On the algebraic K-theory of $\mathrm{K}(\mathbf{Z}/p^n)$

Quillen introduced algebraic K-theory in [?] and computed the K-groups $\mathrm{K}_*(\mathbf{F}_q)$ in [4]. Except in low degrees, the computation of the K-groups of closely related rings, for example $\mathbf{Z}/4$, has remained out of reach. In this paper, we announce new methods for computations of such rings and outline new results.

We are particularly interested in rings of the form $\mathcal{O}_K/\varpi^k$ where K is a finite extension of $\mathbf{Q}_p$, $\mathcal{O}_K$ is its ring of integers, and ϖ^k is a uniformizer. In particular, $p \in (\varpi^e)$ where e is the degree of ramification of K over $\mathbf{Q}_p$. When relevant, we normalize the p-adic valuation on $\mathcal{O}_K$ so that $v_p(p) = 1$ and hence $v_p(\varpi) = \frac{1}{e}$. The residue field of $\mathcal{O}_K$ is the case $k = 1$ and is a finite field $\mathbf{F}_q = \mathcal{O}_K/\varpi$, where $q = p^f$ for some f, called the residual degree of the extension.

1 History

For any field k, $\mathrm{K}_0(k) \cong \mathbf{Z}$ and $\mathrm{K}_1(k) \cong k^\times$. Quillen showed that if $\mathbf{F}_q$ is the finite field with $q = p^f$ elements, then for $n \geqslant 1$,

$$\mathrm{K}_n(\mathbf{F}_q) \cong \begin{cases} 0 & \text{if } n \text{ is even and} \\ \mathbf{Z}/(q^i - 1) & \text{if } n = 2i - 1. \end{cases}$$

Note in particular that there is no p-torsion in the K-groups of $\mathbf{F}_q$.

For ℓ a prime, Suslin's rigidity theorem implies that if R is a commutative ring which is henselian with respect to an ideal I and if ℓ is invertible in R, then

$$\mathrm{K}(R; \mathbf{Z}_\ell) \simeq \mathrm{K}(R/I; \mathbf{Z}_\ell).$$

Examples of such henselian pairs are the rings of integers $\mathcal{O}$ with the ideal (ϖ) or the quotients $\mathcal{O}/\varpi^k$, again with the ideal (ϖ). It follows that

$$\mathrm{K}(\mathcal{O}; \mathbf{Z}_\ell) \simeq \mathrm{K}(\mathcal{O}/\varpi^k; \mathbf{Z}_\ell) \simeq \mathrm{K}(\mathbf{F}_q; \mathbf{Z}_\ell).$$

These are thus the same as the groups computed by Quillen, at least when $\ell \neq p$.

In the case of the p-adic K-theory of $\mathcal{O}_K$ or $\mathcal{O}_K/\varpi^k$, the situation is very different. In the the case of $\mathcal{O}_K$ a result of Clausen–Mathew–Morrow implies that $\mathrm{K}(\mathcal{O}_K; \mathbf{Z}_p) \simeq \mathrm{TC}(\mathcal{O}_K; \mathbf{Z}_p)$, where $\mathrm{TC}(\mathcal{O}_K; \mathbf{Z}_p)$ is the p-adic topological cyclic homology of $\mathcal{O}_K$. Similarly, a result [?] of Dundass–Goodwillie–McCarthy implies that $\mathrm{K}(\mathcal{O}/\varpi^k; \mathbf{Z}_p) \simeq \mathrm{TC}(\mathcal{O}/\varpi^k; \mathbf{Z}_p)$; this is also a consequence of the theorem of [?]. This makes these groups amenable to calculation using trace methods.

Hesselholt and Madsen carry out this approach in [?] and thereby verify the Quillen–Lichtenbaum conjecture for $\mathcal{O}_K$; this conjecture now follows in general from the proof of the Bloch–Kato conjecture due to Rost and Voevodsky.

The Hesselholt–Madsen computations use logarithmic de Rham–Witt forms and TR, i.e., the classical approach to trace method computations. These have recently been revisited by Liu–Wang who compute $\mathrm{K}_*(\mathcal{O}; \mathbf{F}_p)$, the K-groups with mod p coefficients.

Much less is known about the intermediate rings $\mathcal{O}_K/\varpi^k$. As for fields, $\mathrm{K}_0(\mathcal{O}_K/\varpi^k) \cong \mathbf{Z}$ and $\mathrm{K}_1(\mathcal{O}_K/\varpi^k)$ is isomorphic to the group of units in $\mathcal{O}_K/\varpi^k$.

In [?], Dennis and Stein computed $\mathrm{K}_2(\mathcal{O}_K/\varpi^k)$. No other work we are aware of has addressed the K-groups of general rings of the form $\mathcal{O}_K/\varpi^k$.

In special situations, more is known. First, every ring of the form $\mathbf{F}_q[z]/(z^e)$ is of the form $\mathcal{O}_K/\varpi^e$. The algebraic K-groups of these truncated polynomial rings has been studied by many people, first by Hesselholt–Madsen in [?] using classical trace method techniques, then by Speirs in [?] using the new approach to TC due to Nikolaus–Scholze [3], then by Sulyma in [?] using the approach to TC via syntomic cohomology due to Bhatt–Morrow–Scholze [2] and as outlined in [?].

Second, for unramified extension there are some results in low degrees. In the unramified case, where $e = 1$, $\mathcal{O}_K$ is the ring $W(\mathbf{F}_q)$ of p-typical Witt vectors of the residue field. Brun [?] computed the K-groups of $\mathbf{Z}/p^k$ (i.e., when $e = 1$ and $f = 1$) up to degree $p-3$ and Angeltveit [?] computed the K-groups of $W_k(\mathbf{F}_q) = W(\mathbf{F}_q)/\varpi^k = W(\mathbf{F}_q)/p^k$ up to degree $2p-2$.

Angeltveit also proved an important quantitative result:

$$\frac{\#\mathrm{K}_{2i-1}(W_k(\mathbf{F}_q))}{\#\mathrm{K}_{2i-2}(W_k(\mathbf{F}_q))} = q^{i(k-1)}.$$

Both Brun and Angeltveit use classical trace methods and the p-adic filtration on the truncated Witt vectors to translate part of the problem to the cases of truncated polynomial rings where a complete answer is known.

2 New results

As $\mathrm{K}(\mathcal{O}/\varpi^k;\mathbf{Z}_p) \simeq \mathrm{TC}(\mathcal{O}/\varpi^k;\mathbf{Z}_p)$ by [?], it is enough to compute TC of these rings. To do so, we use the filtration on TC constructed by Bhatt–Morrow–Scholze in [2]. If R is a quasisyntomic ring, there is a decreasing filtration $\mathrm{F}^\star_{\mathrm{syn}}\mathrm{TC}(R;\mathbf{Z}_p)$ with associated graded

$$\mathrm{gr}^i_{\mathrm{syn}}\mathrm{TC}(R;\mathbf{Z}_p) \simeq \mathbf{Z}_p(i)(R)[2i],$$

where $\mathbf{Z}_p(i)(R)$ is the weight i syntomic cohomology of R.

In general, as shown in [1], the weight i syntomic cohomology $\mathbf{Z}_p(i)(R)$ is concentrated in $[0, i+1]$, independent of R; this means that $\mathrm{H}^n(\mathbf{Z}_p(i)(R)) = 0$ for $n \notin [0, i+1]$. In the special case of $\mathcal{O}_K$ or $\mathcal{O}_K/\varpi^k$, a soft argument using the ϖ-adic associated graded implies that in fact the weight i syntomic cohomology is in $[0, 2]$; moreover, for $i \geqslant 1$, $\mathrm{H}^0(\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^k)) = 0$ so the compex has cohomology concentrated in degrees 1 and 2.

This implies that the spectral sequence associated to the syntomic filtration on TC vanishes for $\mathcal{O}/\varpi^k$. Hence,

$$\mathrm{TC}_{2i-1}(\mathcal{O}_K/\varpi^k;\mathbf{Z}_p) \cong \mathrm{H}^1(\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^k))$$

and

$$\mathrm{TC}_{2i-2}(\mathcal{O}_K/\varpi^k;\mathbf{Z}_p) \cong \mathrm{H}^2(\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^k)).$$

Theorem 2.1. *For $i \geqslant 1$, there is a complex*

$$\mathbf{Z}_p(i)^\bullet(\mathcal{O}_K/\varpi^k)\colon$$
$$\left(\mathbf{Z}_p^{d(ik-1)} \xrightarrow{\mathrm{syn}_0} \mathbf{Z}_p^{2d(ik-1)} \xrightarrow{\mathrm{syn}1} \mathbf{Z}_p^{d(ik-1)}\right)$$

which computes $\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^k)$. The terms are free $\mathbf{Z}_p$-modules of ranks $d(ik-1)$, $2d(ik-1)$, and $d(ik-1)$, respectively, in cohomological degrees 0, 1, and 2.

The proof of the existence of this explicit cochain complex model of the syntomic complex will be discussed in Section 4.

The groups $\mathrm{K}_*(\mathcal{O}_K/\varpi_k)$ are torsion for $* > 0$. In particular, the complex above is exact rationally. Thus, to compute the cohomology of $\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^k)$, and hence the K-groups of $\mathcal{O}_K/\varpi^k$, it is enough to compute the matrices syn_0 and syn_1 and their elementary divisors.

Theorem 2.2. *The matrices syn_0 and syn_1 are effectively computatble. Specifically, they can be computed with enough p-adic precision to guarantee computability of the effective divisors.*

We have implemented our algorithm in `SAGE` in the case where $f = 1$, i.e., when the residue field is $\mathbf{F}_p$. This is the totally ramified case and is both the most interesting in our opinion and the most computationally accessible.

Corollary 2.3. *There is an algorithm to compute $\mathrm{K}_n(\mathcal{O}_K/\varpi^k;\mathbf{Z}_p)$ for any K, k, and n.*

Along the way, we extend the result of Angeltveit on the quotients of the orders from the unramified case to any $\mathcal{O}_K/\varpi^k$.

Corollary 2.4. *For any $\mathcal{O}_K/\varpi^k$,*

$$\frac{\#\mathrm{K}_{2i-1}(\mathcal{O}_K/\varpi^k)}{\#\mathrm{K}_{2i-2}(\mathcal{O}_K/\varpi^k)} = q^{i(k-1)},$$

where $q = p^f$ is the order of the residue field of $\mathcal{O}_K$.

In the next section, we outline the results of some of our computations.

3 Computations

Conjecture 3.1 (Even vanishing conjecture)**.** *For any k, $\mathrm{H}^2(\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^k)) = 0$ for $i \gg 0$. In particular, $\mathrm{K}_{2i-2}(\mathcal{O}_K/\varpi^k) = 0$ for $i \gg 0$.*

The conjecture has the following consequence, by Corollary 2.4.

Corollary 3.2. *For $i \gg 0$, $\#\mathrm{K}_{2i-1}(\mathcal{O}_K/\varpi^k) = q^{i(k-1)}$.*

4 δ-ring cohomology

5 The syntomic matrices

References

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