

LAX COLIMITS AND FREE FIBRATIONS IN ∞ -CATEGORIES

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ABSTRACT. We define and discuss lax and weighted colimits of diagrams in ∞ -categories and show that the coCartesian fibration associated to a functor is given by its lax colimit. A key ingredient, of independent interest, is a simple characterization of the free Cartesian fibration associated to a functor of ∞ -categories. As an application of these results, we prove that the total space of a presentable Cartesian fibration between ∞ -categories is presentable, generalizing a theorem of Makkai and Paré to the ∞ -categorical setting. Lastly, in the appendix, we observe, using the Duskin nerve, that pseudofunctors between $(2,1)$ -categories give rise to functors between ∞ -categories.

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1. INTRODUCTION

In the context of ordinary category theory, Grothendieck’s theory of fibrations [Gro63] can be used to give an alternative description of functors to the category \mathbf{Cat} of categories. This has been useful, for example, in the theory of stacks in algebraic geometry, as the fibration setup is usually more flexible. When working with ∞ -categories, however, the analogous notion of *Cartesian* fibrations is far more important: since defining a functor to the ∞ -category \mathbf{Cat}_∞ of ∞ -categories requires specifying an infinite amount of coherence data, it is in general not feasible to “write down” definitions of functors, so that manipulating Cartesian fibrations is often the only reasonable way to define key functors.

For ordinary categories, the *Grothendieck construction* gives a simple description of the fibration associated to a functor $F: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Cat}$; this can also be described formally as a certain weighted colimit, namely the oplax colimit of the functor F . For ∞ -categories, on the other hand, the equivalence between Cartesian fibrations and functors has been proved by Lurie using the *straightening functor*, a certain left Quillen functor between

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model categories. This leaves the corresponding right adjoint, the *unstraightening* functor, quite inexplicit.

One of our main goals in this paper is to show that Lurie’s unstraightening functor is a model for the ∞ -categorical analogue of the Grothendieck construction. More precisely, we introduce ∞ -categorical version of lax and oplax limits and colimits and prove the following:

Theorem 1.1.

- (i) *Suppose $F: \mathcal{C} \rightarrow \text{Cat}_\infty$ is a functor of ∞ -categories, and $\mathcal{E} \rightarrow \mathcal{C}$ is a coCartesian fibration associated to F . Then \mathcal{E} is the lax colimit of the functor F .*
- (ii) *Suppose $F: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_\infty$ is a functor of ∞ -categories, and $\mathcal{E} \rightarrow \mathcal{C}$ is a Cartesian fibration associated to F . Then \mathcal{E} is the oplax colimit of the functor F .*

To prove this we make use of an explicit description of the free Cartesian fibration associated to an arbitrary functor of ∞ -categories. More precisely, the ∞ -category $\text{Cat}_{\infty/\mathcal{C}}^{\text{cart}}$ of Cartesian fibrations over \mathcal{C} is a subcategory of the slice ∞ -category $\text{Cat}_{\infty/\mathcal{C}}$, and we show that the inclusion admits a left adjoint given by a simple formula:

Theorem 1.2. *Let \mathcal{C} be an ∞ -category. For $p: \mathcal{E} \rightarrow \mathcal{C}$ any functor of ∞ -categories, let $F(p)$ denote the map $\mathcal{E} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1} \rightarrow \mathcal{C}$ where the pullback is along the map $\mathcal{C}^{\Delta^1} \rightarrow \mathcal{C}$ given by evaluation at $1 \in \Delta^1$ and the projection is induced by evaluation at 0 . Then F defines a functor $\text{Cat}_{\infty/\mathcal{C}} \rightarrow \text{Cat}_{\infty/\mathcal{C}}^{\text{cart}}$, which is left adjoint to the forgetful functor $\text{Cat}_{\infty/\mathcal{C}}^{\text{cart}} \rightarrow \text{Cat}_{\infty/\mathcal{C}}$.*

The third main result of this paper provides a useful extension of the theory of presentable ∞ -categories in the context of Cartesian fibrations. While the theory of *accessible* and *presentable* categories is an important part of ordinary category theory, when working with ∞ -categories the analogous notions turn out to be indispensable. Whereas, for example, it is often possible to give an explicit construction of colimits in an ordinary category, when working with ∞ -categories we often have to conclude that colimits exist by applying general results on presentable ∞ -categories. Similarly, while for ordinary categories it is often possible to write down an adjoint to a given functor, for ∞ -categories an appeal to the adjoint functor theorem, which is most naturally considered in the presentable context, is often unavoidable.

It is thus very useful to know that various ways of constructing ∞ -categories give accessible or presentable ones; many such results are proved in [Lur09a, §5]. Here, we will use our results on Cartesian fibrations to give a criterion for the source of a Cartesian fibration to be presentable, generalizing a theorem of Makkai and Paré [MP89] to the ∞ -categorical context. More precisely, we show:

Theorem 1.3. *Suppose $p: \mathcal{E} \rightarrow \mathcal{C}$ is a Cartesian and coCartesian fibration such that \mathcal{C} is presentable, the fibres \mathcal{E}_x are presentable for all $x \in \mathcal{C}$, and the associated functor $F: \mathcal{C}^{\text{op}} \rightarrow \widehat{\text{Cat}}_\infty$ preserves κ -filtered limits for some cardinal κ . Then the ∞ -category \mathcal{E} is presentable, and the projection p is an accessible functor (i.e. it preserves λ -filtered colimits for some sufficiently large cardinal λ).*

1.1. Overview. In §2 we briefly review the definitions of twisted arrow ∞ -categories and ∞ -categorical ends and coends, and use these to define weighted (co)limits. Then in §3 we prove our main result for coCartesian fibrations over a simplex, using the *mapping simplex* defined in [Lur09a, §3.2.2]. Before we extend this result to general coCartesian fibrations we devote three sections to preliminary results: in §4 we give a description of the *free Cartesian fibration*, i.e. the left adjoint to the forgetful functor from Cartesian fibrations over \mathcal{C} to the slice ∞ -category $\text{Cat}_{\infty/\mathcal{C}}$; in §5 we prove that the space of natural

transformations between two functors is given by an end, and in §6 we prove that the straightening equivalence extends to an equivalence of the natural enrichments in Cat_∞ of the two ∞ -categories involved. §7 then contains the proof of our main result: Cartesian and coCartesian fibrations are given by weighted colimits of the associated functors. In §8 we apply this to identify the functor associated to a certain simple Cartesian fibration, which is a key step in our proof in §9 that the source of a presentable fibration is presentable. Finally, in appendix A we use Duskin's nerve for strict (2,1)-categories to check that the pseudonaturality of the unstraightening functors on the level of model categories implies that they are natural on the level of ∞ -categories.

1.2. Notation. Much of this paper is based on work of Lurie in [Lur09a, Lur14]; we have generally kept his notation and terminology. In particular, by an ∞ -category we mean an $(\infty, 1)$ -category or more specifically a quasicategory, i.e. a simplicial set satisfying certain horn-filling properties. We also use the following conventions, some of which differ from those of Lurie:

- Generic categories are generally denoted by single capital bold-face letters ($\mathbf{A}, \mathbf{B}, \mathbf{C}$) and generic ∞ -categories by single caligraphic letters ($\mathcal{A}, \mathcal{B}, \mathcal{C}$). Specific categories and ∞ -categories both get names in the normal text font.
- If \mathcal{C} is an ∞ -category, we write $\iota\mathcal{C}$ for the *interior* or *underlying space* of \mathcal{C} , i.e. the largest subspace of \mathcal{C} that is a Kan complex.
- If $f: \mathcal{C} \rightarrow \mathcal{D}$ is left adjoint to a functor $g: \mathcal{D} \rightarrow \mathcal{C}$, we will refer to the adjunction as $f \dashv g$.
- We write Pr^{L} for the ∞ -category of presentable ∞ -categories and functors that are left adjoints, i.e. colimit-preserving functors, and Pr^{R} for the ∞ -category of presentable ∞ -categories and functors that are right adjoints, i.e. accessible functors that preserve limits.
- If \mathcal{C} and \mathcal{D} are ∞ -categories, we will denote the ∞ -category of functors $\mathcal{C} \rightarrow \mathcal{D}$ by both $\text{Fun}(\mathcal{C}, \mathcal{D})$ and $\mathcal{D}^{\mathcal{C}}$.
- If S is a simplicial set, we write

$$\text{St}_S^+ : (\text{Set}_\Delta^+)_{/S^\#} \rightleftarrows \text{Fun}(\mathcal{C}(S)^{\text{op}}, \text{Set}_\Delta^+) : \text{Un}_S^+$$

for the marked (un)straightening Quillen equivalence, as defined in [Lur09a, §3.2].

- If \mathcal{C} is an ∞ -category, we write

$$\text{St}_{\mathcal{C}} : \text{Cat}_{\infty/\mathcal{C}}^{\text{cart}} \rightleftarrows \text{Fun}(\mathcal{C}, \text{Cat}_\infty) : \text{Un}_{\mathcal{C}}$$

for the adjoint equivalence of ∞ -categories induced by the (un)straightening Quillen equivalence.

- If S is a simplicial set, we write

$$\text{St}_S^{+, \text{co}} : (\text{Set}_\Delta^+)_{/S^\#} \rightleftarrows \text{Fun}(\mathcal{C}(S), \text{Set}_\Delta^+) : \text{Un}_S^{+, \text{co}}$$

for the coCartesian marked (un)straightening Quillen equivalence, given by $\text{St}_S^{+, \text{co}}(X) := (\text{St}_{\text{Sop}}^+(X^{\text{op}}))^{\text{op}}$.

- If \mathcal{C} is an ∞ -category, we write

$$\text{St}_{\mathcal{C}}^{\text{co}} : \text{Cat}_{\infty/\mathcal{C}}^{\text{coart}} \rightleftarrows \text{Fun}(\mathcal{C}, \text{Cat}_\infty) : \text{Un}_{\mathcal{C}}^{\text{co}}$$

for the adjoint equivalence of ∞ -categories induced by the coCartesian (un)straightening Quillen equivalence.

- If \mathcal{C} is an ∞ -category, we denote the Yoneda embedding for \mathcal{C} by

$$y_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C}).$$

- We write $\text{Cat}_{\infty/\mathcal{C}}^{\text{cart}}$ for the subcategory of $\text{Cat}_{\infty/\mathcal{C}}$ consisting of Cartesian fibrations over \mathcal{C} and morphisms the functors that preserve Cartesian morphisms, $\text{Map}_{\mathcal{C}}^{\text{Cart}}(-, -)$ for the mapping spaces in $\text{Cat}_{\infty/\mathcal{C}}^{\text{cart}}$ and $\text{Fun}_{\mathcal{C}}^{\text{cart}}(-, -)$ for the full subcategory of $\text{Func}(-, -)$ spanned by the functors that preserve Cartesian morphisms. Similarly we write $\text{Cat}_{\infty/\mathcal{C}}^{\text{cocart}}$ for the ∞ -category of coCartesian fibrations over \mathcal{C} , $\text{Map}_{\mathcal{C}}^{\text{cocart}}(-, -)$ for the mapping spaces in $\text{Cat}_{\infty/\mathcal{C}}^{\text{cocart}}$, and $\text{Fun}_{\mathcal{C}}^{\text{cocart}}(-, -)$ for the full subcategory of $\text{Func}(-, -)$ spanned by the functors that preserve coCartesian morphisms.
- By a *Cartesian* fibration we mean any functor of ∞ -categories that is categorically equivalent to an inner fibration that is Cartesian in the sense of [Lur09a, §2.4.2]. Since a Cartesian fibration in Lurie’s sense is an inner fibration that is Cartesian in our sense, we use the term Cartesian inner fibration to refer to this more restrictive notion.

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2. TWISTED ARROW ∞ -CATEGORIES, (CO)ENDS, AND WEIGHTED (CO)LIMITS

In this section we briefly recall the definitions of twisted arrow ∞ -categories and (co)ends, and then use these to give an obvious definition of weighted (co)limits in the ∞ -categorical setting.

Definition 2.1. Let $\epsilon: \Delta \rightarrow \Delta$ be the functor $[n] \mapsto [n] \star [n]^{\text{op}}$. The *edgewise subdivision* $\text{esd}(S)$ of a simplicial set S is the composite ϵ^*S .

Definition 2.2. Let \mathcal{C} be an ∞ -category. The *twisted arrow ∞ -category* $\text{Tw}(\mathcal{C})$ of \mathcal{C} is the simplicial set $\epsilon^*\mathcal{C}$. Thus in particular

$$\text{Hom}(\Delta^n, \text{Tw}(\mathcal{C})) \cong \text{Hom}(\Delta^n * (\Delta^n)^{\text{op}}, \mathcal{C}).$$

The natural transformations $\Delta^\bullet, (\Delta^\bullet)^{\text{op}} \rightarrow \Delta^\bullet * (\Delta^\bullet)^{\text{op}}$ induce a projection $\text{Tw}(\mathcal{C}) \rightarrow \mathcal{C} \times \mathcal{C}^{\text{op}}$.

Remark 2.3. The twisted arrow ∞ -category, which was originally introduced by Joyal, has previously been extensively used by Barwick [Bar13, Bar14] and collaborators [BGN14] and Lurie [Lur14, §5.2.1]. By [Lur14, Proposition 5.2.1.3] the projection $\text{Tw}(\mathcal{C}) \rightarrow \mathcal{C} \times \mathcal{C}^{\text{op}}$ is a right fibration; in particular, the simplicial set $\text{Tw}(\mathcal{C})$ is an ∞ -category if \mathcal{C} is. The functor $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{S}$ associated to this right fibration is the mapping space functor $\text{Map}_{\mathcal{C}}(-, -)$ by [Lur14, Proposition 5.2.1.11].

Example 2.4. The twisted arrow category $\text{Tw}([n])$ of the category $[n]$ is the partially ordered set with objects (i, j) where $0 \leq i \leq j \leq n$ and with $(i, j) \leq (i', j')$ if $i \leq i' \leq j' \leq j$.

The obvious definition of (co)ends in the ∞ -categorical setting is then the following.

Definition 2.5. If $F: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ is a functor of ∞ -categories, the *end* and *coend* of F are, respectively, the limit and colimit of the composite functor

$$\text{Tw}(\mathcal{C}) \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}.$$

Remark 2.6. The ∞ -categorical notions of ends and coends are also discussed in [Gla14, §2]. In the context of simplicial categories, a homotopically correct notion of coends was extensively used by Cordier and Porter [CP97]; see their paper for a discussion of the history of such definitions.

Now we can consider weighted (co)limits:

Definition 2.7. Let \mathcal{R} be a presentably symmetric monoidal ∞ -category, i.e. a presentable ∞ -category equipped with a symmetric monoidal structure such that the tensor product preserves colimits in each variable, and let \mathcal{M} be a left \mathcal{R} -module ∞ -category in Pr^{L} . Then \mathcal{M} is in particular tensored and cotensored over \mathcal{R} , i.e. there are functors

$$\begin{aligned} (- \otimes -) : \mathcal{R} \otimes \mathcal{M} &\rightarrow \mathcal{M}, \\ (-)^{(-)} : \mathcal{R}^{\text{op}} \otimes \mathcal{M} &\rightarrow \mathcal{M}, \end{aligned}$$

such that for every $x \in \mathcal{R}$ the functor $x \otimes - : \mathcal{M} \rightarrow \mathcal{M}$ is left adjoint to $(-)^x$. Given functors $F : \mathcal{C} \rightarrow \mathcal{M}$ and $W : \mathcal{C}^{\text{op}} \rightarrow \mathcal{R}$, the *W-weighted colimit of F* is the coend $\text{colim}_{\text{Tw}(\mathcal{C})} W(-) \otimes F(-)$. Similarly, given $F : \mathcal{C} \rightarrow \mathcal{M}$ and $W : \mathcal{C} \rightarrow \mathcal{R}$, the *W-weighted limit of F* is the end $\text{lim}_{\text{Tw}(\mathcal{C})} F(-)^{W(-)}$.

We are interested in the case where both \mathcal{R} and \mathcal{M} are the ∞ -category Cat_{∞} of ∞ -categories, with the tensoring given by Cartesian product and the cotensoring by $\text{Fun}(-, -)$. In this case there are two special weights for every ∞ -category \mathcal{C} : we have functors $\mathcal{C}_{-/} : \mathcal{C} \rightarrow \text{Cat}_{\infty}$ and $\mathcal{C}_{-/} : \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_{\infty}$ sending $x \in \mathcal{C}$ to $\mathcal{C}_{/x}$ and $\mathcal{C}_{x/}$, respectively. Precisely, these functors are obtained by straightening the source and target projections $\mathcal{C}^{\Delta^1} \rightarrow \mathcal{C}$, which are respectively Cartesian and coCartesian. Using these functors, we can define lax and oplax (co)limits:

Definition 2.8. Suppose $F : \mathcal{C} \rightarrow \text{Cat}_{\infty}$ is a functor. Then:

- The *lax colimit* of F is the colimit of F weighted by $\mathcal{C}_{-/}$, i.e.

$$\text{colim}_{\text{Tw}(\mathcal{C})} \mathcal{C}_{-/} \times F(-).$$

- The *oplax colimit* of F is the colimit of F weighted by $(\mathcal{C}^{\text{op}})_{-/}$, i.e.

$$\text{colim}_{\text{Tw}(\mathcal{C})} (\mathcal{C}_{-/})^{\text{op}} \times F(-).$$

- The *lax limit* of F is the limit of F weighted by $\mathcal{C}_{-/}$, i.e.

$$\text{lim}_{\text{Tw}(\mathcal{C})} \text{Fun}(\mathcal{C}_{-/}, F(-)).$$

- The *oplax limit* of F is the limit of F weighted by $(\mathcal{C}^{\text{op}})_{-/}$, i.e.

$$\text{lim}_{\text{Tw}(\mathcal{C})} \text{Fun}((\mathcal{C}_{-/})^{\text{op}}, F(-)).$$

3. COCARTESIAN FIBRATIONS OVER A SIMPLEX

In this preliminary section we study coCartesian fibrations over the simplices Δ^n , and observe that in this case the description of a coCartesian fibration as a lax colimit follows easily from results of Lurie in [Lur09a, §3.2]. To see this we first recall the definition of the *mapping simplex* of a functor $\phi : [n] \rightarrow \text{Set}_{\Delta}^+$ and show that its fibrant replacement is a coCartesian fibration associated to the corresponding functor $\Delta^n \rightarrow \text{Cat}_{\infty}$.

Definition 3.1. Let $\phi: [n] \rightarrow \text{Set}_\Delta^+$ be a functor. The *mapping simplex* $M_{[n]}(\phi) \rightarrow \Delta^n$ has k -simplices given by a map $\sigma: [k] \rightarrow [n]$ together with a k -simplex $\Delta^k \rightarrow \phi(\sigma(0))$. In particular, an edge of $M_{[n]}(\phi)$ is given by a pair of integers $0 \leq i \leq j \leq n$ and an edge $f \in \phi(i)$; let S be the set of edges of $M_{[n]}(\phi)$ where the edge f is marked. Then $M_{[n]}^{\natural}(\phi)$ is the marked simplicial set $(M_{[n]}(\phi), S)$. This gives a functor $M_{[n]}^{\natural}: \text{Fun}([n], \text{Set}_\Delta^+) \rightarrow (\text{Set}_\Delta^+)_{/\Delta^n}$, pseudonatural in $\mathbf{\Delta}^{\text{op}}$ (with respect to composition and pullback) — see appendix A for a discussion of pseudonatural transformations.

Definition 3.2. Let $\phi: [n] \rightarrow \text{Set}_\Delta$ be a functor. The *relative nerve* $N_{[n]}(\phi) \rightarrow \Delta^n$ has k -simplices given by a map $\sigma: [k] \rightarrow [n]$ and for every ordered subset $J \subseteq [k]$ with greatest element j , a map $\Delta^J \rightarrow \phi(\sigma(j))$ such that for $J' \subseteq J$ the diagram

$$\begin{array}{ccc} \Delta^{J'} & \longrightarrow & \phi(j') \\ \downarrow & & \downarrow \\ \Delta^J & \longrightarrow & \phi(j) \end{array}$$

commutes. Given a functor $\bar{\phi}: [n] \rightarrow \text{Set}_\Delta^+$ we define $N_{[n]}^+(\bar{\phi})$ to be the marked simplicial set $(N_{[n]}(\phi), M)$ where ϕ is the underlying functor $[n] \rightarrow \text{Set}_\Delta$ of $\bar{\phi}$, and M is the set of edges $\Delta^1 \rightarrow N_{[n]}^+\phi$ determined by

- a pair of integers $0 \leq i \leq j \leq n$,
- a vertex $x \in \phi(i)$,
- a vertex $y \in \phi(j)$ and an edge $\phi(i \rightarrow j)(x) \rightarrow y$ that is marked in $\bar{\phi}(j)$.

This determines a functor $N_{[n]}^+: \text{Fun}([n], \text{Set}_\Delta^+) \rightarrow (\text{Set}_\Delta^+)_{/\Delta^n}$, pseudonatural in $\mathbf{\Delta}^{\text{op}}$.

Remark 3.3. By [Lur09a, Proposition 3.2.5.18], the functor $N_{[n]}^+$ is a right Quillen equivalence from the projective model structure on $\text{Fun}([n], \text{Set}_\Delta^+)$ to the coCartesian model structure on $(\text{Set}_\Delta^+)_{/\Delta^n}$. In particular, if $\phi: [n] \rightarrow \text{Set}_\Delta^+$ is a functor such that $\phi(i)$ is fibrant (i.e. is a quasicategory marked by its set of equivalences) for every i , then $N_{[n]}^+(\phi)$ is a coCartesian fibration.

Definition 3.4. There is a natural transformation $\nu_{[n]}: M_{[n]}^{\natural}(-) \rightarrow N_{[n]}^+(-)$ that sends a k -simplex $(\sigma: [k] \rightarrow [n], \Delta^k \rightarrow \phi(\sigma(0)))$ in $M_{[n]}^{\natural}(\sigma)$ to the k -simplex of $N_{[n]}^+(\phi)$ determined by the composites $\Delta^J \rightarrow \Delta^k \rightarrow \phi(\sigma(0)) \rightarrow \phi(\sigma(j))$. This is clearly pseudonatural in maps in $\mathbf{\Delta}^{\text{op}}$, i.e. we have a pseudofunctor $\mathbf{\Delta}^{\text{op}} \rightarrow \text{Fun}([1], \text{Cat})$ that to $[n]$ assigns $\nu_{[n]}: [1] \times \text{Fun}([n], \text{Set}_\Delta^+) \rightarrow (\text{Set}_\Delta^+)_{/\Delta^n}$.

Proposition 3.5. *Suppose $\phi: [n] \rightarrow \text{Set}_\Delta^+$ is fibrant. Then the natural map $\nu_{[n], \phi}: M_{[n]}^{\natural}(\phi) \rightarrow N_{[n]}^+(\phi)$ is a coCartesian equivalence.*

Proof. Since $N_{[n]}^+(\phi) \rightarrow \Delta^n$ is a coCartesian fibration by [Lur09a, Proposition 3.2.5.18], it follows from [Lur09a, Proposition 3.2.2.14] that it suffices to check that $\nu_{[n], \phi}$ is a “quasi-equivalence” in the sense of [Lur09a, Definition 3.2.2.6]. Thus we need only show that the induced map on fibres $M_{[n]}^{\natural}(\phi)_i \rightarrow N_{[n]}^+(\phi)_i$ is a categorical equivalence for all $i = 0, \dots, n$. But unwinding the definitions we see that this can be identified with the identity map $\phi(i) \rightarrow \phi(i)$ (marked by the equivalences). \square

Let $\mathrm{Un}_{[n]}^{+, \mathrm{co}}: \mathrm{Fun}([n], \mathrm{Set}_{\Delta}^+) \rightarrow (\mathrm{Set}_{\Delta}^+)_{/\Delta^n}$ be the coCartesian version of the marked *unstraightening functor* defined in [Lur09a, §3.2.1]. By [Lur09a, Remark 3.2.5.16] there is a natural transformation $\lambda_{[n]}: N_{[n]}^+ \rightarrow \mathrm{Un}_{[n]}^{+, \mathrm{co}}$, which is a weak equivalence on fibrant objects by [Lur09a, Corollary 3.2.5.20]. Since this is also pseudonatural in Δ^{op} , combining this with Proposition 3.5 we immediately get:

Corollary 3.6. *For every $[n] \in \Delta^{\mathrm{op}}$ there is a natural transformation $\lambda_{[n]}\nu_{[n]}: M_{[n]}^{\sharp}(-) \rightarrow \mathrm{Un}_{[n]}^{+, \mathrm{co}}(-)$, and this is pseudonatural in $[n] \in \Delta^{\mathrm{op}}$. If $\phi: [n] \rightarrow \mathrm{Set}_{\Delta}^+$ is fibrant, then the map $M_{[n]}^{\sharp}(\phi) \rightarrow \mathrm{Un}_{[n]}^{+, \mathrm{co}}(\phi)$ is a Cartesian equivalence.*

It is immediate from the definition that $M_{[n]}^{\sharp}(\phi)$ is the pushout

$$\begin{array}{ccc} \phi(0)^{\sharp} \times (\Delta^{\{1, \dots, n\}})^{\sharp} & \hookrightarrow & \phi(0)^{\sharp} \times (\Delta^n)^{\sharp} \\ \downarrow & & \downarrow \\ M_{[n-1]}^{\sharp}(\phi|_{\{1, \dots, n\}}) & \hookrightarrow & M_{[n]}^{\sharp}(\phi). \end{array}$$

Moreover, since the category of marked simplicial sets is a left proper model category and the top horizontal map is a cofibration, this is a homotopy pushout. Combining this with Corollary 3.6, we get the following:

Lemma 3.7. *Suppose $F: [n] \rightarrow \mathrm{Cat}_{\infty}$ is a functor, and that $\mathcal{E} \rightarrow \Delta^n$ is the associated coCartesian fibration. Let \mathcal{E}' be the pullback of \mathcal{E} along the inclusion $\Delta^{\{1, \dots, n\}} \hookrightarrow \Delta^n$. Then there is a pushout square*

$$\begin{array}{ccc} F(0) \times \Delta^{\{1, \dots, n\}} & \longrightarrow & F(0) \times \Delta^n \\ \downarrow & & \downarrow \\ \mathcal{E}' & \longrightarrow & \mathcal{E} \end{array}$$

in Cat_{∞} .

Unwinding the definition, we see that $M_{[n]}^{\sharp}(\phi)$ is the colimit of the diagram

$$\begin{array}{ccc}
& & \phi(0) \times (\Delta^n)^\# \\
& \nearrow & \\
\phi(0) \times (\Delta^{\{1, \dots, n\}})^\# & & \\
& \searrow & \\
& & \phi(1) \times (\Delta^{\{1, \dots, n\}})^\# \\
& \nearrow & \\
\phi(1) \times (\Delta^{\{2, \dots, n\}})^\# & & \\
& \searrow & \\
& & \phi(2) \times (\Delta^{\{2, \dots, n\}})^\# \\
& & \vdots \\
& & \\
& & \phi(n-1) \times (\Delta^{\{n-1, n\}})^\# \\
& \nearrow & \\
\phi(n-1) \times (\Delta^{\{n\}})^\# & & \\
& \searrow & \\
& & \phi(n) \times (\Delta^{\{n\}})^\#
\end{array}$$

By Example 2.4 the category indexing this colimit is a cofinal subcategory of the twisted arrow category $\mathrm{Tw}([n])$ of $[n]$ — this is easy to check using [Lur09a, Corollary 4.1.3.3] since both categories are partially ordered sets. Hence we may identify $M_{[n]}^\#(\phi)$ with the coend

$$\mathrm{colim}_{\mathrm{Tw}([n])} (\mathbb{N}[n]_{-/} \times \phi(-)).$$

Moreover, since we can write this colimit as an iterated pushout along cofibrations, this is a homotopy colimit. Using the results of appendix A we can then prove the following:

Proposition 3.8. *There is a natural equivalence*

$$\mathrm{colim}_{\mathrm{Tw}([n])} \phi(-) \times [n]_{-/} \xrightarrow{\sim} \mathrm{Un}_{[n]}^{\mathrm{co}}(\phi)$$

of functors $\mathrm{Fun}([n], \mathrm{Cat}_\infty) \rightarrow \mathrm{Cat}_\infty$, natural in $\mathbf{\Delta}^{\mathrm{op}}$.

Lemma 3.9. *Let $G: \mathbf{C} \rightarrow \mathbf{D}$ be a right Quillen functor between model categories, such that \mathbf{C} has functorial fibrant replacements. Suppose $f: X \rightarrow \bar{X}$ and $g: Y \rightarrow \bar{Y}$ are weak equivalences such that \bar{X} and \bar{Y} are fibrant, and $G(f)$ and $G(g)$ are weak equivalences in \mathbf{D} . Then if $h: X \rightarrow Y$ is a weak equivalence, the morphism $G(h)$ is also a weak equivalence in \mathbf{D} .*

Proof. Let $Q: \mathbf{C} \rightarrow \mathbf{C}$ be a fibrant replacement functor, and let $\eta: \text{id} \rightarrow Q$ be a natural weak equivalence. Then we have a commutative diagram

$$\begin{array}{ccc}
 \bar{X} & \xrightarrow{i_{\bar{X}}} & Q\bar{X} \\
 f \uparrow & & \uparrow Qf \\
 X & \xrightarrow{i_X} & QX \\
 \downarrow h & & \downarrow Qh \\
 Y & \xrightarrow{i_Y} & QY \\
 \downarrow g & & \downarrow Qg \\
 \bar{Y} & \xrightarrow{i_{\bar{Y}}} & Q\bar{Y}
 \end{array}$$

Here $i_{\bar{X}}$, $i_{\bar{Y}}$, Qf , Qh , and Qg are all weak equivalences between fibrant objects, and so are taken by the right Quillen functor G to weak equivalences in \mathbf{D} . Since by assumption the same is true for f and g , the maps $G(i_X)$ and $G(i_Y)$ must be weak equivalences by the 2-out-of-3 property, and so for the same reason the map $G(h)$ is also a weak equivalence. \square

Proof of Proposition 3.8. We will prove this by applying Proposition A.29 to a relative Grothendieck fibration constructed in the same way as in Proposition A.30. The only difference is that the mapping simplex of a functor $\phi: [n] \rightarrow \text{Set}_{\Delta}^+$ is not in general fibrant. We must therefore consider a larger relative subcategory of $(\text{Set}_{\Delta}^+)_{/\Delta^n}$ containing the mapping simplices of fibrant functors whose associated ∞ -category is still $\text{Cat}_{\infty/\Delta^n}^{\text{cocart}}$.

By [Lur09a, Proposition 3.2.2.7] every mapping simplex admits a weak equivalence to a fibrant object that is preserved under pullbacks along all morphisms in Δ . We therefore think of $M_{[n]}^{\natural}$ and $\text{Un}_{[n]}^{+, \text{co}}$ as functors from fibrant objects in $\text{Fun}([n], \text{Set}_{\Delta}^+)$ to objects in $(\text{Set}_{\Delta}^+)_{/\Delta^n}$ that admit a weak equivalence to a fibrant object that is preserved by pullbacks — by Lemma 3.9 all weak equivalences between such objects are preserved by pullbacks, so we still get functors of relative categories.

It remains to show that inverting the weak equivalences in this subcategory gives the same ∞ -category as inverting the weak equivalences in the subcategory of fibrant objects. This follows from [BK12, 7.5], since the fibrant replacement functor gives a homotopy equivalence of relative categories. \square

4. FREE FIBRATIONS

Our goal in this section is to prove that for any ∞ -category \mathcal{C} , the forgetful functor

$$\text{Cat}_{\infty/\mathcal{C}}^{\text{cart}} \rightarrow \text{Cat}_{\infty/\mathcal{C}}$$

has a left adjoint, given by the following explicit formula:

Definition 4.1. Let \mathcal{C} be an ∞ -category. For $p: \mathcal{E} \rightarrow \mathcal{C}$ any functor of ∞ -categories, let $F(p)$ denote the map $\mathcal{E} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1} \rightarrow \mathcal{C}$, where the pullback is along the target fibration $\mathcal{C}^{\Delta^1} \rightarrow \mathcal{C}$ given by evaluation at $1 \in \Delta^1$, and the projection $F(p)$ is induced by evaluation at 0. Then F defines a functor $\text{Cat}_{\infty/\mathcal{C}} \rightarrow \text{Cat}_{\infty/\mathcal{C}}$.

The projection $F(p): \mathcal{E} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1} \rightarrow \mathcal{C}$ is called the free Cartesian fibration on $p: \mathcal{E} \rightarrow \mathcal{C}$.

Example 4.2. The free Cartesian fibration on the identity $\mathcal{C} \rightarrow \mathcal{C}$ is the source fibration $F: \mathcal{C}^{\Delta^1} \rightarrow \mathcal{C}$, given by evaluation at $0 \in \Delta^1$.

Lemma 4.3. *The functor F factors through the subcategory $\text{Cat}_{\infty/\mathcal{C}}^{\text{cart}} \rightarrow \text{Cat}_{\infty/\mathcal{C}}$.*

Proof. By [Lur09a, Corollary 2.4.7.12] the projection $F(p) \rightarrow \mathcal{C}$ is a Cartesian fibration for any $p: \mathcal{E} \rightarrow \mathcal{C}$, and a morphism in $F(p)$ is Cartesian if and only if its image in \mathcal{E} is an equivalence. It is thus clear that for any map $\phi: \mathcal{E} \rightarrow \mathcal{F}$ in $\text{Cat}_{\infty/\mathcal{C}}$, the induced map $F(\phi)$ preserves Cartesian morphisms, since the diagram

$$\begin{array}{ccc} \mathcal{E} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1} & \longrightarrow & \mathcal{F} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1} \\ \downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{F} \end{array}$$

commutes. □

Remark 4.4. If $p: \mathcal{E} \rightarrow \mathcal{C}$ is a functor, the objects of $F(p)$ can be identified with pairs $(e, \phi: c \rightarrow p(e))$ where $e \in \mathcal{E}$ and ϕ is a morphism in \mathcal{C} . Similarly, a morphism in $F(p)$ can be identified with the data of a morphism $\alpha: e' \rightarrow e$ in \mathcal{C} and a commutative diagram

$$\begin{array}{ccc} c' & \longrightarrow & p(e') \\ \downarrow & & \downarrow \\ c & \longrightarrow & p(e). \end{array}$$

If (e, ϕ) is an object in $F(p)$ and $\psi: c' \rightarrow c$ is a morphism in \mathcal{C} , the Cartesian morphism over ψ with target (e, ϕ) is the obvious morphism from $(e, \phi\psi)$.

Our goal is then to prove the following:

Theorem 4.5. *Let \mathcal{C} be an ∞ -category. The functor $F: \text{Cat}_{\infty/\mathcal{C}} \rightarrow \text{Cat}_{\infty/\mathcal{C}}^{\text{cart}}$ is left adjoint to the forgetful functor $U: \text{Cat}_{\infty/\mathcal{C}}^{\text{cart}} \rightarrow \text{Cat}_{\infty/\mathcal{C}}$.*

Remark 4.6. Analogues of this result in the setting of ordinary categories (as well as enriched and internal variants) can be found in [Str80] and [Web07].

Composition with the degeneracy $s_0: \Delta^1 \rightarrow \Delta^0$ induces a functor $\mathcal{C} \rightarrow \mathcal{C}^{\Delta^1}$ (which sends an object of \mathcal{C} to the constant functor $\Delta^1 \rightarrow \mathcal{C}$ with that value). Since the composition of this with both of the evaluation maps $\mathcal{C}^{\Delta^1} \rightarrow \mathcal{C}$ is the identity, this obviously gives a natural map $\mathcal{E} \rightarrow \mathcal{E} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1}$ over \mathcal{C} , i.e. a natural transformation

$$\eta: \text{id} \rightarrow UF$$

of functors $\text{Cat}_{\infty/\mathcal{C}} \rightarrow \text{Cat}_{\infty/\mathcal{C}}$. We will show that this is a *unit transformation* in the sense of [Lur09a, Definition 5.2.2.7], i.e. that it induces an equivalence

$$\text{Map}_{\mathcal{C}}^{\text{Cart}}(F(\mathcal{E}), \mathcal{F}) \rightarrow \text{Map}_{\mathcal{C}}(UF(\mathcal{E}), U(\mathcal{F})) \rightarrow \text{Map}_{\mathcal{C}}(\mathcal{E}, U(\mathcal{F}))$$

for all $\mathcal{E} \rightarrow \mathcal{C}$ in $\text{Cat}_{\infty/\mathcal{C}}$ and $\mathcal{F} \rightarrow \mathcal{C}$ in $\text{Cat}_{\infty/\mathcal{C}}^{\text{cart}}$.

We first check this for the objects of $\text{Cat}_{\infty/\mathcal{C}}$ with source Δ^0 and Δ^1 , which generate $\text{Cat}_{\infty/\mathcal{C}}$ under colimits. If a map $\Delta^0 \rightarrow \mathcal{C}$ corresponds to the object $x \in \mathcal{C}$, then its image under F is the projection $\mathcal{C}/_x \rightarrow \mathcal{C}$. Thus in this case we need to show the following:

Lemma 4.7.

- (i) For every $x \in \mathcal{C}$, the map $\text{Map}_{\mathcal{C}}^{\text{Cart}}(\mathcal{C}_{/x}, \mathcal{E}) \rightarrow \text{Map}_{\mathcal{C}}(\{x\}, \mathcal{E}) \simeq \iota\mathcal{E}_x$ is an equivalence.
(ii) More generally, for any $\mathcal{X} \in \text{Cat}_{\infty}$, the map

$$\text{Map}_{\mathcal{C}}^{\text{Cart}}(\mathcal{X} \times \mathcal{C}_{/x}, \mathcal{E}) \rightarrow \text{Map}_{\mathcal{C}}(\mathcal{X} \times \{x\}, \mathcal{E}) \simeq \text{Map}(\mathcal{X}, \mathcal{E}_x)$$

is an equivalence.

Proof. The inclusion of the ∞ -category of right fibrations over \mathcal{C} into $\text{Cat}_{\infty/\mathcal{C}}^{\text{cart}}$ has a right adjoint, which sends a Cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ to its restriction to the subcategory $\mathcal{E}_{\text{cart}}$ of \mathcal{E} where the morphisms are the p -Cartesian morphisms. The map $\text{Map}_{\mathcal{C}}^{\text{Cart}}(\mathcal{C}_{/x}, \mathcal{E}) \rightarrow \iota\mathcal{E}_x$ thus factors through the ∞ -category of right fibrations over \mathcal{C} , which is modelled by the contravariant model structure on $(\text{Set}_{\Delta})_{/\mathcal{C}}$ constructed in [Lur09a, §2.1.4].

By [Lur09a, Propostion 4.4.4.5], the inclusion $\{x\} \rightarrow \mathcal{C}_{/x}$ is a trivial cofibration in this model category. Since this is a simplicial model category by [Lur09a, Proposition 2.1.4.8], it follows immediately that we have an equivalence

$$\text{Map}_{\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{E}_{\text{cart}}) \xrightarrow{\sim} \text{Map}_{\mathcal{C}}(\{x\}, \mathcal{E}_{\text{cart}}).$$

This proves (i). To prove (ii) we simply observe that since the model category is simplicial, the product $K \times \{x\} \rightarrow K \times \mathcal{C}_{/x}$ is also a trivial cofibration for any simplicial set K . \square

For the case of maps $\Delta^1 \rightarrow \mathcal{C}$, the key observation is:

Proposition 4.8. *If $\Delta^1 \rightarrow \mathcal{C}$ corresponds to a map $f: x \rightarrow y$ in \mathcal{C} , then the diagram*

$$\begin{array}{ccc} \mathcal{C}_{/x} & \longrightarrow & \Delta^1 \times \mathcal{C}_{/x} \\ \downarrow & & \downarrow \\ \mathcal{C}_{/y} & \longrightarrow & \mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \Delta^1 \end{array}$$

is a pushout square in $\text{Cat}_{\infty/\mathcal{C}}^{\text{cart}}$, where the top map is induced by the inclusion $\{0\} \hookrightarrow \Delta^1$.

Proof. Since colimits in $\text{Cat}_{\infty/\mathcal{C}}^{\text{cart}} \simeq \text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_{\infty})$ are detected fibrewise, it suffices to show that for every $c \in \mathcal{C}$, the diagram on fibres is a pushout in Cat_{∞} . This diagram can be identified with

$$\begin{array}{ccc} \text{Map}_{\mathcal{C}}(c, x) & \longrightarrow & \Delta^1 \times \text{Map}_{\mathcal{C}}(c, x) \\ \downarrow & & \downarrow \\ \text{Map}_{\mathcal{C}}(c, y) & \longrightarrow & \mathcal{C}_{c/} \times_{\mathcal{C}} \Delta^1. \end{array}$$

This is a pushout by Lemma 3.7, since $\mathcal{C}_{c/} \times_{\mathcal{C}} \Delta^1 \rightarrow \Delta^1$ is the left fibration corresponding to the map of spaces $\text{Map}_{\mathcal{C}}(c, x) \rightarrow \text{Map}_{\mathcal{C}}(c, y)$ induced by composition with f . \square

Corollary 4.9. *For every map $\sigma: \Delta^1 \rightarrow \mathcal{C}$ and every Cartesian fibration $\mathcal{E} \rightarrow \mathcal{C}$, the map*

$$\eta_{\sigma}^*: \text{Map}_{\mathcal{C}}^{\text{Cart}}(\mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \Delta^1, \mathcal{E}) \rightarrow \text{Map}_{\mathcal{C}}(\Delta^1, \mathcal{E})$$

is an equivalence.

Proof. By Proposition 4.8, if the map σ corresponds to a morphism $f: x \rightarrow y$ in \mathcal{C} , we have a pullback square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}^{\mathrm{Cart}}(\mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \Delta^1, \mathcal{E}) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}^{\mathrm{Cart}}(\mathcal{C}_{/y}, \mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}}^{\mathrm{Cart}}(\Delta^1 \times \mathcal{C}_{/x}, \mathcal{E}) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}^{\mathrm{Cart}}(\mathcal{C}_{/x}, \mathcal{E}). \end{array}$$

The map η_{σ}^* fits in an obvious map of commutative squares from this to the square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(\Delta^1, \mathcal{E}) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(\{y\}, \mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}}(\Delta^1 \times \{x\}, \mathcal{E}) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(\{x\}, \mathcal{E}), \end{array}$$

where the right vertical map is given by composition with Cartesian morphisms over f . Since $\mathcal{E} \rightarrow \mathcal{C}$ is a Cartesian fibration, this is also a pullback square (this amounts to saying morphisms in \mathcal{E} over f are equivalent to composites of a morphism in \mathcal{E}_x with a Cartesian morphism over f . But now by Lemma 4.7, we have a natural transformation of pullback squares that's an equivalence everywhere except the top left corner, so the map in that corner is an equivalence too. \square

To complete the proof, we now only need to observe that F preserves colimits:

Lemma 4.10. *F preserves colimits.*

Proof. Colimits in $\mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}$ are detected fibrewise, so we need to show that for every $x \in \mathcal{C}$, the functor $\mathcal{C}_{x/} \times_{\mathcal{C}} (-): \mathrm{Cat}_{\infty/\mathcal{C}} \rightarrow \mathrm{Cat}_{\infty}$ preserves colimits. But $\mathcal{C}_{x/} \rightarrow \mathcal{C}$ is a flat fibration by [Lur14, Example B.3.11], so pullback along it preserves colimits as a functor $\mathrm{Cat}_{\infty/\mathcal{C}} \rightarrow \mathrm{Cat}_{\infty/\mathcal{C}_{x/}}$ (since on the level of model categories the pullback functor is a left Quillen functor by [Lur14, Corollary B.3.15]), and the forgetful functor $\mathrm{Cat}_{\infty/\mathcal{C}_{x/}} \rightarrow \mathrm{Cat}_{\infty}$ also preserves colimits. \square

Proof of Theorem 4.5. By Lemma 4.10 the source and target of the natural map

$$\mathrm{Map}_{\mathcal{C}}^{\mathrm{Cart}}(F(\mathcal{E}), \mathcal{F}) \rightarrow \mathrm{Map}_{\mathcal{C}}(\mathcal{E}, U(\mathcal{F}))$$

both take colimits in \mathcal{E} to limits of spaces, so it suffices to check that this map is an equivalence when $\mathcal{E} = \Delta^0$ and Δ^1 , since the maps $\Delta^0, \Delta^1 \rightarrow \mathcal{C}$ generate $\mathrm{Cat}_{\infty/\mathcal{C}}$ under colimits. Thus the Theorem follows from Lemma 4.7 and Corollary 4.9. \square

Proposition 4.11.

- (i) *Suppose $X \rightarrow S$ is a map of ∞ -categories and K is an ∞ -category. Then there is a natural equivalence $F(K \times X) \simeq K \times F(X)$.*
- (ii) *The unit map $X \rightarrow F(X)$ induces an equivalence of ∞ -categories*

$$\mathrm{Fun}_S^{\mathrm{cart}}(F(X), Y) \xrightarrow{\sim} \mathrm{Fun}_S(X, Y).$$

Proof. (i) is immediate from the definition of F . Then (ii) follows from the natural equivalence

$$\begin{aligned} \text{Map}(K, \text{Fun}_{\mathcal{S}}(A, B)) &\simeq \text{Map}_{\mathcal{S}}(K \times A, B) \simeq \text{Map}_{\mathcal{S}}^{\text{cart}}(F(K \times A), B) \\ &\simeq \text{Map}_{\mathcal{S}}^{\text{cart}}(K \times F(A), B) \simeq \text{Map}(K, \text{Fun}_{\mathcal{S}}^{\text{cart}}(F(A), B)). \quad \square \end{aligned}$$

5. NATURAL TRANSFORMATIONS AS AN END

It is a familiar result from ordinary category theory that for two functors $F, G: \mathbf{C} \rightarrow \mathbf{D}$ the set of natural transformations from F to G can be identified with the *end* of the functor $\mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \text{Set}$ that sends (C, C') to $\text{Hom}_{\mathbf{D}}(F(C), G(C'))$. Our goal in this section is to prove the analogous result for ∞ -categories:

Proposition 5.1. *Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be two functors of ∞ -categories. Then the space $\text{Map}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(F, G)$ of natural transformations from F to G is naturally equivalent to the end of the functor*

$$\mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{(F^{\text{op}}, G)} \mathcal{D}^{\text{op}} \times \mathcal{D} \xrightarrow{\text{Map}_{\mathcal{D}}} \mathcal{S}.$$

A proof of this is also given in [Gla14, Proposition 2.3]; we include a slightly different proof for completeness.

Lemma 5.2. *Suppose $i: \mathcal{C}_0 \hookrightarrow \mathcal{C}$ is a fully faithful functor of ∞ -categories. Then for any ∞ -category \mathcal{X} the functor $\text{Fun}(\mathcal{X}, \mathcal{C}_0) \rightarrow \text{Fun}(\mathcal{X}, \mathcal{C})$ is also fully faithful.*

Proof. A functor $G: \mathcal{A} \rightarrow \mathcal{B}$ is fully faithful if and only if the commutative square of spaces

$$\begin{array}{ccc} \text{Map}(\Delta^1, \mathcal{A}) & \longrightarrow & \text{Map}(\Delta^1, \mathcal{B}) \\ \downarrow & & \downarrow \\ \iota \mathcal{A}^{\times 2} & \longrightarrow & \iota \mathcal{B}^{\times 2} \end{array}$$

is Cartesian. Thus, we must show that for any \mathcal{X} , the square

$$\begin{array}{ccc} \text{Map}(\Delta^1 \times \mathcal{X}, \mathcal{C}_0) & \longrightarrow & \text{Map}(\Delta^1 \times \mathcal{X}, \mathcal{C}) \\ \downarrow & & \downarrow \\ \text{Map}(\mathcal{X}, \mathcal{C}_0)^{\times 2} & \longrightarrow & \text{Map}(\mathcal{X}, \mathcal{C})^{\times 2} \end{array}$$

is Cartesian. But this is equivalent to the commutative square of ∞ -categories

$$\begin{array}{ccc} \mathcal{C}_0^{\Delta^1} & \longrightarrow & \mathcal{C}^{\Delta^1} \\ \downarrow & & \downarrow \\ \mathcal{C}_0^{\times 2} & \longrightarrow & \mathcal{C}^{\times 2} \end{array}$$

being Cartesian. By [Lur09a, Corollary 2.4.7.11] the vertical maps in this diagram are bifibrations in the sense of [Lur09a, Definition 2.4.7.2], so by [Lur09a, Propositions 2.4.7.6 and 2.4.7.7] to prove that this square is Cartesian it suffices to show that for all $x, y \in \mathcal{C}_0$ the induced map on fibres $(\mathcal{C}_0^{\Delta^1})_{(x,y)} \rightarrow (\mathcal{C}^{\Delta^1})_{(ix, iy)}$ is an equivalence. But this can be identified

with the map $\text{Map}_{\mathcal{C}_0}(x, y) \rightarrow \text{Map}_{\mathcal{C}}(ix, iy)$, which is an equivalence as i is by assumption fully faithful. \square

Proof of Proposition 5.1. By [Lur09a, Corollary 3.3.3.4], we can identify the limit of the functor

$$\phi: \text{Tw}(\mathcal{C}) \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{(F^{\text{op}}, G)} \mathcal{D}^{\text{op}} \times \mathcal{D} \xrightarrow{\text{Map}_{\mathcal{D}}} \mathcal{S}$$

with the space of sections of the associated left fibration. By [Lur14, Proposition 5.2.1.11], the left fibration associated to $\text{Map}_{\mathcal{D}}$ is the projection $\text{Tw}(\mathcal{D}) \rightarrow \mathcal{D}^{\text{op}} \times \mathcal{D}$, so the left fibration associated to ϕ is the pullback of this along $\text{Tw}(\mathcal{C}) \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}^{\text{op}} \times \mathcal{D}$. Thus the space of sections is equivalent to the space of commutative diagrams

$$\begin{array}{ccc} \text{Tw}(\mathcal{C}) & \longrightarrow & \text{Tw}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathcal{C}^{\text{op}} \times \mathcal{C} & \longrightarrow & \mathcal{D}^{\text{op}} \times \mathcal{D}, \end{array}$$

i.e. the space of maps from $\text{Tw}(\mathcal{C})$ to the pullback of $\text{Tw}(\mathcal{D})$ in the ∞ -category of left fibrations over $\mathcal{C}^{\text{op}} \times \mathcal{C}$. Using the “straightening” equivalence between this ∞ -category and that of functors $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{S}$ we can identify our limit with the space of maps from $y_{\mathcal{C}}$ to $F^* \circ y_{\mathcal{D}} \circ G$ in $\text{Fun}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathcal{S}) \simeq \text{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{C}))$. Since F^* has a left adjoint $F_!$, we have an equivalence

$$\text{Map}_{\text{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{C}))}(y_{\mathcal{C}}, F^* y_{\mathcal{D}} \circ G) \simeq \text{Map}_{\text{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))}(F_! y_{\mathcal{C}}, y_{\mathcal{D}} \circ G).$$

But by [Lur09a, Proposition 5.2.6.3] the functor $F_! y_{\mathcal{C}}$ is equivalent to $y_{\mathcal{D}} \circ F$, and so the limit is equivalent to $\text{Map}_{\text{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))}(y_{\mathcal{D}} \circ F, y_{\mathcal{D}} \circ G)$. The Yoneda embedding $y_{\mathcal{D}}$ is fully faithful by [Lur09a, Proposition 5.1.3.1], so Lemma 5.2 implies that the functor $\text{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ given by composition with $y_{\mathcal{D}}$ is fully faithful, hence we have an equivalence

$$\text{Map}_{\text{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))}(y_{\mathcal{D}} \circ F, y_{\mathcal{D}} \circ G) \simeq \text{Map}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(F, G),$$

which completes the proof. \square

6. ENHANCED MAPPING FUNCTORS

The Yoneda embedding for ∞ -categories, constructed in [Lur09a, Proposition 5.1.3.1] or [Lur14, Proposition 5.2.1.11], gives for any ∞ -category \mathcal{C} a mapping space functor $\text{Map}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{S}$. In some cases, this is the underlying functor to spaces of an interesting functor $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Cat}_{\infty}$ — in particular, this is the case if \mathcal{C} is the underlying ∞ -category of an $(\infty, 2)$ -category.

Definition 6.1. A mapping ∞ -category functor for an ∞ -category \mathcal{C} is a functor

$$\text{MAP}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Cat}_{\infty}$$

such that the composite $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Cat}_{\infty} \xrightarrow{\iota} \mathcal{S}$ is equivalent to $\text{Map}_{\mathcal{C}}$.

Lemma 6.2. Suppose \mathcal{C} is an $(\infty, 2)$ -category with underlying ∞ -category \mathcal{C}' . Then \mathcal{C}' has a mapping ∞ -category functor that sends (C, D) to the ∞ -category of maps from C to D in \mathcal{C} .

Proof. This follows from the same argument as in [Lur09a, §5.1.3], using the model of $(\infty, 2)$ -categories as categories enriched in marked simplicial sets, cf. [Lur09b]. \square

Example 6.3. The ∞ -category Cat_∞ of ∞ -categories has a mapping ∞ -category functor

$$\text{MAP}_{\text{Cat}_\infty} := \text{Fun},$$

defined using the fact that Cat_∞ is the coherent nerve of the simplicial category of quasi-categories.

Lemma 6.4. *Suppose \mathcal{C} is an ∞ -category with a mapping ∞ -category functor $\text{MAP}_\mathcal{C}$. Then for any ∞ -category \mathcal{D} the functor ∞ -category $\mathcal{C}^\mathcal{D}$ has a mapping ∞ -category functor $\text{MAP}_{\mathcal{C}^\mathcal{D}}$ given by the composite*

$$(\mathcal{C}^\mathcal{D})^{\text{op}} \times \mathcal{C}^\mathcal{D} \rightarrow \text{Fun}(\mathcal{D}^{\text{op}} \times \mathcal{D}, \mathcal{C}^{\text{op}} \times \mathcal{C}) \rightarrow \text{Fun}(\text{Tw}(\mathcal{D}), \text{Cat}_\infty) \xrightarrow{\lim} \text{Cat}_\infty,$$

where the second functor is given by composition with the projection $\text{Tw}(\mathcal{D}) \rightarrow \mathcal{D}^{\text{op}} \times \mathcal{D}$ and $\text{MAP}_\mathcal{C}$.

Proof. We must show that the underlying functor to spaces $\iota \circ \text{MAP}_{\mathcal{C}^\mathcal{D}}$ is $\text{Map}_{\mathcal{C}^\mathcal{D}}$. Since ι preserves limits (being a right adjoint), this follows immediately from Proposition 5.1. \square

Definition 6.5. Suppose \mathcal{C} is an ∞ -category with a mapping ∞ -category functor $\text{MAP}_\mathcal{C}$. We say that \mathcal{C} is *tensoried over Cat_∞* if there is a functor $\otimes: \text{Cat}_\infty \times \mathcal{C} \rightarrow \mathcal{C}$ such that for every $C \in \mathcal{C}$ the functor $- \otimes C: \text{Cat}_\infty \rightarrow \mathcal{C}$ is left adjoint to $\text{MAP}_\mathcal{C}(C, -)$.

Example 6.6. The ∞ -category Cat_∞ is obviously tensoried over Cat_∞ via the Cartesian product $\times: \text{Cat}_\infty \times \text{Cat}_\infty \rightarrow \text{Cat}_\infty$.

Lemma 6.7. *Suppose \mathcal{C} is an ∞ -category with a mapping ∞ -category $\text{MAP}_\mathcal{C}$ that is tensoried over Cat_∞ . Then for any ∞ -category \mathcal{D} , the mapping ∞ -category functor for $\mathcal{C}^\mathcal{D}$ defined in Lemma 6.4 is also tensoried over Cat_∞ , via the composite*

$$\text{Cat}_\infty \times \mathcal{C}^\mathcal{D} \rightarrow \text{Cat}_\infty^\mathcal{D} \times \mathcal{C}^\mathcal{D} \simeq (\text{Cat}_\infty \times \mathcal{C})^\mathcal{D} \rightarrow \mathcal{C}^\mathcal{D}$$

where the first functor is given by composition with the functor $\mathcal{D} \rightarrow *$ and the last by composition with the tensor functor for \mathcal{C} .

Proof. We must show that for every functor $F: \mathcal{D} \rightarrow \mathcal{C}$ there is a natural equivalence

$$\text{Map}_{\mathcal{C}^\mathcal{D}}(\mathcal{X} \otimes F, G) \simeq \text{Map}_{\text{Cat}_\infty}(\mathcal{X}, \text{MAP}_{\mathcal{C}^\mathcal{D}}(F, G)).$$

By Proposition 5.1 and the definition of \otimes for $\mathcal{C}^\mathcal{D}$, there is a natural equivalence

$$\text{Map}_{\mathcal{C}^\mathcal{D}}(\mathcal{X} \otimes F, G) \simeq \lim_{\text{Tw}(\mathcal{D})} \text{Map}_\mathcal{C}(\mathcal{X} \otimes F(-), G(-)).$$

Now using that \mathcal{C} is tensoried over Cat_∞ , this is clearly naturally equivalent to

$$\lim_{\text{Tw}(\mathcal{D})} \text{Map}_{\text{Cat}_\infty}(\mathcal{X}, \text{MAP}_\mathcal{C}(F(-), G(-))).$$

Moving the limit inside, this is

$$\text{Map}_{\text{Cat}_\infty}(\mathcal{X}, \lim_{\text{Tw}(\mathcal{D})} \text{MAP}_\mathcal{C}(F(-), G(-))),$$

which is $\text{Map}_{\text{Cat}_\infty}(\mathcal{X}, \text{MAP}_{\mathcal{C}^\mathcal{D}}(F, G))$ by definition. \square

Example 6.8. For any ∞ -category \mathcal{D} , the ∞ -category $\text{Cat}_\infty^\mathcal{D}$ is tensoried over Cat_∞ : $\mathcal{X} \otimes F$ is the functor $D \mapsto \mathcal{X} \times F(D)$.

In the case where \mathcal{C} is the ∞ -category Cat_∞ of ∞ -categories, Lemma 6.4 gives a mapping ∞ -category functor

$$\text{Nat}_{\mathcal{D}^{\text{op}}} := \text{MAP}_{\text{Fun}(\mathcal{D}^{\text{op}}, \text{Cat}_\infty)}$$

for $\text{Fun}(\mathcal{D}^{\text{op}}, \text{Cat}_\infty)$ for any ∞ -category \mathcal{D} . However, using the equivalence $\text{Fun}(\mathcal{D}^{\text{op}}, \text{Cat}_\infty) \simeq \text{Cat}_{\infty/\mathcal{D}}^{\text{cart}}$ we can construct another such functor: the space of maps from \mathcal{E} to \mathcal{E}' in $\text{Cat}_{\infty/\mathcal{D}}^{\text{cart}}$ is the underlying ∞ -groupoid of the ∞ -category $\text{Fun}_{\mathcal{D}}^{\text{cart}}(\mathcal{E}, \mathcal{E}')$, the full subcategory of $\text{Fun}_{\mathcal{D}}(\mathcal{E}, \mathcal{E}')$ spanned by the functors that preserve Cartesian morphisms. We will now prove that these two functors are equivalent:

Proposition 6.9. *For every ∞ -category \mathcal{C} there is a natural equivalence*

$$\text{Fun}_{\mathcal{C}}^{\text{cart}}(\mathcal{E}, \mathcal{E}') \simeq \text{Nat}_{\mathcal{C}^{\text{op}}}(\text{St}_{\mathcal{C}}\mathcal{E}, \text{St}_{\mathcal{C}}\mathcal{E}').$$

Proof. By the Yoneda Lemma it suffices to show that there are natural equivalences

$$\text{Map}_{\text{Cat}_\infty}(\mathcal{X}, \text{Fun}_{\mathcal{C}}^{\text{cart}}(\mathcal{E}, \mathcal{E}')) \simeq \text{Map}_{\text{Cat}_\infty}(\mathcal{X}, \text{Nat}_{\mathcal{C}^{\text{op}}}(\text{St}_{\mathcal{C}}\mathcal{E}, \text{St}_{\mathcal{C}}\mathcal{E}')).$$

It is easy to see that $\text{Map}_{\text{Cat}_\infty}(\mathcal{X}, \text{Fun}_{\mathcal{C}}^{\text{cart}}(\mathcal{E}, \mathcal{E}'))$ is naturally equivalent to $\text{Map}_{\text{Cat}_{\infty/\mathcal{C}}}(\mathcal{X} \times \mathcal{E}, \mathcal{E}')$ — these correspond to the same components of $\text{Map}_{\text{Cat}_\infty}(\mathcal{X}, \text{Fun}_{\mathcal{C}}(\mathcal{E}, \mathcal{E}'))$. The equivalence $\text{St}_{\mathcal{C}}$ preserves products, so this is equivalent to the mapping space

$$\text{Map}_{\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_\infty)}(\text{St}_{\mathcal{C}}(\mathcal{X} \times \mathcal{C}) \times \text{St}_{\mathcal{C}}\mathcal{E}, \text{St}_{\mathcal{C}}\mathcal{E}').$$

But the projection $\mathcal{X} \times \mathcal{C} \rightarrow \mathcal{C}$ corresponds to the constant functor $c^*\mathcal{X}: \mathcal{C}^{\text{op}} \rightarrow * \rightarrow \text{Cat}_\infty$ with value \mathcal{C} (since the Cartesian fibration associated to this composite is precisely the pullback of $\mathcal{X} \rightarrow *$ along $\mathcal{C} \rightarrow *$). Thus there is a natural equivalence

$$\text{Map}_{\text{Cat}_\infty}(\mathcal{X}, \text{Fun}_{\mathcal{C}}^{\text{cart}}(\mathcal{E}, \mathcal{E}')) \simeq \text{Map}_{\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_\infty)}(c^*\mathcal{X} \times \text{St}_{\mathcal{C}}\mathcal{E}, \text{St}_{\mathcal{C}}\mathcal{E}').$$

But by Lemma 6.7, the ∞ -category $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_\infty)$ is tensored over Cat_∞ , and this is naturally equivalent to $\text{Map}_{\text{Cat}_\infty}(\mathcal{X}, \text{Nat}_{\mathcal{C}^{\text{op}}}(\text{St}_{\mathcal{C}}\mathcal{E}, \text{St}_{\mathcal{C}}\mathcal{E}'))$, as required. \square

7. CARTESIAN AND COCARTESIAN FIBRATIONS AS WEIGHTED COLIMITS

In ordinary category theory it is a familiar fact that the Grothendieck fibration associated to a functor $F: \mathbf{C}^{\text{op}} \rightarrow \text{Cat}$ can be identified with the oplax colimit of F , and the Grothendieck opfibration associated to a functor $F: \mathbf{C} \rightarrow \text{Cat}$ with the lax colimit of F . In this section we will show that Cartesian and coCartesian fibrations admit analogous descriptions.

It is immediate from our formula for the free Cartesian fibration that the sections of a Cartesian fibration are given by the oplax limit of the corresponding functor:

Proposition 7.1. *The ∞ -category of sections of the Cartesian fibration associated to F is given by the oplax limit of F . In other words, there is a natural equivalence*

$$\text{Fun}_{\mathcal{C}}(\mathcal{C}, \text{Un}_{\mathcal{C}}(F)) \simeq \lim_{\text{Tw}(\mathcal{C})} \text{Fun}(\mathcal{C}_{-/}, F(-))$$

of functors $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_\infty) \rightarrow \text{Cat}_\infty$.

Proof. By Theorem 4.5 and Proposition 6.9 we have natural equivalences

$$\text{Fun}_{\mathcal{C}}(\mathcal{C}, \text{Un}_{\mathcal{C}}(F)) \simeq \text{Fun}_{\mathcal{C}}^{\text{cart}}(F(\mathcal{C}), \text{Un}_{\mathcal{C}}(F)) \simeq \text{Nat}_{\mathcal{C}^{\text{op}}}(\mathcal{C}_{-/}, F) \simeq \lim_{\text{Tw}(\mathcal{C}^{\text{op}})} \text{Fun}(\mathcal{C}_{-/}, F(-)).$$

\square

Definition 7.2. Let $F: \mathcal{C} \rightarrow \text{Cat}_\infty$ be a functor, and let $\mathcal{F} \rightarrow \mathcal{C}$ be its associated coCartesian fibration. Given an ∞ -category \mathcal{X} , write $\Phi_{\mathcal{X}}^F$ for the simplicial set over \mathcal{C} with the universal property

$$\text{Hom}(K \times_{\mathcal{C}} \mathcal{F}, \mathcal{X}) \cong \text{Hom}_{\mathcal{C}}(K, \Phi_{\mathcal{X}}^F).$$

By [Lur09a, Corollary 3.2.2.13] the projection $\Phi_{\mathcal{X}}^F \rightarrow \mathcal{C}$ is a Cartesian fibration.

Proposition 7.3. *The Cartesian fibration $\Phi_{\mathcal{X}}^F \rightarrow \mathcal{C}$ classifies the functor*

$$\text{Fun}(F(-), \mathcal{X}): \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_\infty.$$

Proof. We first consider the case where \mathcal{C} is a simplex Δ^n . By Proposition 3.8 there are natural equivalences

$$\text{colim}_{\text{Tw}([n])} \phi(-) \times [n]_{-/} \xrightarrow{\simeq} \text{Un}_{[n]}^{\text{co}}(\phi)$$

for any $\phi: [n] \rightarrow \text{Cat}_\infty$, natural in Δ . Thus by Proposition 7.1 there are natural equivalences

$$\begin{aligned} \text{Fun}_{\Delta^n}(\Delta^n, \Phi_{\mathcal{X}}^\phi) &\simeq \text{Fun}(\text{Un}_{[n]}^{\text{co}}(\phi), \mathcal{X}) \simeq \lim_{\text{Tw}([n])} \text{Fun}([n]_{-/}, \text{Fun}(\phi(-), \mathcal{X})) \\ &\simeq \text{Fun}_{\Delta^n}(\Delta^n, \text{Un}_{[n]}(\text{Fun}(\phi(-), \mathcal{X}))). \end{aligned}$$

Since this equivalence is natural in Δ^{op} and Cat_∞ is a localization of the presheaf ∞ -category $\mathcal{P}(\Delta)$, we get by the Yoneda lemma a natural equivalence

$$\Phi_{\mathcal{X}}^\phi \simeq \text{Un}_{[n]}^{\text{cart}}(\text{Fun}(\phi(-), \mathcal{X})).$$

Since Cat_∞ is an accessible localization of $\mathcal{P}(\Delta)$, any ∞ -category \mathcal{C} is naturally equivalent to the colimit of the diagram $\Delta_{/\mathcal{C}}^{\text{op}} \rightarrow \text{Cat}_{\infty/\mathcal{C}} \rightarrow \text{Cat}_\infty$. Now given $F: \mathcal{C} \rightarrow \text{Cat}_\infty$, we have, since pullback along a Cartesian fibration preserves colimits,

$$\text{Un}_{\mathcal{C}}^{\text{cart}}(\text{Fun}(F(-), \mathcal{X})) \simeq \text{colim}_{\sigma \in \Delta_{/\mathcal{C}}^{\text{op}}} \text{Un}_{[n]}(\text{Fun}(F\sigma(-), \mathcal{X})) \simeq \text{colim}_{\sigma \in \Delta_{/\mathcal{C}}^{\text{op}}} \Phi_{\mathcal{X}}^{F\sigma} \simeq \Phi_{\mathcal{X}}^F,$$

which completes the proof. \square

Theorem 7.4. *The coCartesian fibration associated to a functor $F: \mathcal{C} \rightarrow \text{Cat}_\infty$ is given by the lax colimit of F . In other words, there is a natural equivalence*

$$\text{Un}_{\mathcal{C}}^{\text{co}}(F) \simeq \text{colim}_{\text{Tw}(\mathcal{C})} \mathcal{C}_{-/} \times F(-)$$

of functors $\text{Fun}(\mathcal{C}, \text{Cat}_\infty) \rightarrow \text{Cat}_\infty$.

Proof. Let $F: \mathcal{C} \rightarrow \text{Cat}_\infty$ be a functor. Then by Proposition 7.3, we have a natural equivalence

$$\text{Fun}(\text{Un}_{\mathcal{C}}(F), \mathcal{X}) \simeq \text{Fun}_{\mathcal{C}}(\mathcal{C}, \Phi_{\mathcal{X}}^F).$$

By Proposition 7.1 we have a natural equivalence between the right-hand side and

$$\lim_{\text{Tw}(\mathcal{C})} \text{Fun}(\mathcal{C}_{-/}, \text{Fun}(F(-), \mathcal{X})) \simeq \text{Fun}(\text{colim}_{\text{Tw}(\mathcal{C})} \mathcal{C}_{-/} \times F(-), \mathcal{X}).$$

By the Yoneda lemma, it follows that $\text{Un}_{\mathcal{C}}(F)$ is naturally equivalent to $\text{colim}_{\text{Tw}(\mathcal{C})} \mathcal{C}_{-/} \times F(-)$. \square

Corollary 7.5. *Any ∞ -category \mathcal{C} is the lax colimit of the constant functor $\mathcal{C} \rightarrow \text{Cat}_\infty$ with value $*$.*

Proof. The identity $\mathcal{C} \rightarrow \mathcal{C}$ is the coCartesian fibration associated to this functor. \square

Corollary 7.6. *The Cartesian fibration associated to a functor $F: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_\infty$ is given by the oplax colimit of F . In other words, there is a natural equivalence*

$$\text{Un}_{\mathcal{C}}(F) \simeq \text{colim}_{\text{Tw}(\mathcal{C})} \mathcal{C}_{-/} \times F(-)$$

of functors $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_\infty) \rightarrow \text{Cat}_\infty$.

Proof. We have a natural equivalence $\text{Un}_{\mathcal{C}}(F) \simeq \text{Un}_{\mathcal{C}^{\text{op}}}^{\text{co}}(F^{\text{op}})^{\text{op}}$. Since $(-)^{\text{op}}$ is an automorphism of Cat_∞ it preserves colimits, so by Theorem 7.4 we have

$$\text{Un}_{\mathcal{C}}(F) \simeq \left(\text{colim}_{\text{Tw}(\mathcal{C})} (\mathcal{C}^{\text{op}})_{-/} \times F(-)^{\text{op}} \right)^{\text{op}} \simeq \text{colim}_{\text{Tw}(\mathcal{C})} \mathcal{C}_{-/} \times F(-). \quad \square$$

8. A CARTESIAN FIBRATION IDENTIFIED

The goal of this section is to use the results of §7 to prove a preliminary result we will need in the next section. Specifically, we wish to identify the functor associated to a certain Cartesian fibration:

Definition 8.1. If $p: \mathcal{E} \rightarrow \mathcal{B}$ is a functor of ∞ -categories, we denote by $\mathcal{E}_{\mathcal{B}}^{\triangleright}$ the pushout

$$\mathcal{B} \amalg_{\mathcal{E} \times \{0\}} \mathcal{E} \times \Delta^1.$$

Proposition 8.2. *Let $p: \mathcal{E} \rightarrow \mathcal{B}$ be a functor of ∞ -categories, and let $j: \mathcal{B} \rightarrow \mathcal{E}_{\mathcal{B}}^{\triangleright}$ be the obvious inclusion. Then the functor $j^*: \mathcal{P}(\mathcal{E}_{\mathcal{B}}^{\triangleright}) \rightarrow \mathcal{P}(\mathcal{B})$ is a Cartesian fibration corresponding to the functor $\mathcal{P}(\mathcal{B})^{\text{op}} \simeq \text{RFib}(\mathcal{B})^{\text{op}} \rightarrow \text{Cat}_\infty$ that sends a right fibration $\mathcal{Y} \rightarrow \mathcal{B}$ to $\mathcal{P}(\mathcal{Y} \times_{\mathcal{B}} \mathcal{E})$.*

Before we can prove Proposition 8.2, we need some preliminary observations:

Proposition 8.3. *Suppose given a commutative triangle*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathcal{D} \\ & \searrow p & \swarrow q \\ & & \mathcal{C} \end{array}$$

of functors between ∞ -categories such that:

- (1) p and q are Cartesian fibrations.
- (2) f is an inner fibration that takes p -Cartesian edges to q -Cartesian edges.
- (3) For each object $c \in \mathcal{C}$ the induced map on fibres $f_c: \mathcal{E}_c \rightarrow \mathcal{D}_c$ is a Cartesian fibration.
- (4) Suppose given a commutative square

$$\begin{array}{ccc} \phi^* e' & \xrightarrow{\alpha} & e' \\ \beta \downarrow & & \downarrow \gamma \\ \phi^* e & \xrightarrow{\delta} & e \end{array}$$

in \mathcal{E} lying over the degenerate square

$$\begin{array}{ccc} c' & \xrightarrow{\phi} & c \\ \text{id}_{c'} \downarrow & & \downarrow \phi \\ c' & \xrightarrow{\text{id}_c} & c \end{array}$$

in \mathcal{C} , where α and δ are p -Cartesian edges and γ is f_c -Cartesian. Then β is $f_{c'}$ -Cartesian. (In other words, the induced functor $\phi^*: \mathcal{E}_c \rightarrow \mathcal{E}_{c'}$ takes f_c -Cartesian edges to $f_{c'}$ -Cartesian edges.)

Then f is also a Cartesian fibration.

Proof. By [Lur09a, Proposition 2.4.4.3] we must show that f -Cartesian morphisms exist in \mathcal{E} . More precisely, suppose given $e \in \mathcal{E}$ lying over $d \in \mathcal{D}$ and $c \in \mathcal{C}$ (i.e. $d \simeq f(e)$ and $c \simeq p(e) \simeq q(d)$) and a morphism $\delta: d' \rightarrow d$ in \mathcal{D} lying over $\gamma: c' \rightarrow c$ in \mathcal{C} . Then we must show that there exists an f -Cartesian morphism $e' \rightarrow e$ over δ .

Since p is a Cartesian fibration, there exists a p -Cartesian morphism $\beta: \gamma^*e \rightarrow e$ over γ , and as f takes p -Cartesian edges to q -Cartesian edges, its image in \mathcal{D} is a q -Cartesian edge $f(\beta): \gamma^*d \rightarrow d$. There is then an essentially unique factorization of δ through $f(\beta)$, as

$$d' \xrightarrow{\alpha} \gamma^*d \xrightarrow{f(\beta)} d.$$

Now α is a morphism in $\mathcal{D}_{c'}$, so since $f_{c'}$ is a Cartesian fibration there exists an $f_{c'}$ -Cartesian edge $\epsilon: \alpha^*\gamma^*e \rightarrow \gamma^*e$. We will show that the composite $\beta \circ \epsilon: \alpha^*\gamma^*e \rightarrow \gamma^*e \rightarrow e$ is an f -Cartesian morphism over δ .

To see this, we consider the commutative diagram

$$\begin{array}{ccccc} \text{Map}_{\mathcal{E}}(x, \alpha^*\gamma^*e) & \longrightarrow & \text{Map}_{\mathcal{E}}(x, \gamma^*e) & \longrightarrow & \text{Map}_{\mathcal{E}}(x, e) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Map}_{\mathcal{D}}(f(x), d') & \longrightarrow & \text{Map}_{\mathcal{D}}(f(x), \gamma^*d) & \longrightarrow & \text{Map}_{\mathcal{D}}(f(x), d) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Map}_{\mathcal{C}}(p(x), c') & \xrightarrow{\text{id}} & \text{Map}_{\mathcal{C}}(p(x), c') & \longrightarrow & \text{Map}_{\mathcal{C}}(p(x), c), \end{array}$$

where x is an arbitrary object of \mathcal{E} . By [Lur09a, Proposition 2.4.4.3] to see that $\beta \circ \epsilon$ is f -Cartesian we must show that the composite of the two upper squares is Cartesian. We will prove this by showing that both of the upper squares are Cartesian. By construction β is p -Cartesian and $f(\beta)$ is q -Cartesian, so the composite of the two right squares and the bottom right square are both Cartesian, hence so is the upper right square.

Since a commutative square of spaces is Cartesian if and only if the induced maps on all fibres are equivalences, to see that the upper left square is Cartesian it suffices to show

that the square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}}(x, \alpha^* \gamma^* e)_\mu & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(x, \gamma^* e)_\mu \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{D}}(f(x), d')_\mu & \longrightarrow & \mathrm{Map}_{\mathcal{D}}(f(x), \gamma^* d)_\mu \end{array}$$

obtained by taking the fibre at $\mu: p(x) \rightarrow c'$ is Cartesian for every map μ . Now taking p - and q -Cartesian pullbacks along μ we can (since f takes p -Cartesian morphisms to q -Cartesian morphisms) identify this with the square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}_{p(x)}}(x, \mu^* \alpha^* \gamma^* e) & \longrightarrow & \mathrm{Map}_{\mathcal{E}_{p(x)}}(x, \mu^* \gamma^* e) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{D}_{p(x)}}(f(x), \mu^* d') & \longrightarrow & \mathrm{Map}_{\mathcal{D}_{p(x)}}(f(x), \mu^* \gamma^* d). \end{array}$$

But this is Cartesian since by assumption the map $\mu^* \alpha^* \gamma^* e \rightarrow \mu^* \gamma^* e$ is $f_{p(x)}$ -Cartesian (because ϵ is $f_{c'}$ -Cartesian). \square

Corollary 8.4. *Suppose given a commutative triangle*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ & \searrow p & \swarrow q \\ & & \mathcal{C} \end{array}$$

where p and q are left fibrations and f is an inner fibration. Then f is also a left fibration.

Proof. This follows from (the dual of) Proposition 8.3 since the remaining hypotheses are automatic for maps of spaces. \square

Proposition 8.5. *Suppose $p: \mathcal{K} \rightarrow \mathcal{C}$ is a left fibration of ∞ -categories. Then the functor*

$$p_!: \mathrm{LFib}(\mathcal{K}) \rightarrow \mathrm{LFib}(\mathcal{C})_{/p}$$

given by composition with p is an equivalence. Moreover, this equivalence is natural in $p \in \mathrm{LFib}(\mathcal{C})$.

Proof. By Corollary 8.4 the functor $p_!$ is essentially surjective. We are thus required to show that $p_!$ is fully faithful, i.e. that for left fibrations $\alpha: \mathcal{A} \rightarrow \mathcal{K}$ and $\beta: \mathcal{B} \rightarrow \mathcal{K}$ we get an equivalence

$$\mathrm{Map}_{\mathrm{LFib}(\mathcal{K})}(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{LFib}(\mathcal{C})_{/p}}(p_! \mathcal{A}, p_! \mathcal{B}).$$

But this is clear, since we can identify both sides with $\mathrm{Map}_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$. The naturality is also easy to see, since this map comes from a natural left Quillen functor of simplicial model categories. \square

Corollary 8.6. *Suppose $p: \mathcal{K} \rightarrow \mathcal{C}$ is a left fibration, corresponding to a functor $F: \mathcal{C} \rightarrow \mathcal{S}$. Then the functor $p_!: \mathrm{Fun}(\mathcal{K}, \mathcal{S}) \rightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{S})_{/F}$ given by left Kan extension along p is an equivalence, natural in $p \in \mathrm{LFib}(\mathcal{C})$.*

Proof. This follows from combining Proposition 8.5 with the naturality of the straightening equivalence between left fibrations and functors, which can be proved by the same argument as for Corollary A.31. \square

Proof of Proposition 8.2. We first consider the special case where p is the identity map $\mathcal{E} \rightarrow \mathcal{E}$. Let $i: \mathcal{E} \rightarrow \mathcal{E} \times \Delta^1$ denote the inclusion of the fibre over 0. Then the functor $i^*: \mathcal{P}(\mathcal{E} \times \Delta^1) \rightarrow \mathcal{P}(\mathcal{E})$ can be identified with the projection $\text{ev}_1: \text{Fun}(\Delta^1, \mathcal{P}(\mathcal{E})) \rightarrow \mathcal{P}(\mathcal{E})$. This is the Cartesian fibration for the overcategory functor, which sends a presheaf $\phi \in \mathcal{P}(\mathcal{E})$ to $\mathcal{P}(\mathcal{E})_{/\phi}$. This is naturally equivalent to the functor that sends ϕ to $\mathcal{P}(\mathcal{Y})$, where $\mathcal{Y} \rightarrow \mathcal{E}$ is the associated right fibration, by Corollary 8.6.

Now for a general functor p we have a pullback diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{E}_{\mathcal{B}}^{\triangleright}) & \xrightarrow{j^*} & \mathcal{P}(\mathcal{B}) \\ q^* \downarrow & & \downarrow p^* \\ \mathcal{P}(\mathcal{E} \times \Delta^1) & \xrightarrow{i^*} & \mathcal{P}(\mathcal{E}). \end{array}$$

It follows that that j^* is a Cartesian fibration and corresponds to the composite functor $\mathcal{P}(\mathcal{B}) \simeq \text{RFib}(\mathcal{B}) \rightarrow \text{Cat}_{\infty}$ that sends a right fibration $\pi: \mathcal{Y} \rightarrow \mathcal{B}$ to $\mathcal{P}(\mathcal{Y} \times_{\mathcal{B}} \mathcal{E})$. \square

Corollary 8.7. *Let p and j be as above.*

- (i) *If p is a Cartesian fibration corresponding to a functor $F: \mathcal{B}^{\text{op}} \rightarrow \text{Cat}_{\infty}$, then j^* corresponds to the functor $\mathcal{P}(\mathcal{B})^{\text{op}} \simeq \text{RFib}(\mathcal{B})^{\text{op}} \rightarrow \text{Cat}_{\infty}$ that sends a right fibration $\mathcal{Y} \rightarrow \mathcal{B}$ to $\text{Fun}_{\mathcal{B}^{\text{op}}}(\mathcal{Y}^{\text{op}}, \mathcal{P}_{\mathcal{B}}(\mathcal{E}))$, where $\mathcal{P}_{\mathcal{B}}(\mathcal{E}) \rightarrow \mathcal{B}^{\text{op}}$ is the Cartesian fibration corresponding to the functor $\mathcal{P} \circ F: \mathcal{B} \rightarrow \text{Cat}_{\infty}$ that sends $b \in \mathcal{B}$ to $\mathcal{P}(F(b))$.*
- (ii) *If p is a coCartesian fibration corresponding to a functor $F: \mathcal{B} \rightarrow \text{Cat}_{\infty}$, then j^* corresponds to the functor $\mathcal{P}(\mathcal{B})^{\text{op}} \simeq \text{RFib}(\mathcal{B})^{\text{op}} \rightarrow \text{Cat}_{\infty}$ that sends a right fibration $\mathcal{Y} \rightarrow \mathcal{B}$ to $\text{Fun}_{\mathcal{B}^{\text{op}}}(\mathcal{Y}^{\text{op}}, \mathcal{P}_{\mathcal{B}}(\mathcal{E}))$, where $\mathcal{P}_{\mathcal{B}}(\mathcal{E}) \rightarrow \mathcal{B}^{\text{op}}$ is the coCartesian fibration corresponding to the functor $\mathcal{P} \circ F: \mathcal{B}^{\text{op}} \rightarrow \text{Cat}_{\infty}$ that sends $b \in \mathcal{B}$ to $\mathcal{P}(F(b))$.*

Proof. This follows from combining Proposition 8.2 with Proposition 7.3. \square

9. PRESENTABLE FIBRATIONS ARE PRESENTABLE

In ordinary category theory, an *accessible fibration* is a Grothendieck fibration $p: \mathbf{E} \rightarrow \mathbf{C}$ such that \mathbf{C} is an accessible category, the corresponding functor $F: \mathbf{C}^{\text{op}} \rightarrow \widehat{\text{Cat}}$ factors through the category of accessible categories and accessible functors, and F preserves κ -filtered limits for κ sufficiently large.

In [MP89], Makkai and Paré prove that if p is an accessible fibration, then its source \mathbf{E} is also an accessible category, and p is an accessible functor. The goal of this section is to prove an ∞ -categorical variant of this result. As it makes the proof much clearer we will, however, restrict ourselves to considering only *presentable fibrations* of ∞ -categories, defined as follows:

Definition 9.1. A *presentable fibration* is a Cartesian fibration $p: \mathcal{E} \rightarrow \mathcal{B}$ such that \mathcal{B} is a presentable ∞ -category, the corresponding functor $F: \mathcal{B}^{\text{op}} \rightarrow \widehat{\text{Cat}}_{\infty}$ factors through the ∞ -category Pr^{R} of presentable ∞ -categories and right adjoints, and F preserves κ -filtered limits for κ sufficiently large.

Remark 9.2. Suppose $p: \mathcal{E} \rightarrow \mathcal{B}$ is a presentable fibration. Since the morphisms of \mathcal{B} are all mapped to right adjoints under the associated functor, it follows that p is also a coCartesian fibration.

The goal of this section is then to prove the following:

Theorem 9.3. *Let $p: \mathcal{E} \rightarrow \mathcal{B}$ be a presentable fibration. Then \mathcal{E} is a presentable ∞ -category.*

As in Makkai and Paré’s proof of [MP89, Theorem 5.3.4], we will prove this by explicitly describing the total space of the presentable fibration associated to a special class of functors as an accessible localization of a presheaf ∞ -category. To state this result we must introduce some notation:

Definition 9.4. Suppose κ is an ordinal. Let Cat_∞^κ be the category of small ∞ -categories that have all κ -small colimits, and κ -small-colimit-preserving functors between them. Then Ind_κ gives a functor from Cat_∞^κ to the ∞ -category $\text{Pr}^{L,\kappa}$ of κ -presentable ∞ -categories and colimit-preserving functors that preserve κ -compact objects. Using the equivalence $\text{Pr}^L \simeq (\text{Pr}^R)^{\text{op}}$ we may equivalently regard this as a functor $\text{Ind}_\kappa^{\text{op}}: \text{Cat}_\infty^{\kappa,\text{op}} \rightarrow \text{Pr}^{R,\kappa}$, where $\text{Pr}^{R,\kappa}$ is the ∞ -category of κ -presentable ∞ -categories and right adjoints that preserve κ -filtered colimits.

The key step in the proof of Theorem 9.3 can then be stated as follows:

Proposition 9.5. *Let $p: \mathcal{C} \rightarrow \mathcal{B}$ be a coCartesian fibration whose corresponding functor $F: \mathcal{B} \rightarrow \text{Cat}_\infty$ factors through Cat_∞^κ . Write $\mathcal{C}_\mathcal{B}^\triangleleft$ for $\mathcal{B} \amalg_{\mathcal{C} \times \{0\}} \mathcal{C} \times \Delta^1$ and let $j: \mathcal{B} \rightarrow \mathcal{C}_\mathcal{B}^\triangleleft$ be the obvious inclusion. Suppose \mathcal{E} is the full subcategory of $\mathcal{P}(\mathcal{C}_\mathcal{B}^\triangleleft)$ spanned by presheaves \mathcal{F} such that:*

(1) *For every map $f: x \rightarrow y$ in \mathcal{C} , lying over $g: a \rightarrow b$ in \mathcal{B} , the diagram*

$$\begin{array}{ccc} \mathcal{F}(y) & \longrightarrow & \mathcal{F}(x) \\ \downarrow & & \downarrow \\ \mathcal{F}(-\infty_b) & \longrightarrow & \mathcal{F}(-\infty_a) \end{array}$$

is a pullback square in \mathcal{S} .

(2) *Suppose K is a κ -small simplicial set and $\bar{q}: (K^{\text{op}})^\triangleright \rightarrow \mathcal{C}_b$ is a colimit diagram in \mathcal{C}_b for some b . Then \mathcal{F} takes the composite*

$$\bar{q}^{\text{op},\triangleleft}: K^\heartsuit \rightarrow (\mathcal{C}_b^\triangleleft)^{\text{op}} \rightarrow (\mathcal{C}_\mathcal{B}^\triangleleft)^{\text{op}}$$

to a limit diagram in \mathcal{S} .

Then:

- (i) *The functor $j^*|_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{P}(\mathcal{B})$ is a Cartesian fibration.*
- (ii) *The functor $\mathcal{P}(\mathcal{B})^{\text{op}} \rightarrow \text{Cat}_\infty$ associated to $j^*|_{\mathcal{E}}$ is the unique limit-preserving functor extending $\text{Ind}_\kappa^{\text{op}} \circ F: \mathcal{B}^{\text{op}} \rightarrow \text{Cat}_\infty$.*
- (iii) *The ∞ -category \mathcal{E} is an accessible ∞ -category and the functor $j^*|_{\mathcal{E}}$ is an accessible functor.*

We will prove Proposition 9.5 using Corollary 8.7 together with the following simple observations:

Lemma 9.6. *Suppose $\mathcal{E} \rightarrow \mathcal{B}$ is a coCartesian fibration. The functor $\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart,op}} \rightarrow \text{Cat}_{\infty}$ given by $\mathcal{C} \rightarrow \mathcal{B} \mapsto \text{Fun}_{\mathcal{B}}^{\text{cocart}}(\mathcal{C}, \mathcal{E})$ preserves limits.*

Proof. The functor $\text{Fun}_{\mathcal{B}}^{\text{cocart}}(\mathcal{C}, -)$ is clearly right adjoint to the functor $\text{Cat}_{\infty} \rightarrow \text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}$ that sends \mathcal{D} to the composite $\mathcal{D} \times \mathcal{C} \rightarrow \mathcal{C} \rightarrow \mathcal{B}$. In particular, we have a natural equivalence

$$\text{Map}(\mathcal{D}, \text{Fun}_{\mathcal{B}}^{\text{cocart}}(\mathcal{C}, \mathcal{E})) \simeq \text{Map}_{\mathcal{B}}^{\text{cocart}}(\mathcal{D} \times \mathcal{C}, \mathcal{E}),$$

so to see that $\text{Fun}_{\mathcal{B}}^{\text{cocart}}(\mathcal{C}, \mathcal{E})$ takes colimits in \mathcal{C} to limits, it suffices to show that the functor sending $\mathcal{C} \rightarrow \mathcal{B}$ to $\mathcal{D} \times \mathcal{C} \rightarrow \mathcal{B}$ preserves colimits. But this is obvious, since using the equivalence $\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}} \simeq \text{Fun}(\mathcal{B}, \text{Cat}_{\infty})$ it is clear that colimits are computed fibrewise, and the Cartesian product in Cat_{∞} preserves colimits in each variable. \square

Lemma 9.7. *Suppose \mathcal{C} is a small ∞ -category, and let $S = \{\bar{p}_{\alpha} : K_{\alpha}^{\triangleright} \rightarrow \mathcal{C}\}$ be a small set of diagrams in \mathcal{C} . Then the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by presheaves that take the diagrams in S to limit diagrams in S is accessible, and the inclusion of this into $\mathcal{P}(\mathcal{C})$ is an accessible functor.*

Proof. Let $y_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$ denote the Yoneda embedding. A presheaf $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{S}$ takes $\bar{p}_{\alpha}^{\text{op}}$ to a limit diagram if and only if it is local with respect to the map of presheaves

$$\text{colim}(y_{\mathcal{C}} \circ \bar{p}|_{K_{\alpha}}) \rightarrow y_{\mathcal{C}}(\infty),$$

where ∞ denotes the cone point. Thus if S' is the set of these morphisms for $\bar{p}_{\alpha} \in S$, the subcategory in question is precisely the full subcategory of S' -local objects. Since S , and hence S' , is by assumption a small set, it follows that this subcategory is an accessible localization of $\mathcal{P}(\mathcal{C})$. In particular, it is itself accessible and the inclusion into $\mathcal{P}(\mathcal{C})$ is an accessible functor. \square

Proof of Proposition 9.5. By Proposition 8.2 the functor $j^* : \mathcal{P}(\mathcal{C}_{\mathcal{B}}^{\triangleleft}) \rightarrow \mathcal{P}(\mathcal{B})$ is a Cartesian fibration. To see that $j^*|_{\mathcal{E}}$ is a Cartesian fibration it therefore suffices to show that if $\mathcal{F} \in \mathcal{E}$ and $\eta : \gamma \rightarrow \phi := j^*\mathcal{F}$ is a morphism in $\mathcal{P}(\mathcal{B})$, then $\eta^*\mathcal{F}$ is also in \mathcal{E} . The presheaf $\eta^*\mathcal{F}$ is determined by the diagrams

$$\begin{array}{ccc} \eta^*\mathcal{F}(x) & \longrightarrow & \mathcal{F}(x) \\ \downarrow & & \downarrow \\ \gamma(b) & \longrightarrow & \phi(b) \end{array}$$

being pullbacks for all $b \in \mathcal{B}$, $x \in \mathcal{C}_b$.

To see that $\eta^*\mathcal{F}$ satisfies condition (1) we must check that for $f : x \rightarrow y$ in \mathcal{C} lying over $f : a \rightarrow b$ in \mathcal{B} the diagram

$$\begin{array}{ccc} \eta^*\mathcal{F}(y) & \longrightarrow & \eta^*\mathcal{F}(x) \\ \downarrow & & \downarrow \\ \gamma(b) & \longrightarrow & \gamma(a) \end{array}$$

is a pullback square. But this is clear since \mathcal{F} satisfies (1) and the objects $\eta^*\mathcal{F}(y)$ and $\eta^*\mathcal{F}(x)$ are pullbacks.

To check that $\eta^*\mathcal{F}$ satisfies condition (2), suppose that K is a κ -small simplicial set and $\bar{q}: (K^{\text{op}})^{\triangleright} \rightarrow \mathcal{C}_b$ is a colimit diagram in \mathcal{C}_b for some b . Then we wish to show that $\eta^*\mathcal{F}$ takes the composite

$$\bar{q}^{\text{op}, \triangleleft}: K^{\triangleleft} \rightarrow (\mathcal{C}_b^{\triangleleft})^{\text{op}} \rightarrow (\mathcal{C}_{\mathcal{B}}^{\triangleleft})^{\text{op}}$$

to a limit diagram in \mathcal{S} . This is true if and only if the associated diagram $K^{\triangleleft} \rightarrow \mathcal{C}_b^{\text{op}} \rightarrow \mathcal{S}_{/\gamma(b)}$ is a limit diagram. But this diagram is the pullback of the corresponding diagram for \mathcal{F} along $\eta_b: \gamma(b) \rightarrow \phi(b)$, hence this is true since \mathcal{F} satisfies condition (2).

This completes the proof of (i). To prove (ii), we first observe that, by Corollary 8.7(ii), the Cartesian fibration j^* corresponds to the functor $\mathcal{P}(\mathcal{B})^{\text{op}} \simeq \text{RFib}(\mathcal{B})^{\text{op}} \rightarrow \text{Cat}_{\infty}$ that sends a right fibration $\mathcal{Y} \rightarrow \mathcal{B}$ to $\text{Fun}_{\mathcal{B}^{\text{op}}}(\mathcal{Y}^{\text{op}}, \mathcal{P}_{\mathcal{B}}(\mathcal{C}))$, where $\mathcal{P}_{\mathcal{B}}(\mathcal{C}) \rightarrow \mathcal{B}^{\text{op}}$ is the coCartesian fibration corresponding to the functor $\mathcal{P} \circ F: \mathcal{B}^{\text{op}} \rightarrow \text{Cat}_{\infty}$ that sends $b \in \mathcal{B}$ to $\mathcal{P}(F(b))$. Unwinding the equivalence between $\mathcal{P}(\mathcal{C}_{\mathcal{B}}^{\triangleleft})_{\phi}$ and $\text{Fun}_{\mathcal{B}^{\text{op}}}(\mathcal{Y}^{\text{op}}, \mathcal{P}_{\mathcal{B}}(\mathcal{C}))$, where $\phi \in \mathcal{P}(\mathcal{B})$ corresponds to the right fibration $\mathcal{Y} \rightarrow \mathcal{B}$, we see that a presheaf $\mathcal{F} \in \mathcal{P}(\mathcal{C}_{\mathcal{B}}^{\triangleleft})$ over ϕ corresponds to the functor $G: \mathcal{Y}^{\text{op}} \rightarrow \mathcal{P}_{\mathcal{B}}(\mathcal{C})$ that sends $(b, x \in \phi(b))$ to the presheaf $G(b, x): \mathcal{C}_b^{\text{op}} \rightarrow \mathcal{S}$ that sends c to the fibre of $\mathcal{F}(c) \rightarrow \phi(b)$ at x . Then it is clear that G preserves coCartesian arrows if and only if \mathcal{F} satisfies condition (1) and that $G(b, x)$ preserves κ -small limits if and only if \mathcal{F} satisfies condition (2).

As \mathcal{C}_b has κ -small colimits, we may identify $\text{Ind}_{\kappa}(\mathcal{C}_b)$ with the full subcategory of $\mathcal{P}(\mathcal{C}_b)$ spanned by presheaves that preserve κ -small limits, hence $j^*|_{\mathcal{E}}$ corresponds to the functor that sends $\mathcal{Y} \rightarrow \mathcal{B}$ to $\text{Fun}_{\mathcal{B}^{\text{op}}}^{\text{cocart}}(\mathcal{Y}^{\text{op}}, \text{Ind}_{\kappa, \mathcal{B}}^{\text{op}}(\mathcal{C}))$, where $\text{Ind}_{\kappa, \mathcal{B}}^{\text{op}}(\mathcal{C}) \rightarrow \mathcal{B}^{\text{op}}$ is the coCartesian fibration associated to $\text{Ind}_{\kappa}^{\text{op}} \circ F$. This functor preserves limits by Lemma 9.6, since the full subcategory of left fibrations are closed under colimits in $\text{Cat}_{\infty/\mathcal{B}^{\text{op}}}^{\text{cocart}}$, as \mathcal{S} is closed under colimits in Cat_{∞} . Moreover, by the Yoneda lemma the restriction of this functor to \mathcal{B}^{op} along the Yoneda embedding is given by $\text{Ind}_{\kappa}^{\text{op}} \circ F$. This proves (ii).

It follows from Lemma 9.7 that \mathcal{E} is accessible and the inclusion $\mathcal{E} \hookrightarrow \mathcal{P}(\mathcal{C}_{\mathcal{B}}^{\triangleleft})$ is an accessible functor, since \mathcal{E} can be described as the full subcategory of $\mathcal{P}(\mathcal{C}_{\mathcal{B}}^{\triangleleft})$ that preserves a set of limit diagrams — since \mathcal{C} is small, the limits considered in (1) form a set, and for (2) it suffices to consider the set of diagrams coming from pushout diagrams and κ -small coproducts in each fibre. The functor $j^*: \mathcal{P}(\mathcal{C}_{\mathcal{B}}^{\triangleleft}) \rightarrow \mathcal{P}(\mathcal{B})$ preserves colimits, since these are computed pointwise, hence the composite $\mathcal{E} \rightarrow \mathcal{P}(\mathcal{B})$ is also an accessible functor, which proves (iii). \square

To complete the proof of Theorem 9.3 we now just need two easy Lemmas:

Lemma 9.8. *Suppose $\pi: \mathcal{E} \rightarrow \mathcal{B}$ is a coCartesian fibration such that both \mathcal{B} and the fibres \mathcal{E}_b for all $b \in \mathcal{B}$ admit small colimits, and the functors $f_b: \mathcal{E}_b \rightarrow \mathcal{E}_{b'}$ preserve colimits for all morphisms $f: b \rightarrow b'$ in \mathcal{B} . Then \mathcal{E} admits small colimits.*

Proof. The coCartesian fibration π satisfies the conditions of [Lur09a, Corollary 4.3.1.11] for all small simplicial sets K , and so in every diagram

$$\begin{array}{ccc} K & \xrightarrow{p} & \mathcal{E} \\ \downarrow & \nearrow \bar{p} & \downarrow \pi \\ K^{\triangleright} & \xrightarrow{\bar{q}} & \mathcal{B} \end{array}$$

there exists a lift \bar{p} that is a π -colimit of p . Given a diagram $p: K \rightarrow \mathcal{E}$ we can apply this with \bar{q} a colimit of $\pi \circ p$ to get a colimit $\bar{p}: K^{\triangleright} \rightarrow \mathcal{E}$ of p . \square

Lemma 9.9. *Suppose $F: \mathcal{C} \rightleftarrows \mathcal{D} : U$ is an adjunction. Then:*

- (i) *If the right adjoint U preserves κ -filtered colimits, then F preserves κ -compact objects.*
- (ii) *If in addition \mathcal{C} is κ -accessible, then U preserves κ -filtered colimits if and only if F preserves κ -compact objects.*

Proof. For (i), suppose $X \in \mathcal{C}$ is a κ -compact object and $p: K \rightarrow \mathcal{D}$ is a κ -filtered diagram. Then we have

$$\begin{aligned} \mathrm{Map}_{\mathcal{D}}(F(X), \mathrm{colim} p) &\simeq \mathrm{Map}_{\mathcal{C}}(X, G(\mathrm{colim} p)) \simeq \mathrm{Map}_{\mathcal{C}}(X, \mathrm{colim} G \circ p) \\ &\simeq \mathrm{colim} \mathrm{Map}_{\mathcal{C}}(X, G \circ p) \simeq \mathrm{colim} \mathrm{Map}_{\mathcal{D}}(F(X), p). \end{aligned}$$

Thus $\mathrm{Map}_{\mathcal{D}}(F(X), -)$ preserves κ -filtered colimits, i.e. $F(X)$ is κ -compact. For the second claim, suppose F preserves κ -compact objects, and $p: K \rightarrow \mathcal{D}$ is a κ -filtered diagram; we wish to prove that the natural map $\mathrm{colim} G \circ p \rightarrow G(\mathrm{colim} p)$ is an equivalence. Since \mathcal{C} is κ -accessible, to prove this it suffices to show that the induced map

$$\mathrm{Map}_{\mathcal{C}}(X, \mathrm{colim} G \circ p) \rightarrow \mathrm{Map}_{\mathcal{C}}(X, G(\mathrm{colim} p))$$

is an equivalence for all κ -compact objects $X \in \mathcal{C}$. But when X is κ -compact, we have equivalences

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}}(X, G(\mathrm{colim} p)) &\simeq \mathrm{Map}_{\mathcal{D}}(F(X), \mathrm{colim} p) \simeq \mathrm{colim} \mathrm{Map}_{\mathcal{D}}(F(X), p) \\ &\simeq \mathrm{colim} \mathrm{Map}_{\mathcal{C}}(X, G \circ p) \simeq \mathrm{Map}_{\mathcal{C}}(X, \mathrm{colim} G \circ p), \end{aligned}$$

so this is true. \square

Proof of Theorem 9.3. It follows from Lemma 9.8 that \mathcal{E} has small colimits. It thus remains to prove that \mathcal{E} is accessible and p is an accessible functor. Let $F: \mathcal{B}^{\mathrm{op}} \rightarrow \widehat{\mathrm{Cat}}_{\infty}$ be a functor associated to p . Choose a cardinal κ so that \mathcal{B} is κ -presentable and F preserves κ -filtered limits. Since \mathcal{B} is κ -presentable, $\mathcal{B} \simeq \mathrm{Ind}_{\kappa} \mathcal{B}^{\kappa}$ is the full subcategory of $\mathcal{P}(\mathcal{B}^{\kappa})$ spanned by the presheaves that preserve κ -small limits. Let $\widehat{F}: \mathcal{P}(\mathcal{B}^{\kappa})^{\mathrm{op}} \rightarrow \widehat{\mathrm{Cat}}_{\infty}$ be the unique limit-preserving functor extending $F|_{\mathcal{B}^{\kappa, \mathrm{op}}}$. If $\widehat{p}: \widehat{\mathcal{E}} \rightarrow \mathcal{P}(\mathcal{B}^{\kappa})$ is a Cartesian fibration associated to \widehat{F} then we have a pullback square

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \widehat{\mathcal{E}} \\ \downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & \mathcal{P}(\mathcal{B}^{\kappa}), \end{array}$$

where the bottom map preserves κ -filtered colimits, so by [Lur09a, Proposition 5.4.6.6] it suffices to show that $\widehat{\mathcal{E}}$ is accessible and \widehat{p} is an accessible functor.

Since \mathcal{B}^{κ} is a small ∞ -category, we can choose a cardinal λ such that $F|_{\mathcal{B}^{\kappa, \mathrm{op}}}$ factors through the ∞ -category $\mathrm{Pr}^{\mathrm{R}, \lambda}$ of λ -presentable ∞ -categories and right adjoints that preserve λ -filtered colimits. Equivalently, by Lemma 9.9, we can think of this, via the equivalence $\mathrm{Pr}^{\mathrm{R}} \simeq (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}}$ as a functor from \mathcal{B}^{κ} to the ∞ -category $\mathrm{Pr}^{\mathrm{L}, \lambda}$ of λ -presentable ∞ -categories and functors that preserve colimits and λ -compact objects. Taking λ -compact objects defines a functor $(-)^{\lambda}: \mathrm{Pr}^{\mathrm{L}, \lambda} \rightarrow \mathrm{Cat}_{\infty}$. Then, defining $F_0: (\mathcal{B}^{\kappa})^{\mathrm{op}} \rightarrow \mathrm{Cat}_{\infty}$ to be $(F|_{\mathcal{B}^{\kappa}})^{\lambda}$, we see that $F \simeq \mathrm{Ind}_{\lambda} F_0$, and so $\widehat{\mathcal{E}}$ is accessible and \widehat{p} is an accessible functor by Proposition 9.5. \square

APPENDIX A. PSEUDOFUNCTORS AND THE NATURALITY OF UNSTRAIGHTENING

At several points in this paper we will need to know that the unstraightening functors $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_{\infty}) \rightarrow \text{Cat}_{\infty/\mathcal{C}}^{\text{cart}}$, and a number of similar constructions, are natural as we vary the ∞ -category \mathcal{C} . The obvious way to prove this is to consider the naturality of the unstraightening $\text{Un}_S^{\dagger}: \text{Fun}(\mathfrak{C}(S), \text{Set}_{\Delta}^{\dagger}) \rightarrow (\text{Set}_{\Delta}^{\dagger})/S$ as we vary the simplicial set S . However, since pullbacks are only determined up to canonical isomorphism, these functors are not natural “on the nose”, but only up to natural isomorphism — i.e. they are only *pseudo-natural*. In the body of the paper we have swept such issues under the rug, but in this appendix we indulge ourselves in a bit of 2-category theory to prove that pseudo-naturality on the level of model categories does indeed give naturality on the level of ∞ -categories. We begin by reviewing Duskin’s nerve of bicategories [Dus02] and its basic properties. However, we will only need to consider the case of *strict* 2- and (2,1)-categories:

Definition A.1. A *strict 2-category* is a category enriched in Cat , and a *strict (2,1)-category* is a category enriched in Gpd . We write Cat_2 for the category of strict 2-categories and $\text{Cat}_{(2,1)}$ for the category of strict (2,1)-categories.

Definition A.2. Suppose \mathbf{C} and \mathbf{D} are strict 2-categories. A *normal oplax functor* $F: \mathbf{C} \rightarrow \mathbf{D}$ consists of the following data:

- (a) for each object $x \in \mathbf{C}$, an object $F(x) \in \mathbf{D}$,
- (b) for each 1-morphism $f: x \rightarrow y$ in \mathbf{C} , a 1-morphism $F(f): F(x) \rightarrow F(y)$,
- (c) for each 2-morphism $\phi: f \rightrightarrows g$ in $\mathbf{C}(x, y)$, a 2-morphism $F(\phi): F(f) \rightrightarrows F(g)$ in $\mathbf{D}(F(x), F(y))$,
- (d) for each pair of composable 1-morphism $f: x \rightarrow y$, $g: y \rightarrow z$ in \mathbf{C} , a 2-morphism $\eta_{f,g}: F(g \circ f) \rightrightarrows F(g) \circ F(f)$,

such that:

- (i) for every object $x \in \mathbf{C}$, the 1-morphism $F(\text{id}_x) = \text{id}_{F(x)}$,
- (ii) for every 1-morphism $f: x \rightarrow y$ in \mathbf{C} , the 2-morphism $F(\text{id}_f) = \text{id}_{F(f)}$,
- (iii) for composable 2-morphisms $\phi: f \rightrightarrows g$, $\psi: g \rightrightarrows h$ in $\mathbf{C}(x, y)$, we have $F(\psi \circ \phi) = F(\psi) \circ F(\phi)$,
- (iv) for every morphism $f: x \rightarrow y$, the morphisms $\eta_{\text{id}_x, f}$ and $\eta_{f, \text{id}_y}: F(f) \rightarrow F(f)$ are both $\text{id}_{F(f)}$,
- (v) if $\phi: f \rightrightarrows f'$ is a 2-morphism in $\mathbf{C}(x, y)$ and $\psi: g \rightarrow g'$ is a 2-morphism in $\mathbf{C}(y, z)$, then the diagram

$$\begin{array}{ccc}
 F(g \circ f) & \xrightarrow{\eta_{f,g}} & F(g) \circ F(f) \\
 \downarrow F(\psi \circ \phi) & & \downarrow F(\psi) \circ F(\phi) \\
 F(g' \circ f') & \xrightarrow{\eta_{f',g'}} & F(g') \circ F(f')
 \end{array}$$

commutes,

(vi) for composable triples of 1-morphisms $f: x \rightarrow y$, $g: y \rightarrow z$, $h: z \rightarrow w$, the diagram

$$\begin{array}{ccc} F(h \circ g \circ f) & \xrightarrow{\eta_{f,hg}} & F(h \circ g) \circ F(f) \\ \eta_{gf,h} \downarrow & & \downarrow \eta_{g,h} \circ \text{id} \\ F(h) \circ F(g \circ f) & \xrightarrow{\text{id} \circ \eta_{f,g}} & F(h) \circ F(g) \circ F(f) \end{array}$$

We say a normal lax functor F from \mathbf{C} to \mathbf{D} is a *normal pseudofunctor* if the 2-morphisms $\eta_{f,g}$ are all isomorphisms. In particular, if the 2-category \mathbf{C} is a (2,1)-category, all normal lax functors $\mathbf{C} \rightarrow \mathbf{D}$ are normal pseudofunctors.

Remark A.3. In 2-category theory one typically considers the more general notions of (not necessarily normal) oplax functors and pseudofunctors, which do not satisfy $F(\text{id}_x) = \text{id}_{F(x)}$ but instead include the data of natural maps $F(\text{id}_x) \rightarrow \text{id}_{F(x)}$ (which are isomorphisms for pseudofunctors). We only consider the normal versions because, as we will see below, these correspond to maps of simplicial sets between the *nerves* of strict 2- and (2,1)-categories.

Before we recall the definition of the nerve of a strict 2-category, we first review the definition of nerves for ordinary categories and simplicial categories:

Definition A.4. Let $N: \text{Cat} \rightarrow \text{Set}_\Delta$ be the usual nerve of categories, i.e. if \mathbf{C} is a category then NC_k is the set $\text{Hom}([n], \mathbf{C})$ where $[n]$ is the category corresponding to the partially ordered set $\{0, \dots, n\}$.

Remark A.5. Since Cat has colimits, the functor N has a left adjoint $C: \text{Set}_\Delta \rightarrow \text{Cat}$, which is the unique colimit-preserving functor such that $C(\Delta^n) = [n]$.

Lemma A.6. *The functor $C: \text{Set}_\Delta \rightarrow \text{Cat}$ takes inner anodyne morphisms to isomorphisms.*

Proof. Since C preserves colimits, it suffices to prove that C takes the inner horn inclusions $\Lambda_i^n \rightarrow \Delta^n$ ($i = 1, \dots, n-1$) to isomorphisms. Let Sp^n denote the n -spine, i.e. the simplicial set $\Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \dots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}$. We first observe that C takes the inclusion $\text{Sp}^n \hookrightarrow \Delta^n$ to an isomorphism: Since C preserves colimits, this is the map of categories

$$[1] \amalg_{[0]} \dots \amalg_{[0]} [1] \rightarrow [n].$$

But the category $[n]$ is the free category on the graph with vertices $0, \dots, n$ and $i \rightarrow (i+1)$, which obviously decomposes as a colimit in this way, and the free category functor on graphs preserves colimits. Now the inclusion $\text{Sp}^n \hookrightarrow \Lambda_i^n$ for any inner horn can be written as the composite of pushouts along spine inclusions $\text{Sp}^k \rightarrow \Delta^k$ (for $k < n$), so C takes this inclusion to an isomorphism. By the 2-out-of-3 property, it follows that C also takes the inclusion $\Lambda_i^n \rightarrow \Delta^n$ to an isomorphism. \square

Proposition A.7. *The functor $C: \text{Set}_\Delta \rightarrow \text{Cat}$ preserves products.*

Proof. Since C preserves colimits and the Cartesian products in Cat and Set_Δ both commute with colimits in each variable, it suffices to check that the natural map $C(\Delta^n \times \Delta^m) \rightarrow C(\Delta^n) \times C(\Delta^m)$ is an isomorphism for all n, m . Since products of inner anodyne maps are inner anodyne by [Lur09a, Corollary 2.3.2.4], the inclusion $\text{Sp}^n \times \text{Sp}^m \rightarrow \Delta^n \times \Delta^m$ is inner

anodyne. Thus in the diagram

$$\begin{array}{ccc} \mathbf{C}(\mathrm{Sp}^n \times \mathrm{Sp}^m) & \longrightarrow & \mathbf{C}(\mathrm{Sp}^n) \times \mathbf{C}(\mathrm{Sp}^m) \\ \downarrow & & \downarrow \\ \mathbf{C}(\Delta^n \times \Delta^m) & \longrightarrow & \mathbf{C}(\Delta^n) \times \mathbf{C}(\Delta^m) \end{array}$$

the vertical maps are isomorphisms by Lemma A.6. It hence suffices to prove that the upper horizontal map is an isomorphism, which, since \mathbf{C} preserves colimits, reduces to showing that $\mathbf{C}(\Delta^n \times \Delta^m) \rightarrow \mathbf{C}(\Delta^n) \times \mathbf{C}(\Delta^m)$ is an isomorphism when n and m are 0 or 1. The cases where n or m is 0 are trivial, so it only remains to show that $\mathbf{C}(\Delta^1 \times \Delta^1) \rightarrow [1] \times [1]$ is an isomorphism. The simplicial set $\Delta^1 \times \Delta^1$ is the pushout $\Delta^2 \amalg_{\Delta^{\{0,2\}}} \Delta^2$, so this amounts to showing that the analogous functor $[2] \amalg_{[1]} [2] \rightarrow [1] \times [1]$ is an isomorphism, or equivalently that for any category \mathbf{C} , the square

$$\begin{array}{ccc} \mathrm{Hom}([1] \times [1], \mathbf{C}) & \longrightarrow & \mathrm{Hom}([2], \mathbf{C}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}([2], \mathbf{C}) & \longrightarrow & \mathrm{Hom}([1], \mathbf{C}) \end{array}$$

is Cartesian. But this just amounts to a commutative square in \mathbf{C} being two compatible commutative triangles, which is obvious. \square

Definition A.8. Let $\mathfrak{C}(\Delta^n)$ denote the simplicial category with objects $0, \dots, n$ and with morphisms given by $\mathfrak{C}(\Delta^n)(i, j) = \emptyset$ if $i > j$ and $NP_{i,j}$, where $P_{i,j}$ is the partially ordered set of subsets of $\{i, i+1, \dots, j\}$ containing i and j , otherwise. Composition of morphisms is induced by union of such subsets.

Remark A.9. The simplicial set $NP_{i,j}$ is isomorphic to $(\Delta^1)^{\times(j-i-1)}$ for $j > i$.

Definition A.10. The *coherent nerve* is the functor $\mathfrak{N}: \mathrm{Cat}_\Delta \rightarrow \mathrm{Set}_\Delta$ defined by

$$\mathfrak{N}\mathbf{C}_k = \mathrm{Hom}(\mathfrak{C}(\Delta^k), \mathbf{C}).$$

This has a left adjoint $\mathfrak{C}: \mathrm{Set}_\Delta \rightarrow \mathrm{Cat}_\Delta$, which is the unique colimit-preserving functor extending the cosimplicial simplicial category $\mathfrak{C}(\Delta^\bullet)$.

Definition A.11. The functor N preserves products, being a right adjoint, and so induces a functor $N_*: \mathrm{Cat}_2^{\mathrm{str}} \rightarrow \mathrm{Cat}_\Delta$, given by applying N on the mapping spaces; this has a left adjoint $C_*: \mathrm{Cat}_\Delta \rightarrow \mathrm{Cat}_2^{\mathrm{str}}$ given by composition with \mathfrak{C} , since \mathfrak{C} preserves products by Proposition A.7.

Definition A.12. Let $N_2: \mathrm{Cat}_2^{\mathrm{str}} \rightarrow \mathrm{Set}_\Delta$ denote the composite

$$\mathrm{Cat}_2^{\mathrm{str}} \xrightarrow{N_*} \mathrm{Cat}_\Delta \xrightarrow{\mathfrak{N}} \mathrm{Set}_\Delta.$$

This functor has a left adjoint C_2 , which is the composite

$$\mathrm{Set}_\Delta \xrightarrow{\mathfrak{C}} \mathrm{Cat}_\Delta \xrightarrow{C_*} \mathrm{Cat}_2^{\mathrm{str}}.$$

Remark A.13. It is obvious from the definitions given in [Dus02] that the functor N_2 as we have defined it is simply the restriction of Duskin's nerve for bicategories to strict 2-categories. The results on this nerve we will now discuss can all be found in [Dus02] in the more general setting of bicategories.

Remark A.14. We can describe the strict 2-category $\mathbf{C}_2(\Delta^n)$ as follows: its objects are $0, \dots, n$. For $i > j$, the category $\mathbf{C}_2(\Delta^n)(i, j)$ is empty, and for $j > i$ it is the partially ordered set $P_{i,j}$ (which is isomorphic to $[1]^{\times(j-i-1)}$ if $j > i$). We can thus describe the low-dimensional simplices of the nerve $\mathbf{N}_2\mathbf{C}$ of a strict 2-category \mathbf{C} as follows:

- The 0-simplices are the objects of \mathbf{C} .
- The 1-simplices are the 1-morphisms of \mathbf{C} .
- A 2-simplex in $\mathbf{N}_2\mathbf{C}$ is given by objects x_0, x_1, x_2 , 1-morphisms $f_{01}: x_0 \rightarrow x_1$, $f_{12}: x_1 \rightarrow x_2$, $f_{02}: x_0 \rightarrow x_2$, and a 2-morphism $\phi_{012}: f_{02} \Rightarrow f_{12} \circ f_{01}$.
- A 3-simplex is given by
 - objects x_0, x_1, x_2, x_3 ,
 - 1-morphisms $f_{ij}: x_i \rightarrow x_j$ for $0 \leq i < j \leq 3$,
 - 2-morphisms $\phi_{012}: f_{02} \Rightarrow f_{12} \circ f_{01}$, $\phi_{123}: f_{13} \Rightarrow f_{23} \circ f_{12}$, $\phi_{023}: f_{03} \Rightarrow f_{23} \circ f_{02}$ and $\phi_{013}: f_{03} \Rightarrow f_{13} \circ f_{01}$, such that the square

$$\begin{array}{ccc} f_{03} & \xrightarrow{\phi_{013}} & f_{13} \circ f_{01} \\ \phi_{023} \downarrow & & \downarrow \phi_{123} \circ \text{id} \\ f_{23} \circ f_{02} & \xrightarrow{\text{id} \circ \phi_{012}} & f_{23} \circ f_{12} \circ f_{01} \end{array}$$

commutes.

Definition A.15. Let $\Delta_{\leq k}$ denote the full subcategory of Δ spanned by the objects $[n]$ for $n \leq k$. The restriction $\text{sk}_k: \text{Set}_\Delta \rightarrow \text{Fun}(\Delta_{\leq k}^{\text{op}}, \text{Set})$ has a left adjoint

$$\text{cosk}_k: \text{Fun}(\Delta_{\leq k}^{\text{op}}, \text{Set}) \rightarrow \text{Set}_\Delta.$$

We say a simplicial set X is k -coskeletal if it is in the image of the functor cosk_k (which is fully faithful since $\Delta_{\leq k}$ is a full subcategory of Δ). Equivalently, X is k -coskeletal if every map $\partial\Delta^n \rightarrow X$ extends to a unique n -simplex $\Delta^n \rightarrow X$ when $n > k$.

Proposition A.16. *For every strict 2-category \mathbf{C} , the simplicial set $\mathbf{N}_2\mathbf{C}$ is 3-coskeletal.*

Proof. We must show that every map $\partial\Delta^k \rightarrow \mathbf{N}_2\mathbf{C}$ extends to a unique map from Δ^k if $k > 3$. Equivalently, we must show that given a map $\mathfrak{C}(\partial\Delta^k) \rightarrow \mathbf{N}_*\mathbf{C}$ it has a unique extension to $\mathfrak{C}(\Delta^k)$ for $k > 3$. We can describe the simplicial category $\mathfrak{C}(\partial\Delta^k)$ and its map to $\mathfrak{C}(\Delta^k)$ as follows:

- the objects of $\mathfrak{C}(\partial\Delta^k)$ are $0, \dots, k$,
- the maps $\mathfrak{C}(\partial\Delta^k)(i, j) \rightarrow \mathfrak{C}(\Delta^k)(i, j)$ are isomorphisms except when $i = 0$ and $j = k$,
- the simplicial set $\mathfrak{C}(\partial\Delta^k)(0, k)$ is the boundary of the $(k-1)$ -cube $\mathfrak{C}(\Delta^k)(0, k) \cong (\Delta^1)^{\times(k-1)}$.

Thus extending a map $F: \mathfrak{C}(\partial\Delta^k) \rightarrow \mathbf{N}_*\mathbf{C}$ to $\mathfrak{C}(\Delta^k)$ amounts to extending the map $\mathfrak{C}(\partial\Delta^k)(0, k) \rightarrow \mathbf{NC}(F(0), F(k))$ to $\mathfrak{C}(\Delta^k)(0, k)$. But the inclusion

$$\mathfrak{C}(\partial\Delta^k)(0, k) \rightarrow \mathfrak{C}(\Delta^k)(0, k)$$

is a composition of pushouts of inner horn inclusions and the inclusion $\partial\Delta^{k-1} \rightarrow \Delta^{k-1}$, and if $k-1 > 2$ the nerve of a category has unique extensions along these. \square

Theorem A.17 (Duskin). *Suppose \mathbf{C} and \mathbf{D} are strict 2-categories. Then the maps of simplicial sets $\mathbf{N}_2\mathbf{C} \rightarrow \mathbf{N}_2\mathbf{D}$ can be identified with the normal oplax functors $\mathbf{C} \rightarrow \mathbf{D}$.*

Remark A.18. This is the main result of [Dus02]. We do not include a complete proof here, but we will now briefly indicate how a map of nerves gives rise to a normal oplax functor. By Proposition A.16, a map $N_2\mathbf{C} \rightarrow N_2\mathbf{D}$ can be identified with a map $F: \text{sk}_3 N_2\mathbf{C} \rightarrow \text{sk}_3 N_2\mathbf{D}$. Using Remark A.14 we can identify this with the data of a normal oplax functor as given in Definition A.2:

- The 0-simplices of $N_2\mathbf{C}$ are the objects of \mathbf{C} , so F assigns an object $F(c) \in \mathbf{D}$ to every $c \in \mathbf{C}$, which gives (a)
- The 1-simplices of $N_2\mathbf{C}$ are the 1-morphisms in \mathbf{C} , with sources and targets given by the face maps $[0] \rightarrow [1]$, so F assigns a 1-morphism $F(f): F(x) \rightarrow F(y)$ to every 1-morphism $f: x \rightarrow y$ in \mathbf{C} , which gives (b).
- Moreover, identity 1-morphisms correspond to degenerate edges in $N_2\mathbf{C}$, so since these are preserved by any map of simplicial sets we get $F(\text{id}_x) = \text{id}_{F(x)}$, i.e. (i).
- The 2-simplices of $N_2\mathbf{C}$ are given by three 1-morphisms $f: x \rightarrow y$, $g: y \rightarrow z$, $h: z \rightarrow w$ (corresponding to the three face maps), and a 2-morphism $\phi: h \Rightarrow g \circ f$. In particular:
 - Considering 2-simplices where the second edge is degenerate, which correspond to 2-morphisms in \mathbf{C} , we see that F assigns a 2-morphism $F(\phi): F(h) \Rightarrow F(g)$ to every $\phi: h \Rightarrow g$ in \mathbf{C} , which gives (c).
 - Considering 2-simplices where the 2-morphism ϕ is the identity, we see (as this condition is not preserved by F) that F assigns a 2-morphism $F(g \circ f) \Rightarrow F(g) \circ F(f)$ to all composable pairs of 1-morphisms, which gives (d).
- Since F preserves degenerate 2-simplices, which correspond to identity 2-morphisms of the form $f \circ \text{id} \Rightarrow f$ and $\text{id} \circ f \rightarrow f$, we get (ii) and (iv).
- The 3-simplices of $N_2\mathbf{C}$ are given by
 - objects x_0, x_1, x_2, x_3 ,
 - 1-morphisms $f_{ij}: x_i \rightarrow x_j$ for $0 \leq i < j \leq 3$,
 - 2-morphisms $\phi_{012}: f_{02} \Rightarrow f_{12} \circ f_{01}$, $\phi_{123}: f_{13} \Rightarrow f_{23} \circ f_{12}$, $\phi_{023}: f_{03} \Rightarrow f_{23} \circ f_{02}$ and $\phi_{013}: f_{03} \Rightarrow f_{13} \circ f_{01}$, such that the square

$$\begin{array}{ccc}
 f_{03} & \xrightarrow{\phi_{013}} & f_{13} \circ f_{01} \\
 \phi_{023} \downarrow & & \downarrow \phi_{123} \circ \text{id} \\
 f_{23} \circ f_{02} & \xrightarrow{\text{id} \circ \phi_{012}} & f_{23} \circ f_{12} \circ f_{01}
 \end{array}$$

commutes.

In particular, we have:

- If $x_1 = x_2 = x_3$, $f_{12} = f_{13} = f_{23} = \text{id}_{x_1}$, and $\phi_{123} = \text{id}_{\text{id}_{x_1}}$, then this says $\phi_{013} = \phi_{012} \circ \phi_{023}$, and since F preserves identities this gives (iii).
- In the case where the 2-morphisms are all identities, we get (vi).
- To get (v), we consider the 3-simplices where $f_{12} = \text{id}$, $\phi_{023} = \text{id}$, and ϕ_{013} is the composite of the ϕ_{012} and ϕ_{123} .

Corollary A.19. *If \mathbf{C} and \mathbf{D} are strict 2-categories, lax normal functors $\mathbf{C} \rightarrow \mathbf{D}$ can be identified with functors $C_2 N_2\mathbf{C} \rightarrow \mathbf{D}$.*

Definition A.20. The inclusion $\text{Gpd} \hookrightarrow \text{Cat}$ of the category of groupoids preserves products, and so induces a functor $\text{Cat}_{(2,1)}^{\text{str}} \rightarrow \text{Cat}_2^{\text{str}}$; we write $N_{(2,1)}$ for the composite

$$\text{Cat}_{(2,1)}^{\text{str}} \rightarrow \text{Cat}_2^{\text{str}} \xrightarrow{N_2} \text{Set}_\Delta.$$

Corollary A.21. *If \mathbf{C} and \mathbf{D} are strict $(2,1)$ -categories, then a morphism of simplicial sets $N_{(2,1)}\mathbf{C} \rightarrow N_{(2,1)}\mathbf{D}$ can be identified with a normal pseudofunctor $\mathbf{C} \rightarrow \mathbf{D}$.*

Definition A.22. Recall that a *relative category* is a category \mathbf{C} equipped with a subcategory W containing all isomorphisms; see [BK12] for a more extensive discussion. A functor of relative categories $f: (\mathbf{C}, W) \rightarrow (\mathbf{C}', W')$ is a functor $f: \mathbf{C} \rightarrow \mathbf{C}'$ that takes W into W' . We write $\text{RelCat}_{(2,1)}$ for the strict $(2,1)$ -category of relative categories, functors of relative categories, and all natural isomorphisms between these.

We now want to prove that a normal pseudofunctor to $\text{RelCat}_{(2,1)}$ determines a map of ∞ -categories to Cat_∞ via the following construction:

Definition A.23. If (\mathbf{C}, W) is a relative category, let $L(\mathbf{C}, W) \in \text{Set}_\Delta^+$ be the marked simplicial set $(N\mathbf{C}, NW_1)$. This defines a simplicial functor $N_*\text{RelCat}_{(2,1)} \rightarrow \text{Set}_\Delta^+$.

Lemma A.24. *Let \mathbf{C} be a strict $(2,1)$ -category, and let F be a normal pseudofunctor $F: \mathbf{C} \rightarrow \text{RelCat}_{(2,1)}$. If W is a collection of 1-morphisms in \mathbf{C} such that F takes the morphisms in W to weak equivalences of relative categories, then F determines a functor of ∞ -categories $\mathbf{C}[W^{-1}] \rightarrow \text{Cat}_\infty$, where $\mathbf{C}[W^{-1}]$ is a fibrant replacement for the marked simplicial set $(N_{(2,1)}\mathbf{C}, W)$, which sends $x \in \mathbf{C}$ to $\mathbf{E}_x[W_x^{-1}]$ where $F(x) = (\mathbf{E}_x, W_x)$.*

Proof. By Proposition A.21 the normal pseudofunctor F corresponds to a map of simplicial sets $N_{(2,1)}\mathbf{C} \rightarrow N_{(2,1)}\text{RelCat}_{(2,1)}$. Composing this with the map $\mathfrak{N}(L): N_{(2,1)}\text{RelCat}_{(2,1)} \rightarrow \mathfrak{N}\text{Set}_\Delta^+$ we get a map $N_{(2,1)}\mathbf{C} \rightarrow \mathfrak{N}\text{Set}_\Delta^+$. We may regard this as a map of marked (large) simplicial sets

$$(N_{(2,1)}\mathbf{C}, W) \rightarrow (\mathfrak{N}\text{Set}_\Delta^+, W'),$$

where W' is the collection of marked equivalences in Set_Δ^+ . Now invoking [Lur14, Theorem 1.3.4.20] we conclude that Cat_∞ is a fibrant replacement for the marked simplicial set $(\mathfrak{N}\text{Set}_\Delta^+, W')$, so this map corresponds to a map $\mathbf{C}[W^{-1}] \rightarrow \text{Cat}_\infty$ in the ∞ -category $\widehat{\text{Cat}}_\infty$ underlying the model category of (large) marked simplicial sets. \square

We will now make use of Grothendieck's description of pseudofunctors to the $(2,1)$ -category of categories to get a way of constructing pseudofunctors to $\text{RelCat}_{(2,1)}$:

Theorem A.25 (Grothendieck [Gro63]). *Let \mathbf{C} be a category. Then pseudofunctors from \mathbf{C}^{op} to the strict 2-category CAT correspond to Grothendieck fibrations over \mathbf{C} .*

Remark A.26. Let us briefly recall how a pseudofunctor is constructed from a Grothendieck fibration, as this is the part of Grothendieck's theorem we will actually use. A *cleavage* of a Grothendieck fibration $p: \mathbf{E} \rightarrow \mathbf{B}$ is the choice, for each $(e \in \mathbf{E}, f: b \rightarrow p(e))$, of a single Cartesian morphism over f with target e ; cleavages always exist, by the axiom of choice. Given a choice of cleavage of p , we define the pseudofunctor $\mathbf{C}^{\text{op}} \rightarrow \text{CAT}$ by assigning the fibre \mathbf{E}_b to each $b \in \mathbf{B}$, and for each $f: b \rightarrow b'$ the functor f^* assigns to $e \in \mathbf{E}_b$ the source of the Cartesian morphism over f with target e in the cleavage. Clearly, this pseudofunctor will be normal precisely when the cleavage is *normal* in the sense that the Cartesian morphisms over the identities in \mathbf{B} are all chosen to be identities in \mathbf{E} . Obviously, every Grothendieck fibration has a normal cleavage, so from any Grothendieck fibration we can construct a normal pseudofunctor.

Definition A.27. A *relative Grothendieck fibration* is a Grothendieck fibration $p: \mathbf{E} \rightarrow \mathbf{C}$ together with a subcategory \mathbf{W} of \mathbf{E} containing all the p -Cartesian morphisms. In particular, the restricted projection $\mathbf{W} \rightarrow \mathbf{C}$ is also a Cartesian fibration. Moreover, for

every $x \in \mathbf{C}$ the fibres $(\mathbf{E}_x, \mathbf{W}_x)$ are relative categories, and the functor f^* induced by each f in \mathbf{C} is a functor of relative categories. If (\mathbf{C}, U) is a relative category, we say that the relative Grothendieck fibration is *compatible with U* if this functor $f^*: (\mathbf{E}_q, \mathbf{W}_q) \rightarrow (\mathbf{E}_p, \mathbf{W}_p)$ is a weak equivalence of relative categories for every $f: p \rightarrow q$ in U .

The following is then an obvious consequence of Theorem A.25:

Lemma A.28. *Relative Grothendieck fibrations over a category \mathbf{C} correspond to normal pseudofunctors $\mathbf{C}^{\text{op}} \rightarrow \text{RelCat}_{(2,1)}$.*

Proposition A.29. *Let (\mathbf{E}, \mathbf{W}) be a relative Grothendieck fibration over \mathbf{C} compatible with a collection U of morphisms in \mathbf{C} . Then this induces a functor of ∞ -categories*

$$\mathbf{C}[U^{-1}]^{\text{op}} \rightarrow \text{Cat}_{\infty}$$

that sends $p \in \mathbf{C}$ to $\mathbf{E}_p[\mathbf{W}_p^{-1}]$.

Proof. Combine Lemmas A.28 and A.24. □

All the maps whose naturality we are interested in can easily be constructed as relative Grothendieck fibrations. We will explicitly describe this in the case of the unstraightening equivalence, and leave the other cases to the reader.

Proposition A.30. *The unstraightening functors $\text{Un}_S^+: \text{Fun}_{\Delta}(\mathfrak{C}[S], \text{Set}_{\Delta}^+)^{\text{fib}} \rightarrow (\text{Set}_{\Delta}^+)_{/S}^{\text{fib}}$ define a relative Grothendieck fibration over $\text{Set}_{\Delta} \times \Delta^1$ compatible with the categorical equivalences in Set_{Δ} .*

Proof. Let \mathbf{E} be the category whose objects are triples (i, S, X) where $i = 0$ or 1 , $S \in \text{Set}_{\Delta}$, and X is a fibrant simplicial functor $\mathfrak{C}(S) \rightarrow \text{Set}_{\Delta}^+$ if $i = 0$ and a fibrant map $Y \rightarrow S^{\sharp}$ in Set_{Δ}^+ if $i = 1$; the morphisms $(i, S, X) \rightarrow (j, T, Y)$ consist of a morphism $i \rightarrow j$ in $[1]$, a morphism $f: S \rightarrow T$ in Set_{Δ} , and the following data:

- if $i = j = 1$, $X: \mathfrak{C}(S) \rightarrow \text{Set}_{\Delta}^+$ and $Y: \mathfrak{C}(T) \rightarrow \text{Set}_{\Delta}^+$, a simplicial natural transformation $X \rightarrow \mathfrak{C}(f) \circ Y$,
- if $i = j = 0$, X is $E \rightarrow S$ and Y is $F \rightarrow T$, a commutative square

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ S^{\sharp} & \longrightarrow & T^{\sharp} \end{array}$$

in Set_{Δ}^+ ,

- if $i = 0$ and $j = 1$, X is $E \rightarrow S$ and $Y: \mathfrak{C}(T) \rightarrow \text{Set}_{\Delta}^+$, a commutative square

$$\begin{array}{ccc} E & \longrightarrow & \text{Un}_S^+(X) \\ \downarrow & & \downarrow \\ S & \longrightarrow & T \end{array}$$

Composition is defined in the obvious way, using the natural maps of [Lur09a, Proposition 3.2.1.4]. We claim that the projection $\mathbf{E} \rightarrow \Delta^1 \times \text{Set}_{\Delta}$ is a Grothendieck fibration. It suffices to check that Cartesian morphisms exist for morphisms of the form (id_i, f) and $(0 \rightarrow 1, \text{id}_S)$, which is clear. □

Corollary A.31. *There is a functor of ∞ -categories $\text{Cat}_\infty \rightarrow \text{Fun}(\Delta^1, \widehat{\text{Cat}}_\infty)$ that sends \mathcal{C} to the unstraightening equivalence*

$$\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_\infty) \xrightarrow{\sim} \text{Cat}_{\infty/\mathcal{C}}^{\text{cart}}.$$

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