$ when it seems no confusion can arise.

Variante 10.43. If (A^*, R^*) is a graded δ -pair, then there is a graded version of the construction above, producing a prismatic package $\underline{\Delta}_{R^*/A^*} \in \mathcal{C}_{\mathrm{GrSpf} A^*}$. However, because the Frobenius map $c\varphi$ is p -graded in the sense that it takes weight i into weight pi , the induced syntomic cohomology

$$\mathbf{Z}_p(i)(R^*/A^*) = \mathrm{fib} \left(\bigoplus_{* \in \mathbf{Z}} \mathrm{gr}^* \mathbf{N}^{\geq *}\widehat{\Delta}_{R^*/A^*}^{(1)}\{i\} \xrightarrow{\mathrm{can}-c\varphi} \bigoplus_{* \in \mathbf{Z}} \mathrm{gr}^* \widehat{\Delta}_{R^*/A^*}^{(1)}\{i\} \right)$$

is filtered, not graded. The filtration is given by

$$F^m \mathbf{Z}_p(i)(R^*/A^*) = \mathrm{fib} \left(\bigoplus_{* \geq m} \mathrm{gr}^* \mathbf{N}^{\geq *}\widehat{\Delta}_{R^*/A^*}^{(1)}\{i\} \xrightarrow{\mathrm{can}-c\varphi} \bigoplus_{* \geq m} \mathrm{gr}^* \widehat{\Delta}_{R^*/A^*}^{(1)}\{i\} \right).$$

Proposition 10.44. *Let (F^*A, F^*R) be a relatively quasisyntomic filtered δ -pair. The pullback maps $1^*: \mathcal{C}_{\mathrm{FSpf} A} \rightarrow \mathcal{C}_{\mathrm{Spf} A}$ and $0^*: \mathcal{C}_{\mathrm{FSpf} A} \rightarrow \mathcal{C}_{\mathrm{GrSpf} \mathrm{gr}^* A}$ induce natural equivalences*

- (a) $1^* \underline{\Delta}_{F^*R/F^*A} \simeq \underline{\Delta}_{R/A}$ and
- (b) $0^* \underline{\Delta}_{F^*R/F^*A} \simeq \underline{\Delta}_{\mathrm{gr}^*R/\mathrm{gr}^*A}$.

Proof. This follows from base change for the prismatic package, Theorem 1.2(7). □

Corollary 10.45. *For each $i \in \mathbf{Z}$ and all relatively quasisyntomic filtered δ -pairs $A \rightarrow R$, there are natural equivalences*

- (i) $\mathrm{gr}^* \mathbf{Z}_p(i)(F^*R/F^*A) \simeq \mathrm{gr}^* \mathbf{Z}_p(i)(\mathrm{gr}^*R/\mathrm{gr}^*A)$ and
- (ii) $F^0 \mathbf{Z}_p(i)(F^*R/F^*A) \simeq \mathbf{Z}_p(i)(F^0R/F^0A)$.

²⁰A map of filtered objects $F^*M \rightarrow F^*N$ is p -filtered if it arises from a map $F^*F^*M \rightarrow F^*N$ where F is the Frobenius on $\mathbf{A}^1/\mathbf{G}_m$.

²¹Here we use that a p -filtered map of decreasing \mathbf{N} -indexed filtration forgets to a filtered map since $F^{pi}M$ maps canonically to F^iM for $i \geq 0$.

A Some formal stack theory

We give some background on quasi-coherent sheaf theory on p -adic formal stacks in [A.1](#) culminating in the establishment of some results on base change for quasi-coherent cohomology in [Section A.2](#). We claim no originality for this material, but do not know of a suitable reference.

A.1 Quasi-coherent sheaves on formal stacks

Our goal in this section is to compare several possible definitions of a quasi-coherent sheaf on $\mathrm{Spf} R$. While in the body of this paper, all constructions are by default derived unless specified otherwise, given an abelian group M , we will need to work both with the derived reduction modulo p of M , which we will write as $M//p$, and the non-derived version $M/p = H_0(M//p)$.

As in the rest of this paper, we work with formal prestacks, i.e., presheaves of spaces (aka anima) on the category of spectra of p -nilpotent rings. Formal stacks are sheaves of anima or spectra with respect to the faithfully flat topology on spectra of p -nilpotent rings.

Remark A.1. Our formal stacks are formal analogues of higher stacks as opposed to derived stacks (see [\[29\]](#) for an overview of the distinction and further references). Quasi-lci conditions guarantee that the Cartier–Witt theory developed here using formal higher stacks agrees with the derived analogue, as in [\[8\]](#).

Example A.2 (Formal spectra). Given a commutative ring R , $\mathrm{Spf} R$ denotes the formal stack given by the restriction of $\mathrm{Spec} R$ to a functor on spectra of p -nilpotent commutative rings. If R has bounded p -power torsion, then the natural map $\mathrm{Spf} R_p^\wedge \rightarrow \mathrm{Spf} R$ is an equivalence of formal stacks.

Warning A.3. We will write $\widehat{\mathbf{A}}^1$ for $\mathrm{Spf} \mathbf{Z}[t]$ and $\widehat{\mathbf{G}}_m$ for $\mathrm{Spf} \mathbf{Z}[t^{\pm 1}]$. Note that this diverges from common notation in the literature, where for example $\widehat{\mathbf{A}}^1$ denotes the prestack on all affine schemes, which to any $\mathrm{Spec} R$ assigns the non-unital ring of nilpotent elements in R . By contrast, for us, formal will always mean formal with respect to the p -adic topology.

Definition A.4 (Quasicoherent sheaves on formal stacks). Given any formal (pre)stack \mathcal{X} , one defines $D(\mathcal{X}) = \lim_{\mathrm{Spf} R \rightarrow \mathcal{X}} D(R)$ where the limit ranges over all p -nilpotent commutative rings R and all $\mathrm{Spf} R$ points of \mathcal{X} . In all cases studied in this paper, the limit above is computed as the limit over a small category and so we will not get into size considerations here.

For an object $\mathcal{F} \in D(\mathcal{X})$ we will refer to the ‘value’ on $x: \mathrm{Spf} R \rightarrow \mathcal{X}$ as the pullback and write it as $x^*\mathcal{F} \in D(R)$.

Example A.5. If p is nilpotent in R , then $D(\mathrm{Spf} R) \simeq D(R)$.

Definition A.6. Let \mathcal{X} be a formal prestack and let $\mathcal{F} \in D(\mathcal{X})$ be a quasi-coherent sheaf.

- (i) We say \mathcal{F} is perfect if for every p -nilpotent commutative ring R and every point $x: \mathrm{Spf} R \rightarrow \mathcal{X}$ the pullback $x^*\mathcal{F} \in D(R)$ is perfect. Write $\mathrm{Perf}(\mathcal{X}) \subseteq D(\mathcal{X})$ for the full subcategory of perfect objects.
- (ii) We say \mathcal{F} has p -complete Tor-amplitude in $[a, b]$ for $a, b \in \mathbf{Z} \sqcup \{\pm\infty\}$ if for every $x: \mathrm{Spf} R \rightarrow \mathcal{X}$ the pullback $x^*\mathcal{F}$ belongs to $D(R)_{[a, b]}$, i.e., $H_i(x^*\mathcal{F}) = 0$ for $i \notin [a, b]$.

Now, we study p -complete \mathbf{Z} -linear stable presentable ∞ -categories.

Definition A.7 (p -completion). Given a $D(\mathbf{Z})$ -module \mathcal{C} in Pr^{L} , the p -completion of \mathcal{C} is $\mathcal{C}_p^\wedge = \mathcal{C} \otimes_{D(\mathbf{Z})} D(\mathbf{Z})_p^\wedge$.

Remark A.8. The object $D(\mathbf{Z})_p^\wedge$ is idempotent in the ∞ -category of $D(\mathbf{Z})$ -modules in Pr^L . Indeed, consider the localization sequence

$$D(\mathbf{Z}[1/p]) \rightarrow D(\mathbf{Z}) \rightarrow D(\mathbf{Z})_p^\wedge$$

of $D(\mathbf{Z})$ -modules. Tensoring with $D(\mathbf{Z})_p^\wedge$ results in another localization sequence, for example by [1, Cor. 3.5]. However, $D(\mathbf{Z})_p^\wedge$ is compactly generated by \mathbf{F}_p so that $D(\mathbf{Z})_p^\wedge \simeq D(\mathrm{End}_{\mathbf{Z}}(\mathbf{F}_p))$ and hence $D(\mathbf{Z}[1/p]) \otimes_{D(\mathbf{Z})} D(\mathbf{Z})_p^\wedge \simeq D(\mathbf{Z}[1/p] \otimes_{\mathbf{Z}} \mathrm{End}_{\mathbf{Z}}(\mathbf{F}_p)) \simeq 0$. Thus, $D(\mathbf{Z})_p^\wedge \rightarrow D(\mathbf{Z})_p^\wedge \otimes_{D(\mathbf{Z})} D(\mathbf{Z})_p^\wedge$ is an equivalence. It follows that $\mathrm{Mod}_{D(\mathbf{Z})_p^\wedge}(\mathrm{Pr}^L)$ is a full subcategory of $\mathrm{Mod}_{D(\mathbf{Z})}(\mathrm{Pr}^L)$, that the inclusion has a left adjoint, and that it is given by p -completion in the sense above.

Note also that we could have worked over the sphere spectrum as opposed to \mathbf{Z} here, but we will not need this generality.

Definition A.9 (p -completeness). A $D(\mathbf{Z})$ -module \mathcal{C} in Pr^L is p -complete if the natural map $\mathcal{C} \rightarrow \mathcal{C}_p^\wedge$ is an equivalence. By the remark above, this property is equivalent to saying that \mathcal{C} admits the structure of a $D(\mathbf{Z})_p^\wedge$ -module in Pr^L . Moreover, by idempotentness of $D(\mathbf{Z})_p^\wedge$, the p -completion of any \mathcal{C} in $\mathrm{Mod}_{D(\mathbf{Z})}(\mathrm{Pr}^L)$ is p -complete.

Our philosophy is that formal geometry is algebraic geometry done in $D(\mathbf{Z})_p^\wedge$ -modules in Pr^L . Recall from [23, Sec. 23.1.2] that a compactly assembled presentable ∞ -category \mathcal{C} is one which is a retract, in Pr^L , of a compactly generated ∞ -category. This is equivalent to the dualizability of \mathcal{C} in $\mathrm{Pr}_{\mathrm{st}}^L$. If \mathcal{C} is a $D(\mathbf{Z})$ -module in $\mathrm{Pr}_{\mathrm{st}}^L$, then it is dualizable as a $D(\mathbf{Z})$ -module if and only if it is dualizable in $\mathrm{Pr}_{\mathrm{st}}^L$ by [23, Thm. D.7.0.7]. Moreover, a $D(\mathbf{Z})$ -linear version of [23, Prop. D.7.3.1] implies that dualizability relative as a $D(\mathbf{Z})$ -module in $\mathrm{Pr}_{\mathrm{st}}^L$ is equivalent to being a retract, in $D(\mathbf{Z})$ -modules in $\mathrm{Pr}_{\mathrm{st}}^L$, of a compactly generated $D(\mathbf{Z})$ -linear presentable ∞ -category.

Lemma A.10. *For any compactly assembled $D(\mathbf{Z})$ -module \mathcal{C} in Pr^L , the natural map $\mathcal{C} \otimes_{D(\mathbf{Z})} D(\mathbf{Z})_p^\wedge \rightarrow \lim_n \mathcal{C} \otimes_{D(\mathbf{Z})} D(\mathbf{Z}/p^n)$ is an equivalence.*

Proof. The case of $D(\mathbf{Z})_p^\wedge \simeq \lim_n D(\mathbf{Z}/p^n)$ follows from Proposition A.11 below. Now being dualizable relative to $D(\mathbf{Z})$ implies that the functor $\mathcal{C} \otimes_{D(\mathbf{Z})} -$ has a left adjoint given by tensoring with the dual \mathcal{C}^\vee . Thus it preserves limits. \square

Proposition A.11. *For a commutative ring R with bounded p -power torsion, there are natural equivalences*

$$D(\mathrm{Spf} R) \simeq \lim_n D(R//p^n) \simeq D(R)_{R//p\text{-cpl}} \simeq D(R)_p^\wedge \simeq \mathrm{Mod}_R(D(\mathbf{Z})_p^\wedge),$$

where $D(R)_{R//p\text{-cpl}}$ is the ∞ -category of $R//p$ -local objects in $D(R)$.

Proof. The fourth equivalence holds more generally: if \mathcal{C} is any presentable $D(\mathbf{Z})$ -linear stable ∞ -category, then $D(R) \otimes_{D(\mathbf{Z})} \mathcal{C} \simeq \mathrm{Mod}_R(\mathcal{C})$. This follows from the definition of the tensor product on Pr^L ; see [22, Prop. 4.8.1.17]. The third equivalence follows from the equivalence between complete and torsion objects: $D(R)_{R//p\text{-cpl}} \simeq D(R)_{p\text{-tors}}$, where the torsion category is the full subcategory of R -module spectra M where each $x \in \pi_* M$ is annihilated by some power of p (depending on x); see [25, Thm. 3.9] or [15, Thm. 3.3.5]. On the other hand, $D(R)_{p\text{-tors}}$ is compactly generated by $R//p$, for example by [25, Prop. 3.7]. It follows that

$$D(R)_{R//p\text{-cpl}} \simeq D(R)_{p\text{-tors}} \simeq D(R \otimes_{\mathbf{Z}} \mathrm{End}_{\mathbf{Z}}(\mathbf{F}_p)) \simeq D(R) \otimes_{D(\mathbf{Z})} D(\mathrm{End}_{\mathbf{Z}}(\mathbf{F}_p)) \simeq D(R) \otimes_{D(\mathbf{Z})} D(\mathbf{Z})_p^\wedge,$$

functorially in R .

Now, we prove the second and first equivalences. If R is an animated commutative ring, then $D(R)_{R//p\text{-cpl}} \simeq \lim D(R//p^n)$. This is closely related to [25, Prop. 2.21], which implies that if $S^0 = R//p$ and if S^\bullet is the descent complex associated to $R \rightarrow S^0$, then $D(R)_{R//p\text{-cpl}} \simeq \lim \mathrm{Tot}(D(S^\bullet))$. One can rewrite the limit

as $\lim_n \text{Tot}^n(\mathbf{D}(S^\bullet))$, the limit of the finite totalizations. As each Tot^n is a finite limit, the natural map $\mathbf{D}(\text{Tot}^n(S^\bullet)) \rightarrow \text{Tot}^n(\mathbf{D}(S^\bullet))$ is fully faithful and the map $\mathbf{D}(R) \rightarrow \text{Tot}^n(\mathbf{D}(S^\bullet))$ canonically factors through the subcategory $\mathbf{D}(\text{Tot}^n(S^\bullet))$ generated by the unit. One has $\text{Tot}^n(S^\bullet) \simeq S//p^{n+1}$ for all $n \geq 0$ by [25, Prop. 2.14] and hence there are functors

$$\mathbf{D}(R)_{R//p\text{-cpl}} \rightarrow \lim_n \mathbf{D}(R//p^{n+1}) \rightarrow \lim_n \text{Tot}^n(\mathbf{D}(S^\bullet)),$$

where the composition is an equivalence and the right-hand map is fully faithful. It follows that the left-hand map is fully faithful and essentially surjective, i.e., an equivalence.

Note that $\mathbf{D}(\text{Spf } R) \simeq \lim_n \mathbf{D}(R/p^n)$ as every $R \rightarrow S$ where p is nilpotent in S factors through some R/p^n . To complete the proof of the lemma, we now show that if R is a commutative ring with bounded p -torsion, then natural map $\lim_n \mathbf{D}(R//p^n) \rightarrow \lim_n \mathbf{D}(R/p^n)$ is an equivalence. As $R//p$ is a compact generator of $\mathbf{D}(R)_p^\wedge$ and hence gives a generator of $\lim_n \mathbf{D}(R//p^n)$ by the equivalence above, it suffices to check that it determines a compact generator of $\lim_n \mathbf{D}(R/p^n)$ and that the induced map on endomorphism ring spectra is an equivalence. First, we check that it is a compact generator. Suppose that $\{M_n\}$ is a tower of $\{R/p^n\}$ -modules defining an object of the limit. We claim first that M_1 is non-zero. To see this, it is enough to assume inductively that M_2 is non-zero. Then, $M_1 \simeq M_2 \otimes_{R/p^2} R/p$. But, $R/p^2 \rightarrow R/p$ is descendable by [24, Prop. 3.35] and hence $\mathbf{D}(R/p^2) \rightarrow \mathbf{D}(R/p)$ is conservative by [24, Prop. 3.22]. As $\lim \mathbf{D}(R/p^n) \xrightarrow{\{M_n\} \mapsto M_1} \mathbf{D}(R/p)$ commutes with all colimits, it follows that this functor reflects colimits. Now, given a tower $\{M_n\}$ as above,

$$\begin{aligned} \text{Map}_{\lim_n \mathbf{D}(R/p^n)}(\{R/p^n \otimes_R R//p\}, \{M_n\}) &\simeq \lim_n \text{Map}_{R/p^n}(R/p^n \otimes_R R//p, M_n) \\ &\simeq \lim_n (M_n \otimes_R R//p)[-1] \\ &\simeq \lim_n (M_n \otimes_{R/p^n} R/p^n \otimes_R R//p)[-1]. \end{aligned}$$

Each $R/p^n \otimes_R R//p$ fits into a canonical fiber sequence

$$R/p^n[p][1] \rightarrow R/p^n \otimes_R R//p \rightarrow R/p$$

where the outer terms are canonically R/p -modules. We see that each $M_n \otimes_{R/p^n} R/p^n \otimes_R R//p$ fits into a fiber sequence

$$M_1 \otimes_{R/p} R/p^n[p][1] \rightarrow M_n \otimes_{R/p^n} R/p^n \otimes_R R//p \rightarrow M_1.$$

As R has bounded p -power torsion, the limit of the left-hand terms vanishes because it is pro-zero. The limit of the right-hand terms is M_1 . Thus, the mapping spectrum above is naturally equivalent to $M_1[-1]$. By what we have already said, the functor $\{M_n\} \mapsto M_1$ is conservative and reflects filtered colimits, which completes the proof that $R//p$ maps to a compact generator in $\lim_n \mathbf{D}(R/p^n)$. The proof that the endomorphism ring spectra agree is an exercise in the Mittag-Leffler condition using boundedness of p -power torsion; it is left to the reader. \square

Remark A.12. This is one point where the derived approach is more natural. If one works with formal spectra of p -nilpotent animated commutative rings in the context of derived formal stacks, then for any animated commutative ring R there is an equivalence $\mathbf{D}(\text{Spf } R) \simeq \mathbf{D}(R)_p^\wedge$.

Corollary A.13. *Let \mathcal{X} be a (non-formal) higher stack. If $\mathbf{D}(\mathcal{X})$ is compactly generated and $\mathbf{D}(\mathcal{X})^\omega$ has bounded p -power torsion, then the natural map $\mathbf{D}(\mathcal{X}) \rightarrow \mathbf{D}(\widehat{\mathcal{X}})$ is equivalent to the morphism $\mathbf{D}(\mathcal{X}) \rightarrow \mathbf{D}(\mathcal{X})_p^\wedge \simeq \lim_n \mathbf{D}(\mathcal{X}) \otimes_{\mathbf{D}(\mathbf{Z})} \mathbf{D}(\mathbf{Z}/p^n)$. Here $\widehat{\mathcal{X}}$ is the restriction of \mathcal{X} to a functor on p -nilpotent commutative rings.*

Proof. Our assumption on $D(\mathcal{X})$ implies that $D(\mathcal{X})^\wedge \simeq \lim_n D(\mathcal{X}) \otimes_{D(\mathbf{Z})} D(\mathbf{Z}/p^n)$ by Lemma A.10. On the other hand, $\operatorname{colim} \mathcal{X} \times_{\mathbf{Z}} \operatorname{Spf} \mathbf{Z}/p^n \simeq \widehat{\mathcal{X}}$, so $D(\widehat{\mathcal{X}}) \simeq \lim_n D(\mathcal{X} \times_{\mathbf{Z}} \operatorname{Spf} \mathbf{Z}/p^n)$. The limits agree under our assumption on bounded p -power torsion. \square

Example A.14. Recall that $D(\mathbf{BG}_m) \simeq \operatorname{GrD}(\mathbf{Z})$, the stable ∞ -category of graded \mathbf{Z} -module spectra. We have $\operatorname{GrD}(\mathbf{Z}) \simeq \prod_{i \in \mathbf{Z}} D(\mathbf{Z})$, so

$$\begin{aligned} D(\widehat{\mathbf{BG}}_m) &\simeq \operatorname{GrD}(\mathbf{Z}) \otimes_{D(\mathbf{Z})} D(\mathbf{Z})_p^\wedge \\ &\simeq \left(\prod_{i \in \mathbf{Z}} D(\mathbf{Z}) \right) \otimes_{D(\mathbf{Z})} D(\mathbf{Z})_p^\wedge \\ &\simeq \prod_{i \in \mathbf{Z}} D(\mathbf{Z})_p^\wedge \\ &\simeq \operatorname{Gr}(D(\mathbf{Z})_p^\wedge), \end{aligned}$$

the ∞ -category of graded objects in p -complete \mathbf{Z} -module spectra. Here, one commutes the product with the tensor product either from geometric considerations or from the fact that infinite coproducts and products agree in Pr^L as can be proved using [21, Thm. 5.5.3.18].

Example A.15. Similarly, $D(\mathbf{A}^1/\mathbf{G}_m) \simeq \operatorname{FD}(\mathbf{Z}) \simeq \operatorname{Fun}(\mathbf{Z}^{\operatorname{op}}, D(\mathbf{Z}))$, the stable ∞ -category of filtrations in \mathbf{Z} -module spectra. The p -completion is equivalent to $D(\widehat{\mathbf{A}}^1/\widehat{\mathbf{G}}_m) \simeq \operatorname{Fun}(\mathbf{Z}^{\operatorname{op}}, D(\mathbf{Z})_p^\wedge)$, the ∞ -category of filtrations in p -complete \mathbf{Z} -module spectra.

Remark A.16. Spectral sequences can behave in strange ways in the p -complete world. For example, consider the filtration $p^* \mathbf{Q}_p$ on \mathbf{Q}_p where $p^n \mathbf{Q}_p \subseteq \mathbf{Q}_p$ consists of the elements of p -adic valuation at least n . As a filtration in $D(\mathbf{Z}_p)$, this is complete and exhaustive, i.e., $\operatorname{colim}_{n \rightarrow \infty} p^{-n} \mathbf{Q}_p = \mathbf{Q}_p$. However, as a filtration in $D(\mathbf{Z}_p)_p^\wedge$, valid because each $p^n \mathbf{Q}_p \cong \mathbf{Z}_p$, this filtration is complete but has colimit 0, the p -completion of \mathbf{Q}_p . Thus, this is a complete exhaustive filtration on 0 where the associated graded pieces gr^i are all equivalent to \mathbf{F}_p . The associated spectral sequence degenerates at the E_1 -page, but the $E_\infty = E_1$ -page is very far from the associated graded of the induced filtration on the homotopy groups of the abutment.

Example A.17. Let $B \leftarrow A \rightarrow C$ be a p -completely Tor-independent diagram of bounded commutative rings with pushout $B \otimes_A C$. Then $D(B \otimes_A C) \simeq D(B) \otimes_{D(A)} D(C)$. It follows that

$$D(\operatorname{Spf}(B \otimes_A C)) \simeq \lim_n D(B/p^n \otimes_{A/p^n} C/p^n) \simeq \lim_n D(B/p^n) \otimes_{D(A/p^n)} D(C/p^n) \simeq D(B)_p^\wedge \otimes_{D(A)_p^\wedge} D(C)_p^\wedge.$$

Moreover, this is equivalent to $D(B \otimes_A C) \otimes_{D(\mathbf{Z})} D(\mathbf{Z})_p^\wedge$.

Now, we briefly discuss the standard t -structure on quasi-coherent sheaves on formal stacks.

Definition A.18. Let \mathcal{X} be a formal stack. An object $\mathcal{F} \in D(\mathcal{X})$ is connective if $x^* \mathcal{F} \in D(R)_{\geq 0}$ for all p -nilpotent rings R and all R -points $x: \operatorname{Spf} R \rightarrow \mathcal{X}$. The full subcategory $D(\mathcal{X})_{\geq 0} \subseteq D(\mathcal{X})$ defines the connective part of a t -structure on $D(\mathcal{X})$.

Example A.19. Suppose that R is a bounded p -complete commutative ring and consider $D(\operatorname{Spf} R)_{\geq 0} \subseteq D(R)_p^\wedge$. Recall that $D(\operatorname{Spf} R) \simeq \operatorname{Mod}_R(D(\operatorname{Spf} \mathbf{Z}_p))$. We claim that $\mathcal{F} \in D(\operatorname{Spf} R)$ is connective if and only if the underlying object of $D(\operatorname{Spf} \mathbf{Z}_p)$ is connective, which happens if and only if $H_i(\mathcal{F}) = 0$ for $i < 0$. In other words, pushforward along $\operatorname{Spf} R \rightarrow \operatorname{Spf} \mathbf{Z}_p$ is t -exact. If $H_i(\mathcal{F}) = 0$ for $i < 0$, then for every $R \rightarrow S$ where p is nilpotent in S , one has that $\mathcal{F} \otimes_R S$ is connective as connective objects are closed under tensor products. On the other hand, $\mathcal{F} \simeq \lim_n (\mathcal{F} \otimes_R R//p^n)$ and each term of $\mathcal{F} \otimes_R R//p^n$ is connective as it fits into an exact

sequence $\mathcal{F} \otimes_R R/p^n \otimes_{R/p^n} R[p^n][1] \rightarrow \mathcal{F} \otimes_R R//p^n \rightarrow \mathcal{F} \otimes_R R/p^n$ and the outer terms are connective by hypothesis. Moreover, the fiber of $\mathcal{F} \otimes_R R//p^n \rightarrow \mathcal{F} \otimes_R R//p^{n-1}$ is naturally equivalent to $\mathcal{F} \otimes_R R//p$ and is in particular connective. It follows that in the tower $\{\mathcal{F} \otimes_R R//p^n\}_n$ the induced maps on H_0 are surjective and so the Mittag-Leffler condition applies and the limit is connective. Thus, the t -structure on $D(\mathrm{Spf} R)$ is the standard t -structure; the coconnective objects are precisely those derived p -complete R -module spectra which are coconnective. The heart of $D(\mathrm{Spf} R)$ is the abelian category of derived p -complete R -modules.

Warning A.20. The coconnective objects in $D(\mathrm{Spf} R)$ do not necessarily have the property that their base change to $\mathrm{Spf} S$ is coconnective for S an R -algebra in which p is nilpotent.

A.2 Base change for prismatic crystals

Now, we give a discussion of base change and the projection formula for prismatic cohomology of general prismatic crystals.

Definition A.21 (Base change for quasi-coherent cohomology). Suppose that

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\ \downarrow f' & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array} \quad (7)$$

is a commutative square of formal stacks. We say this square satisfies base change for quasi-coherent cohomology if the commutative diagram

$$\begin{array}{ccc} D(\mathcal{Y}) & \xrightarrow{f^*} & D(\mathcal{X}) \\ \downarrow g^* & & \downarrow g'^* \\ D(\mathcal{Y}') & \xrightarrow{f'^*} & D(\mathcal{X}') \end{array}$$

of left adjoint functors is right adjointable in the sense of [22, Def. 4.7.4.13]. Using the equivalence $\alpha: f'^* \circ g^* \simeq g'^* \circ f^*$ witnessing the commutativity of these squares, this means that the natural transformation of functors $g^* \circ f_* \rightarrow f'_* \circ f'^* \circ g^* \circ f_* \simeq f'_* \circ g'^* \circ f^* \circ f_* \rightarrow f'_* \circ g'^*: D(\mathcal{X}) \rightarrow D(\mathcal{Y}')$ is an equivalence.

Warning A.22. Right adjointability is not a symmetric notion, and hence neither is the notion of a square satisfying base change for quasi-coherent cohomology. In other words, if it holds for a square as above, it might or might not hold for the transposed square.

Variant A.23. (i) We say that (7) satisfies base change for *bounded above* quasi-coherent cohomology if g^* and g'^* preserve bounded above objects and if the natural transformation $g^* \circ f_* \rightarrow f'_* \circ g'^*$ is an equivalence when evaluated on any bounded above object of $D(\mathcal{X})$.

(ii) We say that (7) satisfies base change for \mathcal{O} -cohomology if the natural transformation $g^* \circ f_* \rightarrow f'_* \circ g'^*$ is an equivalence when evaluated on $\mathcal{O}_{\mathcal{X}}$.

There is also the projection formula. See [28, Tag 08EU] for the classical case of schemes.

Definition A.24 (Projection formula). A morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ of formal stacks satisfies the projection formula (for quasi-coherent cohomology) if the induced symmetric monoidal functor $f^*: D(\mathcal{Y}) \rightarrow D(\mathcal{X})$ does, i.e., if for every $\mathcal{F} \in D(\mathcal{X})$ and $\mathcal{G} \in D(\mathcal{Y})$, the natural map $f_*(\mathcal{F}) \otimes \mathcal{G} \rightarrow f_*(\mathcal{F} \otimes f^*(\mathcal{G}))$ is an equivalence.

Variante A.25. We say that f satisfies the projection formula for bounded above quasi-coherent cohomology with respect to a class of objects $\mathcal{A} \subseteq \mathrm{D}(\mathcal{Y})$ if $f_*(\mathcal{F}) \otimes \mathcal{G} \rightarrow f_*(\mathcal{F} \otimes f^*(\mathcal{G}))$ is an equivalence for $\mathcal{F} \in \mathrm{D}(\mathcal{X})$ bounded above and $\mathcal{G} \in \mathcal{A}$. When $\mathcal{G} = \mathcal{O}_{\mathcal{Y}}$, the projection formula morphism is always an equivalence. Thus, f satisfies the projection formula for quasi-coherent cohomology with respect to objects in the thick subcategory generated by the unit $\mathcal{O}_{\mathcal{Y}}$ in $\mathrm{D}(\mathcal{Y})$.

Example A.26. For us, we will be primarily interested in the class \mathcal{A} of objects in $\mathrm{D}(\mathcal{Y})$ which can be written as filtered colimits of uniformly bounded above perfect objects, i.e., the full subcategory of those \mathcal{G} such that $\mathcal{G} \simeq \mathrm{colim}_I \mathcal{G}_i$ where I is filtered and each \mathcal{G}_i is locally of Tor-amplitude in $(-\infty, N]$ for some N independent of i . Call this class $\mathrm{Ind}^+(\mathrm{Perf}(\mathcal{X})) \subseteq \mathrm{D}(\mathcal{X})$.

Lemma A.27 (Colimits and the projection formula). *Suppose that $f: \mathcal{X} \rightarrow \mathcal{Y}$ is a colimit in formal stacks of morphisms $f_i: \mathcal{X}_i \rightarrow \mathcal{Y}_i$. If each f_i satisfies the projection formula and if each of the induced squares*

$$\begin{array}{ccc} \mathcal{X}_i & \xrightarrow{a_i} & \mathcal{X} \\ \downarrow f_i & & \downarrow f \\ \mathcal{Y}_i & \xrightarrow{b_i} & \mathcal{Y} \end{array}$$

satisfies base change for quasi-coherent cohomology, then f satisfies the projection formula.

Proof. Fix $\mathcal{F} \in \mathrm{D}(\mathcal{X})$ and $\mathcal{G} \in \mathrm{D}(\mathcal{Y})$. As $\mathrm{D}(\mathcal{Y}) \simeq \lim_i \mathrm{D}(\mathcal{Y}_i)$, to prove that $f_*(\mathcal{F}) \otimes \mathcal{G} \rightarrow f_*(\mathcal{F} \otimes f^*(\mathcal{G}))$ is an equivalence, it is enough to prove that for each i the natural map $b_i^*(f_*(\mathcal{F}) \otimes \mathcal{G}) \rightarrow b_i^*f_*(\mathcal{F} \otimes f^*(\mathcal{G}))$ is an equivalence. Using symmetric monoidality of the pullback functors as well as base change for quasi-coherent cohomology, this map can be rewritten as the natural map $f_{i*}(a_i^*(\mathcal{F})) \otimes b_i^*(\mathcal{G}) \rightarrow f_{i*}a_i^*(\mathcal{F} \otimes f^*(\mathcal{G})) \simeq f_{i*}(a_i^*(\mathcal{F}) \otimes f_i^*(b_i^*(\mathcal{G})))$, which is an equivalence by the projection formula for f_i applied to $a_i^*(\mathcal{F}) \in \mathrm{D}(\mathcal{X}_i)$ and $b_i^*(\mathcal{G}) \in \mathrm{D}(\mathcal{Y}_i)$. \square

Variante A.28 (Colimits and bounded above projection formula). If in the situation of Lemma A.27 each f_i satisfies the projection formula for bounded above quasi-coherent cohomology with respect to classes $\mathcal{A} \in \mathrm{D}(\mathcal{Y}_i)$, if $\mathcal{A} \subseteq \mathrm{D}(\mathcal{Y})$ is a class of objects which is sent under b_i^* into \mathcal{A}_i , and if a_i^* preserves bounded above objects, then f satisfies base change for bounded above quasi-coherent cohomology with respect to \mathcal{A} .

Lemma A.29 (Colimits and base change). *Suppose that a square \square as in (7) is realized as a colimit in formal stacks of commutative squares*

$$\square_i = \begin{array}{ccc} \mathcal{X}'_i & \xrightarrow{g'_i} & \mathcal{X}_i \\ \downarrow f'_i & & \downarrow f_i \\ \mathcal{Y}'_i & \xrightarrow{g_i} & \mathcal{Y}_i \end{array}$$

over some indexing simplicial set I . For each edge $j \rightarrow i$ in I , let

$$\square_{i,j} = \begin{array}{ccc} \mathcal{X}_j & \longrightarrow & \mathcal{X}_i \\ \downarrow f_j & & \downarrow f_i \\ \mathcal{Y}_j & \longrightarrow & \mathcal{Y}_i \end{array} \quad \text{and} \quad \square'_{i,j} = \begin{array}{ccc} \mathcal{X}'_j & \longrightarrow & \mathcal{X}'_i \\ \downarrow f'_j & & \downarrow f'_i \\ \mathcal{Y}'_j & \longrightarrow & \mathcal{Y}'_i. \end{array}$$

If \square_i , $\square_{i,j}$, and $\square'_{i,j}$ satisfy base change for cohomology for each edge $j \rightarrow i$ in I , then so does $\mathrm{colim}_{i \in I} \square_i \simeq \square$.

Proof. The lemma follows from [22, Cor. 4.7.4.18(2)], to which we refer for more details. At the level of quasi-coherent sheaves, applying $D(-)$ turns these colimit diagrams into limit diagrams in Pr^{L} , i.e., $D(\square) \simeq \lim D(\square_i)$. As i varies, the pullback functors $D(\mathcal{Y}_i) \rightarrow D(\mathcal{X}_i)$ give a functor $D(f^*): I^{\mathrm{op}} \rightarrow \mathrm{Fun}(\Delta^1, \mathrm{Pr}^{\mathrm{L}})$. By hypothesis, this functor factors to give a functor $D(f^*): I^{\mathrm{op}} \rightarrow \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Pr}^{\mathrm{L}})$, where $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Pr}^{\mathrm{L}})$ is the (non-full) subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Pr}^{\mathrm{L}})$ on all left adjoint functors $\mathcal{C} \rightarrow \mathcal{D}$ is Pr^{L} and where the morphisms are natural transformations $(\mathcal{C} \rightarrow \mathcal{D}) \rightarrow (\mathcal{E} \rightarrow \mathcal{F})$ in Pr^{L} corresponding to right adjointable commutative squares

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{F}. \end{array}$$

The same result for the maps induced by $\mathcal{X}'_i \rightarrow \mathcal{Y}'_i$ gives another functor $D(f'^*): I^{\mathrm{op}} \rightarrow \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Pr}^{\mathrm{L}})$. Moreover, the condition on the squares \square_i implies that pulling back along the maps f_i and f'_i gives a natural transformation $D(f^*) \rightarrow D(f'^*)$ of functors $I^{\mathrm{op}} \rightarrow \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Pr}^{\mathrm{L}})$. Taking limits and applying [22, Cor. 4.7.4.18(2)], the result follows. \square

Remark A.30. The proof of Lemma A.29 also shows that the commutative squares

$$\begin{array}{ccc} \mathcal{X}_i & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y}_i & \longrightarrow & \mathcal{Y} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{X}'_i & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathcal{Y}'_i & \longrightarrow & \mathcal{Y}' \end{array}$$

satisfy base change for quasi-coherent cohomology for all $i \in I$ as well.

Remark A.31 (Colimits are universal). Recall the fact that colimits are universal in an ∞ -topos, meaning that they are stable under pullback. In particular, if $\mathrm{colim}_{i \in I} \mathcal{Y}_i \simeq \mathcal{Y}$ as formal stacks and if $\mathcal{X} \rightarrow \mathcal{Y}$ is a map of formal stacks, then $\mathrm{colim}_{i \in I} \mathcal{Y}_i \times_{\mathcal{Y}} \mathcal{X} \rightarrow \mathcal{X}$ is an equivalence. See [21, Sec. 6.1].

Variante A.32. Suppose that $\{\mathcal{X}_i \rightarrow \mathcal{Y}_i\}_{i \in I}$ is an I -indexed diagram of morphisms of formal stacks. If each square

$$\square_{i,j} = \begin{array}{ccc} \mathcal{X}_j & \xrightarrow{a'_{ij}} & \mathcal{X}_i \\ \downarrow f_j & & \downarrow f_i \\ \mathcal{Y}_j & \xrightarrow{a_{ij}} & \mathcal{Y}_i \end{array}$$

satisfies base change for quasi-coherent cohomology, then so does each square

$$\square_i = \begin{array}{ccc} \mathcal{X}_i & \xrightarrow{g'_i} & \mathcal{X} \\ \downarrow f_i & & \downarrow f \\ \mathcal{Y}_i & \xrightarrow{g_i} & \mathcal{Y} \end{array}$$

where $(\mathcal{X} \rightarrow \mathcal{Y}) \simeq \mathrm{colim}_I (\mathcal{X}_i \rightarrow \mathcal{Y}_i)$. The proof is similar to that of Lemma A.29. However, we will need a version which holds under weakened hypotheses in the bounded above case. Specifically, assume that each a'_{ij} and a_{ij} preserve bounded above objects and that each square $\square_{i,j}$ satisfies base change for bounded above quasi-coherent cohomology. Then, so does each square \square_i . To prove this, consider an object $\mathcal{F} \in D(\mathcal{X})^+$ and

note that $\mathcal{F} \simeq \lim g'_i g_i'^* \mathcal{F}$. Let $\mathcal{F}_i = g_i'^* \mathcal{F}$. So, we can view \mathcal{F} as the compatible family of objects $\mathcal{F}_i \in D(\mathcal{X}_i)$, meaning that there are compatible equivalences $\mathcal{F}_j \simeq a_{ij}'^* \mathcal{F}_i$ for all arrows $j \rightarrow i$ in I . Applying f_* one finds that $f_* \mathcal{F} \simeq \lim_I f_* g'_i \mathcal{F}_i \simeq \lim_I g_i^* f_{i*} \mathcal{F}_i$. We want to know that $g_i^* f_* \mathcal{F} \simeq f_{i*} g_i'^* \mathcal{F} \simeq f_{i*} \mathcal{F}_i$. However, there are equivalences $a_{ij}'^* f_{i*} \mathcal{F}_i \simeq f_{j*} a_{ij}'^* \mathcal{F}_i \simeq f_{j*} \mathcal{F}_j$, compatible in I , by our assumption on base change for bounded above cohomology and the preservation of bounded above objects by $a_{ij}'^*$ and a_{ij}^* . Thus, the objects $f_{i*} \mathcal{F}_i$ (together with the given equivalences) determine an object of $\lim D(\mathcal{Y}_i) \simeq D(\mathcal{Y})$ (or a cartesian section of an appropriate fibration corresponding to $i \mapsto D(\mathcal{Y}_i)$). This object is $f_* \mathcal{F}$ and the proof is complete.

Remark A.33. Let $D(\mathcal{X}_*) \rightarrow I^{\text{op}}$ be the cartesian fibration corresponding to $i \mapsto D(\mathcal{X}_i)$ together with the pullback maps. The limit $D(\mathcal{X})$ is the ∞ -category of cartesian sections of the fibration. There is by hypothesis a natural transformation $D(\mathcal{Y}_*) \rightarrow D(\mathcal{X}_*)$ of cartesian fibrations over I^{op} and the pullback functor f^* takes cartesian sections to cartesian sections. The functor $D(\mathcal{X}_*) \rightarrow D(\mathcal{Y}_*)$, corresponding to the right adjoints f_* , need not take cartesian fibrations to cartesian fibrations. However, this is exactly what is guaranteed by our assumption on the squares $\square_{i,j}$.

Lemma A.34. *Let $B \leftarrow A \rightarrow C$ be a diagram of bounded p -complete commutative rings. If the diagram is p -completely Tor-independent (for example if B or C is p -completely flat over A), then*

$$\begin{array}{ccc} \text{Spf}(B \widehat{\otimes}_A C) & \longrightarrow & \text{Spf } C \\ \downarrow & & \downarrow \\ \text{Spf } B & \longrightarrow & \text{Spf } A \end{array}$$

satisfies base change for quasi-coherent cohomology. In addition, $\text{Spf } C \rightarrow \text{Spf } A$ satisfies the projection formula.

Proof. The assumptions imply that this square is the colimit of the squares

$$\begin{array}{ccc} \text{Spf}(B/p^n \otimes_{A/p^n} C/p^n) & \longrightarrow & \text{Spf}(C/p^n) \\ \downarrow & & \downarrow \\ \text{Spf}(B/p^n) & \longrightarrow & \text{Spf}(A/p^n) \end{array}$$

and each of these squares satisfies the projection formula and base change for quasi-coherent cohomology by a standard argument; see for example [28, Tags 08EU, 08IB]. The statement now follows from Lemmas A.27 and A.29. \square

Remark A.35. This simple case of base change can be proved directly. Consider the commutative square

$$\begin{array}{ccc} D(A)_p^\wedge & \longrightarrow & D(C)_p^\wedge \\ \downarrow & & \downarrow \\ D(B)_p^\wedge & \longrightarrow & D(B \widehat{\otimes}_A C)_p^\wedge. \end{array}$$

For a derived p -complete C -module spectrum M , the base change transformation is a morphism $B \widehat{\otimes}_A M \rightarrow (B \widehat{\otimes}_A C) \widehat{\otimes}_C M$, which is an equivalence. However, the lemma gives an illustration of how we will use colimits to produce new cases of base change.

Lemma A.36. *Let $B \leftarrow A \rightarrow C$ be a diagram of bounded prisms. If the diagram is (p, I) -completely Tor-independent (for example if B or C is (p, I) -completely flat over A), then the pullback square*

$$\begin{array}{ccc} \mathrm{Spf}(B \widehat{\otimes}_A C) & \longrightarrow & \mathrm{Spf} C \\ \downarrow & & \downarrow \\ \mathrm{Spf} B & \longrightarrow & \mathrm{Spf} A \end{array}$$

satisfies base change for quasi-coherent cohomology. In addition, $\mathrm{Spf} C \rightarrow \mathrm{Spf} A$ satisfies the projection formula.

Proof. Follow the proof of Lemma A.34 but with $\mathrm{Spf}(-/I^n)$. \square

In order to prove more general cases of base change for prismatic crystals, we need a base case provided by the next proposition.

Proposition A.37. *Suppose that A is a bounded prism and that X is a quasicompact separated formal scheme over $\mathrm{Spf} A$. If X is bounded and $L_{X/\overline{A}}$ has p -complete Tor-amplitude in $[0, 1]$, then for every map of prisms $A \rightarrow B$ of bounded (p, I) -complete Tor-amplitude, the pullback square*

$$\begin{array}{ccc} \mathrm{WCart}_{X_{\overline{B}/B}} & \longrightarrow & \mathrm{WCart}_{X/A} \\ \downarrow & & \downarrow \\ \mathrm{Spf} B & \longrightarrow & \mathrm{Spf} A \end{array}$$

satisfies base change for bounded above quasi-coherent cohomology. Moreover, $\mathrm{WCart}_{X/A} \rightarrow \mathrm{Spf} A$ satisfies the projection formula for $\mathcal{F} \in D(\mathrm{WCart}_{X/A})$ and $\mathcal{G} \in \mathrm{Ind}^+(\mathrm{Perf}(\mathrm{Spf} A))$.

Proof. We have the following lemma, which is not a special case of Lemma A.29.

Lemma A.38. *Suppose that*

$$\begin{array}{ccc} \mathcal{X}'_i & \xrightarrow{g'_i} & \mathcal{X}_i \\ \downarrow f'_i & & \downarrow f_i \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

is a finite diagram of commutative squares each satisfying base change for (bounded above) quasi-coherent cohomology. Then, the colimit diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\ \downarrow f' & & \downarrow f \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

satisfies base change for (bounded above) quasi-coherent cohomology. Moreover, if each f_i satisfies the projection formula, then so does f .

Proof. Write $h_i: \mathcal{X}_i \rightarrow \mathcal{X}$. Fix $\mathcal{F} \in \mathcal{D}(\mathcal{X})$ which by definition is equivalent to $\lim h_{i*} h_i^* \mathcal{F}$. Let $\mathcal{F}_i = h_i^* \mathcal{F}$. Thus,

$$\begin{aligned} g^* f_*(\mathcal{F}) &\simeq g^* f_* \lim h_{i*} \mathcal{F}_i \\ &\simeq g^* \lim f_* h_{i*} \mathcal{F}_i \\ &\simeq \lim g^* f_* h_{i*} \mathcal{F}_i \\ &\simeq \lim g^* f_{i*} \mathcal{F}_i \\ &\simeq \lim f'_{i*} g_i'^* \mathcal{F}_i \\ &\simeq \lim f'_{i*} h_i^* g'^* \mathcal{F} \\ &\simeq f'_* g'^* \mathcal{F}, \end{aligned}$$

where the third equivalence is because g^* commutes with finite limits and the fifth is by the assumption of base change. The proof of the claim about the projection formula is similar using that left adjoint functors between stable ∞ -categories commute with finite limits. \square

Now, one reduces the proof of the proposition using Lemma A.38 to the case where $X = \mathrm{Spf} R$ by induction on the number of affines needed to cover X . Indeed, assume the proposition is true for formal schemes which can be covered by at most n affines, and assume that X can be covered by $n + 1$ affines $\{\mathrm{Spf} R_i\}_{i=1}^{n+1}$. Then, let $Y = \cup_{i=1}^n \mathrm{Spf} R_i$ and $Z = \mathrm{Spf} R_{n+1}$. The intersection $W = Y \cap Z$ satisfies the hypotheses of the lemma, and is covered by n affines. Moreover,

$$\begin{array}{ccc} \mathrm{WCart}_{W/A} & \longrightarrow & \mathrm{WCart}_{Z/A} \\ \downarrow & & \downarrow \\ \mathrm{WCart}_{Y/A} & \longrightarrow & \mathrm{WCart}_{X/A} \end{array}$$

is a pushout in formal stacks and similarly for the base change to B . Now, apply the lemma.

Now, assume that $X = \mathrm{Spf} R$ and let $R \rightarrow R^0$ be a quasisyntomic cover where R^0 is relatively quasiregular semiperfectoid over \bar{A} . Let R^\bullet be the resulting Čech complex. By [8, Thm. 7.17], there are natural equivalences

$$\mathrm{WCart}_{R \widehat{\otimes}_{\bar{A}} \bar{B}/B} \simeq \mathrm{Spf} \Delta_{R \widehat{\otimes}_{\bar{A}} \bar{B}/B} \quad \text{and} \quad \mathrm{WCart}_{R/A} \simeq \mathrm{Spf} \Delta_{R/A},$$

where $\mathrm{Spf} \Delta_{R/A}$ is the functor $\mathrm{Map}_{\mathrm{DAI}g_A}(\Delta_{R/A}, -)$ restricted to (p, I) -nilpotent A -algebras and similarly for $\mathrm{Spf} \Delta_{R \widehat{\otimes}_{\bar{A}} \bar{B}/B}$. Throughout, we use that $\Delta_{R/A} \simeq \mathrm{Tot} \Delta_{R^\bullet/A}$ and that $|\mathrm{Spf} \Delta_{R^\bullet/A}| \simeq |\mathrm{Spf} \Delta_{R/A}|$ and similarly for the base change to B . The first equivalence is by descent for prismatic cohomology; the second is because $\mathrm{WCart}_{R^0/A} \rightarrow \mathrm{WCart}_{R/A}$ is a surjection of flat sheaves (by [8, Lem. 6.3]) and by compatibility of $\mathrm{WCart}_{-/A}$ with pullbacks (which follows from [8, Rem. 3.5]).

Warning A.39. Note that despite the apparent affine behavior of $\mathrm{WCart}_{R/A}$, it is not typically the case that the global sections functor $\mathcal{D}(\mathrm{Spf} \Delta_{R/A}) \rightarrow \mathcal{D}(\Delta_{R/A})_{p,I}^\wedge$ is an equivalence. This complicates the proof of base change. However, when $\Delta_{R/A}$ is discrete, taking global sections is an equivalence.

Thus, the pullback square of the statement of the proposition is equivalent in the affine case to a pullback square

$$\begin{array}{ccc} \mathrm{Spf} \Delta_{R \widehat{\otimes}_{\bar{A}} \bar{B}/B} & \xrightarrow{g'} & \mathrm{Spf} \Delta_{R/A} \\ \downarrow f' & & \downarrow f \\ \mathrm{Spf} B & \xrightarrow{g} & \mathrm{Spf} A. \end{array}$$

We prove a lemma giving some properties of this square.

Lemma A.40. (a) *The map g' satisfies the projection formula.*

(b) *The map f satisfies the projection formula for bounded above objects with respect to $\mathrm{Ind}^+(\mathrm{Perf}(\mathrm{Spf} A))$.*

(c) *The natural map $f^*g_*B \rightarrow g'_*f'^*B$ is an equivalence.*²²

Proof. For parts (a) and (c) we use the commutative diagram

$$\begin{array}{ccccc} \mathrm{Spf} \Delta_{R^\bullet \widehat{\otimes}_A \overline{B}/B} & \longrightarrow & \mathrm{Spf} \Delta_{R \widehat{\otimes}_A \overline{B}/B} & \longrightarrow & \mathrm{Spf} B \\ \downarrow g' & & \downarrow g' & & \downarrow g \\ \mathrm{Spf} \Delta_{R^\bullet/A} & \longrightarrow & \mathrm{Spf} \Delta_{R/A} & \longrightarrow & \mathrm{Spf} A. \end{array}$$

The left vertical arrow represents a simplicial diagram in morphisms of stacks satisfying the projection formula. The induced pullback functors on quasi-coherent sheaves gives a cosimplicial diagram in $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Pr}^{\mathrm{L}})$ with limit the pullback map on quasi-coherent sheaves associated to the middle vertical arrow. It follows from Variant A.32 that the left-hand square satisfies base change for quasi-coherent sheaves in each simplicial degree. Thus, by Lemma A.27, the map g' satisfies the projection formula. As the exterior square satisfies base change for quasi-coherent cohomology, so does the right-hand square by taking a limit in $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Pr}^{\mathrm{L}})$.

For part (b), suppose that $\mathcal{F} \in \mathrm{D}(\mathrm{Spf} \Delta_{R/A})$ and $\mathcal{G} \in \mathrm{Spf} A$ are bounded above and assume that \mathcal{G} can be written as a filtered colimit of uniformly bounded above (p, I) -completely perfect complexes of A -module spectra. The natural map

$$f_*(\mathcal{F}) \widehat{\otimes}_A \mathcal{G} \rightarrow f_*(\mathcal{F} \otimes_{\mathcal{O}_{\mathrm{Spf} \Delta_{R/A}}} f^*(\mathcal{G}))$$

can be written as

$$\mathrm{Tot}(\mathcal{F}^\bullet) \widehat{\otimes}_A \mathcal{G} \rightarrow \mathrm{Tot}(\mathcal{F}^\bullet \widehat{\otimes}_A \mathcal{G}),$$

where \mathcal{F}^\bullet denotes the pullback of \mathcal{F} to $\mathrm{Spf} \Delta_{R^\bullet/A}$. The map is an equivalence for any (p, I) -completely perfect A -module spectrum. As totalizations commute with filtered colimits of uniformly bounded above objects, the map is an equivalence for \mathcal{G} as well. \square

Returning to the proof of the proposition in the case when $X = \mathrm{Spf} R$ is affine, consider the natural map $g^*f_*\mathcal{F} \rightarrow f'_*g'^*\mathcal{F}$ for $\mathcal{F} \in \mathrm{D}(\mathrm{Spf} \Delta_{R/A})$ bounded above. As g_* is conservative, to test if this map is an equivalence, we can apply g_* to obtain $g_*g^*f_*\mathcal{F} \rightarrow g_*f'_*g'^*\mathcal{F}$, which can be rewritten as a map

$$f_*\mathcal{F} \widehat{\otimes}_A B \rightarrow f_*g'_*g'^*\mathcal{F} \simeq f_*(g'_*\mathcal{O}_{\mathrm{Spf} \Delta_{R \widehat{\otimes}_A \overline{B}/B}} \otimes_{\mathcal{O}_{\mathrm{Spf} \Delta_{R/A}}} \mathcal{F}) \simeq f_*(f^*g_*B \otimes_{\mathcal{O}_{\mathrm{Spf} \Delta_{R/A}}} \mathcal{F}) \simeq g_*B \widehat{\otimes}_A f_*\mathcal{F},$$

where the first equivalence is by part (a) of Lemma A.40, the second by part (c), and third by part (b), which uses that B has bounded (p, I) -complete Tor-amplitude as an A -module and hence can be written as a finite iterated extension of shifts of (p, I) -completely flat A -modules, which are filtered colimits of finitely presented projective A -modules by a (p, I) -complete version of Lazard's theorem. One checks that the map is indeed the natural equivalence between the left and right-hand sides, which completes the proof. Finally, the claim about the projection formula is Lemma A.40(b). \square

Corollary A.41. *Suppose that A is a δ -ring and X is a quasisyntomic p -adic formal scheme over $\mathrm{Spf} A$. If X is quasicompact and separated and $\mathrm{L}_{X/A}$ has p -complete Tor-amplitude in $[0, 1]$, then for every transversal prism B^0 the pullback square*

$$\begin{array}{ccc} \mathrm{WCart}_{X_{B^0/A} \widehat{\otimes} B^0} & \longrightarrow & \mathrm{WCart}_{X/A} \\ \downarrow & & \downarrow \\ \mathrm{Spf} B^0 & \longrightarrow & \mathrm{WCart} \end{array}$$

²²That is, the *transposed square* satisfies base change for \mathcal{O} -cohomology.

satisfies base change for bounded above quasi-coherent cohomology. Moreover, $\mathrm{WCart}_{X/A} \rightarrow \mathrm{WCart}$ satisfies the bounded above projection formula with respect to $\mathrm{Ind}^+(\mathrm{Perf}(\mathrm{WCart}))$.

Proof. Let $\mathrm{Spf} B^\bullet \rightarrow \mathrm{WCart}$ be the Čech complex of $\mathrm{Spf} B^0 \rightarrow \mathrm{WCart}$. Pulling back $\mathrm{WCart}_{X/A} \rightarrow \mathrm{WCart}$ along $\mathrm{Spf} B^\bullet \rightarrow \mathrm{WCart}$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{WCart}_{X_{\overline{B^\bullet}/A \widehat{\otimes} B^\bullet}} & \longrightarrow & \mathrm{WCart}_{X/A} \\ \downarrow & & \downarrow \\ \mathrm{Spf} B^\bullet & \longrightarrow & \mathrm{WCart} \end{array}$$

where the vertical arrow on the left defines a simplicial object in morphisms of formal stacks. For each $[m] \rightarrow [n]$ in Δ^{op} the pullback square

$$\begin{array}{ccc} \mathrm{WCart}_{X_{\overline{B^m}/A \widehat{\otimes} B^m}} & \longrightarrow & \mathrm{WCart}_{X_{\overline{B^n}/A \widehat{\otimes} B^n}} \\ \downarrow & & \downarrow \\ \mathrm{Spf} B^m & \longrightarrow & \mathrm{Spf} B^n \end{array}$$

satisfies base change for bounded above quasi-coherent sheaves by Proposition A.37 and the right-hand morphism satisfies the projection formula for bounded above objects. The desired base change and projection formula result by Variant A.32. \square

Corollary A.42. *Let $A \rightarrow R$ be a relatively quasisyntomic δ -pair and let $A \rightarrow B$ be a map of δ -rings which has p -complete bounded Tor-amplitude. The pullback square*

$$\begin{array}{ccc} \mathrm{WCart}_{R \widehat{\otimes}_A B/B} & \xrightarrow{g'} & \mathrm{WCart}_{R/A} \\ \downarrow f' & & \downarrow f \\ \mathrm{Spf} B \times \mathrm{WCart} & \xrightarrow{g} & \mathrm{Spf} A \times \mathrm{WCart} \end{array}$$

satisfies base change for quasi-coherent cohomology and f satisfies the projection formula for bounded above quasi-coherent cohomology with respect to $\mathrm{Ind}^+(\mathrm{Perf}(\mathrm{Spf} A \times \mathrm{WCart}))$.

Proof. Fix a transversal prism C^0 and consider the Čech complex $\mathrm{Spf} C^\bullet \rightarrow \mathrm{WCart}$ of $\rho_{C^0}: \mathrm{Spf} C^0 \rightarrow \mathrm{WCart}$. Pulling back along $\mathrm{Spf} A \times \mathrm{Spf} C^\bullet \rightarrow \mathrm{Spf} A \times \mathrm{WCart}$ for a transversal prism point $\rho_C: \mathrm{Spf} C \rightarrow \mathrm{WCart}$, the resulting cosimplicial pullback square is equivalent to

$$\begin{array}{ccc} \mathrm{WCart}_{R \widehat{\otimes}_A B \widehat{\otimes} C^\bullet / B \widehat{\otimes} C^\bullet} & \xrightarrow{g'^\bullet} & \mathrm{WCart}_{R \widehat{\otimes} C^\bullet / A \widehat{\otimes} C^\bullet} \\ \downarrow f'^\bullet & & \downarrow f^\bullet \\ \mathrm{Spf}(B \widehat{\otimes} C^\bullet) & \xrightarrow{g^\bullet} & \mathrm{Spf}(A \widehat{\otimes} C^\bullet). \end{array}$$

Each of these squares satisfies the hypothesis of Proposition A.37 by our assumption of the p -complete bounded Tor-amplitude of B as an A -module and hence it satisfies base change for bounded above quasi-coherent cohomology. We would like to now apply Lemma A.29, but we need a variant which applies in the bounded

above setting, along the lines of Variant A.32. Fix

$$\begin{aligned} a^\bullet &: \mathrm{WCart}_{R\widehat{\otimes}C^\bullet/A\widehat{\otimes}C} \rightarrow \mathrm{WCart}_{R/A}, \\ b^\bullet &: \mathrm{Spf}(A\widehat{\otimes}C^\bullet) \rightarrow \mathrm{Spf} A \times \mathrm{WCart}, \\ a'^\bullet &: \mathrm{WCart}_{R\widehat{\otimes}_A B\widehat{\otimes}C^\bullet/B\widehat{\otimes}C} \rightarrow \mathrm{WCart}_{R\widehat{\otimes}_A B/B}, \\ b'^\bullet &: \mathrm{Spf}(B\widehat{\otimes}C^\bullet) \rightarrow \mathrm{Spf} B \times \mathrm{WCart}, \end{aligned}$$

Consider $\mathcal{F} \in \mathrm{D}(\mathrm{WCart}_{R/A})$, which is equivalent to $\lim_{[i] \in \Delta} a_*^i a^{i*} \mathcal{F}$. Let $\mathcal{F}_i = a^{i*} \mathcal{F}$. By Variant A.32, $b^{i*} f_* \mathcal{F} \simeq f_{i*} \mathcal{F}_i$ for all i , naturally in i . Thus, the pullback $g^* f_* \mathcal{F}$ is determined by the compatible family $g^{\bullet*} f_{\bullet*} \mathcal{F}_\bullet$ (as an appropriate cartesian section corresponding to an object of $\mathrm{D}(\mathrm{Spf} B \times \mathrm{WCart}) \simeq \lim \mathrm{Spf}(B\widehat{\otimes}C)$). However, by base change for bounded above quasi-coherent cohomology in each (co)simplicial degree, $g^{\bullet*} f_{\bullet*} \mathcal{F}_\bullet \simeq f'_{\bullet*} g'^{\bullet*} \mathcal{F}_\bullet$, which is checked in the same way to yield $f'_* g'^* \mathcal{F}$ upon passage to limits using Variant A.32 again. \square

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