

## Chapter 1

### Bicategories in field theories - an invitation

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We explain some applications of bicategories in both classical and quantum field theory. This includes a modern perspective on some pioneering work of Max Kreuzer and Bert Schellekens on rational conformal field theory.

#### 1. Introduction

The notion of a category undeservedly has the reputation of being abstract. In fact, it just summarizes the combinatorics of arrows. Ignoring set theoretic subtleties, a category consists of two entities: first, objects, which can be represented by points. Second, morphisms are then represented by arrows starting or ending at these points. The basic operation is concatenation of arrows. This composition is required to be associative and to have units. Examples of categories are abundant; the following examples are of particular importance for us:

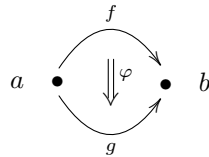
##### Example 1.

- (1) The category  $\mathbf{Vect}$  of vector spaces over a given field  $k$ : objects are vector spaces, morphisms are linear maps.
- (2) The category of, say complex, algebras: objects are complex algebras, morphisms are morphisms of algebras.
- (3) (Smooth oriented) cobordisms  $\mathbf{Cobord}_{d+1,d}$ : objects are smooth oriented  $d$ -dimensional manifolds without boundaries. A morphism  $\Sigma_1 \rightarrow$

$\Sigma_2$  is a (diffeomorphism class of an) oriented  $d+1$ -dimensional manifold  $M$ , together with a diffeomorphism  $\Sigma_1 \sqcup \overline{\Sigma_2} \rightarrow \partial M$ .

Since categories have two layers of structure, objects and morphisms, functors are defined as pairs of compatible maps, on objects and on morphisms. The notion of a functor allows to give a concise definition of a  $(d+1, d)$ -dimensional *topological field theory* that is originally due to Atiyah [Atiyah (1988)]: it is a (symmetric monoidal) functor  $\text{Cobord}_{d+1,d} \rightarrow \text{Vect}$ .

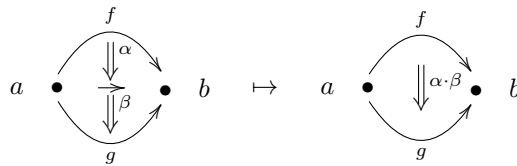
Let us now introduce the central notion of this contribution, the one of a bicategory. For categories, the underlying combinatorics is the one of dots and arrows; for bicategories, the combinatorics of *bigones* is relevant:



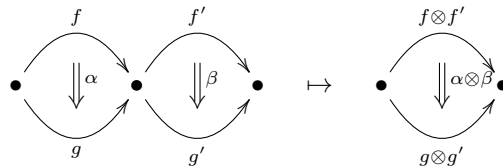
We have now three layers of structure: objects  $a, b, \dots$ , 1-morphisms  $f, g, \dots$ , and 2-morphisms  $\varphi$ . 1-morphisms can be concatenated, in the same way as morphisms in a category:

$$a \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet c \quad \mapsto \quad a \bullet \xrightarrow{f \otimes g} \bullet c$$

We denote the corresponding concatenation with  $\otimes$ ; this concatenation is not necessarily strictly associative - an issue we will not discuss in this contribution. For 2-morphisms, there are two different concatenations: vertically



and horizontally:



Since bicategories have three layers of structures, it is clear that bifunctors relating two bicategories should act on objects, 1-morphisms and 2-morphisms.

Let us give examples of bicategories that will be relevant for us:

**Example 2.**

- (1) The bicategory of complex algebras with algebras as objects, bimodules as 1-morphisms and morphisms of bimodules as 2-morphisms. The concatenation of 1-morphisms is the tensor product of bimodules over the appropriate algebra,  $B \otimes_A B'$ .
- (2) The bicategory  $2\text{-Vect}$  of 2-vector spaces: objects are  $\mathbb{C}$ -linear finitely semi-simple abelian categories. (Bimodules over algebras are in fact a full subcategory of this category.)
- (3) Using ideas from 2-form gauge theories, we assign to any smooth manifold  $M$  a bicategory  $Grbtriv^\nabla(M)$ . Its objects are 2-forms  $B \in \Omega^2(M)$ , i.e. gauge fields, its 1-morphisms  $B \xrightarrow{\lambda} B'$  are 1-forms  $\lambda$  such that  $d\lambda = B' - B$ , i.e. gauge transformations parametrized by 1-forms. The 2-morphisms are gauge transformations of gauge transformations:  $U(1)$ -valued functions  $\varphi$  on  $M$  such that  $\frac{1}{i}d \log \varphi = \lambda' - \lambda$ .
- (4) Our final example are extended cobordisms  $\text{Cobord}_{3,2,1}$ . Its objects are oriented smooth one-dimensional manifolds, i.e. disjoint unions of circles. 1-morphisms are cobordisms of these one-dimensional manifolds. For example the pair of pants, the three-punctured sphere, can be seen as a 1-morphism from two circles to a single circle. 2-morphism finally are manifolds with corners. The following figure shows an example of a 2-morphism:

## 2. Equivariant Hopf algebras from topological field theories

We start by describing an application of bicategories to topological field theory, yielding interesting algebraic and representation theoretic structures. The examples (2) and (4) of bicategories suggests the following

**Definition 1.1.** An *extended topological field theory* is a (symmetric monoidal) bifunctor

$$Z : \text{Cobord}_{3,2,1} \rightarrow 2 - \text{Vect} .$$

The qualifier “extended” can be justified by the observation that  $Z$  can be restricted to an ordinary functor from the endomorphism category of

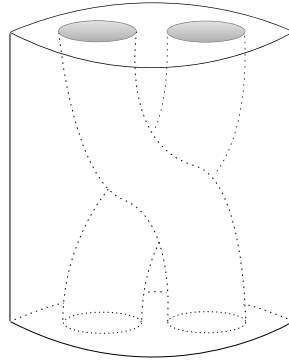


Fig. 1. A 2-morphism between two pair of pants, i.e. three-punctured spheres.

the object  $\emptyset$  to the category of vector spaces. Unpacking the definitions, one finds that this restriction is a topological field theory in the sense of Atiyah.

Dijkgraaf-Witten theory [Dijkgraaf and Witten (1990)] is an example of an extended topological field theory. The following explicit construction is based on the contributions of many authors, in particular of Freed-Quinn [Freed and Quinn (1993)] and of Morton [Morton (2010)]. Let  $G$  be a finite group; for a smooth manifold  $M$  of any dimension, denote by  $\mathcal{A}_G(M)$  the category of  $G$ -bundles on  $M$  which is actually a groupoid, i.e. a category in which all morphisms, namely all gauge transformations, are invertible. From this structure, one can build a bifunctor  $\tilde{\mathcal{A}}_G$  from  $\text{Cobord}_{3,2,1}$  to the category of spans of groupoids. Dijkgraaf-Witten theory is then the concatenation of  $\tilde{\mathcal{A}}_G$  with a 2-linearization:

$$Z_{DW} : \text{Cobord}_{3,2,1} \xrightarrow{\tilde{\mathcal{A}}_G} \text{SpanGrp} \xrightarrow{[-, \text{Vect}]} 2\text{-Vect} .$$

Let us evaluate this functor explicitly in examples:

- The 1-manifold  $S^1$  is sent to the category  $\mathcal{C} := [\mathcal{A}_G(S^1), \text{Vect}]$ . Taking into account that the category  $\mathcal{A}_G(S^1)$  is equivalent to the action groupoid  $G//G$  for the adjoint action (this is of course just “holonomy modulo gauge transformations at a point”), one easily deduces that this category is equivalent to the category of representations of the Drinfeld double,  $\mathcal{C} \cong \mathcal{D}(G)\text{-mod}$ .
- The pair of pants, i.e. the sphere with two ingoing and one outgoing puncture yields a functor  $\mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$  which gives rise to a tensor product.
- 2-morphisms, i.e. 3-manifolds with corners, finally give rise to natural

transformations. For example, the manifold with corners depicted in figure 1 gives rise to a braiding on the category  $\mathcal{C}$ .

This construction can be extended [Maier, Nikolaus and Schweigert (2011)] to an equivariant version: Let  $J$  be another finite group. A weak  $J$ -action on the stack  $\mathcal{A}_G$  amounts to a weak  $J$ -action on the group  $G$ , i.e. a group automorphism  $\varphi_j : G \rightarrow G$  for any  $j \in J$  and an element  $c_{j,k} \in G$  for any pair  $j, k \in J$  such that  $\varphi_j \circ \varphi_k = \text{ad}_{c_{j,k}} \circ \varphi_{jk}$ , i.e. such that the group law of  $J$  is realized only up to inner automorphisms of  $G$ . (The elements  $c_{j,k} \in G$  have to obey appropriate coherence conditions which we omit here.)

By Schreier's [Schreier (1926)] results, this amounts to an extension of groups

$$1 \rightarrow G \rightarrow H \rightarrow J \xrightarrow{\pi} 1$$

that is not necessarily split. (One should also keep in mind that the group  $G$  is not required to be abelian.)

One can now generalize [Maier, Nikolaus and Schweigert (2011)] the construction just described to obtain a  $J$ -equivariant extended topological field theory: to any  $J$ -cover  $P \xrightarrow{J} M$ , one associates a category  $\mathcal{A}_G(P \xrightarrow{J} M)$  of  $G$ -bundles twisted by  $P$ . A  $G$ -bundle twisted by the cover  $P \xrightarrow{J} M$  is a pair  $(Q, \varphi)$ , consisting of an  $H$ -bundle  $Q$  over  $M$  and a smooth map  $\varphi : Q \rightarrow P$  over  $M$  that is required to obey  $\varphi(qh) = \varphi(q) \cdot \pi(h)$  for all  $q \in Q$  and  $h \in H$ . Put differently, a  $P \xrightarrow{J} M$ -twisted  $G$ -bundle is a lift of the  $J$ -cover  $P$  along the group homomorphism  $\pi : H \rightarrow J$ . The next steps in the construction are, exactly as in [Morton (2010)], the extension to spans of groupoids and 2-linearization. In this way, one obtains a  $J$ -equivariant topological field theory from which one can extract in particular a (weakly)  $J$ -modular Hopf algebra.

It should be noted that the normal subgroup  $G$  of  $H$  gives rise to a crossed module. In fact, it can be expected that more general crossed modules give rise to equivariant topological field theories as well, but details remain to be worked out.

### 3. 2-stacks and gerbes

#### 3.1. 2-stacks

While the previous application was about the construction of specific bi-categories, we now turn to sheaves of bicategories. Our goal is to associate

to every smooth manifold  $M$  a bicategory in such a way that for each diffeomorphism, there is a pullback functor for the bicategories in a way that generalizes the way bundles can be pulled back. Moreover, we impose gluing conditions for open covers (and generalizations thereof). Our starting point is the third example in Example 2. This example has pullbacks, but still does not have all properties we wish to have: the problem with the bicategory  $Grbtriv^\nabla(M)$  can be understood by considering the related category  $Buntriv^\nabla(M)$  of trivial  $U(1)$ -bundles with connection on  $M$  with global 1-forms  $A \in \Omega^1(M)$  as objects and gauge transformations  $\frac{1}{i}d \log \varphi = A' - A$  as morphisms  $A \xrightarrow{\varphi} A'$ . Bundles are only *locally* trivial, but can be glued together to globally non-trivial bundles. Global properties of bundles with connections have numerous well-known physical implications; global properties are equally important for the objects we are about to construct.

In the case of 2-form gauge theories, the correct global geometric objects are also obtained by gluing together locally trivial objects:

- The first datum is an open cover  $(U_i)$  of the manifold  $M$ .
- We then chose a trivial object  $B_i \in Grbtriv^\nabla(U_i)$  for each open subset.
- The third datum is a 1-morphism  $A_{ij} : B_j \rightarrow B_i$  for each intersection  $U_i \cap U_j$ .
- The next datum is a choice of 2-morphisms on threefold intersections which are required to obey coherence conditions on fourfold intersections. For details, we refer to [Nikolaus and Schweigert (2011)].

In this way, we obtain an object called an  $(U(1)$ -)bundle gerbe with connection.

The whole construction can be described more systematically as follows: we start with a bifunctor  $Grbtriv^\nabla$  from the (opposed) category of smooth manifolds to bicategories. It is compatible with pullbacks and thus forms a 2-prestack. Recall from geometry that sheafification allows to construct to any presheaf  $\mathcal{X}$  a sheaf  $\mathcal{X}^+$ . One of our results, Theorem 3.3 of [Nikolaus and Schweigert (2011)] asserts that this plus construction can be extended to 2-prestacks. In fact, applying the plus construction to the 2-prestack  $Grbtriv^\nabla$  yields just bicategories of bundle gerbes with connection,

$$(Grbtriv^\nabla M)^+ = Grb(M) ,$$

as defined in [Stevenson (2000); Waldorf (2007)]. Theorem 3.3 of [Nikolaus and Schweigert (2011)] thus shows that the latter form a sheaf on

the category of smooth manifolds (with the topology given by surjective submersions).

This allows us to construct gerbes by gluing local data, even along surjective submersions. We have thus the familiar tools from theory of bundles with connection at our disposal.

The plus construction for 2-stacks relies on quite a few more results. Along the way, we get results of independent interest, both in mathematics and mathematical physics. In particular, our results apply not only to manifolds, but to Lie groupoids, i.e. orbifolds, as well. Since this includes the case of action groupoids, we also obtain equivariant versions of all our results. As a central result, we mention Theorem 2.16 of [Nikolaus and Schweigert (2011)]: the category  $\mathcal{X}(\Lambda)$  associated by a 2-stack  $\mathcal{X}$  to a Lie groupoid  $\Lambda$  is invariant under Morita equivalence of Lie groupoids.

### 3.2. Jandl gerbes

A bundle with connection on a smooth manifold  $M$  leads to the notion of holonomy which associates a group element to any closed curve in  $M$ . The action of electrically charged particles moving in an electromagnetic background field described by the bundle contains terms that can be formulated in terms of the holonomy. Bundle gerbes with connection similarly lead to the notion of a surface holonomy: given an abelian bundle gerbe on  $M$ , one can associate to a closed oriented surface in  $M$  an element of the group  $U(1)$ . Surface holonomies are in fact one motivation to study bundle gerbes. They also have a physical application: they are (exponentials of) Wess-Zumino terms and enter in the action of strings moving in the background described by the bundle gerbe, i.e. in a non-trivial  $B$ -field.

It is by now well-established that sigma models, for example on compact Lie groups, can also be considered for unoriented surfaces. In fact, they play an important role in type I string compactifications. This raises the following question: what geometric data on  $M$  are needed to define unoriented surface holonomy. The first idea that comes to one's mind is to replace 2-forms on  $M$  by 2-densities on  $M$ . These are 2-forms on the orientation cover  $\hat{M}$  of  $M$  that are odd under the orientation reversion involution  $\sigma$  of  $\hat{M}$ , i.e.  $\sigma^*\omega = -\omega$ . Remembering that 2-forms are in fact objects in the 2-category in the third entry of Example 2, we realize that should better not impose equality of objects on the nose, but only up to isomorphism. This is also clear from a more physical point of view: we should impose equality only up to gauge transformations.

To set up the correct category  $JGrbtriv^\nabla(M)$  of trivial objects for un-oriented surface holonomy, we first introduce the category  $JBun^\nabla(M)$  of Jandl bundles: its objects are pairs, consisting of a  $U(1)$ -bundle  $P \rightarrow M$  and a smooth map  $\sigma : M \rightarrow \mathbb{Z}_2$ . Morphisms  $(P, \sigma) \rightarrow (Q, \mu)$  only exist if  $\sigma = \mu$ . The definition

$$(P, \sigma) \otimes (Q, \mu) := (P \otimes Q^\sigma, \sigma\mu)$$

endows this category with the structure of a monoidal category. Here  $Q^\sigma$  is the  $U(1)$ -bundle obtained by taking the inverse of the bundle on the connected components on which  $\sigma$  takes the value  $-1 \in \mathbb{Z}_2$ .

We are now ready to define the bicategory  $JGrbtriv^\nabla(M)$  of trivial Jandl gerbes:

- Its objects  $I_\omega$  are in bijection to 2-forms  $\omega \in \Omega^2(M)$ .
- Its 1-morphisms  $I_\omega \rightarrow I_{\omega'}$  are Jandl bundles  $(P, \sigma)$  with curvature  $\text{Curv}(P) = \sigma\omega' - \omega$ .
- Its 2-morphisms are morphisms of Jandl gerbes which have to obey suitable coherence conditions.

Trivial Jandl gerbes form a 2-prestack on the category of smooth manifolds. We can apply the general stackification procedure of Theorem 3.3 of [Nikolaus and Schweigert (2011)] and obtain a 2-stack of Jandl gerbes. One can describe a Jandl gerbe on a smooth manifold  $M$  explicitly in terms of descent data: a surjective submersion  $Y \rightarrow M$ , a 2-form  $\omega \in \Omega^2(Y)$ , a Jandl bundle  $(P, \sigma)$  on  $Y \times_M Y$  and a certain morphism of Jandl bundles. Since morphisms of Jandl bundles only exist in the case of equality of the  $\mathbb{Z}_2$ -valued functions  $\sigma$ , the data  $(Y, \sigma)$  combine into local data describing a  $\mathbb{Z}_2$ -cover  $\mathcal{O}(\mathcal{G})$  on  $M$ , the orientation cover of the Jandl gerbe. In string theory, the orientation cover is sometimes called the target space; the natural involution on the orientation cover is then called the orientifold involution.

One can also derive an explicit formula for the Wess-Zumino term on an un-oriented surface for a given Jandl gerbe. It can be explicitly written as a combination of integrals of 2-forms and 1-forms and the evaluation of  $U(1)$ -valued functions. Rewriting this formula on the orientation cover yields a formula obtained earlier in [Schreiber, Schweigert and Waldorf (2007)]. Checking that this formula does not depend on the choices involved is a nice exercise using Stokes' theorem. <sup>a</sup> One can also compare Jandl gerbes on compact Lie groups to algebraic results and finds complete agreement.

<sup>a</sup>We are sure that Max would have liked this exercise much more than the framework of bicategories that has lead us to the formula.



## 4. RCFT correlators from TFT

### 4.1. Surface operators

In this last section, we first briefly move to 3-categories, keeping a rather informal style of exposition. In fact, many new developments (see e.g. [Bartels, Douglas, Henriques (2009)]) point to the direction that Chern-Simons theories or chiral conformal field theories with fixed central charge naturally form a 3-category. Similar comments apply to modular tensor categories. Since we want later on to avoid the use of 3-categories, we will restrict to endomorphisms of a fixed object in the 3-category which have the structure of a bicategory.

In the case of three-dimensional topological field theories, this higher categorical structure nicely fits with the structure of defects of various codimension: consider a three-manifold  $M$  that is separated by an embedded surface  $\Sigma \subset M$  into two disjoint parts,  $M = M_1 \sqcup \Sigma \sqcup M_2$ . Suppose that a Chern-Simons theory of type  $\mathcal{C}_i$  is defined on each open submanifold  $M_i$ . In this situation, transition conditions at the interface  $\Sigma$  have to be specified. Such a set of conditions has been called a surface operator in [Kapustin and Saulina (2010)].

Let us make surface operators slightly more concrete for the case when the topological field theories on  $M_i$  are abelian Chern-Simons theories and thus given by even lattices  $L_i \subset \mathbb{R}^n$ . Then (types of) topological surface operators are in bijection to Lagrangian subgroups  $S \subset L_1^*/L_1 \times \overline{L_2^*/L_2}$ , where  $L^*$  denotes the lattice dual to the lattice  $L$ .

Two surface operators  $S_1$  and  $S_2$  between the same pair of Chern-Simons theories  $\mathcal{C}_1$  and  $\mathcal{C}_2$  might in turn be separated by a defect of codimension 2, a line. This is in fact a generalization of the well-known concept of a Wilson line: in case the two Chern-Simons theories are identical,  $\mathcal{C}_1 = \mathcal{C}_2$ , there is a distinguished surface operator, the invisible surface operator  $\text{Inv}$ . It corresponds to the identity 1-morphism of  $\mathcal{C}_1$  in the underlying 3-category. In the case of an abelian Chern-Simons theory, it corresponds to the diagonal subgroup  $S \subset L \times \overline{L}$ . Line operators in the invisible surface operator are just ordinary Wilson lines.

We thus arrive at a (monoidal) bicategory that can be assigned to any Chern-Simons theory or, more generally, to any chiral conformal field theory or modular tensor category: its objects are topological surface operators, 1-morphisms are generalized Wilson lines, 2-morphisms label insertions on the generalized Wilson lines.

#### 4.2. The Kreuzer-Schellekens bihomomorphism

We make this picture more concrete in the following situation: consider a generalized Wilson line  $W$  separating the invisible surface operator for a modular tensor category  $\mathcal{C}$  from some general surface operator  $S$ . One can argue [Kapustin and Saulina (2010)] that this corresponds to a Frobenius algebra  $A_W$  in the modular tensor category  $\mathcal{C}$ . A Frobenius algebra is an object  $A$  in  $\mathcal{C}$ , together with a unital associative multiplication  $\mu : A \otimes A \rightarrow A$  and a counital coassociative comultiplication  $\Delta : A \rightarrow A \otimes A$  such that  $\Delta$  is a morphism of  $A$ -bimodules. For pictures, we refer to section 2.2 of [Kapustin and Saulina (2010)]

For a general tensor category, even a modular tensor category, Frobenius algebras are hard to classify. There is however a systematic construction of a subclass of such algebras [Fuchs, Runkel and Schweigert (2004)]. Choose a subgroup  $H$  of the Picard group  $\text{Pic}(\mathcal{C})$  of the modular tensor category  $\mathcal{C}$ . This is the finite abelian group consisting of isomorphism classes of invertible objects in  $\mathcal{C}$ , in other words of simple currents [Schellekens and Yankielowicz (1990)]. The associativity constraint  $\mathcal{C}$  on the category yields a closed 3-cochain  $\psi$  on the Picard group  $\text{Pic}(\mathcal{C})$  and, by restriction, on the subgroup  $H$ . Any 2-cochain

$$\omega : H \times H \rightarrow \mathbb{C}^\times$$

such that  $d\omega = \psi$  gives the structure of an associative algebra on the object  $\bigoplus_{h \in H} U_h$  of  $\mathcal{C}$ . Two cochains  $\omega$  and  $\omega'$  that differ by an exact cochain,  $\omega' - \omega = d\eta$  for some 1-cochain  $\eta$  give rise to isomorphic algebras.

Let us first assume that the associator  $\psi$  is trivial. In this case, isomorphism classes of such algebras are classified by the cohomology group  $H^2(A, \mathbb{C}^\times)$  for the abelian group  $A = H$ . It is a classical result that this group is isomorphic to the group  $AB(A, \mathbb{C}^\times)$  of alternating complex-valued bihomomorphisms on the group  $A$  via

$$H^2(A, \mathbb{C}^\times) \xrightarrow{\sim} AB(A, \mathbb{C}^\times)$$

$$[\omega] \mapsto \xi(g, h) = \frac{\omega(g, h)}{\omega(h, g)}$$

For the Picard subcategory of a modular tensor category, the associator  $\psi$  is, in general, non-trivial. Max Kreuzer and Bert Schellekens realized [Kreuzer and Schellekens (1994)] in which way the condition on the bihomomorphism to be alternating has to be modified; the corresponding bihomomorphisms have therefore been called Kreuzer-Schellekens bihomomorphisms [Fuchs, Runkel and Schweigert (2004)].

An alternating bihomomorphism  $\xi \in AB(A, \mathbb{C}^\times)$  gives rise to a twisted group algebra  $\mathbb{C}_\xi A$ , i.e. a complex algebra with distinguished basis  $(b_a)_{a \in A}$  in which the group law of  $A$  is realized only projectively,  $b_{a_1} \cdot b_{a_2} = \omega(a_1, a_2) b_{a_1 \cdot a_2}$ . The way a Kreuzer-Schellekens bihomomorphism on a subgroup  $A$  of the Picard group of a tensor category  $\mathcal{C}$  gives rise to a (special symmetric) Frobenius algebra  $A_\xi$  in  $\mathcal{C}$  is a generalization of this classical situation to a braided setting.

Kreuzer-Schellekens algebras  $A_\xi$  have the important property that on the corresponding surface operator, one can punch out holes for free. Pictorially,

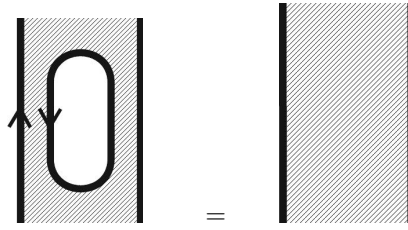


Fig. 2. Punching out a hole

This follows algebraically from the fact that the coproduct  $\Delta$  of a Kreuzer-Schellekens algebra  $A_\xi$  can be normalized in such a way that it is a right inverse to the product,  $\mu \circ \Delta = \text{id}_{A_\xi}$ . In particular, Kreuzer-Schellekens algebras are separable.

#### 4.3. RCFT correlators and the Kreuzer-Schellekens partition function

Another important result in rational conformal field theory due to Max Kreuzer and Bert Schellekens [Kreuzer and Schellekens (1994)] is the most general form [Kreuzer and Schellekens (1994)] of a modular invariant partition function of simple current type.

We place this result into the context of more recent developments. The partition function is just one specific correlator; by now the TFT approach to RCFT correlators provides a model-independent construction of all correlators (see e.g. [Fuchs, Runkel and Schweigert (2006)] for a review). Let  $\Sigma$  be an oriented world sheet. Suppose that the chiral data for the rational conformal field theory in question are described by the modular tensor category  $\mathcal{C}$ . According to the results of Reshetikhin and Turaev, there is a

three-dimensional topological field theory

$$\text{tft}_{\mathcal{C}} : \text{Cobord}_{3,2,1} \rightarrow 2\text{Vect}$$

based on the modular tensor category. The principle of holomorphic factorization states that the correlator  $\text{Cor}(\Sigma)$  is an element in the space of conformal blocks  $\text{tft}_{\mathcal{C}}(\hat{\Sigma})$  which is assigned by the topological field theory to the orientation double  $\hat{\Sigma}$  of  $\Sigma$ . The vector  $\text{Cor}(\Sigma)$  has to be invariant under the action of the mapping class group  $\text{Map}(\Sigma)$  and to obey factorization constraints.

Let us sketch the TFT construction of RCFT correlators; for simplicity, we restrict to the case when the worldsheet  $\Sigma$  has empty boundary,  $\partial\Sigma = \emptyset$ . In this case, we consider the three-manifold  $M_{\Sigma} := \Sigma \times [-1, 1]$  with a surface operator placed at the surface  $\Sigma \times \{0\}$ . Generalized Wilson lines are placed along the intervals  $\{p\} \times [-1, 1]$  for each bulk insertion at a point  $p \in \Sigma$ . Since  $M_{\Sigma}$  can be seen as a cobordism  $\emptyset \rightarrow \Sigma \sqcup \bar{\Sigma} \cong \hat{\Sigma}$ , applying the tft functor gives an element

$$\text{Cor}(\Sigma) := \text{tft}_{\mathcal{C}}(M_{\Sigma})1 \in \text{tft}_{\mathcal{C}}(\partial M_{\Sigma}) = \text{tft}_{\mathcal{C}}(\hat{\Sigma})$$

that depends, of course, on the choice of the surface operator on the surface  $\Sigma \times \{0\}$ . Suppose that the surface operator has the property that one can punch out holes for free, as in figure 2. Then, by punching enough holes, we reduce the surface operator to a network of strips and replace it effectively by the dual of a triangulation of  $\Sigma \times \{0\}$ . This triangulation is labeled with the Frobenius algebra corresponding to the generalized Wilson line separating the surface operator and the invisible surface operator  $\text{Inv}$ . This yields an expression of the RCFT correlator in terms of the invariant of a three-manifold  $M_{\Sigma}$  with an embedded ribbon graph to which the Reshetikhin-Turaev construction can be applied.

It can now be shown [Fjelstad, Fuchs, Runkel and Schweigert (2005)] that the elements  $\text{Cor}(\Sigma)$  are invariant under the action of the mapping class group of  $\Sigma$  and obey factorization constraints. It can even be shown that any consistent set of correlators for a given chiral rational conformal field theory is of this form [Fjelstad, Fuchs, Runkel and Schweigert (2008)].

The formalism is also well suited to explicit calculations: one can compute structure constants and partition functions for boundary fields, bulk fields and defect fields. One can apply the RCFT construction in particular to the case of a Kreuzer-Schellekens algebra and compute the partition function of bulk fields. One obtains exactly the result first derived by Max Kreuzer and Bert Schellekens, based in particular on Max Kreuzer's care-

ful understanding of orbifold theories (from which also one of the present authors benefitted a lot [Kreuzer and Schweigert (1995)]).

We hope that this contribution helps to convince practitioners of field theory that bicategories - and higher categories - are one helpful tool to solve problems in topological, classical and quantum field theories. Max Kreuzer was a fine practitioner of field theory with a keen appreciation of the true problems arising in these theories. He might have looked at bicategories with a somewhat skeptical smile. We like the idea that he would have tried immediately to translate some of the insights sketched in this contribution in a language closer to his heart - and we gratefully remember that this was frequently his way to show his appreciation for the results of his colleagues.

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