

# Lifting Problems and Transgression for Non-Abelian Gerbes

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## Abstract

We discuss various lifting and reduction problems for bundles and gerbes in the context of a strict Lie 2-group. We obtain a geometrical formulation (and a new proof) for the exactness of Breen’s long exact sequence in non-abelian cohomology. We use our geometrical formulation in order to define a transgression map in non-abelian cohomology. This transgression map relates the degree one non-abelian cohomology of a smooth manifold (represented by non-abelian gerbes) with the degree zero non-abelian cohomology of the free loop space (represented by principal bundles). We prove several properties for this transgression map. For instance, it reduces – in case of a Lie 2-group with a single object – to the ordinary transgression in ordinary cohomology. We describe applications of our results to string manifolds: Firstly, we obtain a new comparison theorem for different notions of string structures. Secondly, our transgression map establishes a direct relation between string structures and spin structure on the loop space.

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## 1 Introduction

In the present paper we study categorical and bicategorical extensions of the non-abelian cohomology of a smooth manifold  $M$ . In degree zero, non-abelian cohomology with values in a Lie groupoid  $\Gamma$  is a set denoted  $\check{H}^0(M, \Gamma)$ , in

which a cocycle with respect to a cover of  $M$  by open sets  $U_i$  is a collection of smooth maps

$$\alpha_i : U_i \longrightarrow \Gamma_0 \quad \text{and} \quad g_{ij} : U_i \cap U_j \longrightarrow \Gamma_1$$

such that  $g_{ij}(x)$  is a morphism from  $\alpha_i(x)$  to  $\alpha_j(x)$ , and  $g_{ik}(x) = g_{jk}(x) \circ g_{ij}(x)$ . Here we have denoted by  $\Gamma_0$  the objects of  $\Gamma$ , by  $\Gamma_1$  the morphisms, and by  $\circ$  the composition. The categorical extension of the set  $\check{H}^0(M, \Gamma)$  we want to study is the category  $\mathcal{B}un_\Gamma(M)$  of *principal  $\Gamma$ -bundles* over  $M$ . Such bundles have a total space  $p : P \longrightarrow M$ , an anchor map  $\alpha : P \longrightarrow \Gamma_0$ , and carry a principal “action of  $\Gamma$  along  $\alpha$ ”. The relation between the set  $\check{H}^0(M, \Gamma)$  and the category  $\mathcal{B}un_\Gamma(M)$  is that the first is the set of isomorphism classes of the second; under this relation, the functions  $\alpha_i$  are local versions of the anchor map  $\alpha$ , and the functions  $g_{ij}$  are the transition functions. A review of definitions and results about principal  $\Gamma$ -bundles can be found in [MM03, section 5.7] or [NW, Section 2.2].

If  $G$  is a Lie group, there is a Lie groupoid  $\mathcal{B}G$  with a single object and the group  $G$  as its morphisms. A principal  $\mathcal{B}G$ -bundle is the same as an ordinary principal  $G$ -bundle. Correspondingly,  $\check{H}^0(M, \mathcal{B}G)$  is the classical Čech cohomology  $\check{H}_{cl}^1(M, G)$ .

Going to higher degrees in non-abelian cohomology is never for free. For example, the classical Čech cohomology  $\check{H}_{cl}^{n+1}(M, G)$  can only be defined in degrees  $n > 0$  if  $G$  is abelian. Then, for an abelian Lie group  $A$ , the group  $\check{H}_{cl}^{n+1}(M, A)$  has a nice  $(n + 1)$ -categorical extension by *A-bundle  $n$ -gerbes*. For  $n = 1$ , these are the ordinary *A-bundle gerbes* introduced by Murray (for  $A = \mathbb{C}^\times$ ) [Mur96].

In this paper we want to go to *degree one* in non-abelian cohomology. This requires additional structure on the Lie groupoid  $\Gamma$ : a strict Lie 2-group structure – a certain kind of monoidal structure. Degree one non-abelian cohomology with values in a Lie 2-group  $\Gamma$  is a set denoted  $\check{H}^1(M, \Gamma)$ , and a cocycle is a collection of smooth maps

$$g_{ij} : U_i \cap U_j \longrightarrow \Gamma_0 \quad \text{and} \quad f_{ijk} : U_i \cap U_j \cap U_k \longrightarrow \Gamma_1$$

such that  $f_{ijk}(x)$  is a morphism from  $g_{jk}(x) \otimes g_{ij}(x)$  to  $g_{ik}(x)$ , and  $f_{ijk}$  itself satisfies a higher cocycle condition. The bicategorical extension of the set  $\check{H}^1(M, \Gamma)$  that we want to study is the bicategory  $2\text{-}\mathcal{B}un_\Gamma(M)$  of *principal  $\Gamma$ -2-bundles* over  $M$ . The total space of such a 2-bundle is a Lie groupoid  $\mathcal{P}$  that carries a principal action of the Lie 2-group  $\Gamma$ . The set of isomorphism classes of the bicategory  $2\text{-}\mathcal{B}un_\Gamma(M)$  is in bijection with  $\check{H}^1(M, \Gamma)$ . A detailed account of principal 2-bundles is given in Section [NW, Section 6]. In Section 2 of the present paper we review some important definitions and results.

A Lie 2-group  $\Gamma$  has two interesting invariants denoted by  $\pi_0\Gamma$  and  $\pi_1\Gamma$ . The first is the set of isomorphism classes of objects of  $\Gamma$ , and forms a group under the monoidal structure. The second is the automorphism group of the tensor unit  $1 \in \Gamma_0$ ; it is abelian since the monoidal structure equips it with a second group structure which is a homomorphism for the first – such group structures coincide and are abelian via the Eckmann-Hilton argument. For a nice subclass of Lie 2-groups, called “smoothly separable”, the groups  $\pi_0\Gamma$  and  $\pi_1\Gamma$  are again Lie groups with nice properties (see Definition 3.6).

A classical result of Breen [Bre90] combines the non-abelian cohomology groups  $\check{H}^n(M, \Gamma)$  together with the classical Čech cohomology groups in a long exact sequence. With the abbreviation  $G := \pi_0\Gamma$  and  $A := \pi_1\Gamma$ , this sequence is:

$$0 \longrightarrow \check{H}_{cl}^1(M, A) \longrightarrow \check{H}^0(M, \Gamma) \longrightarrow \check{H}_{cl}^0(M, G) \longrightarrow \check{H}_{cl}^2(M, A) \longrightarrow \check{H}^1(M, \Gamma) \xrightarrow{\pi_*} \check{H}_{cl}^1(M, G) \xrightarrow{\delta} \check{H}_{cl}^3(M, A).$$

In Section 3 we explain this sequence in more detail; in particular we explain how it can be regarded as being induced by a short exact sequence of Lie 2-groups.

Above we have collected (bi-)categorical extensions for all of the involved *sets* in this sequence. The first objective of this paper is to establish accompanying categorical extensions of all occurring *maps*, and of the *exactness* of the sequence. This is done in a sequence of Theorems 4.1.2, 4.2.5, 5.1.2, 5.2.3, and 5.3.7 carried out in Sections 4 and 5, successively dealing with the exactness at all positions. In order to elucidate how such a categorical extension

looks like, we shall concentrate – for the purpose of this introduction – to the exactness at the last but one position, i.e. at  $\check{H}_{cl}^1(M, G)$ .

Exactness at  $\check{H}_{cl}^1(M, G)$  means that for a given principal  $G$ -bundle  $E$  over  $M$  the class  $\delta([E])$  vanishes if and only if there exists a principal  $\Gamma$ -2-bundle  $\mathcal{P}$  over  $M$  such that  $\pi_*([\mathcal{P}]) = [E]$ . Such 2-bundles are called  $\Gamma$ -lifts of  $E$ , and form a bicategory  $\mathcal{L}ift_{\Gamma}(E)$ . Essential for the categorical extension of the exactness is a geometrical understanding of the obstruction class  $\delta([E]) \in \check{H}_{cl}^3(M, A)$ . For this purpose we construct an  $A$ -bundle 2-gerbe  $\mathbb{L}_E$  with characteristic class  $[\mathbb{L}_E] = \delta([E])$  (Definition 5.3.3). Our construction of  $\mathbb{L}_E$  generalizes the one of the Chern-Simons bundle 2-gerbe of Carey et al. [CJM<sup>+</sup>05] using a new relation between Lie 2-groups and multiplicative gerbes that we discover. The categorical extension of the exactness is now:

**Theorem A.** *Let  $\Gamma$  be a smoothly separable Lie 2-group, let  $E$  be a principal  $G$ -bundle over  $M$ , and let  $\mathbb{L}_E$  be the associated lifting bundle 2-gerbe. Then, there is an equivalence of bicategories*

$$\left\{ \begin{array}{l} \text{Trivializations of the} \\ \text{lifting bundle 2-gerbe } \mathbb{L}_E \end{array} \right\} \cong \mathcal{L}ift_{\Gamma}(E).$$

Theorem A reproduces the set-theoretical exactness statement in the sense that equivalent bicategories are either both empty or both non-empty. On top of that, Theorem A specifies *how* the various ways of trivializing the obstruction are related to the various possible lifts. Theorem A is stated and proved in the main text as Theorem 5.3.7. The proof uses the descent theory for bundle gerbes developed in [NS11], as well as a reduction theorem (Theorem 5.2.3) that establishes a categorical extension of the exactness of the sequence one position earlier.

The second objective of this paper is to demonstrate that the promotion of the set-theoretical exactness to an equivalence of bicategories, provided by Theorem A, is an *essential* improvement. We recall that there is a transgression homomorphism

$$\check{H}_{cl}^n(M, A) \longrightarrow \check{H}_{cl}^{n-1}(LM, A) \tag{1.1}$$

in classical Čech cohomology, where  $LM = C^{\infty}(S^1, M)$  is the free loop space of  $M$ . It is defined in the first place for a differential extension, e.g. Deligne cohomology. There, transgression is a chain map between the Deligne cochain complexes [Bry93, GT01]; this chain map induces a well-defined map in the ordinary (non-differential) cohomology.

In a paper [SW] of the second author with Urs Schreiber, a generalization of transgression to non-abelian cohomology using connections on non-abelian gerbes and their formulation by parallel transport 2-functors is discussed. Unfortunately, the method developed there works only for the based loop space  $\Omega M \subset LM$ . Yet, it showed already an important difference between the abelian transgression (1.1) and its non-abelian generalization: in the non-abelian case, the structure group will change.

In the formalism of principal 2-bundles, and with help of the lifting theory of Theorem A, we are able to resolve the problems encountered in [SW]. In Section 6 of the present paper we define, for every smoothly separable, strict Lie 2-group  $\Gamma$  with  $\pi_0\Gamma$  compact, a Fréchet Lie group  $L\Gamma$  which we call the *loop group* of  $\Gamma$  (Definition 6.1.4). This construction uses multiplicative gerbes, connections, and a version of Brylinski's original transgression functor [Bry93]. It is functorial in  $\Gamma$ , so that a (weak) homomorphism  $\Lambda : \Gamma \longrightarrow \Omega$  between Lie 2-groups induces a group homomorphism  $L\Lambda : L\Gamma \longrightarrow L\Omega$ .

**Theorem B.** *Let  $\Gamma$  be a smoothly separable Lie 2-group with  $\pi_0\Gamma$  compact and connected. Then, there is a well-defined transgression map*

$$\check{H}^1(M, \Gamma) \longrightarrow \check{H}_{cl}^1(LM, L\Gamma)$$

*for non-abelian cohomology, which is contravariant in  $M$  and covariant in  $\Gamma$ . Moreover, for  $\Gamma = \mathcal{B}A$  we have  $L\Gamma = A$ , and the classical transgression map (1.1) is reproduced.*

The construction of the transgression map proceeds as follows. Firstly, we represent a non-abelian cohomology class by a principal  $\Gamma$ -2-bundle  $\mathcal{P}$  over  $M$ . We make the tautological observation that  $\mathcal{P}$  is a  $\Gamma$ -lift of the principal

$G$ -bundle  $E := \pi_*(\mathcal{P})$ . By Theorem A, it thus corresponds to a trivialization of the associated lifting bundle 2-gerbe  $\mathbb{L}_E$ , which is an *abelian*  $A$ -bundle 2-gerbe. Secondly, we use the existing functorial transgression for *abelian* bundle 2-gerbes, resulting in an  $A$ -bundle gerbe over  $LM$  together with a trivialization. Thirdly, we re-assemble these into a principal  $L\Gamma$ -bundle over  $LM$  using the theory of (ordinary) lifting bundle gerbes [Mur96]. We explain these steps in detail in Section 6, where Theorem B is stated as Theorem 6.3.4. The main difficulties we encounter there are to eliminate the choices of connections needed to make abelian transgression functorial.

In Section 7 we present an application of Theorems A and B to string structures on a spin manifold  $M$ . The Lie 2-group which is relevant is here is some strict Lie 2-group model for the string group. This is a (possibly infinite-dimensional) strict Lie 2-group  $\text{String}(n)$  with  $\pi_0\text{String}(n) = \text{Spin}(n)$  and  $\pi_1\text{String}(n) = \text{U}(1)$ , such that its geometric realization is a three-connected extension

$$1 \longrightarrow \text{BU}(1) \longrightarrow |\text{String}(n)| \longrightarrow \text{Spin}(n) \longrightarrow 1$$

of topological groups. Strict Lie 2-group models have been constructed in [BCSS07, NSW, Wala]. In the language of Theorem A, we say that a *string structure* on a principal  $\text{Spin}(n)$ -bundle  $E$  is a  $\text{String}(n)$ -lift  $\mathcal{P}$  of  $E$  (Definition 7.2).

We prove that the lifting bundle 2-gerbe  $\mathbb{L}_E$  that represents the obstruction against  $\text{String}(n)$ -lifts of  $E$  coincides with the Chern-Simons 2-gerbe  $\mathbb{CS}_E(\mathcal{G})$  associated to the so-called level one multiplicative bundle gerbe over  $G$  (Lemma 7.4). This bundle 2-gerbe provided another notion of string structures suitable in the context of *string connections*. Previously, it was known that these two notions of string structures coincide on a level of equivalence classes. Theorem A promotes this bijection to an equivalence of bicategories (Theorem 7.5). It so enables to switch in a *functorial way* between the two notions.

In order to apply the transgression map of Theorem B we show that the loop group of  $\text{String}(n)$  is the universal central extension of the loop group  $L\text{Spin}(n)$  (Lemma 7.7). Lifts of the structure group of the looped frame bundle of  $M$  from  $L\text{Spin}(n)$  to this universal extension are usually called *spin structures* on the loop space  $LM$  [McL92]. Previously, it was known that  $LM$  is spin if  $M$  is string [McL92]. Theorem B now permits to transgress a *specific* string structure on  $M$  to a *specific* spin structure on  $LM$  (Theorem 7.9).

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## 2 Non-Abelian Gerbes and 2-Bundles

In this section we recall the notions of 2-groups, 2-bundles and non-abelian gerbes, and the relationship between the latter two, following our paper [NW]. We remark that these objects have also been treated in many other papers, cf. [ACJ05, Bre90, Bar04, Woc]. There is one difference between [NW] and the present paper: here we work with not necessarily finite-dimensional manifolds. More precisely, the manifolds and Lie groups in this paper are modelled on locally convex vector spaces.

**Definition 2.1.** A *Lie groupoid* is a small groupoid  $\Gamma$  whose set of objects  $\Gamma_0$  and whose set of morphisms  $\Gamma_1$  are manifolds, and whose structure maps

$$s, t : \Gamma_1 \longrightarrow \Gamma_0 \quad i : \Gamma_0 \longrightarrow \Gamma_1 \quad \text{and} \quad \circ : \Gamma_1 \times_{\Gamma_0} \Gamma_1 \longrightarrow \Gamma_1$$

are smooth, and  $s, t$  are submersions. A *Lie 2-group* is a Lie groupoid  $\Gamma$  such that  $\Gamma_0$  and  $\Gamma_1$  are Lie groups, and such that all structure maps are Lie group homomorphisms.

We remark that the Lie 2-groups of Definition 2.1 are usually called *strict*. We suppress this adjective since all Lie 2-groups in this article are strict. In principle all theorems and notions discussed here have a counterpart

for non-strict 2-groups (even stacky 2-groups). But since our main examples are strict we decided to avoid these additional complications.

Lie groupoids can be seen as a generalization of Lie groups. Indeed, every Lie group  $G$  gives rise to a Lie groupoid  $\mathcal{B}G$  with one object, whose manifold of automorphisms is  $G$ . A Lie groupoid of the form  $\mathcal{B}G$  is a Lie 2-group if and only if  $G$  is abelian.

Let  $\Gamma$  be a Lie groupoid, not necessarily a Lie 2-group. Crucial for the present paper is the notion of a *principal  $\Gamma$ -bundle* over a manifold  $M$ , see e.g. [MM03, Section 5.7] or [NW, Definition 2.2.1]. It generalizes the notion of a principal bundle for a Lie group. Principal  $\Gamma$ -bundles over  $M$  form a category  $\mathcal{B}un_\Gamma(M)$  with many additional features. For example, one can pullback principal  $\Gamma$ -bundles along smooth maps  $f : N \rightarrow M$ . Further, if  $\Lambda : \Gamma \rightarrow \Omega$  is a *smooth anafunctor* between Lie groupoids [NW, Definition 2.3.1], there is an induced functor

$$\Lambda_* : \mathcal{B}un_\Gamma(M) \rightarrow \mathcal{B}un_\Omega(M)$$

which we call *extension* along  $\Lambda$  [NW, Definition 2.3.7]. This induced functor is an equivalence of categories if  $\Lambda$  is a *weak equivalence* [NW, Definition 2.3.10 & Corollary 2.3.11]

If  $\Gamma$  is a Lie 2-group, the category of principal  $\Gamma$ -bundles over  $M$  is monoidal by means of a tensor product

$$\otimes : \mathcal{B}un_\Gamma(M) \times \mathcal{B}un_\Gamma(M) \rightarrow \mathcal{B}un_\Gamma(M).$$

Using this tensor product one can define *non-abelian bundle gerbes* completely analogous to *abelian* bundle gerbes:

**Definition 2.2** ([NW, Definition 5.1.1]). *Let  $M$  be a manifold, and let  $\Gamma$  be a Lie 2-group. A  $\Gamma$ -bundle gerbe over  $M$  is a surjective submersion  $\pi : Y \rightarrow M$ , a principal  $\Gamma$ -bundle  $P$  over  $Y^{[2]}$  and an associative morphism*

$$\mu : \pi_{23}^* P \otimes \pi_{12}^* P \rightarrow \pi_{13}^* P$$

of  $\Gamma$ -bundles over  $Y^{[3]}$ , where  $\pi_{ij} : Y^{[3]} \rightarrow Y^{[2]}$  denotes projection to the  $i$ -th and  $j$ -th factor.

Another useful concept of a non-abelian gerbe is a principal 2-bundle. The idea of a principal 2-bundle is to mimic the definition of an ordinary principal bundle, with the structure group replaced by a Lie 2-group  $\Gamma$  and the total space replaced by a Lie groupoid. We will use the notation  $M_{dis}$  when a manifold  $M$  is regarded as a Lie groupoid with only identity morphisms.

**Definition 2.3** ([NW, Definition 6.1.5]). *A principal  $\Gamma$ -2-bundle over  $M$  is a Lie groupoid  $\mathcal{P}$ , a smooth functor  $\pi : \mathcal{P} \rightarrow M_{dis}$  that is a surjective submersion on the level of objects, and a smooth, strict, right action*

$$R : \mathcal{P} \times \Gamma \rightarrow \mathcal{P}$$

of  $\Gamma$  on  $\mathcal{P}$  that is principal in the sense that the functor  $(\text{pr}_1, R) : \mathcal{P} \times \Gamma \rightarrow \mathcal{P} \times_M \mathcal{P}$  is a weak equivalence.

$\Gamma$ -bundle gerbes as well as principal  $\Gamma$ -2-bundles can be arranged into bicategories which we denote by  $\mathcal{G}rb_\Gamma(M)$  and  $2\text{-}\mathcal{B}un_\Gamma(M)$ , respectively [NW, Sections 5.1 and 6.1]. Moreover, smooth maps  $f : N \rightarrow M$  induce pullback 2-functors for both bicategories.

**Theorem 2.4** ([NW, Theorem 7.1]). *The bicategories  $\mathcal{G}rb_\Gamma(M)$  and  $2\text{-}\mathcal{B}un_\Gamma(M)$  are equivalent. Furthermore, this equivalence is compatible with pullbacks. More precisely  $\mathcal{G}rb_\Gamma$  and  $2\text{-}\mathcal{B}un_\Gamma$  form equivalent 2-stacks over the site of smooth manifolds.*

Finally, we note that a Lie 2-group homomorphism  $\Lambda : \Gamma \rightarrow \Omega$ , see [NW, Eq. 2.4.4], induces 2-functors

$$\Lambda_* : \mathcal{G}rb_\Gamma(M) \rightarrow \mathcal{G}rb_\Omega(M) \quad \text{and} \quad \Lambda_* : 2\text{-}\mathcal{B}un_\Gamma(M) \rightarrow 2\text{-}\mathcal{B}un_\Omega(M)$$

that we call *extension along  $\Lambda$* . These 2-functors are equivalences if the homomorphism  $\Lambda$  is a weak equivalence [NW, Theorems 5.2.2 and 6.2.2].

### 3 The Long Exact Sequence in Non-abelian Cohomology

The degree zero “non-abelian cohomology” of a Lie groupoid  $\Gamma$  is a set  $\check{H}^0(M, \Gamma)$  that generalizes ordinary Čech cohomology in the following way: if  $G$  is a Lie group, then the degree one Čech cohomology group with coefficients the sheaf of smooth,  $G$ -valued functions,  $\check{H}^1(M, G)$ , coincides with  $\check{H}^0(M, \mathcal{B}G)$ . In Section 1 we have outlined an explicit cocycle definition of  $\check{H}^0(M, \Gamma)$ ; a more substantiated discussion can be found in [NW, Section 3].

If  $\Gamma$  is a Lie 2-group, the set  $\check{H}^0(M, \Gamma)$  inherits the structure of a (in general non-abelian) group. Further, the Lie 2-group structure permits to define *degree one* non-abelian cohomology,  $\check{H}^1(M, \Gamma)$ , as outlined in Section 1. Non-abelian cohomology is important for this article because it classifies the geometrical objects of the previous Section 2:

**Theorem 3.5** ([NW, Theorems 3.3, 5.3.2, 7.1]). *Let  $\Gamma$  be a Lie groupoid and  $M$  a paracompact manifold. Then, there is a bijection*

$$\check{H}^0(M, \Gamma) \cong \left\{ \begin{array}{l} \text{Isomorphism} \\ \text{classes of principal} \\ \Gamma\text{-bundles over } M \end{array} \right\}.$$

Moreover, if  $\Gamma$  is a Lie 2-group, there are bijections

$$\check{H}^1(M, \Gamma) \cong \left\{ \begin{array}{l} \text{Isomorphism} \\ \text{classes of } \Gamma\text{-bundle} \\ \text{gerbes over } M \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Isomorphism} \\ \text{classes of principal} \\ \Gamma\text{-2-bundles over } M \end{array} \right\}.$$

For  $\Gamma$  a Lie 2-group, we denote by  $\pi_0\Gamma$  the group of isomorphism classes of objects in  $\Gamma$  and by  $\pi_1\Gamma$  the group of automorphisms of  $1 \in \Gamma_0$ . We remark that  $\pi_1\Gamma$  is abelian due to its two commuting group structures (multiplication and composition). We also remark that the sequence

$$1 \longrightarrow \pi_1\Gamma \xrightarrow{i} \ker(s) \xrightarrow{t} \Gamma_0 \xrightarrow{\pi} \pi_0\Gamma \longrightarrow 0 \quad (3.1)$$

is exact as a sequence of groups.

**Definition 3.6.** *A Lie 2-group  $\Gamma$  is called smoothly separable if  $\pi_1\Gamma$  is a split Lie subgroup of  $\Gamma_1$  and the group  $\pi_0\Gamma$  is a Lie group such that the projection  $\Gamma_0 \longrightarrow \pi_0\Gamma$  is a submersion.*

For a smoothly separable Lie 2-group  $\Gamma$ , the sequence (3.1) is exact as a sequence of Lie groups: the inclusion  $i$  is an embedding, and the map  $t$  is a submersion onto its image. Further, we have the following sequence of Lie 2-groups and smooth functors:

$$\mathcal{B}\pi_1\Gamma \longrightarrow \Gamma \longrightarrow \pi_0\Gamma_{dis}. \quad (3.2)$$

This sequence is – in a certain sense – an exact sequence of Lie 2-groups, although it is not exact on the level of morphisms. More precisely, it is a fibre sequence. We are not going to give a rigorous treatment of fibre sequences in Lie 2-groups, but the reader may keep this in mind as a guiding principle throughout this paper.

**Proposition 3.7** ([Bre90, 4.2.2]). *Let  $M$  be a paracompact manifold, and let  $\Gamma$  be a smoothly separable Lie 2-group. Then, the sequence (3.2) induces a long exact sequence in non-abelian cohomology:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \check{H}^0(M, \mathcal{B}\pi_1\Gamma) & \longrightarrow & \check{H}^0(M, \Gamma) & \longrightarrow & \check{H}^0(M, \pi_0\Gamma_{dis}) \\ & & & & & & \uparrow \\ & & & & & & \check{H}^1(M, \pi_0\Gamma_{dis}) \\ & & & & & & \uparrow \\ & & & & & & \check{H}^1(M, \Gamma) \\ & & & & & & \uparrow \\ & & & & & & \check{H}^1(M, \mathcal{B}\pi_1\Gamma) \\ & & & & & & \uparrow \\ & & & & & & \check{H}^2(M, \mathcal{B}\pi_1\Gamma). \end{array}$$

The last set in the long exact sequence of Proposition 3.7 is defined as the classical Čech cohomology group  $\check{H}^2(M, \underline{\pi}_1\Gamma)$  with values in the abelian Lie group  $\underline{\pi}_1\Gamma$ . In contrast to this there is no way to define  $\check{H}^2$  (or higher) for general Lie 2-groups  $\Gamma$ , so that the sequence stops there. Many variants of Proposition 3.7 can be found in the literature. The case of the automorphism 2-group  $\text{AUT}(H)$  [NW, Example 2.4.4] is treated in [Gir71]. Discrete crossed modules and algebraic variants are treated in [Ded60] and [DF66].

According to Theorem 3.5 the various cohomology sets occurring in the sequence of Proposition 3.7 have natural geometric interpretations in terms of  $\Gamma$ -bundles or  $\Gamma$ -2-bundles. The main goal of this paper is to understand the exactness of the sequence geometrically in terms of reduction problems and lifting problems of these bundles. In particular, our results give an independent proof of Proposition 3.7. Moreover, we will promote the (set-theoretical) exactness statements to statements about (bi-)categories of (2-)bundles.

We remark that distinguishing between *lifting* and *reduction* problems is purely a matter of taste: a lift is defined exactly in the same way as a reduction. We decided to use the terminology *reduction* in situations where an object comes from a “simpler” structure (e.g. a principal groupoid bundle *reduces* to an ordinary abelian bundle, see Section 4.1), while we use the terminology *lift* when it comes from a “more complicated” structure (e.g. a principal bundle *lifts* to a non-abelian 2-bundle, see Section 5.3). Furthermore, we will formulate the “reduction theorems” in a more general form than the “lifting theorems”, simply because we use the first ones to prove the latter ones.

## 4 Reduction and Lifting for Groupoid Bundles

In this section we work over a paracompact manifold  $M$ , and with a smoothly separable Lie 2-group  $\Gamma$ . We discuss reduction and lifting problems for principal groupoid bundles. In particular, we prove that the first row of the sequence of Proposition 3.7 is exact:

$$0 \longrightarrow \check{H}^0(M, \mathcal{B}\underline{\pi}_1\Gamma) \longrightarrow \check{H}^0(M, \Gamma) \longrightarrow \check{H}^0(M, \underline{\pi}_0\Gamma_{dis}) \longrightarrow \check{H}^1(M, \mathcal{B}\underline{\pi}_1\Gamma). \quad (4.1)$$

We use this section mainly as a warmup for Section 5, and to introduce some concepts that we will use there. Another bundle-theoretical proof of the exactness has also appeared in [MRS, Section 3].

### 4.1 Reduction of Groupoid Bundles to Abelian Bundles

We recall that we have the functor sequence (3.2):

$$\mathcal{B}\underline{\pi}_1\Gamma \xrightarrow{i} \Gamma \xrightarrow{\pi} \underline{\pi}_0\Gamma_{dis}.$$

Extension of principal bundles along  $i$  and  $\pi$  gives an induced sequence of functors

$$\mathcal{B}un_{\mathcal{B}\underline{\pi}_1\Gamma}(M) \xrightarrow{i_*} \mathcal{B}un_{\Gamma}(M) \xrightarrow{\pi_*} C^\infty(M, \underline{\pi}_0\Gamma)_{dis}, \quad (4.1.1)$$

where we have used the canonical equivalence  $\mathcal{B}un_{G_{dis}}(M) \cong C^\infty(M, G)_{dis}$ , see [NW, Example 2.2.4]. More explicitly, if  $P$  is a principal  $\Gamma$ -bundle over  $M$ , the associated map  $\pi_*P : M \rightarrow \underline{\pi}_0\Gamma$  is given as follows: one lifts a point  $x \in M$  to an element  $p$  in the total space  $P$ , takes its anchor  $\alpha(p) \in \Gamma_0$ , and projects to its equivalence class in  $\underline{\pi}_0\Gamma$ . The sequence (4.1.1) realizes geometrically the part

$$\check{H}^0(M, \mathcal{B}\underline{\pi}_1\Gamma) \longrightarrow \check{H}^0(M, \Gamma) \longrightarrow \check{H}^0(M, \underline{\pi}_0\Gamma_{dis})$$

of the sequence (4.1).

Theorem 4.1.2 below states that the sequence (4.1.1) is a fibre sequence in categories. We denote by  $1 \in C^\infty(M, \underline{\pi}_0\Gamma)$  the constant function with value  $1 \in \underline{\pi}_0\Gamma_0$ , and start with the following observation:

**Lemma 4.1.1.** *Let  $P$  be a principal  $\pi_1\Gamma$ -bundle over  $M$ . Then,  $\pi_*(i_*(P)) = 1$ .*

Proof. The assertion is clear because the composition  $\pi \circ i$  is the constant functor. Since we will later need the notation anyway, we give a more explicit proof. Let us recall the definition of  $i_*$  [NW, Definition 2.3.8]. The total space of  $i_*(P)$  is  $(P \times t^{-1}(1)) / \sim$ , where  $(p, \gamma \circ \omega) \sim (p \cdot \gamma, \omega)$  for all  $p \in P$ ,  $\omega \in t^{-1}(1) \subseteq \Gamma_1$  and  $\gamma \in \pi_1\Gamma$ . The bundle projection is  $(p, \omega) \mapsto \pi(p)$ , the anchor is  $(p, \omega) \mapsto s(\omega)$ , and the  $\Gamma$ -action is  $(p, \omega) \circ \omega' = (p, \omega \circ \omega')$  for  $\omega \in \Gamma_1$  with  $s(\omega) = t(\omega')$ . In particular, since  $s(\omega) = t(\omega) = 1$  in  $\pi_0(\Gamma)$ ,  $\pi_*(i_*(P)) = 1$ .  $\square$

Principal  $\Gamma$ -bundles of the form  $i_*(P)$  are examples of *reducible* bundles. More generally, let  $P$  be a principal  $\Gamma$ -bundle  $P$  over  $M$ . A *reduction of  $P$  to an abelian bundle* is a principal  $\mathcal{B}\pi_1\Gamma$ -bundle  $P_{\text{red}}$  over  $M$  and a bundle isomorphism  $i_*(P_{\text{red}}) \cong P$ .

Lemma 4.1.1 states that a principal  $\Gamma$ -bundle  $P$  which admits a reduction to an abelian bundle satisfies  $\pi_*(P) = 1$ . Theorem 4.1.2 below shows that the converse is also true. In order to prepare this statement, we denote by  $\mathcal{B}un_\Gamma(M)^1$  the full subcategory of  $\mathcal{B}un_\Gamma(M)$  over those principal  $\Gamma$ -bundles  $P$  with  $\pi_*(P) = 1$ . By Lemma 4.1.1, the functor  $i_*$  factors through a functor

$$i_* : \mathcal{B}un_{\mathcal{B}\pi_1\Gamma}(M) \longrightarrow \mathcal{B}un_\Gamma(M)^1. \quad (4.1.2)$$

We have the following ‘‘reduction theorem’’ for principal  $\Gamma$ -bundles:

**Theorem 4.1.2.** *The functor (4.1.2) establishes an equivalence of categories:*

$$\mathcal{B}un_{\mathcal{B}\pi_1\Gamma}(M) \cong \mathcal{B}un_\Gamma(M)^1.$$

*In particular, a principal  $\Gamma$ -bundle  $P$  over  $M$  admits a reduction to an abelian bundle if and only if  $\pi_*(P) = 1$ ; in this case the reduction is unique up to unique isomorphisms.*

Proof. Let  $P$  be a principal  $\Gamma$ -bundle over  $M$  with anchor  $\alpha : P \rightarrow \Gamma_0$  and  $\pi_*(P) = 1$ . Let  $P_{\text{red}} \subseteq P$  denote the subset of points with  $\alpha(p) = 1$ . Let  $s : U \rightarrow P$  be a local section, and consider the composition  $\alpha \circ s : U \rightarrow \Gamma_0$ . By assumption, the image of  $\alpha \circ s$  is contained in the kernel of  $\pi : \Gamma_0 \rightarrow \pi_0\Gamma$ , which is – by exactness of (3.1) – the image of the submersion  $t : \ker(s) \rightarrow \Gamma_0$ . Thus – after a possible refinement of  $U$  –  $\alpha \circ s$  lifts to a smooth map  $\gamma : U \rightarrow \ker(s)$ , i.e.  $t \circ \gamma = \alpha \circ s$ . Now consider the new section  $\tilde{s} := s \circ \gamma$ . Since  $\alpha \circ \tilde{s} = 1$ ,  $\tilde{s}$  is a section into  $P_{\text{red}}$ . It remains to notice that the action of  $\Gamma$  on  $P$  restricts to a transitive and free action of  $\pi_1\Gamma$  on  $P_{\text{red}}$ . This shows that  $P_{\text{red}}$  is a principal  $\pi_1\Gamma$ -bundle over  $M$ .

We claim that  $i_*(P_{\text{red}}) \cong P$ , which proves that the functor (4.1.2) is essentially surjective. Indeed, in the notation of the proof of Lemma 4.1.1, an isomorphism is given by

$$\varphi : i_*(P_{\text{red}}) \rightarrow P : (p, \omega) \mapsto p \circ \omega;$$

this is well-defined, smooth, fibre preserving and  $\Gamma$ -equivariant. In order to see that the functor (4.1.2) is also full and faithful, we recall that the extension of a bundle morphism  $\varphi : Q_1 \rightarrow Q_2$  is  $(i_*\varphi)(q, \omega) = (\varphi(q), \omega)$ . Now let  $\eta : i_*(Q_1) \rightarrow i_*(Q_2)$  be a bundle morphism. Then, we define  $\varphi : Q_1 \rightarrow Q_2$  by  $\eta(q, 1) = (\varphi(q), 1)$ . It is straightforward to check that this is well-defined and that the two assignments are inverses of each other.  $\square$

We conclude with reducing Theorem 4.1.2 to isomorphism classes of objects:

**Corollary 4.1.3.** *The sequence (4.1) is exact at  $\check{H}^0(M, \mathcal{B}\pi_1\Gamma)$  and  $\check{H}^0(M, \Gamma)$ .*

## 4.2 Liftings of Functions to Groupoid Bundles

We start with the following observation about the smoothly separable Lie 2-group  $\Gamma$ : the projection functor  $\pi : \Gamma \rightarrow \pi_0\Gamma_{\text{dis}}$  together with the 2-group multiplication

$$m : \Gamma \times \mathcal{B}\pi_1\Gamma \rightarrow \Gamma$$



is a principal  $\mathcal{B}_{\pi_1}\Gamma$ -2-bundle. The  $\mathcal{B}_{\pi_1}\Gamma$ -bundle gerbe that corresponds to the 2-bundle  $\Gamma$  under the equivalence of Theorem 2.4 is denoted by  $\mathcal{G}_\Gamma$ . Following [NW, Section 7.1] it consists of the following data:

1. its surjective submersion is the projection  $\pi : \Gamma_0 \rightarrow \pi_0\Gamma$ .
2. its principal  $\pi_1\Gamma$ -bundle over  $\Gamma_0^{[2]}$  is  $(s, t) : \Gamma_1 \rightarrow \Gamma_0^{[2]}$ , with the action given by multiplication.
3. its bundle gerbe product  $\text{pr}_{23}^*\Gamma_1 \otimes \text{pr}_{12}^*\Gamma_1 \rightarrow \text{pr}_{13}^*\Gamma_1$  is the composition in  $\Gamma$ , i.e. the product of  $\gamma_{12} \in \text{pr}_{12}^*\Gamma_1$  and  $\gamma_{23} \in \text{pr}_{23}^*\Gamma_1$  is  $\gamma_{12} \circ \gamma_{23}$ .

For preparation, we continue with two lemmata concerned with the bundle gerbe  $\mathcal{G}_\Gamma$ . In the following we denote by  $\mathbf{I}_f := M_f \times_t \Gamma_1$  the trivial principal  $\Gamma$ -bundle over  $M$  with anchor  $f$  [NW, Example 2.2.3].

**Lemma 4.2.1.** *There exists an isomorphism  $\tau : i_*(\Gamma_1) \rightarrow \mathbf{I}_\Delta$  of  $\Gamma$ -bundles over  $\Gamma_0^{[2]}$ , where  $\Delta : \Gamma_0^{[2]} \rightarrow \Gamma_0$  is the difference map  $\Delta(g_1, g_2) := g_2^{-1}g_1$ , such that the composition in  $\Gamma$  is respected in the sense that the diagram*

$$\begin{array}{ccc} \text{pr}_{23}^*i_*(\Gamma_1) \otimes \text{pr}_{12}^*i_*(\Gamma_1) & \xrightarrow{\text{pr}_{23}^*\tau \times \text{pr}_{12}^*\tau} & \text{pr}_{23}^*\mathbf{I}_\Delta \otimes \text{pr}_{12}^*\mathbf{I}_\Delta \\ \downarrow i_*(\circ) & & \parallel \\ \text{pr}_{13}^*i_*(\Gamma_1) & \xrightarrow{\tau} & \text{pr}_{13}^*\mathbf{I}_\Delta \end{array}$$

is commutative.

*Proof.* We construct a section  $s : \Gamma_0^{[2]} \rightarrow i_*(\Gamma_1)$  such that  $\alpha \circ s = \Delta$ . Any such section induces the claimed isomorphism. In order to construct the section, we note that the composition of  $\Delta$  with the projection  $\pi : \Gamma_0 \rightarrow \pi_0\Gamma$  is trivial, so that  $\Delta$  lifts locally to a smooth map  $\gamma : U \rightarrow \ker(s)$  by the exactness of (3.1), i.e.  $t \circ \gamma = \Delta$ . Consider  $\text{id}_{\text{pr}_2} : U \rightarrow \Gamma_1$  and – in the notation introduced in the proof of Lemma 4.1.1 – the smooth map

$$s := (\text{id}_{\text{pr}_2} \cdot \gamma, \gamma^{-1}) : U \rightarrow i_*(\Gamma_1).$$

It is straightforward to check that this is a local section and satisfies  $\alpha \circ s = \Delta$ . The difficult part is to check that the definition of  $s$  does not depend on the choice of the lift  $\gamma : U \rightarrow \ker(s)$ ; this implies that  $s$  is in fact a global section. Let  $\gamma' : U \rightarrow \ker(s)$  be another section. By exactness of the sequence (3.1) there is a smooth map  $d : U \rightarrow \pi_1\Gamma$  such that  $\gamma' = \gamma \cdot d$ . Then,

$$(\text{id}_{\text{pr}_2} \cdot \gamma', \gamma'^{-1}) = (\text{id}_{\text{pr}_2} \cdot \gamma \cdot d, \gamma^{-1} \cdot d^{-1}) \sim (\text{id}_{\text{pr}_2} \cdot \gamma, d \circ (\gamma^{-1} \cdot d^{-1})) = (\text{id}_{\text{pr}_2} \cdot \gamma, \gamma^{-1}),$$

where the last step uses the “exchange law” between the composition and the multiplication in the 2-group  $\Gamma$ . Finally, the commutativity of the diagram is equivalent to the identity

$$i_*(\circ)(\text{pr}_{23}^*s, \text{pr}_{12}^*s) = \text{pr}_{13}^*s \tag{4.2.1}$$

for the section  $s$ . Indeed, over a point  $(g_1, g_2, g_3)$ , and for local sections  $\gamma_{12}$  around  $(g_1, g_3)$  and  $\gamma_{23}$  around  $(g_2, g_3)$ , we claim that  $\gamma_{13} := \text{id}_{g_3}^{-1} \cdot ((\text{id}_{g_2} \cdot \gamma_{12}) \circ (\text{id}_{g_3} \cdot \gamma_{23}))$  is a valid local section around  $(g_1, g_3)$ . In order to see this, it suffices to check that  $s(\gamma_{13}) = 1$  and  $t(\gamma_{13}) = \Delta(g_1, g_3)$ . With these choices, the identity (4.2.1) is straightforward to check.  $\square$

We recall that a trivialization of a  $\Gamma$ -bundle gerbe  $\mathcal{H}$  over a manifold  $X$  is a 1-isomorphism  $\mathcal{T} : \mathcal{H} \rightarrow \mathcal{I}$ , where  $\mathcal{I}$  is the trivial  $\Gamma$ -bundle gerbe, consisting of the identity submersion  $\text{id}_X$ , the trivial principal  $\Gamma$ -bundle  $\mathbf{I}_1$  for the constant map  $1 : X \rightarrow \Gamma_0$ , and the identity bundle gerbe product.

**Lemma 4.2.2.** *The  $\Gamma$ -bundle gerbe  $i_*(\mathcal{G}_\Gamma)$  is trivializable.*

Proof. A trivialization is given by the principal  $\Gamma$ -bundle  $T := \mathbf{I}_i$  over  $\Gamma_0$ , for  $i : \Gamma_0 \rightarrow \Gamma_0$  the inversion of the group  $\Gamma_0$ , and by the bundle isomorphism

$$\mathbf{I}_1 \otimes \pi_1^* T \cong \mathbf{I}_{\text{pr}_2} \otimes \mathbf{I}_\Delta \xrightarrow{\text{id} \otimes \tau^{-1}} \pi_2^* T \otimes i_*(\Gamma_1)$$

over  $\Gamma_0^{[2]}$  which is defined using the bundle isomorphism  $\tau$  of Lemma 4.2.1. The required compatibility with the bundle gerbe product is precisely given by the commutativity of the diagram in Lemma 4.2.1.  $\square$

Now we start using the  $\mathcal{B}\pi_1\Gamma$ -bundle gerbe  $\mathcal{G}_\Gamma$ . For a smooth map  $f : M \rightarrow \pi_0\Gamma$ , we denote the  $\mathcal{B}\pi_1\Gamma$ -bundle gerbe  $f^*\mathcal{G}_\Gamma$  over  $M$  by  $\mathcal{L}_f$ , and call it the *lifting bundle gerbe* associated to  $f$ . The assignment  $f \mapsto \mathcal{L}_f$  induces a map

$$\check{H}^0(M, \pi_0\Gamma_{dis}) \longrightarrow \check{H}^1(M, \mathcal{B}\pi_1\Gamma)$$

in non-abelian cohomology; this is the connecting homomorphism of the long exact sequence (3.7). In this section we are interested in the following structure:

**Definition 4.2.3.** *Let  $f : M \rightarrow \pi_0\Gamma$  be a smooth map. A  $\Gamma$ -lift of  $f$  is a principal  $\Gamma$ -bundle  $P$  over  $M$  such that  $\pi_*(P) = f$ .  $\Gamma$ -lifts of  $f$  form a full subcategory of  $\text{Bun}_\Gamma(M)$  that we denote by  $\text{Lift}_\Gamma(f)$ .*

We shall construct  $\Gamma$ -lifts of  $f$  from trivialization of  $\mathcal{L}_f$ . Suppose  $\mathcal{T}$  is a trivialization of  $\mathcal{L}_f$ , consisting of a principal  $\mathcal{B}\pi_1\Gamma$ -bundle  $Q$  over  $Z := M \times_{f \times \pi} \Gamma_0$ , and of an isomorphism

$$\chi : \text{pr}_1^* Q \rightarrow \text{pr}_2^* Q \otimes (\beta \times \beta)^*(\Gamma_1)$$

of principal  $\mathcal{B}\pi_1\Gamma$ -bundles over  $Z^{[2]} := Z \times_M Z$ , where  $\beta := \text{pr}_2 : Z \rightarrow \Gamma_0$ . We further use the notation  $\zeta := \text{pr}_1 : Z \rightarrow M$ . We claim:

**Lemma 4.2.4.** *Consider the principal  $\Gamma$ -bundle  $P := i_*(Q) \otimes \mathbf{I}_\beta$  over  $Z$ . Then:*

(i) *The isomorphism*

$$\begin{array}{ccc} \text{pr}_1^* P \cong \text{pr}_1^* i_*(Q) \otimes \text{pr}_1^* \mathbf{I}_\beta & \xrightarrow{\chi \otimes \text{id}} & \text{pr}_2^* i_*(Q) \otimes i_*((\beta \times \beta)^*(\Gamma_1)) \otimes \text{pr}_1^* \mathbf{I}_\beta \\ & & \downarrow \text{id} \otimes (\beta \times \beta)^* \tau \otimes \text{id} \\ & & \text{pr}_2^* i_*(Q) \otimes \text{pr}_2^* \mathbf{I}_\beta \cong \text{pr}_2^* P \end{array}$$

*defines a descent structure on  $P$  for the surjective submersion  $\zeta : Z \rightarrow M$ . Here,  $\tau$  is the bundle isomorphism of Lemma 4.2.1.*

(ii) *The quotient bundle  $\zeta_!(P)$  is a  $\Gamma$ -lift of  $f$ .*

Proof. For (i) we have to show that the given isomorphism satisfies the cocycle condition over  $Z^{[3]}$ . This follows from the compatibility of  $\chi$  with the bundle gerbe product, and from the commutativity of the diagram in Lemma 4.2.1. In order to prove (ii) we have to show  $\pi_*(\zeta_!(P)) = f$ , which is equivalent to  $\pi_*(P) = f \circ \zeta = \pi \circ \beta$ . Indeed,

$$\pi_*(P) = \pi_*(i_*(Q) \otimes \mathbf{I}_\beta) = \pi_*(i_*(P)) \cdot \pi_*(\mathbf{I}_\beta) = \pi \circ \beta,$$

using the fact that the extension  $\pi_*$  is a monoidal functor and using Lemma 4.1.1.  $\square$

We denote the quotient bundle  $\zeta_!(P)$  of Lemma 4.2.4 by  $P_\mathcal{T}$ . It is easy to see that a 2-morphism  $\mathcal{T}_1 \Rightarrow \mathcal{T}_2$  between trivializations induces a bundle morphism  $P_{\mathcal{T}_1} \rightarrow P_{\mathcal{T}_2}$ . Thus, we have defined a functor

$$\text{Triv}(\mathcal{L}_f) \rightarrow \text{Lift}_\Gamma(f). \quad (4.2.2)$$

This functor underlies the following ‘‘lifting theorem’’:

**Theorem 4.2.5.** *Let  $f : M \rightarrow \pi_0\Gamma$  be a smooth map. Then, the functor (4.2.2) is an equivalence of categories:*

$$\mathcal{Triv}(\mathcal{L}_f) \cong \mathcal{Lift}_\Gamma(f).$$

*In particular, there exists a  $\Gamma$ -lift of  $f$  if and only if the  $\mathcal{B}_{\pi_1}\Gamma$ -bundle gerbe  $\mathcal{L}_f$  is trivializable.*

Proof. Let  $\mathcal{Triv}(i_*(\mathcal{L}_f))^1$  denote the full subcategory of trivializations of  $i_*(\mathcal{L}_f)$  where all principal  $\Gamma$ -bundles  $Q$  satisfy  $\pi_*(Q) = 1$ . By Theorem 4.1.2 the functor

$$i_* : \mathcal{Triv}(\mathcal{L}_f) \rightarrow \mathcal{Triv}(i_*(\mathcal{L}_f))^1$$

is an equivalence of categories. Let  $\mathcal{Desc}_\zeta(\mathcal{Bun}_\Gamma)$  be the category of descent data for the sheaf  $\mathcal{Bun}_\Gamma$  of principal  $\Gamma$ -bundles and the surjective submersion  $\zeta$ , and let  $\mathcal{Desc}_\zeta(\mathcal{Bun}_\Gamma)^{\pi \circ \beta}$  denote the full subcategory where all principal  $\Gamma$ -bundles  $Q$  have  $\pi_*(Q) = \pi \circ \beta$ . The calculations of Lemma 4.2.4 define a functor

$$- \otimes \mathbf{I}_\beta : \mathcal{Triv}(i_*(\mathcal{L}_f))^1 \rightarrow \mathcal{Desc}_\zeta(\mathcal{Bun}_\Gamma)^{\pi \circ \beta},$$

which is an equivalence since  $- \otimes \mathbf{I}_{\beta^{-1}}$  is an inverse functor. Finally, descent theory provides another equivalence  $\mathcal{Lift}_\Gamma(f) \cong \mathcal{Desc}_\zeta(\mathcal{Bun}_\Gamma)^{\pi \circ \beta}$ . By construction, the functor (4.2.2) is the composition of the equivalences collected above, and thus an equivalence.  $\square$

We conclude with a consequence of Theorem 4.2.5 for non-abelian cohomology:

**Corollary 4.2.6.** *The sequence (4.1) is exact at  $\check{H}^0(M, \pi_0\Gamma_{dis})$ .*

## 5 Reduction and Lifting for Non-Abelian Gerbes

In this section we discuss reduction and lifting problems for 2-bundles over a paracompact manifold  $M$ , and for a smoothly separable Lie 2-group  $\Gamma$ . In particular, we give a geometrical proof of the exactness of the second row in the long exact sequence of Proposition 3.7:

$$\check{H}^0(M, \pi_0\Gamma_{dis}) \rightarrow \check{H}^1(M, \mathcal{B}_{\pi_1}\Gamma) \rightarrow \check{H}^1(M, \Gamma) \rightarrow \check{H}^1(M, \pi_0\Gamma_{dis}) \rightarrow \check{H}^2(M, \mathcal{B}_{\pi_1}\Gamma). \quad (5.1)$$

Although we formulate and prove the results of this section in the language of  $\Gamma$ -bundle gerbes, all results carry over to principal 2-bundles via the equivalence of Theorem 2.4.

### 5.1 Reduction of Abelian Gerbes to Functions

We recall from Section 4.2 that there is a  $\mathcal{B}_{\pi_1}\Gamma$ -bundle gerbe  $\mathcal{G}_\Gamma$  over  $\pi_0\Gamma$  associated the Lie 2-group  $\Gamma$ , and that a smooth function  $f : M \rightarrow \pi_0\Gamma$  defines the lifting bundle gerbe  $\mathcal{L}_f := f^*\mathcal{G}_\Gamma$  over  $M$ . We have seen in Lemma 4.2.2 that the extension  $i_*(\mathcal{G}_\Gamma)$  of  $\mathcal{G}_\Gamma$  along  $i : \mathcal{B}_{\pi_1}\Gamma \rightarrow \Gamma$  has a canonical trivialization. Since the extension functor  $i_*$  commutes with pullbacks, we get:

**Corollary 5.1.1.** *Let  $f : M \rightarrow \pi_0\Gamma$  be a smooth map. Then,  $i_*(\mathcal{L}_f)$  is canonically trivializable.*

Let  $\mathcal{G}$  be a  $\mathcal{B}_{\pi_1}\Gamma$ -bundle gerbe over  $M$ . A *reduction of  $\mathcal{G}$  to a function* is a smooth function  $f : M \rightarrow \pi_0\Gamma$  such that  $\mathcal{G} \cong \mathcal{L}_f$ . Corollary 5.1.1 and the functoriality of the extension functor  $i_*$  imply that  $i_*(\mathcal{G})$  is trivializable for every bundle gerbe  $\mathcal{G}$  that can be reduced to a function. The converse is also true in the sense stated below as Theorem 5.1.2. In order to formalize this situation, we define the following bicategory  $\mathcal{Grb}_{\mathcal{B}_{\pi_1}\Gamma}(M)^{i-or}$  of  *$i$ -oriented  $\mathcal{B}_{\pi_1}\Gamma$ -bundle gerbes over  $M$* :

- (i) The objects are  $\mathcal{B}_{\pi_1}\Gamma$ -bundle gerbes  $\mathcal{G}$  over  $M$ , together with a trivialization  $\mathcal{T} : i_*(\mathcal{G}) \rightarrow \mathcal{I}$  of their extension along  $i$ .
- (ii) The 1-morphisms are 1-morphisms  $\mathcal{A} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$  between  $\mathcal{B}_{\pi_1}\Gamma$ -bundle gerbes over  $M$ , together with 2-morphisms

$$\begin{array}{ccc}
 i_*(\mathcal{G}_1) & \xrightarrow{i_*(\mathcal{A})} & i_*(\mathcal{G}_2) \\
 \searrow \tau_1 & \swarrow \varphi & \swarrow \tau_2 \\
 & \mathcal{I} & 
 \end{array}$$

- (iii) The 2-morphisms are 2-morphisms whose extension is compatible with the 2-morphisms of the involved 1-morphisms in the evident way.

Corollary 5.1.1 implies that we get a 2-functor

$$C^\infty(M, \pi_0\Gamma)_{dis} \rightarrow \mathcal{G}rb_{\mathcal{B}_{\pi_1}\Gamma}(M)^{i-or} : f \mapsto (\mathcal{L}_f, f^*\mathcal{T}_\Gamma), \quad (5.1.1)$$

where  $C^\infty(M, \pi_0\Gamma)_{dis}$  is regarded as a bicategory with only identity 1-morphisms and only identity 2-morphisms, and  $\mathcal{T}_\Gamma$  is the trivialization constructed in Lemma 4.2.2. Given this 2-functor we have the following ‘‘reduction theorem’’:

**Theorem 5.1.2.** *The 2-functor (5.1.1) establishes an equivalence of bicategories:*

$$C^\infty(M, \pi_0\Gamma)_{dis} \cong \mathcal{G}rb_{\mathcal{B}_{\pi_1}\Gamma}(M)^{i-or}.$$

*In particular, every  $i$ -orientation of a  $\mathcal{B}_{\pi_1}\Gamma$ -bundle gerbe  $\mathcal{G}$  determines a reduction of  $\mathcal{G}$  to a function.*

*Proof.* We start with the proof that the functor is essentially surjective. Let  $(\mathcal{G}, \mathcal{T})$  be an object in  $\mathcal{G}rb_{\mathcal{B}_{\pi_1}\Gamma}(M)^{i-or}$ . The first part is to construct a preimage  $f$ . Suppose the bundle gerbe  $\mathcal{G}$  consists of a surjective submersion  $\pi : Y \rightarrow M$ , a principal  $\mathcal{B}_{\pi_1}\Gamma$ -bundle  $P$  over  $Y$ <sup>[2]</sup>, and a bundle gerbe product  $\mu$ , and suppose that the trivialization  $\mathcal{T}$  consists of a principal  $\Gamma$ -bundle  $Q$  over  $Y$ , and of an isomorphism

$$\chi : \text{pr}_1^*Q \rightarrow \text{pr}_2^*Q \otimes i_*(P)$$

of  $\Gamma$ -bundles over  $Y$ <sup>[2]</sup>. With Lemma 4.1.1 we get  $\text{pr}_1^*(\pi_*Q) = \text{pr}_2^*(\pi_*Q)$ , so that  $\pi_*Q : Y \rightarrow \pi_0\Gamma$  descends to a unique smooth map  $f : M \rightarrow \pi_0\Gamma$ . The second part is to show that  $f$  is an essential preimage of  $(\mathcal{G}, \mathcal{T})$ , i.e. we have to construct a 1-morphism  $(\mathcal{L}_f, f^*\mathcal{T}_\Gamma) \cong (\mathcal{G}, \mathcal{T})$ . The common refinement of the surjective submersions of  $\mathcal{L}_f$  and  $\mathcal{G}$  is  $Z := \Gamma_0 \pi \times_{\pi_*(Q)} Y$ , which comes with the projections  $y : Z \rightarrow Y$  and  $g : Z \rightarrow \Gamma_0$ . We consider the principal  $\Gamma$ -bundle  $W := \mathbf{I}_{g^{-1}} \otimes y^*Q$  over  $Z$ , which satisfies  $\pi_*W = 1$  and thus reduces to a principal  $\mathcal{B}_{\pi_1}\Gamma$ -bundle  $W_{\text{red}}$  by Theorem 4.1.2. Notice that

$$\begin{array}{ccc}
 i_*(\Gamma_1) \otimes \text{pr}_1^*W & \xrightarrow{\tau \otimes \text{id}} & \mathbf{I}_{\text{pr}_2^*g^{-1}, \text{pr}_1^*g} \otimes \text{pr}_1^*W \\
 & & \parallel \\
 & & \text{pr}_2^*\mathbf{I}_{g^{-1}} \otimes \text{pr}_1^*Q \xrightarrow{\text{id} \otimes \chi} \text{pr}_2^*\mathbf{I}_{g^{-1}} \otimes \text{pr}_2^*Q \otimes i_*(Q) = \text{pr}_2^*W \otimes i_*(P)
 \end{array}$$

is an isomorphism in the category  $\mathcal{B}un_\Gamma(Z \times_M Z)^1$ , and hence determines by Theorem 4.1.2 an isomorphism

$$\alpha : \text{pr}_2^*W_{\text{red}} \otimes i_*(P) \rightarrow i_*(\Gamma) \otimes \text{pr}_1^*W$$

of  $\mathcal{B}_{\pi_1}\Gamma$ -bundles over  $Z \times_M Z$ . The pair  $(W_{\text{red}}, \alpha)$  is a 1-isomorphism  $\mathcal{A} : \mathcal{L}_f \rightarrow \mathcal{G}$ . The condition that it exchanges the trivializations  $f^*\mathcal{T}_\Gamma$  and  $\mathcal{T}$  is now straightforward to check.

We continue with checking that the functor (5.1.1) is an equivalence on Hom-categories. Assume first that  $f_1, f_2 : M \rightarrow \pi_0\Gamma$  are different smooth maps, so that the Hom-category between  $f_1$  and  $f_2$  is empty. We show that there exists no 1-morphism  $\mathcal{A} : \mathcal{L}_{f_1} \rightarrow \mathcal{L}_{f_2}$  that exchanges the trivialization  $f_1^*\mathcal{T}_\Gamma$  with  $f_2^*\mathcal{T}_\Gamma$ . Indeed, this can be seen by extending the involved bundles along  $\pi$ : the bundle of the 1-morphism  $\mathcal{A}$  has the trivial map, while the bundles of the two trivializations have the different maps  $f_1$  and  $f_2$ .

Now we assume that the two maps are equal,  $f_1 = f_2 = : f$ , in which case the Hom-category between  $f_1$  and  $f_2$  has one object and one morphism. The single object is sent to the identity  $\text{id} : \mathcal{L}_f \rightarrow \mathcal{L}_f$ . Suppose  $\mathcal{A} : \mathcal{L}_f \rightarrow \mathcal{L}_f$  is another 1-morphism. Then  $i_*(\mathcal{A}) \cong i_*(\text{id})$ . Let  $P$  be the principal  $\Gamma$ -bundle on  $M$  which marks the difference between  $\text{id}$  and  $\mathcal{A}$ . Then,  $i_*(P)$  is trivializable, and so is  $P$ . Thus,  $\text{id} \cong \mathcal{A}$ . This shows that the functor between Hom-categories is essentially surjective. That it is full and faithful follows because the 2-morphisms in  $\mathcal{G}rb_{\mathcal{B}_{\pi_1}\Gamma}(M)^{i-or}$  are unique.  $\square$

Theorem 5.1.2 induces on isomorphism classes of objects:

**Corollary 5.1.3.** *The sequence (5.1) is exact at  $\check{H}^1(M, \mathcal{B}_{\pi_1}\Gamma)$ .*

## 5.2 Reduction of Non-Abelian Gerbes to Abelian Gerbes

In this section we look at the sequence

$$\mathcal{G}rb_{\mathcal{B}_{\pi_1}\Gamma}(M) \xrightarrow{i_*} \mathcal{G}rb_\Gamma(M) \xrightarrow{\pi_*} \mathcal{B}un_{\mathcal{B}_{\pi_0}\Gamma}(M)_{dis}. \quad (5.2.1)$$

of bicategories and 2-functors induced by the functors  $i$  and  $\pi$ , where we implicitly used the canonical equivalence  $\mathcal{G}rb_{\pi_0\Gamma_{dis}}(M) \cong \mathcal{B}un_{\mathcal{B}_{\pi_0}\Gamma}(M)_{dis}$  [NW, Example 5.1.9].

We want to prove that this sequence is a “fibre sequence” in bicategories. To formulate this properly we note that the category  $\mathcal{B}un_{\mathcal{B}_{\pi_0}\Gamma}(M)$  has a canonical “base point”, namely the trivial principal  $\mathcal{B}_{\pi_0}\Gamma$ -bundle  $\mathbf{I}_1$ . Next we give an explicit definition of the “homotopy fibre” of this base point:

**Definition 5.2.1.**

- (i) Let  $\mathcal{G}$  be a  $\Gamma$ -bundle gerbe over  $M$ . A  $\pi$ -orientation of  $\mathcal{G}$  is a global section of the principal bundle  $\pi_*(\mathcal{G})$ .
- (ii) A 1-morphism  $\varphi : \mathcal{G} \rightarrow \mathcal{G}'$  between  $\pi$ -oriented  $\Gamma$ -bundle gerbes is called  $\pi$ -orientation preserving if the induced morphism  $\pi_*(\varphi)$  of  $\pi_0\Gamma$ -bundles preserves the sections, i.e.  $\pi_*(\varphi) \circ s = s'$ .

The bicategory consisting of all  $\pi$ -oriented  $\Gamma$ -bundle gerbes, all  $\pi$ -orientation-preserving morphisms, and all 2-morphisms is denoted  $\mathcal{G}rb_\Gamma(M)^{\pi-or}$ . The composition  $\pi \circ i$  in sequence (5.2.1) is the trivial 2-group homomorphism. This implies directly:

**Lemma 5.2.2.** *The 2-functor  $\pi_* \circ i_*$  is canonically equivalent to the trivial 2-functor that sends each object to the trivial principal  $\mathcal{B}_{\pi_0}\Gamma$ -bundle and each morphism to the trivial morphism. In particular, the 2-functor  $i_*$  lifts to a 2-functor*

$$i_{or} : \mathcal{G}rb_{\mathcal{B}_{\pi_1}\Gamma}(M) \rightarrow \mathcal{G}rb_\Gamma(M)^{\pi-or}. \quad (5.2.2)$$

Let  $\mathcal{G}$  be a  $\Gamma$ -bundle gerbe over  $M$ . A *reduction of  $\mathcal{G}$  to an abelian gerbe* is a  $\mathcal{B}_{\pi_1}\Gamma$ -bundle gerbe  $\mathcal{G}_{\text{red}}$  such that  $\mathcal{G} \cong i_*(\mathcal{G}_{\text{red}})$ . Lemma 5.2.2 shows one part of the “exactness” of sequence (5.2.1): reducible bundle gerbes are  $\pi$ -orientable. The other part is to show that the homotopy fibre  $\mathcal{G}rb_\Gamma(M)^{\pi-or}$  agrees with  $\mathcal{G}rb_{\mathcal{B}_{\pi_1}\Gamma}(M)$ .

**Theorem 5.2.3.** *The 2-functor (5.2.2) is an equivalence of bicategories:*

$$\mathcal{G}rb_{\mathcal{B}_{\pi_1}\Gamma}(M) \cong \mathcal{G}rb_\Gamma(M)^{\pi-or}.$$

In particular, every  $\pi$ -orientation of a  $\Gamma$ -bundle gerbe  $\mathcal{G}$  determines (up to isomorphism) a reduction of  $\mathcal{G}$  to an abelian gerbe.

Proof. We show first that  $i_{or}$  is essentially surjective. Let  $\mathcal{G}$  be a  $\Gamma$ -bundle gerbe with  $\pi$ -orientation  $s$ . We denote by  $\zeta : Y \rightarrow M$  the surjective submersion of  $\mathcal{G}$ , by  $P$  its principal  $\Gamma$ -bundle over  $Y^{[2]}$ , and by  $\mu$  its bundle gerbe product. The  $\pi$ -orientation  $s$  can be described locally as a map  $s : Y \rightarrow \pi_0\Gamma$  satisfying the condition

$$\pi_*(P) \cdot \zeta_1^* s = \zeta_2^* s \quad (5.2.3)$$

where  $\pi_*(P) : Y^{[2]} \rightarrow \pi_0\Gamma$ . We may assume that  $s$  lifts along  $\pi : \Gamma_0 \rightarrow \pi_0\Gamma$  to a smooth map  $t : Y \rightarrow \Gamma_0$ , otherwise we pass to an isomorphic bundle gerbe by a refinement of the surjective submersion  $\zeta$ . Now we consider the following  $\mathcal{B}\pi_1\Gamma$ -bundle gerbe  $\mathcal{G}_{red}$ . Its surjective submersion is  $\zeta$ . The principal  $\Gamma$ -bundle

$$P' := \mathbf{I}_{(\zeta_3^* t)^{-1}} \otimes P \otimes \mathbf{I}_{\zeta_1^* t}$$

satisfies  $\pi_*(P') = 1$  due to (5.2.3), and so defines a principal  $\mathcal{B}\pi_1\Gamma$ -bundle  $P'_{red}$ ; this is the principal bundle of  $\mathcal{G}_{red}$ . Finally, Theorem 4.1.2 shows that the isomorphism

$$\mu' := \text{id} \otimes \mu \otimes \text{id} : \mathbf{I}_{(\zeta_3^* t)^{-1}} \otimes \zeta_{23}^* P \otimes \zeta_{12}^* P \otimes \mathbf{I}_{\zeta_1^* t} \rightarrow \mathbf{I}_{(\zeta_3^* t)^{-1}} \otimes \zeta_{13}^* P \otimes \mathbf{I}_{\zeta_1^* t}$$

of principal  $\Gamma$ -bundles over  $Y^{[3]}$  reduces to a bundle gerbe product for  $\mathcal{G}_{red}$ . We claim that  $\mathcal{G}$  and  $i_*(\mathcal{G}_{red})$  are isomorphic in  $\mathcal{G}rb_\Gamma(M)^{\pi-or}$ . Indeed, an orientation preserving isomorphism  $\mathcal{G} \rightarrow i_*(\mathcal{G}_{red})$  is given by the pair  $(\mathbf{I}_{t^{-1}}, \text{id})$ .

It remains to show that  $i_{or}$  is fully faithful. We consider two  $\mathcal{B}\pi_1\Gamma$ -bundle gerbes  $\mathcal{G}_1$  and  $\mathcal{G}_2$ . Suppose  $\mathcal{A} : i_*(\mathcal{G}_1) \rightarrow i_*(\mathcal{G}_2)$  is a 1-morphism that respects the canonical  $\pi$ -orientations of Lemma 5.2.2. If  $\mathcal{A}$  consists of a principal  $\Gamma$ -bundle over some common refinement  $Z$ , it follows that  $\pi_*(Q)$  descends to a smooth map  $q : M \rightarrow \pi_0\Gamma$ . The condition that  $\mathcal{A}$  preserves the  $\pi$ -orientations requires  $q$  to be the constant map. This in turn shows that  $\pi_*Q = 1$ , which implies that  $\mathcal{A} \cong i_*(\mathcal{A}_{red})$  by Theorem 4.1.2. This shows that  $i_{or}$  is essentially surjective on Hom-categories. That it is fully faithful on Hom-categories follows again by Theorem 4.1.2: the 2-morphisms between 1-morphisms  $i_*(\mathcal{B}_1)$  and  $i_*(\mathcal{B}_2)$  are exactly the 2-morphisms between  $\mathcal{B}_1$  and  $\mathcal{B}_2$ .  $\square$

**Corollary 5.2.4.** *The sequence (5.1) is exact at  $\check{H}^1(M, \Gamma)$ .*

**Example 5.2.5.** The terminology “ $\pi$ -orientation” is inspired by the example of *Jandl gerbes*, which play an important role in unoriented sigma models [SSW07, NS11]. The Jandl-2-group  $\mathcal{J}U(1)$  is the 2-group induced by the crossed module

$$U(1) \xrightarrow{0} \mathbb{Z}/2$$

with  $\mathbb{Z}/2$  acting on  $U(1)$  by inversion, see [NW, Section 2.4]. The fibre sequence (3.2) is here

$$\mathcal{B}U(1) \rightarrow \mathcal{J}U(1) \rightarrow \mathbb{Z}/2.$$

A  $\mathcal{J}U(1)$ -bundle gerbe  $\mathcal{G}$  is called *Jandl gerbe* [NS11] (there is also a connection whose discussion we omit here). Now,  $\pi_*(\mathcal{G})$  is a  $\mathbb{Z}/2$ -bundle called the *orientation bundle* of the Jandl gerbe. It is crucial for the definition of unoriented surface holonomy. Hence Theorem 5.2.3 shows that an oriented Jandl gerbe is just an ordinary  $\mathcal{B}U(1)$ -bundle gerbe.

### 5.3 Liftings of Principal Bundles to Non-Abelian Gerbes

In this section we come to the last bit of the exact sequence (5.1), namely to the connecting homomorphism

$$\check{H}^1(M, \pi_0\Gamma) \rightarrow \check{H}^2(M, \mathcal{B}\pi_1\Gamma).$$

The geometric objects that represent classes in  $\check{H}^2(M, \mathcal{B}\pi_1\Gamma)$  are  $\pi_1\Gamma$ -*bundle 2-gerbes* [Ste04]. A bundle 2-gerbe is a higher analogue of a bundle gerbe, and can be defined for a general abelian Lie-group  $A$ .

**Definition 5.3.1.** An *A-bundle 2-gerbe* over  $M$  is a surjective submersion  $\pi : Y \rightarrow M$ , a  $\mathcal{B}A$ -bundle gerbe  $\mathcal{G}$  over  $Y^{[2]}$ , a 1-isomorphism

$$\mathcal{M} : \pi_{23}^* \mathcal{G} \otimes \pi_{12}^* \mathcal{G} \rightarrow \pi_{13}^* \mathcal{G}$$

of  $\mathcal{B}A$ -bundle gerbes over  $Y^{[3]}$  and a 2-isomorphism

$$\begin{array}{ccc} \pi_{34}^* \mathcal{G} \otimes \pi_{23}^* \mathcal{G} \otimes \pi_{12}^* \mathcal{G} & \longrightarrow & \pi_{34}^* \mathcal{G} \otimes \pi_{13}^* \mathcal{G} \\ \downarrow & \swarrow \mu & \downarrow \\ \pi_{24}^* \mathcal{G} \otimes \pi_{12}^* \mathcal{G} & \longrightarrow & \pi_{14}^* \mathcal{G} \end{array}$$

over  $Y^{[4]}$  satisfying the pentagon axiom.

**Remark 5.3.2.**

1. Following the terminology of Definition 2.2 and the cohomological count it would be more logical to call this a  $\mathcal{B}\mathcal{B}A$ -bundle 2-gerbe. But we have decided to follow the naming used in the literature, also because bundle 2-gerbes can not be defined for general Lie 2-groups  $\Gamma$ .
2. As pointed out above, every  $A$ -bundle 2-gerbe  $\mathbb{G}$  has a characteristic class  $[\mathbb{G}] \in \check{H}^2(M, \mathcal{B}A)$ . Bundle 2-gerbes are up to isomorphism classified by this class [Ste04].

We recall from Section 4.2 that the Lie 2-group  $\Gamma$  determines a  $\mathcal{B}\pi_1\Gamma$ -bundle gerbe  $\mathcal{G}_\Gamma$  over  $\pi_0\Gamma$ : its submersion is the projection  $\pi : \Gamma_0 \rightarrow \pi_0\Gamma$ , its principal  $\mathcal{B}\pi_1\Gamma$ -bundle over  $\Gamma_0^{[2]}$  is  $\Gamma_1$ , and the multiplication  $\mu$  is given by the composition in  $\Gamma$ . We infer that  $\mathcal{G}_\Gamma$  is multiplicative in the sense of [CJM<sup>+</sup>05]:

(i) There is an isomorphism

$$\mathcal{M}_\Gamma : p_1^* \mathcal{G}_\Gamma \otimes p_2^* \mathcal{G}_\Gamma \rightarrow m^* \mathcal{G}_\Gamma$$

over  $\pi_0\Gamma \times \pi_0\Gamma$ , where  $p_1, p_2$  are the projections and  $m$  is the multiplication.

In order to construct  $\mathcal{M}_\Gamma$  we recall that  $\mathcal{G}_\Gamma$  is the bundle gerbe associated to the principal  $\mathcal{B}\pi_1\Gamma$ -2-bundle  $\Gamma$  over  $\pi_0\Gamma$  via the 2-functor  $\mathcal{E}$  of [NW, Section 7.1]. Since  $\pi_1\Gamma$  is central in  $\Gamma_1$ , the 2-group multiplication  $M : \Gamma \times \Gamma \rightarrow \Gamma$  can be seen as a 2-bundle morphism

$$M : P_1^* \Gamma \otimes P_2^* \Gamma \rightarrow m^* \Gamma$$

over  $\pi_0\Gamma \times \pi_0\Gamma$ . Since the 2-functor  $\mathcal{E}$  is moreover a morphism between pre-2-stacks [NW, Proposition 7.1.8], it converts this 2-bundle morphism into the claimed isomorphism  $\mathcal{M}_\Gamma$ .

(ii) The isomorphism  $\mathcal{M}_\Gamma$  is associative in the sense that there is a 2-isomorphism

$$\begin{array}{ccc} p_{34}^* \mathcal{G} \otimes p_{23}^* \mathcal{G} \otimes p_{12}^* \mathcal{G} & \longrightarrow & p_{34}^* \mathcal{G} \otimes p_{13}^* \mathcal{G} \\ \downarrow & \swarrow \mu_\Gamma & \downarrow \\ p_{24}^* \mathcal{G} \otimes p_{12}^* \mathcal{G} & \longrightarrow & p_{14}^* \mathcal{G} \end{array}$$

that satisfies the pentagon identity. This 2-isomorphism is simply induced by the fact that the 2-group multiplicative  $M$  is strictly associative.

Now let  $E$  be a principal  $\pi_0\Gamma$ -bundle over  $M$ . The idea is to define a  $\pi_1\Gamma$ -bundle 2-gerbe  $\mathbb{L}_E$  whose surjective submersion is the bundle projection  $E \rightarrow M$ . We denote by  $\delta_n : E^{[n+1]} \rightarrow \pi_0\Gamma^n$  the ‘‘difference maps’’ given by  $e_0 \cdot \text{pr}_i(\delta_n(e_0, e_1, \dots, e_n)) = e_i$ .

**Definition 5.3.3.** Let  $E$  be a principal  $\pi_0\Gamma$ -bundle. The lifting bundle 2-gerbe  $\mathbb{L}_E$  is given by the surjective submersion  $E \rightarrow M$ , the bundle gerbe  $\delta_1^*\mathcal{G}_\Gamma$ , the multiplication  $\delta_2^*\mathcal{M}_\Gamma$  and the associator  $\delta_3^*\mu_\Gamma$ .

**Remark 5.3.4.** In the context of Chern-Simons theories with a gauge group  $G$ , there exists a bundle 2-gerbe  $\mathbb{CS}_E(\mathcal{G})$  constructed from a principal  $G$ -bundle  $E$  over  $M$  and a multiplicative  $\mathcal{B}U(1)$ -bundle gerbe  $\mathcal{G}$  over  $G$  [CJM<sup>+</sup>05, Wal10]. A similar construction has also been proposed in [Jur]. In the particular case that  $G = \pi_0\Gamma$  and  $U(1) = \pi_1\Gamma$ , these constructions coincide with the one of Definition 5.3.3, i.e.

$$\mathbb{CS}_E(\mathcal{G}_\Gamma) = \mathbb{L}_E.$$

Next we justify the name ‘‘lifting bundle 2-gerbe’’: we show that  $\mathbb{L}_E$  is the obstruction to lift the bundle  $E$  to a  $\Gamma$ -bundle gerbe. Moreover, the possible lifts of  $E$  correspond to trivializations of  $\mathbb{L}_E$ . In order to produce a precise statement, we first give an explicit definition of the homotopy fibre of the 2-functor

$$\pi_* : \mathcal{G}rb_\Gamma(M) \rightarrow \mathcal{B}un_{\mathcal{B}\pi_0\Gamma}(M)_{dis}.$$

**Definition 5.3.5.** Let  $E$  be a principal  $\pi_0\Gamma$ -bundle. The bicategory  $\mathcal{L}ift_\Gamma(E)$  is defined as follows:

- An object is a  $\Gamma$ -lift of  $E$ : a pair  $(\mathcal{G}, \varphi)$  consisting of a  $\Gamma$ -bundle gerbe  $\mathcal{G}$  and of a bundle isomorphism  $\varphi : \pi_*(\mathcal{G}) \rightarrow E$ .
- A 1-morphism between objects  $(\mathcal{G}_1, \varphi_1)$  and  $(\mathcal{G}_2, \varphi_2)$  is a 1-morphism  $\mathcal{B} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$  such that  $\varphi_2 \circ \pi_*(\mathcal{B}) = \varphi_1$ .
- A 2-morphism between 1-morphisms  $\mathcal{B}$  and  $\mathcal{B}'$  is just a 2-morphism.

**Remark 5.3.6.** For the trivial  $\pi_0\Gamma$ -bundle  $E = M \times \pi_0\Gamma$  the category  $\mathcal{L}ift_\Gamma(E)$  agrees with the category of  $\pi$ -oriented  $\Gamma$ -bundle gerbes introduced in Definition 5.2.1.

The bicategory we want to compare with  $\mathcal{L}ift_\Gamma(E)$  is the 2-groupoid  $\mathcal{T}riv(\mathbb{L}_E)$  of trivializations of the lifting gerbe  $\mathbb{L}_E$ , see [Ste04, Definition 11.1] and [Walb, Section 5.1].

**Theorem 5.3.7.** Let  $E$  be a principal  $\pi_0\Gamma$ -bundle over  $M$ , and let  $\mathbb{L}_E$  be the associated lifting bundle 2-gerbe  $\mathbb{L}_E$ . Then there is an equivalence of bicategories

$$\mathcal{T}riv(\mathbb{L}_E) \cong \mathcal{L}ift_\Gamma(E).$$

In particular,  $E$  admits a  $\Gamma$ -lift if and only if  $\mathbb{L}_E$  is trivializable.

Theorem 5.3.7 is proved in Section 5.4 below. Before that we want to present two corollaries. First we recall that the bicategory of trivializations of an  $A$ -bundle 2-gerbe is a torsor over the monoidal bicategory of  $\mathcal{B}A$ -bundle gerbes [Walb, Lemma 2.2.5]. From Theorem 5.3.7 we get the following two implications:

**Corollary 5.3.8.** The bicategory  $\mathcal{L}ift_\Gamma(E)$  is a torsor over the monoidal bicategory of  $\mathcal{B}\pi_1\Gamma$ -bundle gerbes over  $M$ , i.e. the  $\mathcal{B}\pi_1\Gamma$ -bundle gerbes over  $M$  act on the  $\Gamma$ -lifts of  $E$  in such a way that on isomorphism classes of objects a free and transitive action is induced.

**Corollary 5.3.9.** The sequence (5.1) is exact at  $\check{H}^1(M, \pi_0\Gamma_{dis})$ .

## 5.4 Proof of Theorem 5.3.7

Our strategy to prove Theorem 5.3.7 is to reduce it to Theorem 5.2.3 using descent theory. First we need the following preliminaries.



- (i) Let  $\mathcal{G}$  and  $\mathcal{G}'$  be  $\pi$ -oriented  $\Gamma$ -bundle gerbes over a manifold  $X$ . For a 1-morphism  $\mathcal{A} : \mathcal{G} \rightarrow \mathcal{G}'$  we obtain a smooth map  $h_{\mathcal{A}} : X \rightarrow \pi_0\Gamma$  determined by

$$\pi_*(\mathcal{A})(s(x)) \cdot h_{\mathcal{A}}(x) = s'(x),$$

where  $s$  and  $s'$  are the  $\pi$ -orientations of  $\mathcal{G}$  and  $\mathcal{G}'$ , respectively. We have  $h_{\mathcal{A}} \equiv 1$  if and only if  $\mathcal{A}$  is  $\pi$ -oriented.

- (ii) For a smooth map  $h : X \rightarrow \pi_0\Gamma$  we denote by  $\text{Hom}^h(\mathcal{G}, \mathcal{G}')$  the full subcategory of the Hom-category  $\text{Hom}_{\mathcal{G}r\Gamma(X)}(\mathcal{G}, \mathcal{G}')$  over those 1-morphisms  $\mathcal{A}$  with  $h_{\mathcal{A}} = h$ .

**Lemma 5.4.1.** *Let  $h : X \rightarrow \pi_0\Gamma$  be a smooth map and  $\mathcal{G}$  and  $\mathcal{G}'$  be  $\mathcal{B}\pi_1\Gamma$ -bundle gerbes over  $X$ .*

- (i) *We have an equivalence of categories*

$$t : \text{Hom}(\mathcal{G} \otimes h^*\mathcal{G}_{\Gamma}, \mathcal{G}') \rightarrow \text{Hom}^h(i_{or}(\mathcal{G}), i_{or}(\mathcal{G}')).$$

- (ii) *Let  $h' : X \rightarrow \pi_0\Gamma$  be another smooth map. The diagram*

$$\begin{array}{ccc} \text{Hom}(\mathcal{G}' \otimes h'^*\mathcal{G}_{\Gamma}, \mathcal{G}'') \times \text{Hom}(\mathcal{G} \otimes h^*\mathcal{G}_{\Gamma}, \mathcal{G}') & \longrightarrow & \text{Hom}(\mathcal{G} \otimes (hh')^*\mathcal{G}_{\Gamma}, \mathcal{G}'') \\ \downarrow t \times t & & \downarrow t \\ \text{Hom}^{h'}(i_{or}(\mathcal{G}'), i_{or}(\mathcal{G}'')) \times \text{Hom}^h(i_{or}(\mathcal{G}), i_{or}(\mathcal{G}')) & \xrightarrow{\circ} & \text{Hom}^{hh'}(i_{or}(\mathcal{G}), i_{or}(\mathcal{G}'')) \end{array}$$

*of functors, in which the arrow in the first row is given by*

$$(\mathcal{B}, \mathcal{A}) \mapsto \mathcal{B} \circ (\mathcal{A} \otimes \text{id}) \circ (\text{id}_{\mathcal{G}} \otimes (h \times h')^*\mathcal{M}_{\Gamma}^{-1}),$$

*is commutative up to an associative natural equivalence.*

**Proof.** If  $\mathcal{G}$  is a  $\pi$ -oriented  $\Gamma$ -bundle gerbe over  $X$  and  $h : X \rightarrow \pi_0\Gamma$  is a smooth map, we obtain another  $\pi$ -oriented  $\Gamma$ -bundle gerbe denoted  $\mathcal{G}_h$ , which is the same  $\Gamma$ -bundle gerbe equipped with the new  $\pi$ -orientation is  $s_h := s \cdot h$ , where  $s$  is the original  $\pi$ -orientation. We note that

$$\text{Hom}^h(\mathcal{G}, \mathcal{G}') = \text{Hom}^1(\mathcal{G}_h, \mathcal{G}'). \quad (5.4.1)$$

In order to prove (i) we shall construct an orientation-preserving 1-isomorphism

$$\mathcal{C}_h : i_{or}(\mathcal{G} \otimes h^*\mathcal{G}_{\Gamma}) \rightarrow i_{or}(\mathcal{G})_h,$$

which is by (5.4.1) the same as an 1-isomorphism of  $\Gamma$ -bundle gerbes with  $h_{\mathcal{C}_h} = h$ . It is indeed clear that the trivialization  $\mathcal{T}_{\Gamma}$  of  $i_*(\mathcal{G}_{\Gamma})$  constructed in Lemma 4.2.2 induces a 1-isomorphism of  $\Gamma$ -bundle gerbes, and it is straightforward to check that this isomorphism satisfies the condition  $h_{\mathcal{C}_h} = h$ . Then, the equivalence  $t$  is the composition of the following equivalences:

$$\begin{array}{ccc} \text{Hom}(\mathcal{G} \otimes h^*\mathcal{G}_{\Gamma}, \mathcal{G}') & \xrightarrow{i_{or}} & \text{Hom}^1(i_{or}(\mathcal{G} \otimes h^*\mathcal{G}_{\Gamma}), i_{or}(\mathcal{G}')) \\ & & \downarrow \text{Hom}(\mathcal{C}_h, -) \\ & & \text{Hom}^1(i_{or}(\mathcal{G})_h, i_{or}(\mathcal{G}')) \xrightarrow{(5.4.1)} \text{Hom}^h(i_{or}(\mathcal{G}), i_{or}(\mathcal{G}')). \end{array}$$

The construction of the natural equivalence in (ii) is now straightforward. □

Now let  $E$  be a principle  $\mathcal{B}\pi_0\Gamma$ -bundle over  $M$  and let  $\mathbb{L}_E$  be the associated lifting bundle 2-gerbe. We first want to give another description of the category  $\mathcal{Lift}_\Gamma(E)$ . Since the projection  $\zeta : E \rightarrow M$  is a surjective submersion and  $\Gamma$ -bundle gerbes form a stack [NS11, Theorem 3.3], we have an equivalence

$$\mathcal{Grb}_\Gamma(M) \cong \mathcal{Desc}_\zeta(\mathcal{Grb}_\Gamma)$$

between the bicategory  $\Gamma$ -bundle gerbes over  $M$ , and the bicategory of descent data with respect to  $\zeta : E \rightarrow M$ ; for the notation we refer to Section 2 of [NS11]. An object in  $\mathcal{Desc}_\zeta(\mathcal{Grb}_\Gamma)$  is:

1. a  $\Gamma$ -bundle gerbe  $\mathcal{G}$  over  $E$ ,
2. a 1-isomorphism  $\mathcal{A} : \text{pr}_2^*\mathcal{G} \rightarrow \text{pr}_1^*\mathcal{G}$  over  $E^{[2]}$ ,
3. a 2-isomorphism  $\chi : \text{pr}_{23}^*\mathcal{A} \circ \text{pr}_{12}^*\mathcal{A} \Rightarrow \text{pr}_{13}^*\mathcal{A}$  over  $E^{[3]}$ , and
4. a coherence condition for  $\chi$  over  $E^{[4]}$ .

Analogously, there is an equivalence

$$\mathcal{Bun}_{\mathcal{B}\pi_0\Gamma}(M) \cong \mathcal{Desc}_\zeta(\mathcal{Bun}_{\mathcal{B}\pi_0\Gamma})$$

between the category of principal  $\mathcal{B}\pi_0\Gamma$ -bundles and their descent data, which we have already used in the proof of Theorem 4.2.5. Explicitly, the principal  $\mathcal{B}\pi_0\Gamma$ -bundle  $E$  over  $M$  corresponds to the following descent object:

1. the trivial  $\mathcal{B}\pi_0\Gamma$ -bundle  $\mathbf{I}$  over  $E$ ,
2. the isomorphism  $\text{pr}_2^*\mathbf{I} \rightarrow \text{pr}_1^*\mathbf{I}$  induced by the difference map  $\delta_2 : E \times_M E \rightarrow \pi_0\Gamma$ , and
3. a cocycle condition over  $E^{[3]}$ .

Now we are in position to give a descent-theoretical formulation of the bicategory  $\mathcal{Lift}_\Gamma(E)$ . We obtain a bicategory  $\mathcal{D}$ , in which an object consists of:

- (a) a  $\pi$ -oriented  $\Gamma$ -bundle gerbe  $\mathcal{G}$  over  $E$ ,
- (b) a 1-isomorphism  $\mathcal{A} : \text{pr}_2^*\mathcal{G} \rightarrow \text{pr}_1^*\mathcal{G}$  of  $\Gamma$ -bundle gerbes over  $E^{[2]}$  such that  $h_{\mathcal{A}} = \delta$ , i.e. an object in  $\text{Hom}^\delta(\text{pr}_2^*\mathcal{G}, \text{pr}_1^*\mathcal{G})$ ,
- (c) a 2-isomorphism  $\mu : \text{pr}_{23}^*\mathcal{A} \circ \text{pr}_{12}^*\mathcal{A} \Rightarrow \text{pr}_{13}^*\mathcal{A}$  over  $E^{[3]}$ , i.e. a morphism in  $\text{Hom}^{\delta'}(\text{pr}_3^*\mathcal{G}, \text{pr}_1^*\mathcal{G})$  where  $\delta'$  denotes the map  $\text{pr}_{23}^*\delta \cdot \text{pr}_{12}^*\delta = \text{pr}_{13}^*\delta$ , and
- (d) a coherence condition for  $\mu$  over  $E^{[4]}$ .

The 1-morphisms and 2-morphisms in the bicategory  $\mathcal{D}$  are defined in the same evident way. By [NS11, Theorem 3.3] the two bicategories are equivalent:  $\mathcal{Lift}_\Gamma(E) \cong \mathcal{D}$ . Functors in both directions are given by pullback and descent along  $\zeta : E \rightarrow M$ .

The next step is to translate the structure of the bicategory  $\mathcal{D}$  using the equivalences of Theorem 5.2.3 and Lemma 5.4.1. We obtain yet another bicategory  $\mathcal{E}$  whose objects consist of

- (a') a  $\mathcal{B}\pi_1\Gamma$ -gerbe  $\mathcal{H}$  over  $E$ , which corresponds to the  $\pi$ -oriented  $\Gamma$ -bundle gerbe  $\mathcal{G}$  of (a) under the equivalence of Theorem 5.2.3,
- (b') a 1-isomorphism  $\mathcal{B} : \text{pr}_2^*\mathcal{H} \otimes \delta^*\mathcal{G}_\Gamma \rightarrow \text{pr}_1^*\mathcal{H}$  over  $E^{[2]}$ , which corresponds to the 1-isomorphism  $\mathcal{A}$  of (b) under the equivalence of Lemma 5.4.1 (i),

(c') a 2-isomorphism

$$\begin{array}{ccc}
\mathrm{pr}_3^* \mathcal{H} \otimes \delta_{23}^* \mathcal{G}_\Gamma \otimes \delta_{12}^* \mathcal{G}_\Gamma & \xrightarrow{\mathrm{pr}_{23}^* \mathcal{B} \otimes \mathrm{id}} & \mathrm{pr}_2^* \mathcal{H} \otimes \delta_{12}^* \mathcal{G}_\Gamma \\
\mathrm{id} \otimes \mathcal{M}_\Gamma \downarrow & \alpha \swarrow \! \! \! \swarrow & \downarrow \mathrm{pr}_{12}^* \mathcal{B} \\
\mathrm{pr}_3^* \mathcal{H} \otimes \delta_{13}^* \mathcal{G}_\Gamma & \xrightarrow{\mathrm{pr}_{13}^* \mathcal{B}} & \mathrm{pr}_1^* \mathcal{H}
\end{array}$$

over  $E^{[3]}$ , which corresponds essentially to the 2-isomorphism  $\mu$  of (c) under the equivalence of Lemma 5.4.1 (i), with an additional whiskering using the diagram of Lemma 5.4.1 (ii), and

(d') a coherence condition for  $\alpha$  over  $E^{[4]}$ .

Since the translations we applied are *2-functorial* (Theorem 5.2.3 and Lemma 5.4.1), a similar description can easily be given for 1- and 2-morphisms of the bicategory  $\mathcal{E}$ . Since they are *equivalences*, the bicategories  $\mathcal{D}$  and  $\mathcal{E}$  are equivalent.

In order to finish the proof of Theorem 5.3.7 it remains to notice that the bicategory  $\mathcal{E}$  is the bicategory  $\mathcal{T}riv(\mathbb{L}_E)$  of trivializations of  $\mathbb{L}_E$ , just as defined in [Walb, Section 5.1].

## 6 Transgression for Non-Abelian Gerbes

In this section,  $M$  is a smooth, finite-dimensional manifold. Transgression requires to work with connections on bundle gerbes, and we refer to [Wal10, Walc] and references therein for a complete discussion using the same notation we are going to use here. Generally, if  $A$  is an abelian Lie group, and  $\mathcal{G}$  is a  $\mathcal{B}A$ -bundle gerbe with connection, there is a monoidal 2-functor

$$\mathcal{T} : \mathcal{G}rb_{\mathcal{B}A}^\nabla(M) \longrightarrow \mathcal{B}un_{\mathcal{B}A}(LM)_{dis}$$

called *transgression*, which takes  $\mathcal{B}A$ -bundle gerbes with connection over  $M$  to principal  $\mathcal{B}A$ -bundles over the free loop space  $LM := C^\infty(S^1, M)$ . We remark that the bundles in the image of  $\mathcal{T}$  are actually equipped with more structure [Walc], which we do not need here. On the level of isomorphism classes, the dependence on the connections drops out, and the 2-functor  $\mathcal{T}$  induces a group homomorphism

$$\tau : \check{H}^1(M, \mathcal{B}A) \longrightarrow \check{H}^0(LM, \mathcal{B}A).$$

The goal of the present section is to generalize this transgression homomorphism from  $\mathcal{B}A$  to a general, smoothly separable Lie 2-group with  $\pi_0 \Gamma$  compact and connected.

### 6.1 The Loop Group of a Lie 2-Group

We consider a smoothly separable Lie 2-group  $\Gamma$  with  $\pi_0 \Gamma$  compact. We denote by  $\mathcal{G}_\Gamma$  the associated  $\mathcal{B}\pi_1 \Gamma$ -bundle gerbe over  $\pi_0 \Gamma$ , see Section 4.2. In Section 5.3 we have equipped  $\mathcal{G}_\Gamma$  with a multiplicative structure, consisting of a 1-isomorphism  $\mathcal{M}_\Gamma$  over  $G \times G$  and of a 2-isomorphism  $\alpha$  over  $G^3$ . Since  $\pi_0 \Gamma$  is compact,  $\mathcal{G}_\Gamma$  admits a *multiplicative connection* [Wal10, Proposition 2.3.8].

We recall in more generality from [Wal10, Definition 1.3] that a multiplicative connection on a multiplicative  $\mathcal{B}A$ -bundle gerbe  $(\mathcal{G}, \mathcal{M}, \alpha)$  over a Lie group  $G$  is a connection  $\lambda$  on  $\mathcal{G}$ , (together denoted by  $\mathcal{G}^\lambda$ ), a 2-form  $\rho \in \Omega^2(G^2, \mathfrak{a})$  with values in the Lie algebra  $\mathfrak{a}$  of  $A$ , and a connection  $\eta$  on  $\mathcal{M}$  (together denoted by  $\mathcal{M}^\eta$ ), such that

$$\mathcal{M}^\eta : \mathrm{pr}_1^* \mathcal{G}^\lambda \otimes \mathrm{pr}_2^* \mathcal{G}^\lambda \longrightarrow m^* \mathcal{G}^\lambda \otimes \mathcal{I}_\rho \quad (6.1.1)$$

is a connection-preserving 1-isomorphism, and  $\alpha$  is a connection-preserving 2-isomorphism. In (6.1.1) we have denoted by  $\mathcal{I}_\rho$  the trivial bundle gerbe equipped with the curving  $\rho$ . The 2-form  $\rho$  is in fact determined by  $\lambda$  and  $\eta$  and the assumption that (6.1.1) is connection-preserving.

Applying the transgression 2-functor  $\mathcal{T}$ , we obtain:

- (a) a principal  $\mathcal{B}A$ -bundle  $\mathcal{T}_{\mathcal{G}^\lambda}$  over  $LG$ .
- (b) a bundle isomorphism

$$\mathrm{pr}_1^* \mathcal{T}_{\mathcal{G}^\lambda} \otimes \mathrm{pr}_2^* \mathcal{T}_{\mathcal{G}^\lambda} \xrightarrow{\mathcal{T}_{\mathcal{M}^\eta}} m^* \mathcal{T}_{\mathcal{G}^\lambda} \otimes \mathcal{T}_{\mathcal{I}_\rho} \xrightarrow{\mathrm{id} \otimes t_\rho} m^* \mathcal{T}_{\mathcal{G}^\lambda}$$

over  $LG^2$ , using the fact that the transgression of the trivial bundle gerbe  $\mathcal{I}_\rho$  has a canonical trivialization  $t_\rho$ .

- (c) an associativity condition for the bundle isomorphism of (b) over  $LG^3$ , coming from the transgression of the 2-isomorphism  $\alpha$ .

We recall the following result:

**Theorem 6.1.1** ([Wal10, Theorem 3.1.7]). *If  $G$  is finite-dimensional, the bundle isomorphism of (b) equips the total space of  $\mathcal{T}_{\mathcal{G}^\lambda}$  with the structure of a Fréchet Lie group, which we denote by  $LG^{\lambda,\eta}$ . Moreover,  $LG^{\lambda,\eta}$  is a central extension*

$$1 \longrightarrow A \longrightarrow LG^{\lambda,\eta} \longrightarrow LG \longrightarrow 1 \quad (6.1.2)$$

of Fréchet Lie groups.

We apply Theorem 6.1.1 to the multiplicative  $\mathcal{B}\pi_1\Gamma$ -bundle gerbe  $\mathcal{G}_\Gamma$ , and some multiplicative connection  $(\lambda, \eta)$  on it (note that  $G = \pi_0\Gamma$  is by assumption compact, in particular finite-dimensional). Since we understand the connection as an auxiliary structure, we have to control different choices. For this purpose we need the following technical lemma.

**Lemma 6.1.2.** *Let  $(\mathcal{G}, \mathcal{M}, \alpha)$  be a multiplicative  $A$ -bundle gerbe over  $G$ , and let  $(\lambda, \eta)$  and  $(\lambda', \eta')$  be multiplicative connections, determining 2-forms  $\rho, \rho' \in \Omega^2(G \times G, \mathfrak{a})$ . Then, there is a 2-form  $\beta \in \Omega^2(G, \mathfrak{a})$  satisfying  $\rho = \rho' + \Delta\beta$ , and a connection  $\epsilon$  on the identity 1-isomorphism  $\mathrm{id} : \mathcal{G} \rightarrow \mathcal{G}$  such that*

$$\mathrm{id}^\epsilon : \mathcal{G}^\lambda \rightarrow \mathcal{G}^{\lambda'} \otimes \mathcal{I}_\beta$$

is connection-preserving, and the canonical 2-isomorphism

$$\begin{array}{ccc} \mathrm{pr}_1^* \mathcal{G}^\lambda \otimes \mathrm{pr}_2^* \mathcal{G}^\lambda & \xrightarrow{\mathcal{M}^\eta} & m^* \mathcal{G}^\lambda \otimes \mathcal{I}_\rho \\ \mathrm{pr}_1^* \mathrm{id}_\epsilon \otimes \mathrm{pr}_2^* \mathrm{id}_\epsilon \downarrow & & \downarrow m^* \mathrm{id}_\epsilon \otimes \mathrm{id}_0 \\ \mathrm{pr}_1^* (\mathcal{G}^{\lambda'} \otimes \mathcal{I}_\beta) \otimes \mathrm{pr}_2^* (\mathcal{G}^{\lambda'} \otimes \mathcal{I}_\beta) & & m^* (\mathcal{G}^{\lambda'} \otimes \mathcal{I}_\beta) \otimes \mathcal{I}_{\rho'} \\ \parallel & \swarrow & \parallel \\ \mathrm{pr}_1^* \mathcal{G}^{\lambda'} \otimes \mathrm{pr}_2^* \mathcal{G}^{\lambda'} \otimes \mathcal{I}_{\mathrm{pr}_1^* \beta + \mathrm{pr}_2^* \beta} & \xrightarrow{\mathcal{M}^{\eta'} \otimes \mathrm{id}_0} & m^* \mathcal{G}^{\lambda'} \otimes \mathcal{I}_{\rho'} \otimes \mathcal{I}_{\mathrm{pr}_1^* \beta + \mathrm{pr}_2^* \beta} \end{array}$$

is connection-preserving.

Proof. We refer to [Walb, Section 5.2] for a general overview about connections on bundle gerbes. The connection on  $\mathcal{G}$  consists of two parts: a 2-form  $B \in \Omega^2(Y, \mathfrak{a})$ , where  $\pi : Y \rightarrow G$  is the surjective submersion of  $\mathcal{G}$ , and a connection  $\omega$  on the principal  $\mathcal{B}A$ -bundle  $P$  over  $Y$ <sup>[2]</sup>. The 1-isomorphism  $\mathcal{M}$  consists of a principal  $\mathcal{B}A$ -bundle  $Q$  over the fibre product  $Z := (Y \times Y)_{m \circ (\pi \times \pi)} \times_{\pi} Y$ , and a connection on  $\mathcal{M}$  is a connection  $\eta$  on  $Q$ . On  $Z$  we have the three projections  $\text{pr}_i : Z \rightarrow Y$ , and the surjective submersion  $\zeta := (\pi \times \pi) \circ (\text{pr}_1, \text{pr}_2) : Z \rightarrow G \times G$ . One of the various conditions that relate all these differential forms is

$$\text{pr}_1^* B + \text{pr}_2^* B = \text{pr}_3^* B + \zeta^* \rho + d\eta; \quad (6.1.3)$$

in particular,  $\rho$  is uniquely determined by  $B$  and  $\eta$ . We claim that the remaining conditions imply the following statement: for  $(B, \omega, \eta)$  and  $(B', \omega', \eta')$  two choices of a multiplicative connection on  $(\mathcal{G}, \mathcal{M}, \alpha)$ , there exist a 2-form  $\beta \in \Omega^2(G)$  and a 1-form  $\epsilon \in \Omega^1(Y)$  such that

$$B' = B - \pi^* \beta + d\epsilon \quad , \quad \omega' = \omega + \text{pr}_2^* \epsilon - \text{pr}_1^* \epsilon \quad \text{and} \quad \eta' = \eta + \text{pr}_1^* \epsilon + \text{pr}_2^* \epsilon - \text{pr}_3^* \epsilon. \quad (6.1.4)$$

Indeed, by Lemma [Walb, Lemma 3.3.5] the two connections  $(B, \omega)$  and  $(B', \omega')$  on  $\mathcal{G}$  differ by a pair  $(\beta, \epsilon)$  in the way stated above. A priori,  $(\beta, \epsilon)$  are defined up to  $\alpha \in \Omega^1(G, \mathfrak{a})$  acting by  $(\beta + d\alpha, \epsilon + \pi^* \alpha)$ . The connections  $\eta$  and  $\eta'$  differ a priori by a uniquely defined 1-form  $\delta \in \Omega^1(Z, \mathfrak{a})$ . The condition that the bundle isomorphism of  $\mathcal{M}$  preserves the new as well as the old connections imposes the condition

$$\zeta_1^*(\delta + \text{pr}_1^* \epsilon + \text{pr}_2^* \epsilon - \text{pr}_3^* \epsilon) = \zeta_2^*(\delta + \text{pr}_1^* \epsilon + \text{pr}_2^* \epsilon - \text{pr}_3^* \epsilon)$$

over  $Z \times_{G \times G} Z$ , where  $\zeta_1, \zeta_2$  are the two projections. Thus, there exists a unique 1-form  $\gamma \in \Omega^1(G \times G, \mathfrak{a})$  such that

$$\zeta^* \gamma = \delta + \text{pr}_1^* \epsilon + \text{pr}_2^* \epsilon - \text{pr}_3^* \epsilon.$$

The condition that the 2-isomorphism  $\alpha$  respect both the old and the new connections imposes the condition that  $\Delta \gamma = 0$ , where  $\Delta$  is the alternating sum over the pullbacks along the face maps in the simplicial manifold  $G^\bullet$ , forming the complex

$$\Omega^1(G) \rightarrow \Omega^1(G \times G) \rightarrow \Omega^1(G \times G \times G) \rightarrow \dots$$

This complex has no cohomology in degree two since  $G$  is compact (see the proof of [Wal10, Proposition 2.3.8]). Thus, there exists a 1-form  $\kappa \in \Omega^1(G)$  such that  $\Delta \kappa = \gamma$ . Now, the new pair  $(\beta + d\kappa, \epsilon + \pi^* \kappa)$  satisfies (6.1.4). One can now check that the corresponding 2-forms  $\rho$  and  $\rho$  determined by (6.1.3) satisfy the claimed identity.  $\square$

Using Lemma 6.1.2 we get:

**Proposition 6.1.3.** *The central extension  $L\mathcal{G}^{\lambda, \eta}$  of Theorem 6.1.1 is independent of the choice of the multiplicative connection  $(\lambda, \eta)$  up to equivalences of central extensions.*

Proof. Let  $(\lambda', \eta')$  be another multiplicative connection on  $(\mathcal{G}, \mathcal{M}, \alpha)$ . By Lemma 6.1.2 there exists a connection  $\epsilon$  on the identity  $\text{id} : \mathcal{G} \rightarrow \mathcal{G}$  and a 2-form  $\beta \in \Omega^2(G, \mathfrak{a})$  such that

$$\text{id}_\epsilon : \mathcal{G}^\lambda \rightarrow \mathcal{G}^{\lambda'} \otimes \mathcal{I}_\beta$$

is a connection-preserving 1-isomorphism. We transgress to the loop space, and use the canonical trivializations  $t_\rho$  of the transgression of a trivial bundle  $\mathcal{I}_\rho$ . We get an isomorphism

$$\varphi_\epsilon := \mathcal{I}_{\text{id}_\epsilon} : \mathcal{T}_{\mathcal{G}^\lambda} \rightarrow \mathcal{T}_{\mathcal{G}^{\lambda'}} \quad (6.1.5)$$

of principal  $\pi_1 \Gamma$ -bundles over  $LM$ , and a commutative diagram

$$\begin{array}{ccc} \text{pr}_1^* \mathcal{T}_{\mathcal{G}^\lambda} \otimes \text{pr}_2^* \mathcal{T}_{\mathcal{G}^\lambda} & \xrightarrow{\mathcal{T}_{\mathcal{M}^\eta}} & m^* \mathcal{T}_{\mathcal{G}^\lambda} \\ \text{pr}_1^* \varphi_\epsilon \otimes \text{pr}_2^* \varphi_\epsilon \downarrow & & \downarrow m^* \varphi_\epsilon \\ \text{pr}_1^* \mathcal{T}_{\mathcal{G}^{\lambda'}} \otimes \text{pr}_2^* \mathcal{T}_{\mathcal{G}^{\lambda'}} & \xrightarrow{\mathcal{T}_{\mathcal{M}^{\eta'}}} & m^* \mathcal{T}_{\mathcal{G}^{\lambda'}} \end{array}$$

where we have used that the transgression of the identity  $\text{id}_0 : \mathcal{I}_\rho \rightarrow \mathcal{I}_\rho$  between trivial bundle gerbes obviously exchanges the canonical trivializations. Summarizing,  $\varphi_\epsilon$  is an equivalence of central extensions.  $\square$

According to Proposition 6.1.3 we may write  $L\mathcal{G}$  for the central extension of Theorem 6.1.1 without mentioning the choice of the multiplicative connection on  $\mathcal{G}$ .

**Definition 6.1.4.** *Let  $\Gamma$  be a smoothly separable Lie 2-group with  $\pi_0\Gamma$  compact. The Fréchet Lie group  $L\mathcal{G}_\Gamma$  is denoted by  $L\Gamma$  and called the loop group of  $\Gamma$ .*

**Example 6.1.5.** For the following two “extremal” examples one can see by just looking at the sequence (6.1.2) what the loop group is:

- (i) If  $\Gamma = \mathcal{B}A$  for an abelian Lie group  $A$ , then  $L\Gamma = A$ .
- (ii) If  $\Gamma = G_{dis}$  for a Lie group  $G$ , then  $L\Gamma = LG$ , i.e.  $L\Gamma$  is the ordinary loop group.

A Lie 2-group homomorphism between Lie 2-groups  $\Gamma$  and  $\Omega$  is a smooth anafunctor  $\Lambda : \Gamma \rightarrow \Omega$  which is compatible with the multiplication functors up to a coherent transformation, see [NW, Eq. 2.4.4]. We recall from [NSW, Appendix B] that a Lie 2-group homomorphism induces smooth maps

$$\pi_0\Lambda : \pi_0\Gamma \rightarrow \pi_0\Omega \quad \text{and} \quad \pi_1\Lambda : \pi_1\Gamma \rightarrow \pi_1\Omega.$$

In order to study the relation between the loop groups  $L\Gamma$  and  $L\Omega$  we need the following two technical lemmata for preparation:

**Lemma 6.1.6.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be multiplicative bundle gerbes over  $G$ , let  $\mathcal{A} : \mathcal{G} \rightarrow \mathcal{H}$  be a multiplicative 1-isomorphism, and let  $(\lambda, \eta)$  be a multiplicative connection on  $\mathcal{G}$ . Then, there exists a multiplicative connection  $(\lambda', \eta')$  on  $\mathcal{H}$ , and a connection  $\epsilon$  on  $\mathcal{A}$ , such that  $\mathcal{A}^\epsilon : \mathcal{G}^\lambda \rightarrow \mathcal{H}^{\lambda'}$  is connection-preserving.*

Proof. First of all, there exists a connection  $\epsilon$  on  $\mathcal{A}$  and a connection  $\lambda'$  on  $\mathcal{H}$  such that  $\mathcal{A}^\epsilon : \mathcal{G}^\lambda \rightarrow \mathcal{H}^{\lambda'}$  is connection-preserving [Walb, Lemma 5.2.4]. Part of the structure of a multiplicative 1-isomorphism is a 2-isomorphism

$$\begin{array}{ccc} \text{pr}_1^*\mathcal{G}^\lambda \otimes \text{pr}_2^*\mathcal{G}^\lambda & \xrightarrow{\mathcal{M}} & m^*\mathcal{G}^\lambda \otimes \mathcal{I}_\rho \\ \text{pr}_1^*\mathcal{A} \otimes \text{pr}_2^*\mathcal{A} \downarrow & \swarrow \gamma & \downarrow m^*\mathcal{A} \\ \text{pr}_1^*\mathcal{H}^{\lambda'} \otimes \text{pr}_2^*\mathcal{H}^{\lambda'} & \xrightarrow{\mathcal{M}'} & m^*\mathcal{H}^{\lambda'} \otimes \mathcal{I}_\rho, \end{array}$$

which goes between 1-isomorphisms between bundle gerbes with connections. The 1-isomorphisms on three sides are in fact equipped with compatible connections. By [Walb, Lemma 5.2.5] there exists a compatible connection  $\eta'$  on  $\mathcal{M}'$  such that  $\gamma$  is connection-preserving. Since the associator  $\alpha'$  of  $\mathcal{H}$  is uniquely determined by the associator  $\alpha$  of  $\mathcal{G}$  and  $\gamma$ , it follows that  $\alpha'$  is also connection-preserving. Thus,  $(\lambda', \eta')$  is a multiplicative connection on  $\mathcal{H}$ .  $\square$

**Lemma 6.1.7.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be isomorphic multiplicative bundle gerbes over  $G$ . Let  $(\lambda, \eta)$  and  $(\lambda', \eta')$  be multiplicative connections on  $\mathcal{G}$  and  $\mathcal{H}$ , respectively. Then, there exists an isomorphism  $L\mathcal{G}^{\lambda, \eta} \cong L\mathcal{H}^{\lambda', \eta'}$ .*

Proof. By Lemma 6.1.6 there exists a connection  $(\tilde{\lambda}, \tilde{\eta})$  on  $\mathcal{H}$  such that  $\mathcal{G}^\lambda$  and  $\mathcal{H}^{\tilde{\lambda}}$  are isomorphic as multiplicative gerbes. This isomorphism transgresses to a Lie group isomorphism  $L\mathcal{G}^{\lambda, \eta} \cong L\mathcal{H}^{\tilde{\lambda}, \tilde{\eta}}$ . Together with Proposition 6.1.3, this shows the claim.  $\square$

**Proposition 6.1.8.** *If  $\Lambda : \Gamma \rightarrow \Omega$  is a Lie 2-group homomorphism, there exists a Lie group homomorphism  $L\Lambda : L\Gamma \rightarrow L\Omega$  such that the diagram*

$$\begin{array}{ccccc} \pi_1\Gamma & \longrightarrow & L\Gamma & \longrightarrow & L\pi_0\Gamma \\ \pi_1\Lambda \downarrow & & \downarrow L\Lambda & & \downarrow L\pi_0\Lambda \\ \pi_1\Omega & \longrightarrow & L\Omega & \longrightarrow & L\pi_0\Omega \end{array}$$

is commutative.

*Proof.* We recall that the bundle gerbes  $\mathcal{G}_\Gamma$  and  $\mathcal{G}_\Omega$  correspond to the principal 2-bundles  $\Gamma$  over  $\pi_0\Gamma$  and  $\Omega$  over  $\pi_0\Omega$ , respectively, under the equivalence of Theorem 2.4. The Lie 2-group homomorphism  $\Lambda$  defines a 1-morphism

$$(\pi_1\Lambda)_*\Gamma \rightarrow (\pi_0\Lambda)^*\Omega$$

of  $\mathcal{B}\pi_1\Omega$ -2-bundles over  $\pi_0\Gamma$ , which is compatible with the composition of  $\Gamma$  and  $\Omega$  in exactly such a way that the induced 1-isomorphism

$$\mathcal{D} : (\pi_1\Lambda)_*\mathcal{G}_\Gamma \rightarrow (\pi_0\Lambda)^*\mathcal{G}_\Omega \tag{6.1.6}$$

of  $\mathcal{B}\pi_1\Omega$ -bundle gerbes over  $\pi_0\Gamma$  is multiplicative. Now let  $(\lambda, \eta)$  and  $(\lambda', \eta')$  be multiplicative connections on  $\mathcal{G}_\Gamma$  and  $\mathcal{G}_\Omega$ , respectively. These induce connections on  $(\pi_1\Lambda)_*\mathcal{G}_\Gamma$  and  $(\pi_0\Lambda)^*\mathcal{G}_\Omega$ . Now Lemma 6.1.7 applies and yields an isomorphism  $L((\pi_1\Lambda)_*\mathcal{G}_\Gamma)^{(\pi_1\Lambda)_*(\lambda, \eta)} \cong L((\pi_0\Lambda)^*\mathcal{G}_\Omega)^{(\pi_0\Lambda)^*(\lambda', \eta')}$ . Since the transgression functor  $\mathcal{T}$  is contravariant in the base manifold and covariant in the Lie group, we get an isomorphism

$$(\pi_1\Lambda)_*(L\mathcal{G}_\Gamma^{\lambda, \eta}) \cong L((\pi_1\Lambda)_*\mathcal{G}_\Gamma)^{(\pi_1\Lambda)_*(\lambda, \eta)} \cong L((\pi_0\Lambda)^*\mathcal{G}_\Omega)^{(\pi_0\Lambda)^*(\lambda', \eta')} \cong (\pi_0\Lambda)^*(L\mathcal{G}_\Omega^{\lambda', \eta'}).$$

Such an isomorphism is the same as the claimed homomorphism  $L\Lambda$ . □

**Remark 6.1.9.** One can show that the equivalences of Proposition 6.1.3 and the group homomorphism  $L\Lambda$  of Proposition 6.1.8 are canonically determined up to homotopy.

## 6.2 Transgression of the Lifting Bundle 2-Gerbe

According to Theorem 5.3.7, a  $\Gamma$ -bundle gerbe  $\mathcal{G}$  defines a principal  $\mathcal{B}\pi_0\Gamma$ -bundle  $E := \pi_*(\mathcal{G})$  over  $M$  together with a trivialization  $\mathbb{T}_\mathcal{G}$  of the lifting bundle 2-gerbe  $\mathbb{L}_E$ . The crucial point is that although  $\mathcal{G}$  is a non-abelian gerbe, the lifting bundle 2-gerbe  $\mathbb{L}_E$  and its trivialization  $\mathbb{T}_\mathcal{G}$  are both abelian.

First we remark that  $LE$  is a principal  $L\pi_0\Gamma$ -bundle over  $LM$ . Here we need the assumption that  $\pi_0\Gamma$  is connected [Wal11, Lemma 5.1]. The obstruction to lift the structure group  $LE$  from  $L\pi_0\Gamma$  to the loop group  $L\Gamma$  is represented by the lifting bundle gerbe  $\mathcal{L}_{LE, L\Gamma}$  over  $LM$ . In short, there is a canonical equivalence of categories

$$\mathcal{T}riv(\mathcal{L}_{LE, L\Gamma}) \cong \mathcal{L}ift_{LE, L\Gamma} \tag{6.2.1}$$

which is an analogue of Theorem 5.3.7 for ordinary bundles [Mur96, Wal11]. The plan is to transgress the lifting bundle 2-gerbe  $\mathbb{L}_E$  to the loop space and identify this transgression  $\mathcal{T}_{\mathbb{L}_E}$  with the lifting gerbe. The main result of this section is:

**Proposition 6.2.1.** *Let  $\Gamma$  be a smoothly separable Lie 2-group with  $\pi_0\Gamma$  compact and connected, and let  $E$  be a principal  $\pi_0\Gamma$ -bundle over  $M$ .*

(a) *There is a canonical isomorphism*

$$\phi_{E, \Gamma} : \mathcal{T}_{\mathbb{L}_E} \rightarrow \mathcal{L}_{LE, L\Gamma}$$

*between the transgression of the lifting bundle 2-gerbe  $\mathbb{L}_E$  and the lifting bundle gerbe for the problem of lifting the structure group of  $LE$  from  $L\pi_0\Gamma$  to  $L\Gamma$ .*

(b) Suppose  $\Omega$  is another Lie 2-group, and  $\Lambda : \Gamma \rightarrow \Omega$  is a Lie 2-group homomorphism. Let  $L\Lambda : L\Gamma \rightarrow L\Omega$  be the Lie group homomorphism of Proposition 6.1.3, let  $E' := (\pi_0\Lambda)_*(E)$ , and let  $\mathcal{L}_{L\Lambda} : \mathcal{L}_{LE, L\Gamma} \rightarrow \mathcal{L}_{LE', L\Omega}$  be the induced isomorphism between lifting bundle gerbes. Then, there exists an isomorphism

$$\mathbb{C} : (\pi_1\Lambda)_*(\mathbb{L}_E) \rightarrow \mathbb{L}_{E'}$$

of  $\mathcal{B}\pi_1\Omega$ -bundle 2-gerbes over  $M$  and a 2-isomorphism

$$\begin{array}{ccc} (\pi_1\Lambda)_*(\mathcal{T}_{\mathbb{L}_E}) & \xrightarrow{(\pi_1\Lambda)_*(\phi_{E,\Gamma})} & (\pi_1\Lambda)_*(\mathcal{L}_{LE, L\Gamma}) \\ \mathcal{T}_{\mathbb{C}} \downarrow & \nearrow & \downarrow \mathcal{L}_{L\Lambda} \\ \mathcal{T}_{\mathbb{L}_{E'}} & \xrightarrow{\phi_{E',\Omega}} & \mathcal{L}_{LE', L\Omega} \end{array}$$

of  $\mathcal{B}\pi_1\Omega$ -bundle gerbes over  $LM$ . In other words, the canonical isomorphism of (a) is compatible with Lie 2-group homomorphisms.

Proposition 6.2.1 is proved by the following lemmata, in which we carefully deal with the dependence of  $L\Gamma$  and  $\mathcal{T}_{\mathbb{L}_E}$  on choices of connections. For this discussion, we assume a connected compact Lie group  $G$ , an abelian Lie group  $A$ , and some multiplicative  $\mathcal{B}A$ -bundle gerbe  $(\mathcal{G}, \mathcal{M}, \alpha)$  over  $G$ . Let  $\mathbb{L}_E$  be the corresponding lifting bundle 2-gerbe. We equip it with a connection following [Wal10, Section 3.2]. Let  $(\lambda, \eta)$  be a multiplicative connection on  $\mathcal{G}$  with associated 2-form  $\rho$ . Then, using the exact sequence of [Mur96, Section 8] there exist a 2-form  $\omega \in \Omega^2(E^{[2]})$  with  $\Delta\omega = -\delta_2^*\rho$  and a 3-form  $C \in \Omega^3(E)$  with  $\Delta C = \delta_1^*\text{curv}(\mathcal{G}^\lambda) + d\omega$ .

We equip the bundle gerbe  $\delta_1^*\mathcal{G}$  over  $E^{[2]}$  with the connection  $\delta_1^*\lambda + \omega$ . For simplicity, we shall denote the resulting bundle gerbe with connection by  $\mathcal{H}_{\lambda,\omega} := \delta_1^*\mathcal{G}^\lambda \otimes \mathcal{I}_\omega$ . It follows that the 1-isomorphism  $\mathcal{N}_\eta$  defined by

$$\begin{array}{ccc} \text{pr}_{23}^*\mathcal{H}_{\lambda,\omega} \otimes \text{pr}_{12}^*\mathcal{H}_{\lambda,\omega} & \xlongequal{\quad} & \text{pr}_{23}^*(\delta_1^*\mathcal{G}^\lambda) \otimes \text{pr}_{12}^*(\delta_1^*\mathcal{G}^\lambda) \otimes \mathcal{I}_{\text{pr}_{12}^*\omega + \text{pr}_{23}^*\omega} \\ & & \downarrow \delta_2^*\mathcal{M}_\eta \otimes \text{id} \\ & & \text{pr}_{13}^*(\delta_1^*\mathcal{G}^\lambda \otimes \mathcal{I}_\rho) \otimes \mathcal{I}_{\text{pr}_{12}^*\omega + \text{pr}_{23}^*\omega} \xlongequal{\quad} \text{pr}_{13}^*\mathcal{H}_{\lambda,\omega} \end{array}$$

is connection-preserving, and that the 2-isomorphism  $\delta_3^*\alpha \otimes \text{id}$  is also connection-preserving. This means that  $\chi := (C, \delta_1^*\lambda + \omega, \delta_2^*\eta)$  is a connection on  $\mathbb{L}_E$ , together denoted  $\mathbb{L}_E^\chi$ . We say that the pair  $(C, \omega)$  is an *extension* of the multiplicative connection  $(\lambda, \eta)$  to a connection  $\chi$  on  $\mathbb{L}_E$ . Given the connection  $\chi$ , we define the following  $\mathcal{B}A$ -bundle gerbe  $\mathcal{T}_{\mathbb{L}_E^\chi}$  over  $LM$ :

- (a) its surjective submersion is  $LE \rightarrow LM$ . Note that  $(LE)^{[k]} = L(E^{[k]})$ .
- (b) its principal  $\mathcal{B}A$ -bundle over  $LE^{[2]}$  is  $\mathcal{T}_{\mathcal{H}_{\lambda,\omega}}$ .
- (c) its bundle gerbe product is  $\mathcal{T}_{\mathcal{N}_\eta}$ . Its associativity is guaranteed by the 2-isomorphism  $\delta_3^*\alpha$ .

We remark that we have not used the 3-form  $C$ ; it is so far only included for completeness. We have the following ‘‘lifting commutes with transgression’’ result:

**Lemma 6.2.2.** *There is a canonical isomorphism  $\phi_\omega : \mathcal{T}_{\mathbb{L}_E^\chi} \rightarrow \mathcal{L}_{LE, LG^{\lambda,\eta}}$  of  $A$ -bundle gerbes over  $LM$ .*

Proof. Both bundle gerbes have the same surjective submersion,  $LE \rightarrow LM$ . The claimed isomorphism comes from an isomorphism

$$\phi_\omega : \mathcal{T}_{\mathcal{H}_{\lambda,\omega}} \rightarrow \delta_1^*LG^{\lambda,\eta}$$



of principal  $A$ -bundles over  $LE^{[2]}$ , where  $\delta_1 : LE^{[2]} \rightarrow LG$  is the difference map analogous to the one used in Definition 5.3.3 and  $\delta_1^* LG^{\lambda, \eta}$  is the principal bundle of  $\mathcal{L}_{LE}$ . Indeed, since  $\mathcal{H}_{\lambda, \omega} = \delta_1^* \mathcal{G}^\lambda \otimes \mathcal{I}_\omega$  by definition, the isomorphism  $\phi_\omega$  is given by the canonical trivialization  $t_\omega$  of  $\mathcal{T}_{\mathcal{I}_\omega}$ . It remains to ensure that  $\phi_\omega$  is compatible with the bundle gerbes products:  $\mathcal{T}_{\mathcal{N}^\eta}$  on  $\mathcal{T}_{\mathbb{L}_E^\chi}$  and  $\delta_2^* \mathcal{T}_{\mathcal{M}^\eta}$  on  $\mathcal{L}_{LE, LG^{\lambda, \eta}}$ . This follows immediately from the definition of  $\mathcal{N}^\eta$ .  $\square$

The following lemma investigates the dependence of the isomorphism of Lemma 6.2.2 under a change of connections. We suppose that  $(\lambda, \eta)$  and  $(\lambda', \eta')$  are multiplicative connections on  $\mathcal{G}$ , and that  $(C, \omega)$  and  $(C', \omega')$  are extensions to connections  $\chi$  and  $\chi'$  on  $\mathbb{L}_E$ , respectively. We assume that

$$\varphi : LG^{\lambda, \eta} \rightarrow LG^{\lambda', \eta'}$$

is one of the equivalences of central extensions of Proposition 6.1.3, and remark that it induces a 1-isomorphism

$$\mathcal{L}_\varphi : \mathcal{L}_{LE, LG^{\lambda, \eta}} \rightarrow \mathcal{L}_{LE, LG^{\lambda', \eta'}}$$

of lifting bundle gerbes over  $LM$ .

**Lemma 6.2.3.** *There exists a 3-form  $F \in \Omega^3(M, \mathfrak{a})$ , a connection  $\kappa$  on the identity isomorphism  $\text{id} : \mathbb{L}_E \rightarrow \mathbb{L}_E$  such that*

$$\text{id}^\kappa : \mathbb{L}_E^\chi \rightarrow \mathbb{L}_E^{\chi'} \otimes \mathbb{I}_F$$

*is a connection-preserving isomorphism between  $A$ -bundle 2-gerbes over  $M$ , and a 2-isomorphism*

$$\begin{array}{ccc} \mathcal{T}_{\mathbb{L}_E^\chi} & \xrightarrow{\phi_\omega} & \mathcal{L}_{LE, LG^{\lambda, \eta}} \\ \mathcal{T}_{\text{id}^\kappa} \downarrow & \swarrow & \downarrow \mathcal{L}_\varphi \\ \mathcal{T}_{\mathbb{L}_E^{\chi'}} & \xrightarrow{\phi_{\omega'}} & \mathcal{L}_{LE, LG^{\lambda', \eta'}} \end{array}$$

*between bundle gerbe isomorphisms over  $LM$ .*

*Proof.* We let  $\text{id}_\epsilon : \mathcal{G}^\lambda \rightarrow \mathcal{G}^{\lambda'} \otimes \mathcal{I}_\beta$  be a connection-preserving 1-isomorphism as in Lemma 6.1.2, so that  $\varphi = t_\beta \circ \mathcal{T}_{\text{id}_\epsilon}$ . We recall that  $\rho' + \Delta\beta = \rho$ , and calculate that  $\Delta(\omega' - \omega + \delta_2^* \beta) = 0$ . Thus, there exists  $\kappa \in \Omega^2(E, \mathfrak{a})$  such that

$$\Delta\kappa = \omega' - \omega + \delta_2^* \beta. \quad (6.2.2)$$

We use  $\kappa$  to construct the 1-isomorphism  $\text{id}^\kappa$ . It consists of the trivial bundle gerbe  $\mathcal{I}_\kappa$  over  $E$ . We define  $F$  such that

$$d\kappa = \pi^* F + C' - C,$$

which is the required compatibility condition for the 3-curvings. The 1-isomorphism  $\text{id}^\kappa$  consists further of the connection-preserving 1-isomorphism

$$\text{pr}_1^* \mathcal{I}_\kappa \otimes \mathcal{H}_{\lambda', \omega'} = \mathcal{I}_{\text{pr}_1^* \kappa + \omega'} \otimes \delta_1^* \mathcal{G}^{\lambda'} \xrightarrow{\text{id} \otimes \delta_1^* \text{id}_\epsilon} \mathcal{I}_{\text{pr}_1^* \kappa + \omega'} \otimes \delta_1^* \mathcal{G}^{\lambda'} \otimes \mathcal{I}_{\delta_1^* \beta} = \mathcal{H}_{\lambda, \omega} \otimes \text{pr}_2^* \mathcal{I}_\kappa.$$

The latter satisfies the higher coherence conditions because of the commutative diagram in Lemma 6.1.2. The 2-isomorphism is given by the trivialization  $t_\kappa$  of  $\mathcal{T}_{\mathcal{I}_\kappa}$ ; (6.2.2) provides the necessary compatibility relation.  $\square$

Now we return to a smoothly separable Lie 2-group  $\Gamma$  with  $\pi_0 \Gamma$  compact and connected, and look at the situation where  $G := \pi_0 \Gamma$ ,  $A := \pi_1 \Gamma$ , and  $\mathcal{G} := \mathcal{G}_\Gamma$ . Then, Lemma 6.2.2 yields the isomorphism  $\phi_{E, \Gamma}$  claimed in Proposition

6.2.1 (a), and Lemma 6.2.3 assures that it is well-defined under different choices of connections. Next we consider a Lie 2-group homomorphism  $\Lambda : \Gamma \rightarrow \Omega$ , and write  $\tilde{\mathcal{G}} := (\pi_1 \Lambda)_*(\mathcal{G}_\Gamma)$ . Let

$$\mathcal{D} : \tilde{\mathcal{G}} \rightarrow (\pi_0 \Lambda)^* \mathcal{G}_\Omega \quad (6.2.3)$$

be the 1-isomorphism (6.1.6) defined by  $\Lambda$ . Let  $(\lambda', \eta')$  be a connection on  $\mathcal{G}_\Omega$ . By Lemma 6.1.6 there exists a multiplicative connection  $(\tilde{\lambda}, \tilde{\eta})$  on  $\tilde{\mathcal{G}}$  and a connection  $\epsilon$  on  $\mathcal{D}$  such that  $\mathcal{D}^\epsilon$  is connection-preserving. Let  $E$  be a principal  $\pi_0 \Gamma$ -bundle over  $M$ , and let  $E' := (\pi_0 \Lambda)_*(E)$ . There is a canonical map  $f : E \rightarrow E'$  which is equivariant along  $\pi_0 \Lambda$ . This can be rephrased as the commutativity of the diagram

$$\begin{array}{ccc} E^{[n+1]} & \xrightarrow{\delta_n} & \pi_0 \Gamma^n \\ \downarrow f^{n+1} & & \downarrow \pi_0 \Lambda \\ E'^{[n+1]} & \xrightarrow{\delta_n} & \pi_0 \Omega^n \end{array} \quad (6.2.4)$$

for all  $n$ . We get an induced isomorphism

$$\mathcal{L}_{\mathcal{D}^\epsilon} : \mathcal{L}_{LE, L\tilde{\mathcal{G}}^{\tilde{\lambda}, \tilde{\eta}}} \rightarrow \mathcal{L}_{LE', L\mathcal{G}_\Omega^{\lambda', \eta'}}$$

Let  $(C', \omega')$  be an extension of  $(\lambda', \eta')$  to a connection  $\chi'$  on  $\mathbb{L}_{E'}$ . Then,  $\tilde{C} := (\pi_0 \Lambda)^* C'$  and  $\tilde{\omega} := (\pi_0 \Lambda)^* \omega'$  is an extension of  $(\tilde{\lambda}, \tilde{\eta})$  to a connection  $\tilde{\chi}$  on  $\tilde{\mathbb{L}}_E$ , the lifting bundle 2-gerbe formed by the  $\pi_0 \Gamma$ -bundle  $E$  and the  $\mathcal{B}_{\pi_1} \Omega$ -bundle gerbe  $\tilde{\mathcal{G}}$  over  $\pi_0 \Gamma$ . The commutativity of (6.2.4) implies that the 1-isomorphism  $\mathcal{D}^\epsilon$  induces a connection-preserving 1-isomorphism

$$\mathbb{L}_{\mathcal{D}^\epsilon} : \tilde{\mathbb{L}}_E^{\tilde{\chi}} \rightarrow \mathbb{L}_{E'}^{\chi'}$$

It is straightforward to check that the diagram

$$\begin{array}{ccc} \mathcal{T}_{\tilde{\mathbb{L}}_E^{\tilde{\chi}}} & \xrightarrow{\phi_{\tilde{\omega}}} & \mathcal{L}_{LE, L\tilde{\mathcal{G}}^{\tilde{\lambda}, \tilde{\eta}}} \\ \downarrow \mathcal{T}_{\mathcal{D}^\epsilon} & \swarrow & \downarrow \mathcal{L}_{\mathcal{D}^\epsilon} \\ \mathcal{T}_{\mathbb{L}_{E'}^{\chi'}} & \xrightarrow{\phi_{\omega'}} & \mathcal{L}_{LE', L\mathcal{G}_\Omega^{\lambda', \eta'}} \end{array} \quad (6.2.5)$$

of 1-isomorphisms between  $\mathcal{B}_{\pi_1} \Omega$ -bundle gerbes over  $LM$  is strictly commutative.

**Lemma 6.2.4.** *Let  $\Lambda : \Gamma \rightarrow \Omega$  be a Lie 2-group homomorphism, let  $(\lambda, \eta)$  and  $(\lambda', \eta')$  be connections on  $\mathcal{G}_\Gamma$  and  $\mathcal{G}_\Omega$ , and let  $\chi$  and  $\chi'$  be extensions to connections on  $\mathbb{L}_E$  and  $\mathbb{L}_{E'}$ , respectively. Then, there exists a 3-form  $F \in \Omega^3(M, \mathfrak{a})$ , a connection-preserving isomorphism*

$$\mathbb{C}^\rho : (\pi_1 \Lambda)_*(\mathbb{L}_E^\chi) \rightarrow \mathbb{L}_{E'}^{\chi'} \otimes \mathbb{I}_F,$$

and a 2-isomorphism

$$\begin{array}{ccc} (\pi_1 \Lambda)_*(\mathcal{T}_{\mathbb{L}_E^\chi}) & \xrightarrow{(\pi_1 \Lambda)_*(\phi_\omega)} & (\pi_1 \Lambda)_*(\mathcal{L}_{LE, L\mathcal{G}_\Gamma^{\lambda, \eta}}) \\ \downarrow \mathcal{T}_{\mathbb{C}^\rho} & \swarrow & \downarrow \mathcal{L}_\varphi \\ \mathcal{T}_{\mathbb{L}_{E'}^{\chi'}} & \xrightarrow{\phi_{\omega'}} & \mathcal{L}_{LE, L\mathcal{G}_\Omega^{\lambda', \eta'}} \end{array}$$

between bundle gerbe isomorphisms over  $LM$ .

Proof. On the multiplicative bundle gerbe  $\tilde{\mathcal{G}} = (\pi_1\Lambda)_*(\mathcal{G}_\Gamma)$  we have two connections:  $(\tilde{\lambda}, \tilde{\eta})$  and  $(\pi_1\Lambda)_*(\lambda, \eta)$ . Accordingly, on  $\tilde{\mathbb{L}}_E = (\pi_1\Lambda)_*(\mathbb{L}_E)$  we have the two extensions  $(\tilde{C}, \tilde{\omega})$  and  $(\pi_1\Lambda)_*(C, \omega)$  to connections  $\tilde{\chi}$  and  $(\pi_1\Lambda)_*\chi$ . Thus, Lemma 6.2.3 applies, and provides a 2-isomorphism

$$\begin{array}{ccc} (\pi_1\Lambda)_*(\mathcal{T}_{\mathbb{L}_E^\chi}) & \xrightarrow{(\pi_1\Lambda)_*(\phi_\omega)} & (\pi_1\Lambda)_*(\mathcal{L}_{LE, L\mathcal{G}_\Gamma^{\lambda, \eta}}) \\ \mathcal{T}_{\text{id}^\kappa} \downarrow & \swarrow & \downarrow \mathcal{L}_\varphi \\ \mathcal{T}_{\tilde{\mathbb{L}}_E^{\tilde{\chi}}} & \xrightarrow{\phi_\omega} & \mathcal{L}_{LE, L\tilde{\mathcal{G}}^{\tilde{\lambda}, \tilde{\eta}}} \end{array}$$

Now the commutative diagram (6.2.5) extends this 2-isomorphism to the claimed one.  $\square$

Lemma 6.2.4 proves Proposition 6.2.1 (b).

### 6.3 Transgression of Trivializations

Now we come to the trivialization  $\mathbb{T}_\mathcal{G}$  of  $\mathbb{L}_E$  associated to the  $\Gamma$ -bundle gerbe  $\mathcal{G}$ . Since we have equipped  $\mathbb{L}_E$  with a connection  $\chi$ , it follows that there exists a compatible connection  $\rho$  on the trivialization  $\mathbb{T}_\mathcal{G}$  [Walb, Proposition 3.3.1]. If  $\mathbb{T}_\mathcal{G}$  consists of a  $\mathcal{B}\pi_1\Gamma$ -bundle gerbe  $\mathcal{S}$  over  $E$ , of a 1-isomorphism

$$\mathcal{C} : \text{pr}_1^*\mathcal{S} \otimes \delta_1^*\mathcal{G}_\Gamma \longrightarrow \text{pr}_2^*\mathcal{S}$$

over  $E^{[2]}$ , and of a 2-isomorphism  $\zeta$  over  $E^{[3]}$ , the connection  $\rho$  is a pair  $\rho = (\gamma, \nu)$  of a connection  $\gamma$  on  $\mathcal{S}$  and of a connection  $\nu$  on  $\mathcal{C}$ , such that  $\mathcal{C}$  and  $\zeta$  are connection-preserving. As described in [Walb, Section 4.2], the trivialization  $\mathbb{T}_\mathcal{G}^\rho$  with connection can be transgressed to a trivialization  $\mathcal{T}_{\mathbb{T}_\mathcal{G}^\rho}$  of  $\mathcal{T}_{\mathbb{L}_E^\chi}$ . It consists of:

- (i) The principal  $\mathcal{B}\pi_1\Gamma$ -bundle  $\mathcal{T}_{\mathcal{S}^\gamma}$  over  $LE$ ,
- (ii) the bundle isomorphism

$$\mathcal{T}_{\mathcal{C}^\nu} : \text{pr}_1^*\mathcal{T}_{\mathcal{S}^\gamma} \otimes \mathcal{T}_{\mathcal{H}_{\lambda, \omega}} \longrightarrow \text{pr}_2^*\mathcal{T}_{\mathcal{S}^\gamma}$$

over  $LE^{[2]}$ , which is compatible with the bundle gerbe product  $\mathcal{T}_{\mathcal{N}^\eta}$  due to the existence of the 2-isomorphism  $\zeta$ .

Under the canonical identification of Proposition 6.2.1 and the equivalence (6.2.1),  $\mathcal{T}_{\mathbb{T}_\mathcal{G}^\rho}$  determines a principal  $BL\mathcal{G}_\Gamma^{\lambda, \eta}$ -bundle over  $LM$ , which we denote by  $\mathcal{T}_\mathcal{G}^{\chi, \rho}$ .

**Lemma 6.3.1.** *The  $BL\mathcal{G}_\Gamma^{\lambda, \eta}$ -bundle  $\mathcal{T}_\mathcal{G}^{\chi, \rho}$  over  $LM$  is independent of the choice of the connection  $\rho$  up to bundle isomorphisms.*

Proof. We may regard the trivialization as an isomorphism  $\mathbb{T}_\mathcal{G}^\rho : \mathbb{L}_E^\chi \longrightarrow \mathbb{I}_H$ , for some  $H \in \Omega^3(M)$ . Another connection  $\rho'$  corresponds to another isomorphism  $\mathbb{T}_\mathcal{G}^{\rho'} : \mathbb{L}_E^\chi \longrightarrow \mathbb{I}_{H'}$ . The difference between  $\mathbb{T}_\mathcal{G}^\rho$  and  $\mathbb{T}_\mathcal{G}^{\rho'}$  can be compensated in a connection on the identity isomorphism  $\text{id} : \mathbb{I} \longrightarrow \mathbb{I}$ . Up to 2-isomorphisms, such a connection is given by a 2-form  $\xi$  with  $d\xi = H' - H$ . Then, there exists a 2-isomorphism

$$\begin{array}{ccc} & \mathbb{L}_E^\chi & \\ \mathbb{T}_\mathcal{G}^\rho \swarrow & & \searrow \mathbb{T}_\mathcal{G}^{\rho'} \\ \mathbb{I}_H & \xrightarrow{\text{id}^\xi} & \mathbb{I}_{H'} \end{array}$$

Since  $\text{id}^\xi$  transgresses to the identity between trivial gerbes over  $LM$ , the 2-isomorphism transgresses to an isomorphism between trivializations of  $\mathcal{T}_{\mathbb{L}_E^X}$ .  $\square$

**Lemma 6.3.2.** *Let  $\Lambda : \Gamma \rightarrow \Omega$  be a Lie 2-group homomorphism, let  $(\lambda, \eta)$  and  $(\lambda', \eta')$  be connections on  $\mathcal{G}_\Gamma$  and  $\mathcal{G}_\Omega$ , and let  $\chi$  and  $\chi'$  be extensions to connections on  $\mathbb{L}_E$  and  $\mathbb{L}_{E'}$ , respectively. Let  $\rho$  and  $\rho'$  be connections on  $\mathbb{T}_\mathcal{G}$  compatible with  $\chi$  and  $\chi'$ , respectively. Then, there exists an isomorphism*

$$(\Lambda)_*(\mathcal{T}_\mathcal{G}^{\chi, \rho}) \cong \mathcal{T}_{\Lambda_*(\mathcal{G})}^{\chi', \rho'}.$$

Proof. Let

$$\mathbb{C}^\rho : (\pi_1 \Lambda)_*(\mathbb{L}_E^X) \rightarrow \mathbb{L}_{E'}^{X'} \otimes \mathbb{I}_F,$$

be the isomorphism of Lemma 6.2.4. With the arguments of Lemma 6.3.1 there exists a 2-isomorphism

$$\begin{array}{ccc} (\pi_1 \Lambda)_*(\mathbb{L}_E^X) & \xrightarrow{(\pi_1 \Lambda)_*(\mathbb{T}_\mathcal{G}^\rho)} & (\pi_1 \Lambda)_*(\mathbb{I}_H) \\ \mathbb{C}^\rho \downarrow & \swarrow & \downarrow \text{id}^\xi \\ \mathbb{L}_{E'}^{X'} \otimes \mathbb{I}_F & \xrightarrow{\mathbb{T}_{\Lambda_* \mathcal{G}}^{\rho'} \otimes \text{id}} & \mathbb{I}_{H'+F} \end{array}$$

Its transgression, together with Lemma 6.3.1, yields the claim.  $\square$

Notice that Lemma 6.3.2 proves for  $\Lambda = \text{id}_\Gamma$  that  $\mathcal{T}_\mathcal{G}^{\chi, \rho}$  is independent of the choices of all connections, up to bundle isomorphisms. We may hence denote it simply by  $\mathcal{T}_\mathcal{G}$ .

**Definition 6.3.3.** *Let  $\Gamma$  be a smoothly separable Lie 2-group with  $\pi_0 \Gamma$  compact and connected, and let  $\mathcal{G}$  be a  $\Gamma$ -bundle gerbe over  $M$ . Then, the principal  $L\Gamma$ -bundle  $\mathcal{T}_\mathcal{G}$  over  $LM$  is called the transgression of  $\mathcal{G}$ .*

Summarizing the results collected above, we have:

**Theorem 6.3.4.** *Let  $\Gamma$  be a smoothly separable Lie 2-group with  $\pi_0 \Gamma$  compact and connected. Then, the assignment  $\mathcal{G} \mapsto \mathcal{T}_\mathcal{G}$  defines a map*

$$\mathcal{T} : \check{H}^1(M, \Gamma) \rightarrow \check{H}^0(LM, BL\Gamma)$$

with the following properties:

- (i) it is contravariant in  $M$  and covariant in  $\Gamma$ ,
- (ii) for  $\Gamma = \mathcal{B}A$ , it reduces to the ordinary transgression homomorphism

$$\tau : \check{H}^1(M, \mathcal{B}A) \rightarrow \check{H}^0(LM, \mathcal{B}A),$$

- (iii) for  $\Gamma = G_{dis}$ , it reduces to the looping of bundles

$$L : \check{H}^1(M, G_{dis}) \rightarrow \check{H}^0(LM, BLG).$$

Proof. The well-definedness of the map  $\mathcal{T}$  as well as the covariance in (i) follow from Lemma 6.3.2. The contravariance in  $M$  is evident. In (ii) we have, in the notation used above,  $E = M$  and correspondingly  $LE = LM$ . This means that both the lifting bundle 2-gerbe  $\mathbb{L}_E$  and the lifting gerbe  $\mathcal{L}_{LE, L\Gamma}$  are canonically trivial. Under these canonical identifications, we can identify  $\mathcal{G} = \mathcal{S}$ , where  $\mathcal{S}$  is the bundle gerbe in  $\mathbb{T}_\mathcal{G}$ . Also, we can identify  $\mathcal{T}_\mathcal{S}$  with  $\mathcal{T}_\mathcal{G}$ . But  $\mathcal{T}_\mathcal{S}$  is the ordinary, abelian transgression which underlies the homomorphism  $\tau$ . In (iii), we have

$A = *$ , so that both lifting problems are trivial. In particular,  $LE$  is the lift of the structure group of  $LE$  from  $LG$  to  $LG$ , i.e.  $LE = \mathcal{T}_G$ .  $\square$

For completeness, we include:

**Corollary 6.3.5.** *The following diagram is commutative:*

$$\begin{array}{ccccccccc}
\check{H}^0(M, \pi_0 \Gamma_{dis}) & \longrightarrow & \check{H}^1(M, B\pi_1 \Gamma) & \longrightarrow & \check{H}^1(M, \Gamma) & \longrightarrow & \check{H}^1(M, \pi_0 \Gamma_{dis}) & \longrightarrow & \check{H}^2(M, B\pi_1 \Gamma) \\
\downarrow L & & \downarrow \tau & & \downarrow \mathcal{T} & & \downarrow L & & \downarrow \tau \\
\check{H}^0(LM, L\pi_0 \Gamma_{dis}) & \longrightarrow & \check{H}^0(LM, B\pi_1 \Gamma) & \longrightarrow & \check{H}^0(LM, B\Gamma) & \longrightarrow & \check{H}^0(LM, B\pi_0 \Gamma) & \longrightarrow & \check{H}^1(LM, B\pi_1 \Gamma)
\end{array}$$

Proof. The two diagrams in the middle are commutative because of Theorem 6.3.4. The commutativity of the outer diagrams is a statement in ordinary Čech cohomology and straightforward to verify.  $\square$

## 7 Application to String Structures

We recall that  $\text{String}(n)$  is a topological group defined up to homotopy equivalence by requiring that it is a 3-connected cover of  $\text{Spin}(n)$ . It is known that  $\text{String}(n)$  cannot be realized as a finite-dimensional Lie group, but as an infinite-dimensional Fréchet Lie group [NSW]. For several reasons, however, it is more attractive to work with *Lie 2-group* models for  $\text{String}(n)$ .

**Definition 7.1** ([NSW, Definition 4.10]). *A 2-group model for  $\text{String}(n)$  is a smoothly separable Lie 2-group  $\Gamma$  such that*

$$\pi_0 \Gamma = \text{Spin}(n) \quad \text{and} \quad \pi_1 \Gamma = \text{U}(1),$$

*and such that the geometric realization  $|\Gamma|$  has the homotopy type of  $\text{String}(n)$ .*

The first 2-group model for  $\text{String}(n)$  has been constructed in [BCSS07] using central extensions of loop groups. Another model has been provided in [NSW] based on the above-mentioned Fréchet Lie group realization of  $\text{String}(n)$ . A further construction appears in [Wala]. We remark that the constructions of [Hen08, SP] are not 2-group models in the sense of Definition 7.1, since they are not *strict* 2-groups.

The major motivation to look at the group  $\text{String}(n)$  comes from string theory; in particular, from fermionic sigma models. 2-group models for  $\text{String}(n)$  are so attractive because they lead directly to (non-abelian) gerbes, which are in turn intimately related to string theory. Non-abelian gerbes for 2-group models for  $\text{String}(n)$  have been considered in [Ste06, Jur], and can be treated with the theory developed in [NW] and the present article.

In the following we describe an application of the lifting theory developed in Section 5.3 to string structures. Let  $M$  be a spin manifold of dimension  $n$ , i.e. the structure group of the frame bundle  $FM$  of  $M$  is lifted to  $\text{Spin}(n)$ . Topologically, a *string structure on  $M$*  is a further lift of the structure group of  $FM$  to  $\text{String}(n)$ . Homotopy theory shows that the obstruction against this further lift is a certain class  $\frac{1}{2}p_1(M) \in H^4(M, \mathbb{Z})$ , and that equivalence classes of string structures form a torsor over  $H^3(M, \mathbb{Z})$ .

The topological definition of a string structure has an evident 2-group-counterpart:

**Definition 7.2** ([Ste06, Jur]). *Suppose  $\Gamma$  is a 2-group model for  $\text{String}(n)$ . Then, a string structure on  $M$  is a  $\Gamma$ -lift of  $FM$  in the sense of Definition 5.3.5, i.e. a  $\Gamma$ -bundle gerbe  $\mathcal{S}$  over  $M$  together with an isomorphism  $\varphi : \pi_*(\mathcal{S}) \rightarrow FM$  of  $\text{Spin}(n)$ -bundles over  $M$ .*

String structures in the sense of Definition 7.2 form a bicategory  $\mathcal{L}ift_{\Gamma}(FM)$ . Via the equivalence between non-abelian gerbes and classifying maps [NW, Section 4] one can easily show that Definition 7.2 is a refinement of the topological notion of a string structure.

Another way to define string structures is to look at the obstruction class  $\frac{1}{2}p_1(M)$ . It can be represented by a Chern-Simons 2-gerbe  $\mathbb{C}\mathbb{S}_E(\mathcal{G})$  [Walb, Theorem 1.1.3]. We recall (also see Remark 5.3.4) that a Chern-Simons 2-gerbe receives as input data a principal  $G$ -bundle  $E$  over  $M$  and a multiplicative  $\mathcal{B}U(1)$ -bundle gerbe  $\mathcal{G}$  over  $G$ . Here, with  $G = \text{Spin}(n)$ , the  $G$ -bundle  $E$  is the frame bundle  $FM$ , and  $\mathcal{G}$  can be any multiplicative  $\mathcal{B}U(1)$ -bundle gerbe over  $\text{Spin}(n)$  with level one, i.e. characteristic class  $[\mathcal{G}] = 1 \in \mathbb{Z} = H^3(\text{Spin}(n), \mathbb{Z})$ . Now, the idea of the following definition is that a string structure is a trivialization of the obstruction against string structures:

**Definition 7.3** ([Walb, Definition 1.1.5]). *Let  $\mathcal{G}$  be a multiplicative  $\mathcal{B}U(1)$ -bundle gerbe over  $\text{Spin}(n)$  with level one. Then, a string structure on  $M$  is a trivialization of the Chern-Simons 2-gerbe  $\mathbb{C}\mathbb{S}_{FM}(\mathcal{G})$ .*

String structures in the sense of Definition 7.3 form a bicategory  $\mathcal{T}riv(\mathbb{C}\mathbb{S}_{FM}(\mathcal{G}))$ . We remark that a priori no string group or 2-group model for  $\text{String}(n)$  is involved in Definition 7.3. However, it depends on the input of the multiplicative bundle gerbe  $\mathcal{G}$ .

As explained in [Walb, Wala] there exists a canonical way to produce such a multiplicative  $\mathcal{B}U(1)$ -bundle gerbe over  $\text{Spin}(n)$  with level one: one starts with the basic gerbe  $\mathcal{G}_{bas}$  over  $\mathcal{B}U(1)$ , which enjoys a finite-dimensional, Lie-theoretical construction [Mei02, GR02]. The multiplicative structure can be obtained by a transgression-regression procedure [Wala].

Another method to obtain the multiplicative bundle gerbe  $\mathcal{G}$  is to start with a 2-group model for  $\text{String}(n)$ . We infer from [NSW, Remark 4.11]:

**Lemma 7.4.** *If  $\Gamma$  is a 2-group model for  $\text{String}(n)$ , then the multiplicative  $\mathcal{B}U(1)$ -bundle gerbe  $\mathcal{G}_{\Gamma}$  introduced in Sections 4.2 and 5.3 has level one.*

As noted in Remark 5.3.4, we have the coincidence

$$\mathbb{C}\mathbb{S}_{FM}(\mathcal{G}_{\Gamma}) = \mathbb{L}_{FM},$$

i.e. the Chern-Simons 2-gerbe is the lifting bundle 2-gerbe for the problem of lifting the structure group of  $FM$  from  $\text{Spin}(n)$  to  $\Gamma$ . Now, Theorem 5.3.7 becomes:

**Theorem 7.5.** *The two notions of string structures from Definition 7.2 and Definition 7.3 coincide. More precisely, for any 2-group model  $\Gamma$  for  $\text{String}(n)$  there is an equivalence of bicategories:*

$$\mathcal{T}riv(\mathbb{C}\mathbb{S}_{FM}(\mathcal{G}_{\Gamma})) \cong \mathcal{L}ift_{\Gamma}(FM).$$

A third (inequivalent) definition is to say that a string structure on  $M$  is the same as a *spin structure* on the free loop space  $LM$  [McL92]. Here we require  $n > 1$  in order to make  $\text{Spin}(n)$  connected.

**Definition 7.6.** *A spin structure on  $LM$  is a lift of the structure group of the looped frame bundle  $LFM$  from  $L\text{Spin}(n)$  to the universal central extension*

$$1 \longrightarrow U(1) \longrightarrow \widehat{L\text{Spin}(n)} \longrightarrow L\text{Spin}(n) \longrightarrow 1.$$

The relation between string structures on  $M$  and spin structures on  $LM$  is based on the following fact concerning the loop group of a 2-group model for the string group, see Definition 6.1.4:

**Lemma 7.7.** *Let  $\Gamma$  be a 2-group model for  $\text{String}(n)$ . Then,  $L\Gamma \cong \widehat{L\text{Spin}(n)}$ .*

Proof. By Lemma 7.4 the multiplicative bundle gerbe  $\mathcal{G}_\Gamma$  has level one. By [Wal10, Corollary 3.1.9], its transgression  $\mathcal{T}_{\mathcal{G}_\Gamma} =: L\Gamma$  is the universal central extension of  $L\text{Spin}(n)$ .  $\square$

Let us first recall how string structures in the sense of Definition 7.3 induce a spin structure on  $LM$ . For this purpose, we simply reduce the procedure described in Section 6 to the case where  $\Gamma$  is a 2-group model for  $\text{String}(n)$ , reproducing a description given in [Wal09]. We choose a multiplicative connection on  $\mathcal{G}_\Gamma$ , and a connection on the frame bundle  $FM$ , for instance the Levi-Cevita connection. By Proposition 6.2.1, the transgression  $\mathcal{T}_{\mathbb{C}\mathcal{S}_{FM}(\mathcal{G})}$  is the lifting bundle gerbe for the problem of lifting the structure group of  $LFM$  from  $L\text{Spin}(n)$  to  $L\Gamma$ .

If now  $\mathbb{T}$  is a trivialization of  $\mathbb{C}\mathcal{S}_{FM}(\mathcal{G}_\Gamma)$ , it admits a connection compatible with the connection on  $\mathbb{C}\mathcal{S}_{FM}(\mathcal{G}_\Gamma)$ , and transgresses to a trivialization  $\mathcal{T}_{\mathbb{T}}$  of  $\mathcal{T}_{\mathbb{C}\mathcal{S}_{FM}(\mathcal{G}_\Gamma)}$ , which is precisely a spin structure on  $LM$ . Lemma 6.3.2 shows that we get a well-defined map

$$\left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{trivializations of } \mathbb{C}\mathcal{S}_{FM}(\mathcal{G}_\Gamma) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{spin structures on } LM \end{array} \right\}. \quad (7.1)$$

**Remark 7.8.** It is not possible to upgrade this map to a functor between categories, because connections cannot be chosen in a functorial way. However, one can include the connections into the structure on both hand sides, and so obtain a functor between a category of *geometric string structures* on  $M$  and a category of *geometric spin structures* on  $LM$ .

Now suppose we have a string structure in the sense of Definition 7.2, i.e. a  $\Gamma$ -bundle gerbe  $\mathcal{S}$  together with a bundle morphism  $\varphi : \pi_*(\mathcal{S}) \rightarrow FM$ . In non-abelian cohomology, this is a class  $[\mathcal{S}] \in \check{H}^1(M, \Gamma)$  such that  $\pi_*([\mathcal{G}]) = [FM] \in \check{H}^0(M, \mathcal{B}\text{Spin}(n))$ . Applying the transgression map

$$\mathcal{T} : \check{H}^1(M, \Gamma) \longrightarrow \check{H}^0(LM, \widehat{\mathcal{B}L\text{Spin}(n)})$$

of Theorem 6.3.4 produces a class  $\mathcal{T}([\mathcal{S}]) \in \check{H}^0(LM, \widehat{\mathcal{B}L\text{Spin}(n)})$ . Theorem 6.3.4 (ii) and (iv) imply that the extension of this class is  $[LFM] \in \check{H}^0(LM, \mathcal{B}\text{Spin}(n))$ , i.e.  $\mathcal{T}([\mathcal{S}])$  is an isomorphism class of spin structures on  $LM$ . Summarizing, we have a map

$$\left\{ \begin{array}{l} \text{Isomorphism classes} \\ \text{of } \Gamma\text{-lifts of } FM \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{spin structures on } LM \end{array} \right\}. \quad (7.2)$$

The two maps (7.1) and (7.2) are compatible with the equivalence of Theorem 7.5 in the following sense:

**Theorem 7.9.** *There is a commutative diagram:*

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{string structures on } M \\ \text{in the sense of Definition 7.3} \end{array} \right\} & \xrightleftharpoons{\text{Theorem (7.5)}} & \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{string structures on } M \\ \text{in the sense of Definition 7.2} \end{array} \right\} \\ & \searrow (7.1) & \swarrow (7.2) \\ & \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{spin structures on } LM \end{array} \right\} & \end{array}$$

Proof. The diagram is commutative because the transgression map in non-abelian cohomology was defined exactly in this way.  $\square$

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