

Rational and p -adic homotopy theory

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Abstract

These are notes for a course taught at the Max-Planck Institute for Mathematics in Bonn in Summer 2016. They are in extremely preliminary form. Comments, corrections and remarks are welcome.

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1 Introduction

Definition 1.1. Algebraic topology studies spaces through algebraic invariants.

Example 1.2. 1. The homology groups $H_*(X; \mathbb{Z})$ of a space are graded abelian groups. They can be used to show that S^n is not homotopy equivalent to S^m for $n \neq m$.

2. Can not distinguish all spaces, e.g. $\mathbb{C}P^2$ and $S^2 \vee S^4$ since

$$H_*(\mathbb{C}P^2; \mathbb{Z}) \cong H_*(S^2 \vee S^4; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } * = 0, 2, 4 \\ 0 & \text{else} \end{cases}$$

have both the same homology groups. Here we can use the cohomology ring $H^*(X, \mathbb{Z})$ as a graded ring. For example we get that

$$H^*(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}[x]/x^3 \quad |x| = 2$$

but

$$H^*(S^2 \vee S^4; \mathbb{Z}) \cong \mathbb{Z}[x, y]/x^2 = y^2 = xy = 0 \quad |x| = 2, |y| = 4$$

3. What about $\Sigma\mathbb{C}P^2$ and $S^3 \vee S^5$? Again isomorphic cohomology groups and also rings (since multiplication trivial in positive degrees for suspensions). Use Steenrod operations

$$Sq^i: H^n(X; \mathbb{Z}/2) \rightarrow H^{n+i}(X; \mathbb{Z}/2)$$

which are also homotopy invariants. We have that

$$H^*(\Sigma\mathbb{C}P^2; \mathbb{Z}/2) \cong H^*(S^3 \vee S^5; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & \text{for } * = 0, 3, 5 \\ 0 & \text{else} \end{cases}$$

but $Sq^2: H^3(\Sigma\mathbb{C}P^2; \mathbb{Z}/2) \rightarrow H^5(\Sigma\mathbb{C}P^2; \mathbb{Z}/2)$ non-trivial, whereas $Sq^2: H^3(S^3 \vee S^5; \mathbb{Z}/2) \rightarrow H^5(S^3 \vee S^5; \mathbb{Z}/2)$ is trivial.

4. What about $F_2L_{7,1}$ and $F_2L_{7,2}$? Here $F_2X = X \times X \setminus \Delta$ is the configuration space of two points and $L_{p,q} = S^3/(\mathbb{Z}/p)$ the 3-dimensional Lens space. Isomorphic cohomology rings and Steenrod actions. Use triple Massey products

$$\langle x, y, z \rangle \subseteq H^*(X, \mathbb{Z}) \quad \text{non-empty if } xy = yz = 0$$

Salvatore, Longoni: All Massey products for $\tilde{F}_2L_{7,1}$ vanish (i.e. contain 0) but for $\tilde{F}_2L_{7,2}$ there is a non-trivial Massey product.

Massey products are a consequence of the existence of a chain level multiplication on $C^*(X, \mathbb{Z})$. So the last observation suggests that one should use the cohomology equipped with Massey products or even the DGA of cochains $C^*(X, \mathbb{Z})$ as an invariant of X . Note that this works over all rings, in particular we could have also worked over \mathbb{Q} .

Question 1.3. What are good invariants?

There are two requirements that we impose

- The invariants are of algebraic nature, i.e. assign to X something like a group, ring etc. which can be studied by algebraic means.
- They are powerful, i.e. a great deal of information about X can be extracted from them and many spaces can be distinguished by them!

More formally we consider functors

$$F : \text{Ho}(\text{Top}) \rightarrow \mathcal{B}$$

where \mathcal{B} is some ‘algebraic category’ which are as close as possible to being fully faithful. If that can’t be achieved we hope that F reflects equivalence and is full, so from $F(X) \simeq F(Y)$ we can conclude $X \simeq Y$. This sound very ambitious, but surprisingly there are indeed some good invariants that come close to the requirements. Now let us outline the content of the lecture.

1.1 Rational Homotopy theory

We sketch the main foundational results of rational homotopy theory (following Quillen and Sullivan). Therefore denote by CDGA_R the category of commutative differential graded algebras over a ring R which are non-negatively graded as cochain complexes.

Theorem 1.4 (Quillen, Sullivan). *There exists a functor*

$$\mathcal{A}_{PL} : \text{Top}^{op} \rightarrow \text{CDGA}_{\mathbb{Q}} \quad (\text{piecewise linear differential forms})$$

such that for a space X the underlying rational DGA of $\mathcal{A}_{PL}(X)$ is connected by a zig-zag of multiplicative quasi-isomorphisms to the DGA of rational cochains $C^(X, \mathbb{Q})$. It induces a functor*

$$\text{Ho}(\mathcal{A}_{PL}) : \text{Ho}(\text{Top}) \rightarrow \text{Ho}(\text{CDGA}_{\mathbb{Q}})$$

which is fully faithful when restricted to the full subcategory of $\text{Ho}(\text{Top})$ spanned by the simply connected, rational spaces of finite type.

Here a simply connected space X is called rational of finite type when either of the following equivalent conditions are satisfied:

1. all homotopy groups $\pi_n(X)$ are finite dimensional rational vector spaces.
2. all homology groups $\tilde{H}_*(X, \mathbb{Z})$ are finite dimensional \mathbb{Q} -vector spaces

Idea for construction of \mathcal{A}_{PL} : if X was a manifold and we worked over \mathbb{R} then we could use $\Omega^*(X)$ as a CDGA model for $C^*(X, \mathbb{R})$. This idea can be used to define \mathcal{A}_{PL} for an arbitrary space by doing if for simplices (where rational forms make sense) and then left Kan extending the functor. \square

The situation is even better, there is a functor back $\text{Ho}(\text{CDGA}_{\mathbb{Q}}) \rightarrow \text{Ho}(\text{Top})$ which sends A^* to the space $\text{Map}_{\text{CDGA}_{\mathbb{Q}}}(A^*, \mathbb{Q})$. It is right adjoint, i.e. we have an adjunction

$$\text{Ho}(\text{Top}) \rightleftarrows \text{Ho}(\text{CDGA}_{\mathbb{Q}})^{op}$$

and the adjunction unit $X \rightarrow \text{Map}_{\text{CDGA}_{\mathbb{Q}}}(\mathcal{A}_{PL}(X), \mathbb{Q})$ is an equivalence for X simply connected, rational and of finite type. For general simply connected spaces of finite type it is the rationalization! In particular we can recover the rational homotopy groups and all rational invariants from the CDGA

Question 1.5. How algebraic is the category $\text{Ho}(\text{CDGA}_{\mathbb{Q}})$?

Definition 1.6. A minimal Sullivan algebra is a CDGA over \mathbb{Q} of the form $(\Lambda V, d)$ where $V = \bigoplus_{n \geq 0} V_n$ is a non-negatively graded rational vector space plus some minimality condition that we spell out later.

The category $\text{Ho}(\text{CDGA}_{\mathbb{Q}}^{\min}) \subseteq \text{Ho}(\text{CDGA}_{\mathbb{Q}})$ is the full subcategory spanned by the minimal Sullivan algebras.

Theorem 1.7 (Sullivan). *The canonical inclusion functor $\text{Ho}(\text{CDGA}_{\mathbb{Q}}^{\min}) \rightarrow \text{Ho}(\text{CDGA}_{\mathbb{Q}})$ is an equivalence of categories. Every morphism in $\text{Ho}(\text{CDGA}_{\mathbb{Q}}^{\min})$ is represented by a morphism of CDGA's (not a zig-zag) and is an isomorphism in $\text{Ho}(\text{CDGA}_{\mathbb{Q}}^{\min})$ (i.e. a quasi-iso of CDGA's) if and only if it is an isomorphism of CDGA's.*

In particular for every CDGA over \mathbb{Q} there is a minimal Sullivan model of it that is unique up to isomorphism.

Corollary 1.8. *There is a functor*

$$\text{Ho}(\text{Top}^{op}) \rightarrow \text{Ho}(\text{CDGA}_{\mathbb{Q}}^{\min})$$

that is fully faithful when restricted to the full subcategory of rational spaces. In particular rational spaces X and Y are homotopy equivalent if and only if the associated minimal Sullivan models are isomorphic.

For a minimal model $A^* = (\Lambda V, d)$ we can compute the homotopy groups of the associated space $\text{Map}_{\text{CDGA}}(A^*, \mathbb{Q})$ as

$$\text{Hom}_{\text{Ho}(\text{CDGA}_{\mathbb{Q}}^{\min})}^*((\Lambda V, d), \mathbb{Q}) \simeq \text{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$$

Thus if X is a simply connected space of finite type, then we can compute $\pi_*(X) \otimes \mathbb{Q}$ as the dual space of V for the minimal Sullivan model $(\Lambda V, d)$ of $\mathcal{A}_{PL}(X)$.

Example 1.9. *Let's consider the sphere S^n . First for n odd, choose a representative $x_n \in \mathcal{A}_{PL}^n(S^n)$ of a generator $H^n(S^n, \mathbb{Q}) \cong \mathbb{Q}$. Get morphism of CDGA's*

$$\Lambda(x_n) \rightarrow \mathcal{A}_{PL}^*(S^n)$$

which is quasi-iso and thus a minimal model! For n even do the same, but $\Lambda(x_n) \rightarrow \mathcal{A}_{PL}^(S^n)$ not quasi-iso. Choose $y_{2n-1} \in \mathcal{A}_{PL}^{2n-1}(S^n)$ so that $dy_{2n-1} = x_n^2$. Then*

$$\Lambda(x_n, y_{2n-1}) \rightarrow \mathcal{A}_{PL}^*(S^n)$$

is quasi iso. Thus we can compute

$$\pi_*(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{for } * = n \\ \mathbb{Q} & \text{for } * = 2n - 1 \text{ if } n = 2k \\ 0 & \text{else} \end{cases}$$

We will prove all the results stated here. In fact we prove much more, namely we show how to get rid of the simply connected and finite type assumptions.

1.2 p -adic homotopy theory

Recall that \mathcal{A}_{PL} was a strictly commutative model for $C^*(X, \mathbb{Q})$.

Question 1.10. Can we repeat the story for $C^*(X, \mathbb{F}_p)$ or even $C^*(X, \mathbb{Z})$?

This can not happen, it is in fact obstructed by the existence of Steenrod operations. More precisely we have the following:

Proposition 1.11. *There can not be a functor*

$$\mathcal{A}_{\mathbb{F}_p} : \text{Top}^{op} \rightarrow \text{CDGA}_{\mathbb{F}_p}$$

such that there exists a zig-zag of quasi-isomorphisms of chain complexes over \mathbb{F}_p between $\mathcal{A}_{\mathbb{F}_p}(X)$ and $C^*(X, \mathbb{F}_p)$ that is natural in X (the similar thing is also impossible for $C^*(X, \mathbb{Z})$ or over other non-rational rings).

There are however ways to remedy the situation and get sort of a commutative ring structure:

- One can consider $C^*(X, \mathbb{F}_p)$ as a cosimplicial commutative algebra, that is a functor $\Delta \rightarrow \text{CAlg}_{\mathbb{F}_p}$. The cosimplicial algebra that models $C^*(X, \mathbb{F}_p)$ sends $[n] \in \Delta$ to the commutative algebra of maps $\text{Hom}_{\text{Set}}(\text{Sing}X)_n, \mathbb{F}_p)$ with pointwise multiplication. Here $(\text{Sing}X)_n = \text{Hom}_{\text{Top}}(|\Delta^n|, X)$ is the singular complex of X .
- One can consider $C^*(X, \mathbb{F}_p)$ as being equipped with a multiplication that is commutative up to coherent homotopies. This sort of structure is called an \mathbb{E}_∞ -algebra structure over \mathbb{F}_p and will be discussed carefully in the course.

These two approaches are in fact intimately related. For every ring R there are ‘forgetful’ functors

$$\text{Ho}(\text{CDGA}_R) \xrightarrow{\text{shuffle}} \text{Ho}(\text{csCAlg}_R) \rightarrow \text{Ho}(\mathbb{E}_\infty \text{Alg}_R^{\leq 0})$$

which do not change the underlying chain complex. If R is rational then these functors are equivalences. This explains why we were able to find a rational commutative differential graded algebra $\mathcal{A}_{PL}(X)$ that models $C^*(X, \mathbb{Q})$ with the cosimplicial structure (or the \mathbb{E}_∞ -structure). But for $R = \mathbb{F}_p$ or $R = \mathbb{Z}$ neither of the two functors is an equivalence. We will however see that the second functor is fully faithful on the class of cosimplicial commutative algebras that arise as $C^*(X, R)$ for some space X .

Definition 1.12. We say that a simply connected space X whose homotopy groups are p -complete and finitely generated (in the completed world), is a simply connected, finite type, p -complete space.

One can show (and we will) that for every space X which is simply connected and of finite type there is a p -completion $X \rightarrow X_p^\wedge$ which has the effect of completion on homotopy groups.

Theorem 1.13 (Goerss). *Consider the functor*

$$C^*(-, \overline{\mathbb{F}_p}) : \text{Top}^{op} \rightarrow \text{csCAlg}_{\overline{\mathbb{F}_p}}$$

which assigns to every space its cochains $C^*(X, \overline{\mathbb{F}}_p)$ over the algebraic closure $\overline{\mathbb{F}}_p$ considered as a cosimplicial algebra. Then the associated functor

$$\mathrm{Ho}(\mathrm{Top})^{op} \rightarrow \mathrm{Ho}(\mathrm{csCAlg}_{\overline{\mathbb{F}}_p})$$

is fully faithful on the subcategory of simply connected, finite type, p -complete spaces.

In fact we can recover every such space X as the space $\mathrm{Map}_{\mathrm{csCAlg}_{\overline{\mathbb{F}}_p}}(C^*(X, \overline{\mathbb{F}}_p), \overline{\mathbb{F}}_p)$ from the cochains with value in $\overline{\mathbb{F}}_p$. In fact Goerss proves a much sharper result, namely he works with simplicial coalgebras instead of algebras to remove the finiteness hypothesis on X . Also he considers the class of \mathbb{F}_p -local spaces. The key is a good algebraic understanding of coalgebras over algebraically closed fields. We will explain all of that.

Theorem 1.14. *There are functors*

$$\begin{aligned} C_\sigma^*(-, \mathbb{F}_p) &: \mathrm{Ho}\left(\mathrm{Top}^{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)}\right)^{op} \rightarrow \mathrm{Ho}\left(\mathrm{csCAlg}_{\mathbb{F}_p}\right) \\ C^*(-, \mathbb{F}_p) &: \mathrm{Ho}(\mathrm{Top})^{op} \rightarrow \mathrm{Ho}\left(\mathrm{csCAlg}_{\mathbb{F}_p}^{\mathrm{Frob}}\right) \end{aligned}$$

which are fully faithful on the subcategories of simply connected, finite type p -complete objects.

We will also prove a version over the integers. A similar but much deeper result has been obtained by Mandell

Theorem 1.15 (Mandell). *Consider the functor*

$$C^*(-, \overline{\mathbb{F}}_p) : \mathrm{Ho}(\mathrm{Top}) \rightarrow \mathrm{Ho}\left(\mathbb{E}_\infty \mathrm{Alg}_{\overline{\mathbb{F}}_p}\right).$$

then it is full faithful when on the subcategory of simply connected finite type p -complete spaces.

The key is to use the Postnikov-Tower and find a good \mathbb{E}_∞ -presentation of the cochains on Eilenberg-MacLane spaces. This will involve understanding the Steenrod actions. Generalization over \mathbb{F}_p .

Theorem 1.16 (Lurie). *There is a functor*

$$C_\sigma^*(-, \mathbb{F}_p) : \mathrm{Ho}\left(\mathrm{Top}^{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)}\right)^{op} \rightarrow \mathrm{Ho}\left(\mathbb{E}_\infty \mathrm{Alg}_{\mathbb{F}_p}\right).$$

then it is full faithful when on the subcategory of simply connected finite type p -complete spaces.

Mandell has even a sort of integral version. Therefore recall that a topological space is called nilpotent if its fundamental group is nilpotent and it acts nilpotently on the higher homotopy groups. In particular all simply connected and simple spaces are nilpotent.

Theorem 1.17 (Mandell). *Assume for two nilpotent topological spaces X and Y of finite type the \mathbb{E}_∞ -algebras $C^*(X, \mathbb{Z})$ and $C^*(Y, \mathbb{Z})$ are equivalent as \mathbb{E}_∞ -algebras over \mathbb{Z} . Then X and Y are weakly homotopy equivalent.*

Unfortunately there is no analogue of the Sullivan minimal models. Thus the category of cosimplicial rings, resp. \mathbb{E}_∞ -algebras are still very complicated to understand and maybe not terribly algebraic.

2 Bousfield localization of spaces

When we say space we shall mean a CW complex (or a Kan complex). In particular we will only talk about homotopy equivalence and not weak homotopy equivalence (we will be more careful about this aspect soon). Now let E be a homology theory on the category of spaces.

Definition 2.1. A morphism $X \rightarrow Y$ of spaces is called an E -equivalence if the induced morphism $E_*(X) \rightarrow E_*(Y)$ is an isomorphism. A space Z is called E -local if for every E -equivalence $X \rightarrow Y$ the induced map $\text{Map}(Y, Z) \rightarrow \text{Map}(X, Z)$ is a homotopy equivalence (equivalently for every E -equivalence the induced map on homotopy classes $[Y, Z] \rightarrow [X, Z]$ is an iso). A map $X \rightarrow X_E$ is called E -localization if it is an E -equivalence and X_E is E -local.

One of the main goals of this lecture is to show that an E -localization always exists and try to understand it.

Example 2.2. • Let $E = H\mathbb{Q}$ be rational homology. Then the local objects are called rational spaces and the map $X \rightarrow X_{\mathbb{Q}}$ is called a rationalization. A map $X \rightarrow Y$ is a rational equivalence if and only if the induced map in cohomology $H^*(Y, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$ is an isomorphism as one immediately sees from UCT since $H^*(X, \mathbb{Q}) = \text{Hom}_{\mathbb{Q}}(H_*(X, \mathbb{Q}), \mathbb{Q})$.

- This implies that all Eilenberg-Mac Lane spaces $K(\mathbb{Q}, n)$ are rational. More generally using UCT shows that for a rational vector space V we have

$$H^*(X, V) = \text{Hom}_{\mathbb{Q}}(H_*(X, \mathbb{Q}), V).$$

and thus that $K(V, n)$ is a rational space if V is a rational vector space and $n \geq 1$.

- Rational spaces (as all E -local spaces) are obviously closed under homotopy limits. Assume a space X is simply connected or more generally simple. We have the Postnikov tower

$$X \simeq \lim \tau_{\leq n} X$$

and since each $\tau_{\leq n} X$ sits in a (homotopy) pullback square

$$\begin{array}{ccc} \tau_{\leq n} X & \longrightarrow & \text{pt} \\ \downarrow & & \downarrow \\ \tau_{\leq n-1} X & \longrightarrow & K(\pi_n(X), n+1) \end{array}$$

we conclude inductively that if all homotopy groups of X are rational vector spaces then X is a rational space. We will show the converse below.

Example 2.3. • Let $E = H\mathbb{F}_p$ be ordinary homology with mod p coefficients. The local objects are called \mathbb{F}_p -local spaces.

Sometimes the terminology p -complete and p -adic equivalence is used, but we try to avoid that terminology since it conflicts with other uses of these words. More precisely there is another functor constructed by Sullivan et al. called p -profinite completion of spaces that we will discuss later in the lecture. This agrees with \mathbb{F}_p -localization on spaces with finitely generated homotopy groups. There is also a third p -completion functor used by Bousfield-Kan that agrees with \mathbb{F}_p -localization on nilpotent spaces (or more generally good spaces).

- Again the same argument as above shows that $K(V, n)$ is \mathbb{F}_p -local for every \mathbb{F}_p -vector space V since

$$H^*(X, V) \cong \text{Hom}_{\mathbb{F}_p}(H_*(X, \mathbb{F}_p), V).$$

More generally we find that $K(M, n)$ is \mathbb{F}_p -local for every \mathbb{Z}/p^2 -module M as we have a short exact sequence

$$0 \rightarrow pM \rightarrow M \rightarrow M/pM \rightarrow 0$$

in which both outer modules are \mathbb{F}_p -vector spaces. This induces a long exact sequence

$$\rightarrow H^*(X, pM) \rightarrow H^*(X, M) \rightarrow H^*(X, M/pM) \rightarrow H^{*+1}(X, pM) \rightarrow \dots$$

which by the 5-Lemma implies the claim. Inductively using the decomposition of a \mathbb{Z}/p^k -module M as

$$0 \rightarrow pM \rightarrow M \rightarrow M/pM \rightarrow 0$$

we find that every Eilenberg Mac-Lane space $K(M, n)$ for an \mathbb{Z}/p^n -module M is \mathbb{F}_p -local.

- Now the natural question is if for an abelian group M the Eilenberg-Mac Lane spaces $K(M_p^\wedge, n)$ is \mathbb{F}_p -local. Recall that $M_p^\wedge = \varprojlim_k M/p^k$. We certainly get that the space

$$\varprojlim_k K(M/p^k, n)$$

is p -local. To compute the homotopy groups of that space we use the Milnor sequence and get

$$\pi_*(-) \cong \begin{cases} M_p^\wedge & \text{for } * = n \\ \varprojlim_k^1 M/p^k \cong 0 & \text{for } * = n + 1 \\ 0 & \text{otherwise} \end{cases}$$

This is equivalent to the space $K(M_p^\wedge, n)$. As an example we see that $K(\mathbb{Z}_p^\wedge, n)$ is \mathbb{F}_p -local.

- Finally again using a Postnikov-Tower argument we can conclude that if a simple space X has p -complete homotopy groups then it is \mathbb{F}_p -local.
- Warning: for general M the map $K(M, n) \rightarrow K(M_p^\wedge, n)$ need not be an \mathbb{F}_p -equivalence, but we will see that it is if M is finitely generated! Also we will see that there are lots of spaces that are p -complete but whose homotopy groups are not p -complete in the naive sense.

Proposition 2.4. *An E -localization of X is, if it exists, uniquely up to canonical homotopy equivalence determined by X . The map $X \rightarrow X_E$ admits in $\text{Ho}(\text{Top})_{X/}$ the following universal descriptions*

- It is initial among maps $X \rightarrow Y$ with Y an E -local space ($Y \simeq \lim_{Z \in \text{Ho}(\text{Top})_{X/}^{\text{loc}}} Z$)
- It is terminal among maps $X \rightarrow Y$ which are E -equivalences ($Y \simeq \text{colim}_{Z \in \text{Ho}(\text{Top})_{X/}^{\text{equiv}}} Z$)

A map $X \rightarrow X'$ between E -local spaces is an E -equivalence if and only if it is a (weak) homotopy equivalence.

Proof. Let $\text{Ho}(\text{Top})_E \subseteq \text{Ho}(\text{Top})$ be the full subcategory of E -local spaces. Lets us prove the last claim and leave the other two as an exercise. Assume that $f : X \rightarrow Y$ is an E -equivalence and that X and Y are E -local. Then for every other E -local space Z (e.g. X or Y itself) we get that $[Y, Z] \rightarrow [X, Z]$ is an isomorphism. But this shows that the map on corepresented functors $[X, -], [Y, -] : \text{Ho}(\text{Top})_E \rightarrow \text{Set}$ is a natural iso, which by Yoneda implies that the map $X \rightarrow Y$ is an iso in $\text{Ho}(\text{Top})_E$. \square

Remark 2.5. *Bousfield shows in [Bou74] that if E is connective it induces the same localization as the homology theory $H_*(-, R)$ where R is either $R = \mathbb{Z}[J^{-1}]$ for J a set of primes or $R = \bigoplus_{p \in J} \mathbb{Z}/p$. We will try to understand the cases $R = \mathbb{Q}$ and $R = \mathbb{Z}/p$ explicitly the other cases work exactly similar.*

A particular interesting case is that of the mop p Moore spectrum \mathbb{S}/p . Then localization with respect to that is equivalent to the \mathbb{F}_p -localization as one sees as follows: The AHSS $H_(X, \pi_*(\mathbb{S}/p)) \Rightarrow (\mathbb{S}/p)_*(X)$ implies that every \mathbb{F}_p -equivalence is also an \mathbb{S}/p -equivalence (here we also have to use that $H_*(X, A)$ for A a finite, abelian p -torsion group is an iso if $H_*(-; \mathbb{F}_p)$ is). Conversely since $\mathbb{S}/p \otimes H\mathbb{F}_p \simeq H\mathbb{F}_p \oplus \Sigma H\mathbb{F}_p$ we find that every \mathbb{S}/p -equivalence is also an \mathbb{F}_p -equivalence. That last is a fact which is stably (i.e. for the category of spectra) wrong (Why? Answer: The AHSS does not converge always stably, since it is not clear that the morphism*

$$X \otimes \mathbb{S}/p \simeq \varprojlim (X \otimes \tau_{\leq n} \mathbb{S}/p)$$

is an equivalence. If X is a space this follows since it gets higher and higher connected).

Similarly one shows that localization with respect to \mathbb{Z}_p^\wedge is equivalent to localization with respect to $\mathbb{Z}_{(p)} = \mathbb{Z}[J^{-1}]$ where J contains all primes except for p .

Theorem 2.6. 1. *A simply connected space X is rational if and only if all homotopy groups are rational.*

2. *A morphism $f : X \rightarrow Y$ between simply connected spaces is a rational equivalence precisely if it induces an isomorphism $\pi_*(X) \otimes \mathbb{Q} \rightarrow \pi_*(Y) \otimes \mathbb{Q}$.*

3. *For a simply connected space X there exists a rationalization $X \rightarrow X_{\mathbb{Q}}$ where $X_{\mathbb{Q}}$ is also simply connected*

Proof. One direction of the first assertion has already been proved in Example 2.2. The other direction will follow from the combination of (2) and the proof of (3).

To prove (2) assume that a morphism $f : X \rightarrow Y$ induces an isomorphism on rationalized homotopy groups. We have to show that it also induces an isomorphism in rational homology. We consider the homotopy fibre F which is connected and has torsion homotopy groups. Using the Serre spectral sequence $H_*(Y, H_*(F, \mathbb{Q})) \Rightarrow H_*(X, \mathbb{Q})$ we see that it suffices to show that the rational homology of F vanishes in positive degrees. We prove this by induction over the Postnikov Tower of F . We have $H_n(F, \mathbb{Q}) = H_n(\tau_{\leq n} F, \mathbb{Q})$. Using induction and again the Serre Spectral sequence for the fibre sequence $K(A, n) \rightarrow \tau_{\leq n} F \rightarrow \tau_{\leq n-1} F$ we can reduce to the case where F is an Eilenberg-Mac Lane space $K(A, n)$ for a torsion group A . Using the Serre Spectral sequence for

$$K(A, n) \rightarrow \text{pt} \rightarrow K(A, n-1)$$

(which is $H^*(K(A, n-1), H_*(K(A, n), \mathbb{Q})) \Rightarrow \mathbb{Q}$) we can assume that $n = 1$. Then A is a filtered colimit of finite abelian groups. Thus we are left to show that for A finite abelian

group the homology $H_*(K(A, 1), \mathbb{Q}) = H_*(A, \mathbb{Q})$ vanishes in positive degrees, which is true (by a transfer argument) and thereby proves part (2).

Finally to prove part (3) we want to construct $X_{\mathbb{Q}}$ which has rational homotopy groups. In fact we will prove a slightly more general statement, namely that there is $X_{\mathbb{Q}}$ with rational homotopy groups and a map $X \rightarrow X_{\mathbb{Q}}$ that induces an isomorphism on rational homotopy groups. First assume X is an Eilenberg MacLane space $K(A, n)$. Then we simply set $X_{\mathbb{Q}} := K(A_{\mathbb{Q}}, n)$ and by (1) and (2) this does the job. Assume we have a fibre sequence $X \rightarrow Y \rightarrow Z$ and we already have rationalizations of Y and Z (of the given form) then we claim that the fibre of $Y_{\mathbb{Q}} \rightarrow Z_{\mathbb{Q}}$ is a rationalization of X . To see this call this fibre $X_{\mathbb{Q}}$ then we get a diagram To see this use that we have a long exact sequence

$$\begin{array}{ccccccc} \longrightarrow & \pi_{n+1}(Z) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_n(Y) & \longrightarrow & \pi_n(Z) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & \pi_{n+1}(Z_{\mathbb{Q}}) & \longrightarrow & \pi_n(X_{\mathbb{Q}}) & \longrightarrow & \pi_n(Y_{\mathbb{Q}}) & \longrightarrow & \pi_n(Z_{\mathbb{Q}}) & \longrightarrow \end{array}$$

of exact sequences. From the bottom one it follows that $\pi_{n+1}(X_{\mathbb{Q}})$ is rational and then tensoring the upper one with \mathbb{Q} remains exact and we find that $\pi_*(X_{\mathbb{Q}}) \cong \pi_*(X) \otimes \mathbb{Q}$. Thus we can construct rationalization of the Postnikov sections $\tau_{\leq n}X$ inductively. But then $X_{\mathbb{Q}} := \lim(\tau_{\leq n}X_{\mathbb{Q}})$ is a rationalization of X (the \lim^1 terms vanish since the tower has eventually constant homotopy groups in every degree). \square

Similarly one can show that X is rational iff $\tilde{H}_*(X, \mathbb{Z})$ is rational and that we have an isomorphism $\tilde{H}_*(X_{\mathbb{Q}}; \mathbb{Z}) \cong \tilde{H}_*(X, \mathbb{Q})$. Now we turn to the considerably more complicated case of the p -adic localization.

Definition 2.7. We say that an abelian group is derived p -complete if $\text{Hom}(\mathbb{Z}/p^{\infty}, A) = 0$ and the canonical map $A \rightarrow \text{Ext}(\mathbb{Z}/p^{\infty}, A)$ is an isomorphism where $\mathbb{Z}/p^{\infty} = \varinjlim \mathbb{Z}/p^n \cong \mathbb{Z}[1/p]/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}_{(p)}$ is the Pruefer group.

Remark 2.8. *The Functors $\text{Ext}(\mathbb{Z}/p^{\infty}, -)$ and $\text{Hom}(\mathbb{Z}/p^{\infty}, -)$ are the left derived functors of completion (which is not left exact!). Thus we have for example a short exact sequence*

$$0 \rightarrow \varprojlim^1 \text{Tor}(\mathbb{Z}/p^n, A) \rightarrow \text{Ext}(\mathbb{Z}/p^{\infty}, A) \rightarrow A_p^{\wedge} \rightarrow 0$$

For example if $\pi_n X$ has bounded order of p -torsion (e.g. if it is finitely generated) we get that $\text{Ext}(\mathbb{Z}/p^{\infty}, A) = A_p^{\wedge}$. In particular we get that \mathbb{Z}/p^n and \mathbb{Z}_p^{\wedge} are derived p -complete groups.

Another way of stating the definition to the say that

$$A \rightarrow \text{RHom}(\mathbb{Z}/p^{\infty}[-1], A) \simeq \varprojlim A//p^n$$

is a quasi iso. This is by the ses $\mathbb{Z} \rightarrow \mathbb{Z}[1/p] \rightarrow \mathbb{Z}/p^{\infty}$ equivalent to saying that $\text{RHom}(\mathbb{Z}[1/p], A) \simeq 0$.

It is true (but a bit tricky to prove....) that if $A \rightarrow \text{Ext}^1(\mathbb{Z}/p^{\infty}, A)$ is an isomorphism that then also $\text{Hom}(\mathbb{Z}/p^{\infty}, A) = 0$ so that A is derived p -complete. Thus our definition is slightly redundant. The argument (thanks to Tobi and Achim) is as follows: assume that $A \cong \text{Ext}^1(\mathbb{Z}/p^{\infty}, A)$. Then the long exact sequence

$$0 \rightarrow \text{Hom}(\mathbb{Z}/p^{\infty}, A) \rightarrow \text{Hom}(\mathbb{Z}[1/p], A) \rightarrow \text{Hom}(\mathbb{Z}, A) \rightarrow \text{Ext}(\mathbb{Z}/p^{\infty}, A)$$

implies that the map $\mathrm{Hom}(\mathbb{Z}[1/p], A) \rightarrow \mathrm{Hom}(\mathbb{Z}, A)$ is zero. Similarly you see that also all the maps $\mathrm{Hom}(\mathbb{Z}[1/p], A) \xrightarrow{(p^k)^*} \mathrm{Hom}(\mathbb{Z}, A)$ are zero. Unfolding what it means to have an element in $\mathrm{Hom}(\mathbb{Z}[1/p], A)$ this already shows that the whole group has to be zero. As a consequence also $\mathrm{Hom}(\mathbb{Z}/p^\infty, A)$ is trivial.

Theorem 2.9. 1. A simply connected space X is \mathbb{F}_p -local if and only if all its homotopy groups are derived p -complete.

2. A morphism $X \rightarrow Y$ of simply connected spaces is an \mathbb{F}_p -equivalence precisely if the relative homotopy groups $\pi_n(Y, X) := \pi_{n-1} \mathrm{fib}(X \rightarrow Y)$ are uniquely p -divisible.

3. For a simply connected space X there exists an \mathbb{F}_p -completion $X \rightarrow X_{\mathbb{F}_p}$ where $X_{\mathbb{F}_p}$ is also simply connected and we have a split short exact sequence

$$0 \rightarrow \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_n X) \rightarrow \pi_n X_{\mathbb{F}_p} \rightarrow \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) \rightarrow 0$$

We will prove part of this theorem in a couple of steps.

Lemma 2.10. An abelian group A is uniquely p -divisible precisely if $\mathrm{RHom}(\mathbb{Z}/p^\infty, A) \simeq 0$.

Proof. By the ses $\mathbb{Z} \rightarrow \mathbb{Z}[1/p] \rightarrow \mathbb{Z}/p^\infty$ we see that the vanishing of $\mathrm{RHom}(\mathbb{Z}/p^\infty, A)$ is equivalent to saying that the canonical morphism $\mathrm{RHom}(\mathbb{Z}[1/p], A) \rightarrow A$ is an equivalence. Thus A is a $\mathbb{Z}[1/p]$ -module. Conversely if A is a $\mathbb{Z}[1/p]$ -module then

$$\mathrm{RHom}(\mathbb{Z}/p^\infty, A) \simeq \mathrm{RHom}(\mathbb{Z}/p^\infty, A \otimes^L \mathbb{Z}[1/p]) \simeq \mathrm{RHom}_{\mathbb{Z}[1/p]}(\mathbb{Z}/p^\infty \otimes^L \mathbb{Z}[1/p], A \otimes^L \mathbb{Z}[1/p]) \simeq 0.$$

where $\mathbb{Z}/p^\infty \otimes^L \mathbb{Z}[1/p]$ is easily seen using flatness of $\mathbb{Z}[1/p]$ and the colimit description of \mathbb{Z}/p^∞ . \square

Lemma 2.11. Under the same assumption as in the last lemma we have that $H_*(K(A, 1), \mathbb{F}_p) = 0$ for $* \geq 1$.

Proof. We can write the $\mathbb{Z}[1/p]$ -module A as filtered colimit of its finitely generated submodules $B \subseteq A$. These are all sums of either finite groups of order coprime to p and $\mathbb{Z}[1/p]$. For the finite summands it's clear that the group homology vanishes, thus it remains to show that $H_*(\mathbb{Z}[1/p], \mathbb{F}_p) = 0$. To that end we write $\mathbb{Z}[1/p]$ as the colimit $\varinjlim H_*(\mathbb{Z}, \mathbb{F}_p)$ by multiplication with p in \mathbb{Z} . But since $H_*(\mathbb{Z}, \mathbb{F}_p) = H_*(S^1, \mathbb{F}_p) = \mathbb{F}_p$ for $* = 1$ and 0 else we get that this colimit is zero. \square

Lemma 2.12. Every morphism of simply connected spaces $X \rightarrow Y$ whose relative homotopy groups are uniquely p -divisible is an \mathbb{F}_p -equivalence.

Proof. Let $F \rightarrow X \rightarrow Y$ denote the homotopy fibre of the map which is connected. Using the Serre-Spectral sequence $H_*(Y, H_*(F, \mathbb{F}_p)) \Rightarrow H_*(X, \mathbb{F}_p)$ we see that it suffices to show that $H_*(F, \mathbb{F}_p)$ vanishes in positive degree. We want to show that for every connected space F for which p -acts invertibly on the homotopy groups the \mathbb{F}_p -homology $H_*(F, \mathbb{F}_p)$ vanishes in positive degree. Now we use the Postnikov tower of F and the same tricks as in the rational case to reduce to the case $F = K(A, 1)$. But then it is the statement of Lemma 2.11. \square

Now we want to understand the \mathbb{F}_p -localization of the Eilenberg-Mac Lane spaces $K(A, n)$. Therefore let us define

$$\begin{aligned} K(A, n) // p^k &:= K(0 \leftarrow A \xleftarrow{p^k} A \leftarrow 0 \leftarrow \dots) \\ &\simeq \text{fib}(K(A, n+1) \xrightarrow{p^n} K(A, n+1)) \\ &\simeq K(A/p^k A, n) \times K(A_{p^k\text{-tor}}, n+1) \end{aligned}$$

where the first line is the Eilenberg-Mac Lane space associated to the chain complex that sits in the bracket and which has A in degree n and $n+1$ and 0 otherwise¹. Then we define

$$K(A, n)_{\mathbb{F}_p} := \varprojlim K(A, n) // p^k$$

Lemma 2.13. *The canonical map $K(A, n) \rightarrow K(A, n)_{\mathbb{F}_p}$ exhibits $K(A, n)_{\mathbb{F}_p}$ as the \mathbb{F}_p -localization of $K(A, n)$. Moreover the relative homotopy groups are uniquely p -divisible.*

Proof. First we deduce from the fact that the homotopy groups of $K(A, n) // p^k$ are p -power torsion that it is \mathbb{F}_p -local (see Example 2.3). Therefore also the limit is \mathbb{F}_p -local. Since both spaces are EM-spaces associated to chain complexes the fibre is also an EM space associated to the homotopy fibre of the map of chain complexes $A \rightarrow \text{RHom}(\mathbb{Z}/p^\infty, A)$ which is $\text{RHom}(\mathbb{Z}[1/p], A)$. But this fibre is a $\mathbb{Z}[1/p]$ module, thus its homotopy groups are $\mathbb{Z}[1/p]$ -modules. Then the last lemma finishes the proof. \square

Also note that we have

$$\pi_*(K(A, n)_{\mathbb{F}_p}) = \begin{cases} \text{Ext}(\mathbb{Z}/p^\infty, A) & \text{for } * = n \\ \text{Hom}(\mathbb{Z}/p^\infty, A) & \text{for } * = n+1 \\ 0 & \text{else} \end{cases}$$

Lemma 2.14. *Assume that $X \rightarrow X_{\mathbb{F}_p}$ and $Y \rightarrow Y_{\mathbb{F}_p}$ are \mathbb{F}_p -localizations whose relative homotopy groups are uniquely p divisible. Let $f : X \rightarrow Y$ be a map with fibre F and $F_{\mathbb{F}_p}$ the fibre of the map $X_{\mathbb{F}_p} \rightarrow Y_{\mathbb{F}_p}$. Then the map $F \rightarrow F_{\mathbb{F}_p}$ has the same property.*

Proof. Consider the diagram

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ F_{\mathbb{F}_p} & \longrightarrow & X_{\mathbb{F}_p} & \longrightarrow & Y_{\mathbb{F}_p} \end{array}$$

We get a long exact sequence $\dots \pi_n(F_{\mathbb{F}_p}, F) \rightarrow \pi_n(X_{\mathbb{F}_p}, X) \rightarrow \pi_n(Y_{\mathbb{F}_p}, Y) \rightarrow \dots$ which easily implies that the relative homotopy groups of $(X_{\mathbb{F}_p}, X)$ are $\mathbb{Z}[1/p]$ modules (since $\mathbb{Z}[1/p]$ is flat). \square

Partial Proof of Theorem 2.9. It follows from the last Lemma 2.13 and a Postnikov tower argument that a simply connected space all of whose homotopy groups are derived p -complete is \mathbb{F}_p -local. We also know by Lemma 2.12 that a morphism whose relative homotopy groups are $\mathbb{Z}[1/p]$ modules is an \mathbb{F}_p -equivalence.

¹Question: are the maps zero or Bocksteins?

Now we show that for every simply connected space X there is another simply connected space $X_{\mathbb{F}_p}$ with a map from X such that the fibre is uniquely p -divisible. If X is Eilenberg Mac Lane then this follows from Lemma 2.13. Assume we know it for $\tau_{\leq -1n}X$. Then we use the pullback

$$\begin{array}{ccc} \tau_{\leq n}X & \longrightarrow & \text{pt} \\ \downarrow & & \downarrow \\ \tau_{\leq n-1}X & \longrightarrow & K(\pi_n(X), n+1) \end{array}$$

and the last Lemma to deduce it for $\tau_{\leq n}X$. Finally a limit argument shows if for X .

Finally if we have a map between simply connected spaces $X \rightarrow Y$ then it is an \mathbb{F}_p -equivalence iff the induced map $X_{\mathbb{F}_p} \rightarrow Y_{\mathbb{F}_p}$ is a homotopy equivalence. But the fibre F of the map $X \rightarrow Y$ is also the fibre of the map $F_X \rightarrow F_Y$ where F_X and F_Y are the fibres of $X \rightarrow X_{\mathbb{F}_p}$ and $Y \rightarrow Y_{\mathbb{F}_p}$ as can be seen from the diagram

$$\begin{array}{ccccc} F & \longrightarrow & \text{pt} & & \\ \downarrow & & \lrcorner & & \downarrow \\ F_X & \longrightarrow & F_Y & \longrightarrow & \text{pt} \\ \downarrow & & \lrcorner & & \lrcorner \\ X & \longrightarrow & Y & \longrightarrow & Y_{\mathbb{F}_p} \end{array}$$

and thus it is clearly p -divisible by the long exact sequence. □

We have not proved the converse of part (1) of the Theorem and the formula for the homotopy groups of the localization $X_{\mathbb{F}_p}$. We skip that but it can be done by a careful analysis of the construction of the localization.

Let us finally note that these statements are more generally true than for simply connected spaces. We will only mention these results and not give proofs. The classical reference is the book [BK72] by Bousfield and Kan.

Definition 2.15. A connected space is called simple if the fundamental group is abelian and the action on all higher homotopy groups is trivial. A connected space X is called nilpotent if the fundamental group $\pi_1(X)$ is nilpotent and the action of $\pi_1(X)$ on $\pi_n(X)$ is nilpotent. The first means that the lower central series

$$G = G_0 \supseteq G_1 \supseteq \dots \supseteq G_n = \{e\} \qquad G_i = [G_{i-1}, G]$$

terminates after finitely many steps in the trivial group. A $\pi_1(X)$ -module is nilpotent if it admits finite filtration by $\pi_1(X)$ such that $\pi_1(X)$ acts trivially on each filtration quotient (think of an action in upper triangular form).

Example 2.16. *Every abelian group is nilpotent. Every trivial module is nilpotent. In particular simple spaces are nilpotent. A further special case are simply connected spaces.*

*Upper unitriangular matrices over any field are nilpotent. A group that is not nilpotent is the free group $\mathbb{Z} * \mathbb{Z}$ so $S^1 \vee S^1$ is not a nilpotent space. Also $S^1 \vee S^2$ is not nilpotent (why?).*

A morphism $X \rightarrow Y$ of spaces is called an $K(A, n)$ -principal fibration (extra structure) if it fits into a homotopy pullback square

$$\begin{array}{ccc} X & \longrightarrow & \text{pt} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & K(A, n+1) \end{array}$$

Note that a morphism of spaces $X \rightarrow Y$ with homotopy fibre a $K(A, n)$ is classified by a morphism $Y \rightarrow B(\text{hAut}K(A, n)) \simeq K(A, n+1)_{\text{hAut}_{\text{Ab}}(A)}$ where this denotes the homotopy quotient (we have that $\text{hAut}_{\text{Ab}}(K(A, n)) \simeq \text{Aut}_{\text{Ab}}(A) \times K(A, n+1)$). Thus a priori it fits into a pullback square

$$\begin{array}{ccc} X & \longrightarrow & B\text{Aut}_{\text{Ab}}(A) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & K(A, n+1)_{\text{hAut}_{\text{Ab}}(A)} \end{array}$$

and we seek to lift the classifying map through $Y \rightarrow K(A, n+1)$. This is of course possible if Y is simply connected. We have the following more general result:

Lemma 2.17. *A connected space is simple precisely if the Postnikov tower of X consists of abelian principal fibrations. A connected space X is nilpotent precisely if the Postnikov tower of X can be refined to a tower of principal fibrations. That means that for every n there is a factorization*

$$\tau_{\leq n} X = X_0 \rightarrow X_1 \dots \rightarrow X_k = \tau_{\leq n-1} X$$

in which each map $X_i \rightarrow X_{i+1}$ is a principal $K(A, n)$ -fibration for some abelian group A .

Corollary 2.18. *Every nilpotent space is $H\mathbb{Z}$ -local.*

Proof. We use a similar strategy as employed in Example 2.2. Every nilpotent space can be Lemma 2.17 be obtained as an iterated pullback of Eilenberg MacLane spaces $K(A, n)$ for A abelian. Thus it suffices to show that $K(A, n)$ is $H\mathbb{Z}$ local. Assume $f: X \rightarrow Y$ is a homology equivalence. Using the 5-Lemma and the UCT sequence

$$0 \rightarrow \text{Ext}(H_{n-1}(X, Z), A) \rightarrow H^n(X, A) \rightarrow \text{Hom}(H_n(X, Z), A) \rightarrow 0$$

we conclude that f induces an isomorphism $[Y, K(A, n)] \rightarrow [X, K(A, n)]$. \square

For a nilpotent group G we say that G is Malcev-complete (or rational) if G admits unique roots, i.e. for every $g \in G$ and integer n there exists a unique $h \in G$ such that $h^n = g$. For a nilpotent group G we denote by $\pi_1(G)_{\mathbb{Q}}$ the Malcev completion of $\pi_1(G)$ which universally adds all roots i.e. is uniquely determined by a universal property. The question is whether it exists. This is indeed the case and the map $j: G \rightarrow G_{\mathbb{Q}}$ can be characterized as follows (due to Quillen Corollary 3.8 in the Appendix of Rational homotopy theory)

- $G_{\mathbb{Q}}$ is nilpotent and uniquely divisible
- The kernel of j is the torsion subgroup of G
- For every $g \in G_{\mathbb{Q}}$ there exist n such that $g^n \in \text{im}(j)$.

A construction is given as $G_{\mathbb{Q}} = \text{Grp}(\mathbb{Q}[G]_I^{\wedge})$ where $I = \ker(\mathbb{Q}[G] \rightarrow \mathbb{Q})$ is the augmentation ideal and Grp are the grouplike elements in a (complete) coalgebra, i.e. all elements h with the property that $\Delta(h) = h \widehat{\otimes} h$. The relevant fact is that it induces on the abelian quotients in the central series the old rationalization. Similarly one can make sense of a group $\text{Ext}(\mathbb{Z}/p^{\infty}, G)$ for G nilpotent such that it reduces to the old thing on quotients (see Bousfield-Kan).

Theorem 2.19 (Bousfield-Kan). *Analogues of Theorem 2.6 and Theorem 2.9 remain valid for simple spaces and even for nilpotent spaces.*

2.1 Some ∞ -category theory

Now we want work towards proving that localization for every homology theory E and every space X exists (by the universal property it has to agree with the one we have discussed so far). We will freely use the theory of ∞ -categories as developed by Joyal, Lurie and others in this course. Recall that an ∞ -category is a (large!) inner Kan simplicial set. For any pair of ∞ -categories \mathcal{C} and \mathcal{D} there is a new ∞ -category $\text{Fun}(\mathcal{C}, \mathcal{D})$ which is just the inner Hom in simplicial sets. We will also use that there is a theory of ∞ -categories that works pretty much like ordinary categories (limits, colimits, adjunctions etc.).

Definition 2.20. The ∞ -category \mathcal{S} of spaces is the homotopy coherent nerve of the simplicially enriched category of Kan complexes.

Concretely we have the following description of the simplicial set $\mathcal{S} = \text{hcN}(\text{Kan}^{\Delta})$:

1. a 0-simplex in \mathcal{S} is a Kan complex
2. a 1-simplex consists of two Kan complexes X_0, X_1 together with a simplicial map $X_0 \rightarrow X_1$.
3. a 2-simplex of \mathcal{S} is given by the following triple of data: three spaces X_0, X_1, X_2 together with maps

$$f_{01} : X_0 \rightarrow X_1 \quad f_{02} : X_0 \rightarrow X_2 \quad f_{12} : X_1 \rightarrow X_2$$

and finally a simplicial homotopy between $f_{12} \circ f_{01}$ and f_{02} .

4. The n -simplices of \mathcal{S} are given by simplicial functors $\mathfrak{C}[\Delta^n] \rightarrow \text{Kan}^{\Delta}$ where $\mathfrak{C}[\Delta^n]$ is the simplicial category with objects $0, \dots, n$ and simplicial sets of morphisms given as

$$\text{Map}_{\mathfrak{C}[\Delta^n]}(n, m) = \text{N}(\{M \subset \{n, n+1, \dots, m\} \mid n, m \in M\}, \subseteq)$$

under inclusion. There is a functor

$$\text{NTop} \rightarrow \mathcal{S}$$

induced by the singular complex functor $\text{Sing} : \text{Top} \rightarrow \text{Kan}^{\Delta}$ (the left hand side is the ordinary 1-category of topological spaces considered as a discrete simplicial category).

Definition 2.21. We will say that a functor $\mathcal{C} \rightarrow \mathcal{C}'$ of ∞ -categories exhibits \mathcal{C}' as the *Dwyer-Kan localization* of \mathcal{C} at a class $W \subseteq \mathcal{C}_1$ of weak equivalence if for every other ∞ -category \mathcal{D} it induces an equivalence

$$\text{Fun}(\mathcal{C}', \mathcal{D}) \rightarrow \text{Fun}^W(\mathcal{C}, \mathcal{D})$$

of ∞ -categories where $\text{Fun}^W(\mathcal{C}, \mathcal{D}) \subseteq \text{Fun}(\mathcal{C}, \mathcal{D})$ denotes the full subsimplicial set spanned by those functors that send weak equivalences in \mathcal{C} to equivalences in \mathcal{D} . We write $\mathcal{C}' \simeq \mathcal{C}[W^{-1}]$.

This in particular implies that $\mathrm{Ho}(\mathcal{C})[W^{-1}] \simeq \mathrm{Ho}(\mathcal{C}')$. It is easy to see that this universal property determines \mathcal{C}' uniquely up to equivalence and also that it exists for every large ∞ -category (and is also large). For example one can construct $\mathcal{C}[W^{-1}]$ as the homotopy pushout

$$\begin{array}{ccc} \bigsqcup_{w \in W} \Delta^1 & \xrightarrow{\bigsqcup_{w \in W} w} & \mathcal{C} \\ \downarrow & & \downarrow \\ \bigsqcup_{w \in W} \Delta^0 & \longrightarrow & \mathcal{C}[W^{-1}] \end{array}$$

in the Joyal model structure or using Dwyer-Kan's Hammock localization.

Proposition 2.22. *1. The functor $\mathrm{Sing} : \mathrm{NTop} \rightarrow \mathcal{S}$ exhibits \mathcal{S} as the Dwyer-Kan localization at the weak equivalences.*

2. The ∞ -category \mathcal{S} is presentable

Proof. The first statement is invariant under Quillen equivalence, hence we can replace Top immediately by sSet (i.e. we have immediately $\mathrm{NsSet}[W^{-1}] \simeq \mathrm{NTop}[W^{-1}]$). But then for every simplicial, combinatorial model category \mathcal{M} we have an equivalence

$$\mathrm{NM}[W^{-1}] \simeq \mathrm{hcNM}_{cf}^{\Delta}$$

and this is a presentable ∞ -category as shown in [Lur09]. \square

Presentability is an important concept here. A large ∞ -category \mathcal{C} is called presentable (the analogue of the property that is called locally presentable for ordinary categories) if it admits all small colimits and there exists a regular cardinal κ and an essentially small, full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ of κ -compact objects such that every object in \mathcal{C} can be written as a κ -filtered colimit of objects in \mathcal{C}_0 . For that to make sense recall that an ∞ -category I is called κ -filtered if for every functor

$$f : K \rightarrow I$$

with K κ -small there exists an extension $\bar{f} : K^{\triangleright} \rightarrow I$. An object x in an ∞ -category \mathcal{C} is called κ -compact if for every colimit diagram $p : I^{\triangleright} \rightarrow \mathcal{C}$ with I a κ -filtered ∞ -category (in short: κ -filtered colimit) the induced diagram $\mathrm{Map}_{\mathcal{C}}(x, p)$ is a colimit in spaces. In short: $\mathrm{Map}_{\mathcal{C}}(x, -)$ commutes with κ -filtered colimits. If $\kappa = \omega$ then we will just say filtered and compact and not ω -filtered and ω -compact.

A nontrivial consequence of presentability for \mathcal{C} is that \mathcal{C} then also admits all small limits (!!!) and that it is locally small (for every pair of objects a, b of \mathcal{C} the mapping space $\mathrm{Map}_{\mathcal{C}}(a, b)$ is essentially small).

Example 2.23. *The ordinary category of sets is presentable as we can take finite sets as the (essentially) small subcategory.*

A priori if we invert a class of weak equivalence in an ordinary category \mathcal{D} there is no size control about $\mathrm{ND}[W^{-1}]$.

Proposition 2.24. *Let \mathcal{C} be an ∞ -category. Then the following are essentially equivalent data*

- 1. full subcategories $\mathcal{C}^0 \subseteq \mathcal{C}$ such that the inclusion admits a left adjoint (sometimes called colocalizing subcategories).*

2. endofunctors $L : \mathcal{C} \rightarrow \mathcal{C}$ which admit a transformation $\eta : \text{id}_{\mathcal{C}} \rightarrow L$ such that the two induced maps $LX \rightarrow L^2X$ (which are $\eta \circ L$ and $L \circ \eta$) are equivalences for every $X \in \mathcal{C}$.
3. Classes of weak equivalences S in \mathcal{C} (i.e. $S \subseteq \mathcal{C}_1$) such that the Dwyer-Kan localization $\mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$ admits a right adjoint that is fully faithful (here two such S and S' are considered equivalent if those morphisms in \mathcal{C} that are sent to equivalences in $\mathcal{C}[S^{-1}]$ and $\mathcal{C}[S'^{-1}]$ agree).

Proof. We give a sketch, for details see [Lur09, Section 5.2.7]. First if we start with a colocalizing subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$ then we obtain a functor as in (2) by looking at the composition

$$L : \mathcal{C} \rightarrow \mathcal{C}^0 \subseteq \mathcal{C}$$

and it comes with a transformation $\text{id} \rightarrow L$ as the unit of the adjunction. Then clearly if for an object $X \in \mathcal{C}^0$ we find that the morphism $X \rightarrow L(X)$ is an equivalence. Since $L(Y)$ is in \mathcal{C}^0 for every $Y \in \mathcal{C}$ this shows that the functor L is idempotent.

Vice versa if we have a functor L as in (2) then we define $\mathcal{C}^0 := L(\mathcal{C})$ as the full subcategory of \mathcal{C} spanned by the essential image of L (this is in fact the essential image of L). By definition we have an inclusion $i : L(\mathcal{C}) \subseteq \mathcal{C}$ and an induced functor $L' : \mathcal{C} \rightarrow L(\mathcal{C})$. The transformation above can now be interpreted as a trafo $\text{id} \rightarrow i \circ L$ and it is easy to see that it exhibits an adjunction between L' and the inclusion.

Given an endofunctor L as in (1) and (2) we define S to be the set of morphisms that are sent to equivalences under L . To see that $L' : \mathcal{C} \rightarrow \mathcal{C}^0$ is the Dwyer-Kan localization at S we have to verify the universal property, namely that

$$(L')^* : \text{Fun}(\mathcal{C}^0, \mathcal{D}) \rightarrow \text{Fun}^S(\mathcal{C}, \mathcal{D})$$

is an equivalence for any other ∞ -category \mathcal{D} . An explicit inverse of the functor L^* is given by i^* which follows easily from the fact that the transformation $X \rightarrow LX$ is in S .

If we are conversely given S then we look at the category of S -local objects, which are the objects $Z \in \mathcal{C}$ such that for every $X \rightarrow Y$ the induced morphisms

$$\text{Map}(Y, Z) \rightarrow \text{Map}(X, Z)$$

is an equivalence. We claim that this subcategory is equivalent to the essential image of the right adjoint R of the functor $L : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$. Certainly every RZ' is local since

$$\text{Map}_{\mathcal{C}}(X, RZ') = \text{Map}_{\mathcal{C}[S^{-1}]}(LX, Z')$$

□

If we are given one of these equivalent data, then we say that \mathcal{C}^0 is a *Bousfield localization* of \mathcal{C} . Note also that even if S is an arbitrary class of morphisms in \mathcal{C} it makes sense to speak of S -local objects and S -equivalences: an object X is called S -local if for every morphism $A \rightarrow B$ in S the induced morphism $\text{Map}(B, X) \rightarrow \text{Map}(A, X)$ is a homotopy equivalence. A morphism $A \rightarrow B$ is called S -equivalence if for every S -local object X the induced map $\text{Map}(B, X) \rightarrow \text{Map}(A, X)$ is an equivalence. The set of all S -equivalences is denoted by $\overline{S} \subseteq \mathcal{C}_1$. Clearly $S \subseteq \overline{S}$. Finally note that the situation for homology localizations of $\text{Ho}(\text{Top})$ is exactly situation (3) (clearly a localization of an ∞ -category gives a localization of the homotopy category). The question really is if there is an adjoint to inclusion of local objects.

Theorem 2.25 (Bousfield, Smith, Lurie). *Let S be a class of morphisms in a presentable ∞ -category \mathcal{C} with corresponding full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$ of S -local objects. Then TFAE*

1. $\mathcal{C}^0 \subseteq \mathcal{C}$ is colocalizing and \mathcal{C}^0 is presentable.
2. $\mathcal{C}^0 \subseteq \mathcal{C}$ is colocalizing and the inclusion preserves κ -filtered colimits for some regular cardinal κ
3. There exists a small set $S^0 \subseteq S$ such that an object in \mathcal{C} is S -local precisely if it is S^0 -local (equivalently $\overline{S^0} = \overline{S}$).
4. There exists a colimit preserving functor $F : \mathcal{C} \rightarrow \mathcal{D}$ to a presentable ∞ -category \mathcal{D} such that \overline{S} consists of those morphisms which are sent to equivalences by F .

This is in [Lur09, Section 5.5.4]. Do not confuse the Bousfield localization at S with $\mathcal{C}[S^{-1}]$. The two only agree (under the equivalent condition of the Theorem) if $S = \overline{S}$ or equivalently if S is closed under pushouts, retracts and transfinite compositions.

Remark 2.26. *In the situation of Theorem 2.25 (4) one can not directly characterize the local objects in terms of the functor $F : \mathcal{C} \rightarrow \mathcal{D}$. But one can produce examples. Let $R : \mathcal{D} \rightarrow \mathcal{C}$ denote the right adjoint to the functor F . Then $R(Z)$ is S -local for every object Z of \mathcal{D} since we have*

$$\mathrm{Map}_{\mathcal{C}}(X, RZ) \simeq \mathrm{Map}_{\mathcal{D}}(FX, Z).$$

But certainly in general not all S -local objects are of this form as the example

$$\Sigma_+^\infty : \mathcal{S} \rightarrow \mathrm{Sp}$$

shows. The Σ_+^∞ -equivalences are precisely the $H\mathbb{Z}$ -equivalences (which are exactly the stable equivalences). But $R(Z) = \Omega^\infty(Z)$ is a simple space for every spectrum Z , in particular has abelian π_1 . We have already seen that nilpotent spaces are also $H\mathbb{Z}$ -local.

Remark 2.27. *The topic of this lecture (in its simplest form) can now be abstractly formulated as follows:*

Find colimit preserving functors $f : \mathcal{S} \rightarrow \mathcal{B}$ where \mathcal{B} is a presentable ∞ -category of ‘algebraic’ nature such that the induced functor $\mathcal{S}[S^{-1}] \rightarrow \mathcal{B}$ (equivalently the restriction of $f|_{\mathcal{S}^0} : \mathcal{S}^0 \rightarrow \mathcal{B}$ to the S -local objects $\mathcal{S}^0 \subseteq \mathcal{S}$) is fully faithful.

Under these assumptions the functor then admits a right adjoint $R : \mathcal{B} \rightarrow \mathcal{S}[S^{-1}]$ (induced by the right adjoint to f) and the questions whether $f|_{\mathcal{S}^0}$ is fully faithful is then equivalent to saying that the unit

$$X \rightarrow RF(X) \simeq \mathrm{Map}_{\mathcal{B}}(F(\mathrm{pt}), F(X))$$

is an S -localization (the latter formula for $X \rightarrow RF(X)$ will become clear in Section 3). Equivalently this map is an equivalence for every S -local object.

Example 2.28. *This is for example far from being true for the functor*

$$\mathcal{S} \rightarrow \mathcal{D}(\mathbb{Z}) \quad X \mapsto C_*(X, \mathbb{Z})$$

as the unit is given by

$$X \rightarrow \mathrm{Map}_{\mathcal{D}(\mathbb{Z})}(\mathbb{Z}, C_*(X, \mathbb{Z})) \simeq \prod K(H_n(X, \mathbb{Z}), n)$$

and induces on homotopy groups the Hurewicz map

$$\pi_n(X) \rightarrow H_n(X, \mathbb{Z})$$

2.2 Back to Homology localizations

In the following Sp is the ∞ -category of spectra. We will introduce this carefully in a second but lets first sketch the application to Bousfield localization. The ∞ -category Sp is freely generated from \mathbb{S} as a stable, presentable ∞ -category. As a consequence there is a unique functor $\otimes : \mathrm{Sp} \times \mathrm{Sp} \rightarrow \mathrm{Sp}$ (the smash product) that preserves colimits separately in each variable and such that $\mathbb{S} \otimes \mathbb{S} \simeq \mathbb{S}$.

Example 2.29. For every spectrum E consider the following functor

$$\mathcal{S} \rightarrow \mathrm{Sp} \quad X \mapsto \Sigma_+^\infty X \otimes E$$

This preserves colimits and thus the Bousfield E -localization exists by Theorem 2.25. Clearly the local equivalence are just the E -local equivalence of Definition 2.1.

Definition 2.30. The ∞ -category Sp of spectra is defined as

$$\mathrm{Sp} := \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$$

Here $\mathcal{S}_*^{\mathrm{fin}} \subseteq \mathcal{S}_*$ is the full subcategory of the ∞ -category of pointed spaces $\mathcal{S}_* := \mathrm{hcN}(\mathrm{Kan}_*^{\Delta}) \simeq \mathcal{S}_{\mathrm{pt}/}$ consisting of finite pointed spaces (i.e. those generated under finite colimits from $S^0 = \mathrm{pt} \sqcup \mathrm{pt}$). The ∞ -category $\mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ is the full subcategory consisting of the reduced excisive functors F . A functor F is called reduced if $F(\mathrm{pt}) \simeq \mathrm{pt}$ and excisive if it sends pushout squares to pullback squares.

The ∞ -category Sp is presentable. This can be seen as follows: the functor category $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ is presentable (as every functor category from a small ∞ -category to a presentable ∞ -category). The claim is that Sp is a colocalizing subcategory that is presentable. Using Theorem 2.25 we have to exhibit the reduced excisive functors as the set of S -local objects for a set S . To this end use

$$S = \{\emptyset \rightarrow \underline{\mathrm{pt}}, \quad \underline{X} \cup_{\underline{X} \cup_Y \underline{Z}} \underline{Z} \rightarrow \underline{Y}\}$$

where the underline denotes the corepresented functor and for every pushout square

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \cup_Y Z \end{array}$$

in $\mathcal{S}_*^{\mathrm{fin}}$.

The functor $\Omega^\infty : \mathrm{Sp} \rightarrow \mathcal{S}$ given by evaluation at S^0 admits a left adjoint denoted as $\Sigma_+^\infty : \mathcal{S} \rightarrow \mathrm{Sp}$ (this adjoint can be constructed as the composition $\mathrm{Sp} = \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}) \xrightarrow{\mathrm{ev}_{S^0}} \mathcal{S}$).

Remark 2.31. Here are a couple of equivalent descriptions of Sp :

1. The ∞ -category Sp is the inverse limit

$$\mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \dots$$

of ∞ -categories (the homotopy limit in the Joyal model category). Unfolding the definitions we get that a spectrum of this type is given by a sequence of pointed Kan complexes $(X_n)_{n \geq 0}$ together with pointed homotopy equivalences $X_n \rightarrow \Omega X_{n+1}$.

2. $\mathrm{Sp} \simeq \mathrm{NSp}_{\mathrm{Model}}[W^{-1}]$ where $\mathrm{Sp}_{\mathrm{Model}}$ is one of the model categories of spectra (e.g. sequential, symmetric, orthogonal etc...). If $\mathrm{Sp}_{\mathrm{Model}}$ is moreover a simplicial model category then we also have $\mathrm{Sp} \simeq \mathrm{hcN}(\mathrm{Sp}_{\mathrm{Model}})_{cf}^{\Delta}$.

3.

$$\mathrm{Sp} \simeq \mathrm{Ind}\left(\varinjlim(\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\mathrm{fin}} \rightarrow \dots)\right)$$

Here the colimit in the bracket is the ∞ -categorical avatar of the classical Spanier-Whitehead category. An object in this colimit can be described as a pair consisting of a nonnegative integer n and a pointed finite CW complex X . The space of morphisms between (X, n) and (Y, m) is the homotopy colimit over the spaces $\mathrm{Map}_*(\Sigma^{k-n}X, \Sigma^{k-m}Y)$ as $k \rightarrow \infty$. Then Ind formally adds filtered colimits (so that the resulting category has all colimits).

$\mathrm{Ind}(\mathcal{C})$ for a small ∞ -category \mathcal{C} is defined as the subcategory of $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ consisting of the filtered colimits of representables. One can be more concrete: for any pair of such filtered colimits $\varinjlim X_i$ and $\varinjlim Y_i$ (where we assume for simplicity that they are indexed by $\mathbb{N}(\mathbb{Z}^{\geq 0}, \subseteq)$ and X_i and Y_i are representable (i.e. in $\mathcal{C} \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$)

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\varinjlim X_i, \varinjlim Y_i) \simeq \varinjlim_i \varprojlim_j \mathrm{Map}_{\mathcal{C}}(X_i, Y_j)$$

That is how the Ind -category is usually defined 1-categorically.

Example 2.32. Consider the endofunctor $\mathrm{Sp} \rightarrow \mathrm{Sp}$ given by $X \mapsto X \otimes E$. Then the localization at the E -equivalences exists. This is called the Bousfield localization of spectra at the E -equivalences.

Theorem 2.33. For the spectrum $E = H\mathbb{Q}$ the exact analogue of Theorem 2.6 holds true: A spectrum X is rational if and only if all homotopy groups are rational and a morphism $f : X \rightarrow Y$ of spectra is a rational equivalence precisely if it induces an isomorphism $\pi_*(X) \otimes \mathbb{Q} \rightarrow \pi_*(Y) \otimes \mathbb{Q}$. In fact there is a formula $X_{\mathbb{Q}} \simeq X \otimes H\mathbb{Q}$.

For the spectrum $E = \mathbb{S}/p$ the exact analogue of Theorem 2.6 holds true: A spectrum X is p -complete (aka \mathbb{S}/p -local) if and only if all its homotopy groups are derived p -complete. A morphism $X \rightarrow Y$ of spectra is an \mathbb{F}_p -equivalence precisely if the homotopy fibre has uniquely p -divisible homotopy groups. For the p -completion of any spectrum we have a split short exact sequence

$$0 \rightarrow \mathrm{Ext}(\mathbb{Z}/p^{\infty}, \pi_n X) \rightarrow \pi_n X_p^{\wedge} \rightarrow \mathrm{Hom}(\mathbb{Z}/p^{\infty}, \pi_{n-1} X) \rightarrow 0$$

In fact there is a formula

$$X_p^{\wedge} \simeq \mathrm{map}(\Sigma^{-1}\mathbb{S}/p^{\infty}, X) \simeq \varprojlim X//p^n$$

where \mathbb{S}/p^{∞} is the Moore spectrum for the Prüfer group \mathbb{Z}/p^{∞} which can be obtained as the cofibre of the map $\mathbb{S} \rightarrow \mathbb{S}[1/p]$.

Proof. Everything follows from the formulas for the localization. Thus we need to proof that $X \rightarrow X \otimes H\mathbb{Q}$ is a rationalization. This is obvious since $H\mathbb{Q} \otimes H\mathbb{Q} \simeq H\mathbb{Q}$.

Using the fibre sequence $\Sigma^{-1}\mathbb{S}/p^{\infty} \rightarrow \mathbb{S} \rightarrow \mathbb{S}[1/p]$ we see that the fibre of the map $X \rightarrow \mathrm{map}(\Sigma\mathbb{S}/p^{\infty}, X)$ is given by $\mathrm{map}(\mathbb{S}[1/p], X)$. This fibre is an $\mathbb{S}[1/p]$ -module and we claim more generally that every $\mathbb{S}[1/p]$ -module M is acyclic. To see this we use the fibre

sequence $M \rightarrow M[1/p] \rightarrow M \otimes \mathbb{S}/p^\infty$ in which the first morphism is an equivalence. Finally we have to show that $\varprojlim X//p^n$ is p -complete. This follows exactly as in Example 2.3 by inductively showing that all $X//p^n$ are p -complete.

The deduction of the formula for homotopy groups is left to the audience. Better try to prove the more general fact that for every Moore spectrum MA and every spectrum X there is a short exact sequence

$$0 \rightarrow \text{Ext}(A, \pi_{n+1}X) \rightarrow \pi_n \text{map}(MA, X) \rightarrow \text{Hom}(A, \pi_n X) \rightarrow 0$$

This is in general not splittable as the example $MA = X = \mathbb{S}/2$ of the mod 2 Moore spectrum shows, since in this case the sequence becomes $\mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2$. Thus we need an argument for the splitting. \square

In fact the above proof just shows that rational spectra are exactly modules over $H\mathbb{Q}$ (we have not really spoken about module objects so far...). Note how much easier this result was than the corresponding one for spaces...

Definition 2.34. The ∞ -category of chain complexes over a ring R is defined as the ∞ -category

$$\mathcal{D}(R) := \text{N}(\text{Ch}_R)[q^{-1}]$$

where Ch_R is the ordinary 1-category of R -chain complexes and q is the class of quasi-isomorphisms.

Remark 2.35. Here are a couple of equivalent descriptions of $\mathcal{D}(R)$:

1. The ∞ -category Ch_R is equivalent to the homotopy coherent nerve of the following simplicial category: objects are DG-projective chain complexes and the simplicial set of morphisms from C_\bullet to D_\bullet is given by the simplicial set associated to the chain complex of maps from C_\bullet to D_\bullet . (DG-projective means that the chain complex is levelwise projective and that every map from it to an exact chain complex is chain nullhomotopic. Equivalently that $\text{Hom}(C_\bullet, -)$ preserves quasi-isomorphisms).
2. The ∞ -category Ch_R is equivalent to the homotopy coherent nerve of the following simplicial category: objects are DG-injective chain complexes and the simplicial set of morphisms from C_\bullet to D_\bullet is given by the simplicial set associated to the chain complex of maps from C_\bullet to D_\bullet . (DG-injective for C_\bullet means that every entry C_n is injective and every morphism from an exact chain complex to C_\bullet is chain nullhomotopic. Equivalently that $\text{Hom}(-, C_\bullet)$ preserves quasi-isomorphisms).
3. The ∞ -category Ch_R is equivalent to the ∞ -category of module spectra over the Eilenberg-Mac Lane spectrum HR . This essentially follows by observing that Ch is a presentable, stable ∞ -category with a single compact generator which is the chain complex R and whose endomorphism spectrum is given by HR .

In particular the ∞ -category $\mathcal{D}(R)$ is presentable and stable. The homotopy category of $\mathcal{D}(R)$ is the ordinary unbounded derived category of the ring R .

There is a ‘forgetful functor’

$$\mathcal{D}(\mathbb{Z}) \rightarrow \text{Sp}$$

which has both adjoints. This functor is not fully faithful (there are Steenrod operations...) nor essentially surjective. The image consists exactly of those spectra all of whose k -invariants vanish, i.e. which are products of Eilenberg Mac-Lane spectra for groups. If R is commutative then there is again a functor

$$\otimes_R : \mathcal{D}(R) \times \mathcal{D}(R) \rightarrow \mathcal{D}(R)$$

such that it preserves colimits separately in each variable and such that $R \otimes_R R \simeq R$. It is obtained by deriving the ordinary tensor product but also uniquely (in the appropriate sense) characterized by the properties above.

Example 2.36. *For every chain complex $E \in \mathcal{D}(\mathbb{Z})$ there is an endofunctor given by tensoring with E and thus a resulting Bousfield localization of $\mathcal{D}(\mathbb{Z})$. Again this Bousfield localization for the case $E = \mathbb{Q}[0]$ and $E = \mathbb{F}_p[0]$ satisfies the same result as Theorem 2.33.*

3 Rational differential forms for spaces

Now we want to use the universal property of the ∞ -category of spaces. Roughly speaking it says, that spaces are freely generated under pushouts from the point.

Proposition 3.1 ([Lur09]). *For every large ∞ -category \mathcal{D} which admits all small colimits the functor*

$$\text{Fun}^L(\mathcal{S}, \mathcal{D}) \rightarrow \mathcal{D}$$

induced from the inclusion $\Delta^0 \xrightarrow{\text{pt}} \mathcal{S}$ is an equivalence of ∞ -categories. Here $\text{Fun}^L(\mathcal{S}, \mathcal{D}) \subseteq \text{Fun}(\mathcal{S}, \mathcal{D})$ denotes the full subcategory of $\text{Fun}(\mathcal{S}, \mathcal{D})$ consisting of those functors that preserve all small colimits.

Proof. We sketch the idea. The point is that any space $X \in \mathcal{S}$ can be written as the colimits $\text{colim}_X \text{pt}$ of the constant diagram on the indexing ∞ -category X . Thus the value of any colimit preserving functor $F : \mathcal{S} \rightarrow \mathcal{D}$ on X is necessarily given by $\text{colim}_X F(\text{pt})$ and thus determined by the value on the pt . \square

We shall denote the functor $\mathcal{S} \rightarrow \mathcal{D}$ induced by $d \in \mathcal{D}$ as $X \mapsto X \otimes d$. In fact if \mathcal{D} is locally small then the functor

$$\mathcal{D} \rightarrow \mathcal{S} \quad y \mapsto \text{Map}_{\mathcal{D}}(d, y)$$

is right adjoint to \mathcal{D} .

Example 3.2. *1. If $\mathcal{D} = \mathcal{S}$ the infinity category of spaces then every colimit preserving endofunctor is of the form $X \mapsto X \times D$ for some space D . The right adjoint is given by $\text{Map}_{\mathcal{S}}(D, -)$.*

2. Every colimit preserving functor from spaces to spectra is given by $X \mapsto \Sigma_+^\infty X \otimes E \simeq X \otimes E$ for some spectrum E (i.e. by taking E -homology of X).

What are concrete ways of constructing functors $\mathcal{S} \rightarrow \mathcal{D}$ that are colimit preserving? By the fact that \mathcal{S} is the Dwyer-Kan localization of Top at the weak homotopy equivalences it follows that a functor $\mathcal{S} \rightarrow \mathcal{D}$ can equivalently be described as a functor $N\text{Top} \rightarrow \mathcal{D}$. Let R be a ring.

Proposition 3.3. *The chain functor $C_*(-, R) : \mathcal{S} \rightarrow \mathcal{D}(R)$ induced by the respective functor $\text{Top} \rightarrow \text{Ch}_R$ preserves colimits and the functor $C^*(-, R) : \mathcal{S}^{op} \rightarrow \mathcal{D}(R)$ preserves limits for every ring R .*

Proof. First it is clear that the functor in question, i.e. the factorization of $\text{Top} \rightarrow \text{Ch}_R$ to the ∞ -level exists, since $C_*(-, R) : \text{Top} \rightarrow \text{Ch}_R$ sends weak equivalence of spaces to quasi-isomorphisms. We observe that we have for a disjoint union of spaces $X = \bigsqcup X_i$ that $C_*(X_i, R) \simeq \bigoplus C_*(X_i, R)$, Moreover we have for a pushout square

$$\begin{array}{ccc} F(U \cap V) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \cup V) \end{array}$$

in $\mathcal{D}(R)$ for every open cover by two sets of a space X . This is a formal consequence of the existence of a Mayer-Vietoris sequence since we can just inert the homotopy pushout of chain complexes into the square (say P) and then both, P as well as $F(U \cup V)$ have compatible Mayer-Vietoris sequences. Thus an application of the 5-Lemma proves that $F(U \cup V)$ is quasi-isomorphic to P . For the functor $C^*(-, R)$ we have similarly that it sends disjoint unions to products and pushouts to pullbacks.

Thus it suffices to prove the following: assume a functor $\text{NTop} \rightarrow \mathcal{C}$ for some ∞ -category \mathcal{C} sends weak equivalences to equivalences, open covers $\{U, V\}$ of $X = U \cup V$ to pushouts and disjoint unions to coproducts. Then the induced functor $\mathcal{S} \rightarrow \mathcal{C}$ is colimit preserving.

To see this we use that every pushout in \mathcal{S} can up to equivalence be described as a cellular pushout for cellular inclusions. Choosing open neighbourhoods then shows that every pushout can up to equivalence be described as an open cover pushout. Thus the assumptions imply that the induced functor $\mathcal{S} \rightarrow \mathcal{C}$ preserves pushouts and infinite coproducts. It is a general fact that in a ∞ -category all colimits are generated from pushouts and infinite coproducts. Thus it preserves all colimits. \square

Remark 3.4. *There is a ‘better’ proof of the above fact as follows: the chain functor factors through simplicial sets, i.e. we have a factorization*

$$\text{Top} \xrightarrow{\text{Sing}} \text{sSet} \xrightarrow{C_*(-, R)} \text{Ch}_R$$

and since the first induces an equivalence after Dwyer-Kan localisation we can just prove that the second functor preserves weak equivalences and colimits.

A sufficient criterion for a functor $F : \text{sSet} \rightarrow \mathcal{M}$ where \mathcal{M} is a model category to induce a colimit preserving functor $\mathcal{S} \rightarrow \mathcal{M}_\infty = \text{NM}[w^{-1}]$ is the following: the functor F is left adjoint (as a functor of 1-categories), the induced map $F(\partial\Delta^n) \rightarrow F(\Delta^n)$ is a cofibration and $F(\Lambda_k^n) \rightarrow F(\Delta^n)$ is a trivial cofibration. This is well known. The argument works as follows:

We first argue that F preserves weak equivalences. To this end we first argue that it sends trivial cofibration to trivial cofibrations. Then since every weak equivalence $f : X \rightarrow Y$ can be factored as $a X \xrightarrow{a} X_0 \xrightarrow{b} Y$ with a a trivial cofibration and b a right inverse to a trivial cofibration this implies that F preserves weak equivalence. Now we consider the induced functor $F : \text{NsSet} \rightarrow \mathcal{M}_\infty$. There is also a functor $G : \text{NsSet} \rightarrow \mathcal{M}_\infty$ with a

natural transformation $G \rightarrow F$ which does preserve weak equivalence and induces a colimit preserving functor (it is determined by the value on the point which we set to be $F(\text{pt})$). Now we want to show that for every simplicial set X the map $G(X) \rightarrow F(X)$ is an equivalence. We will do this by induction over the dimension of X . Eventually it suffice to prove that attaching a cell preserves this. Since we have that

$$\begin{array}{ccc} F(\partial\Delta^n) & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ F(\Delta^n) & \longrightarrow & F(\Delta^n \cup_{\partial\Delta^n} X) \end{array}$$

this follows by induction over the cells and finally using the filtered colimit.

This criterion is in the case of the functor $C_*(-, R)$ and the injective model structure (cofibrations being monos) on Ch_R obviously satisfied since $C_*(-, R)$ preserves monos and for a horn is a quasi-iso (as it is for example equivalent to the chain complex generated by the non-degenerate simplices).

As a next goal we want to define the piecwise linear differential forms for a topological space X . We will first do this for simplicial sets. Therefore let us start with the simplicial set Δ^n . Then we define

$$\mathcal{A}_{PL}^*(\Delta^n) := \frac{\Lambda_{\mathbb{Q}}(x_0, \dots, x_n, dx_0, \dots, dx_n)}{\sum_{i=0}^n x_i = 1, \sum_{i=0}^n dx_i = 0}$$

with $\deg x_i = 0$ and $\deg dx_i = 1$. By $\Lambda_{\mathbb{Q}}$ we mean the free rational, commutative graded algebra and it is equipped with a differential $d(x_i) = dx_i$. It can also be described (!?) as the free CDGA on generators x_1, \dots, x_n with $\sum x_i = 1$.

For example in degree 0 it is given by polynomial functions

$$\mathcal{A}_{PL}^0(\Delta^n) = \frac{\mathbb{Q}[x_0, \dots, x_n]}{\sum_{i=0}^n x_i = 1} =: A$$

on the algebraic n -simplex $\text{Spec} A$. Then the piecwise linear differential forms are the algebra of Kähler differentials on that. As a result from this description we immediately get that

$$\mathcal{A}_{PL}^*(\Delta^n) \otimes_{\mathcal{A}_{PL}^0(\Delta^n)} C^\infty(|\Delta^n|) \cong \Omega^*(|\Delta^n|)$$

where we use the obvious inclusion $A \rightarrow C^\infty(|\Delta^n|)$. To see this use that $\Omega^n(|\Delta^n|)$ for a smooth manifold M has the correct rank as a module over $C^*(|\Delta^n|)$.

Then $\mathcal{A}_{PL}(\Delta^\bullet)$ forms a simplicial object in the category $\text{CDGA}_{\mathbb{Q}}$ given by the obvious (!?) face and degeneracy operators which can be described as follows:

$$\partial_i : \mathcal{A}_{PL}^*(\Delta^n) \rightarrow \mathcal{A}_{PL}^*(\Delta^{n-1}) \quad x_k \mapsto \begin{cases} x_k & k < i \\ 0 & k = i \\ x_{k-1} & k > i \end{cases}$$

and

$$s_i : \mathcal{A}_{PL}^*(\Delta^n) \rightarrow \mathcal{A}_{PL}^*(\Delta^{n+1}) \quad x_k \mapsto \begin{cases} x_k & k < i \\ x_k + x_{k+1} & k = i \\ x_{k+1} & k > i \end{cases}$$

Definition 3.5. For a simplicial set X we define

$$\mathcal{A}_{PL}^*(X) := \text{Hom}_{\text{sSet}}(\text{Sing}(X), \mathcal{A}_{PL}^*(\Delta^\bullet))$$

where we consider the target as a simplicial set by forgetting the differential, the grading, the algebra structure and the rational vector space structure. Then this inherits the pointwise structure of a rational CDGA from these structures. In other words the degree n component $\mathcal{A}^n(X)$ is given by $\text{Hom}_{\text{sSet}}(\text{Sing}(X), \mathcal{A}_{PL}^n(\Delta^\bullet))$ for fixed n . For a topological space Y we set $\mathcal{A}_{PL}^*(Y) := \mathcal{A}_{PL}^*(\text{Sing}Y)$.

This defines a functor

$$\mathcal{A}_{PL}^* : \text{Top} \rightarrow \text{CDGA}_{\mathbb{Q}}$$

Remark 3.6. This functor is equivalent to the functor obtained by first forming the right Kan extension

$$\begin{array}{ccc} \Delta^{op} & \xrightarrow{\mathcal{A}_{PL}} & \text{CDGA}_{\mathbb{Q}} \\ \downarrow & \nearrow & \\ \text{sSet}^{op} & & \end{array}$$

and then postcomposing with the singular complex functor $\text{Top}^{op} \rightarrow \text{sSet}^{op}$. The right Kan extension itself can be described as the functor

$$X \mapsto \int_{n \in \Delta} \text{Hom}_{\text{Set}}(X_n, \mathcal{A}_{PL}^*(\Delta^n))$$

which is clearly given by the simplicial maps.

Remark 3.7. The definition of $\mathcal{A}_{PL}(X)$ makes sense over any field and even over rings. There is a priori no characteristic 0 bound. It turns out that in general is really oddly behaved, e.g. the next proposition fails very badly (thanks to Matthew for pointing out that even $\mathcal{A}_{PL}(\Delta^n)$ is non-contractible over \mathbb{F}_p).

Proposition 3.8. If we consider \mathcal{A}_{PL}^* as a functor $\text{Top}^{op} \rightarrow \text{Ch}_{\mathbb{Q}}$ then it is naturally quasi-isomorphic to $C^*(-, \mathbb{Q})$. In particular it sends weak equivalence to quasi-isomorphisms and the induced functor $\mathcal{S}^{op} \rightarrow \mathcal{D}(\mathbb{Q})$ preserves limits (thus it is right adjoint since the target is locally small by the discussion at the beginning of this section).

Proof. We first construct a map

$$\int : \mathcal{A}_{PL}^*(X) \rightarrow C^*(X, \mathbb{Q})$$

natural in the topological space X . By abstract nonsense it suffices to provide a natural map

$$\mathcal{A}_{PL}^*(\Delta^n) \rightarrow C^*(\Delta^n, \mathbb{Q})$$

such a map is provided by the usual integration map which is in this case certainly a quasi-iso (note that Δ^n is the simplicial simplex and not the geometric one). We claim that \mathcal{A} sends $\partial\Delta^n \rightarrow \Delta^n$ to a fibration, i.e. the restriction

$$\mathcal{A}_{PL}(\Delta^n) \rightarrow \mathcal{A}_{PL}(\partial\Delta^n)$$

is a levelwise surjective map of chain complexes. This is clear since it is in level n given by

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \mathcal{A}_{PL}^n(\Delta^\bullet)) \rightarrow \mathrm{Hom}_{\mathrm{sSet}}(\partial\Delta^n, \mathcal{A}_{PL}^n(\Delta^\bullet))$$

and since the underlying simplicial set of $\mathcal{A}_{PL}^n(\Delta^\bullet)$ is a contractible Kan complex this is surjective. To see that it is a contractible Kan complex use ... Now use induction on dimension of X to finish the proof. \square

Now we need to understand the ∞ -categories of CDGA's and of DGA's over a ring R .

Definition 3.9. Let R be a commutative ring. Then we define the ∞ -category of CDGA's over R as

$$\mathrm{CDGA}_R^\infty := \mathrm{N}(\mathrm{CDGA}_R)[q^{-1}]$$

where q are the morphisms of CDGA's whose underlying morphism of chain complexes is a quasi-isomorphism. Similarly we define the category of DGA's over any ring R as

$$\mathrm{DGA}_R^\infty := \mathrm{N}(\mathrm{DGA}_R)[q^{-1}]$$

The following theorem is (in the language of model categories) well known and due to many people.

Theorem 3.10. *The ∞ -categories $\mathrm{CDGA}_{\mathbb{Q}}^\infty$ and DGA_R^∞ for an arbitrary ring are presentable and the canonical forgetful functors*

$$\mathrm{CDGA}_{\mathbb{Q}}^\infty \rightarrow \mathrm{DGA}_{\mathbb{Q}} \rightarrow \mathcal{D}(\mathbb{Q}) \quad \mathrm{DGA}_R^\infty \rightarrow \mathcal{D}(R)$$

are right adjoint, in particular preserve all limits.

This statement follows from the existence of transferred model structures and is standard. We will skip it here since we will discuss these model structures more extensively soon.

Remark 3.11. *The CDGA part of the Theorem fails very badly over field of positive characteristic or rings. We do not know whether $\mathrm{CDGA}_{\mathbb{F}_p}^\infty$ is presentable or has limits/colimits. But even if it has limits then the forgetful map $\mathrm{CDGA}_{\mathbb{F}_p}^\infty \rightarrow \mathcal{D}(\mathbb{F}_p)$ can not be limit preserving which makes would make it useless for our purposes.*

Corollary 3.12. 1. *The functor $C^*(-, R)$ induces a limit preserving functor*

$$\mathcal{S}^{op} \rightarrow \mathrm{DGA}_R^\infty$$

2. *The functor \mathcal{A}_{PL} induces a limit preserving functor*

$$\mathcal{S}^{op} \rightarrow \mathrm{CDGA}_{\mathbb{Q}}^\infty$$

3. *The composition $\mathcal{S}^{op} \xrightarrow{\mathcal{A}_{PL}} \mathrm{CDGA}_{\mathbb{Q}}^\infty \rightarrow \mathrm{DGA}_{\mathbb{Q}}^\infty$ is equivalent to the functor $C^*(-, \mathbb{Q})$. In particular the cohomology algebra of \mathcal{A}_{PL}^* is isomorphic to the cohomology ring $H^*(X, \mathbb{Q})$ for every space X .*

Proof. We know in all cases that the functors preserve limits since they do by what we have earlier shown when considered as functors with value in chain complexes (the forgetful functors reflect equivalences and preserves limits). Thus it suffices again to compare on points, where it is obvious. \square

Remark 3.13. *One does not even need to construct \mathcal{A}_{PL} explicitly to know that there is a CDGA model for $C^*(-, \mathbb{Q})$ and proceed with the abstract theory. The mere existence of limits in $\text{CDGA}_{\mathbb{Q}}^{\infty}$ and the fact that the forgetful functor $\text{CDGA}_{\mathbb{Q}}^{\infty} \rightarrow \text{DGA}_{\mathbb{Q}}^{\infty}$ preserves limits allows to deduce that there is a factorization of $C^*(-, \mathbb{Q})$ through a functor of CDGAs.*

Remark 3.14. *It follows that there exists a zig-zag of natural quasi-isomorphisms between the functors $\mathcal{A}_{PL} : \text{Top}^{op} \rightarrow \text{CDGA}_{\mathbb{Q}} \rightarrow \text{DGA}_{\mathbb{Q}}$ and the functor $C^*(-, \mathbb{Q}) : \text{Top}^{op} \rightarrow \text{DGA}_{\mathbb{Q}}$. This is a non-trivial result. Even for a fixed manifold M and for ordinary differential forms its tricky to exhibit a direct quasi-isomorphism of DGAs between $\Omega^*(M)$ and $C^*(M, \mathbb{R})$ as the usual deRham map is not strictly multiplicative.*

3.1 Cosimplicial algebras

We have seen that the rational approach, namely to refine the DGA $C^*(-, \mathbb{Q})$ to a CDGA does not naively work in the characteristic p -case. Later we will use Steenrod operations to prove a hard no-go theorem for that. Thus we now want to discuss ways around that. There are two possibilities (as already said in the intro): cosimplicial commutative rings or \mathbb{E}_{∞} -algebras. This will be the content of this and the next section.

Definition 3.15. Let R be a commutative ring. The ordinary category of cosimplicial commutative R -algebras is defined as cosimplicial objects in the category of commutative R -algebras

$$\text{csCAlg}_R := \text{Fun}(\Delta, \text{CAlg}_R)$$

Recall that we can picture such a cosimplicial algebra A^{\bullet} as follows:

$$A^0 \begin{array}{c} \xrightarrow{\partial_0} \\ \xleftarrow{\partial_1} \end{array} A^1 \begin{array}{c} \xrightarrow{\partial_0} \\ \xleftarrow{\partial_2} \end{array} A^2 \quad \dots$$

where the ∂_i are the (co)face operators and the s_i are the codegeneracy operators. Every cosimplicial commutative algebra A^{\bullet} has an underlying cohomologically positively graded chain complex (A^{\bullet}, d) whose n -th group is given by A^n and whose differential is the alternating sum of the coface operators

$$d : A^n \rightarrow A^{n+1} \quad d = \sum_{i=0}^{n-1} (-1)^i \partial_i$$

Proposition 3.16. *The singular chain complex functor admits a refinement*

$$C^*(-, R) : \text{sSet}^{op} \rightarrow \text{csCAlg}_R$$

where $C^*(X, R)$ sends $[n] \in \Delta$ to the commutative algebra of maps $\text{Hom}_{\text{Set}}(X_n, R)$ with pointwise multiplication. Under the forgetful functor $\text{csCAlg}_R \rightarrow \text{Ch}_R$ this agrees with the old singular chain complex.

Definition 3.17. We define the ∞ -category of cosimplicial R -algebras as

$$\text{csCAlg}_R^{\infty} := \text{NcsCAlg}[q^{-1}]$$

where q are the morphisms which induce on underlying chain complexes a quasi-isomorphism.

By definition we have a functor $\text{csCAlg}_R^\infty \rightarrow \mathcal{D}(R)$ which reflects equivalences. Thus we immediately see that the functor $C^*(-, R)$ induces a functor

$$\mathcal{S}^{op} \rightarrow \text{csCAlg}_R^\infty$$

Now we want to discuss the relation between cosimplicial commutative algebras, CDGAs and DGAs. We will establish canonical functors

$$\text{CDGA}_R^{\geq 0} \rightarrow \text{csCAlg}_R \rightarrow \text{DGA}_R$$

which preserve the underlying chain complex up to quasi-isomorphism and thus the notion of equivalence. In particular they will immediately refine to functors on the respective ∞ -categories. For this I found [Fre15, Section 5] a good reference. We will start with the second functor:

Let A^\bullet be a cosimplicial R -algebra (actually we will only need associativity of A here). In order to make the associated chain complex into a DGA we have to supply maps

$$R \rightarrow A^0 \quad \text{and} \quad A^k \otimes_R A^l \rightarrow A^{k+l}$$

which are maps of chain complexes and constitute a DGA over R . The first map is given by the unit in A^0 and the second is the Alexander-Whitney map

$$x \otimes y \mapsto (\sigma_f)_* x \cdot (\sigma_b)_* y$$

where $\sigma_f : [k] \rightarrow [k+l]$ is the ‘front face’ given by the map that sends $i \in [k]$ to $i \in [k+l]$. The map $\sigma_b : [l] \rightarrow [k+l]$ is the back face that sends $i \in [l]$ to $i+k \in [l]$. In other words we have the inclusions

$$\{0, 1, \dots, k\} \xrightarrow{\sigma_f} \{0, 1, \dots, k, k+1, \dots, l\} \xleftarrow{\sigma_b} \{0, 1, \dots, l\}$$

that interlap exactly in k .

Lemma 3.18. *This endows (A^\bullet, d) with the structure of a DGA over R and supplies a functor $\text{csCAlg}_R \rightarrow \text{DGA}_R$.*

Proof. Explicit check, left for the reader. □

Note that the Alexander-Whitney map is not symmetric in any sense, so that there is no way to extend this construction to the commutative situation. One can more generally show that the functor $\text{Ch}_R^{\geq 0} \rightarrow \text{Fun}(\Delta, \text{Mod}_R)$ admits a lax monoidal structure (but not lax symmetric monoidal!).

In order to describe the functor $\text{CDGA}_R \rightarrow \text{csCAlg}_R$ we have to review a bit of the Dold-Kan equivalence. So far we have used the functor which sends a cosimplicial abelian group or R -module M^\bullet to the chain complex which is M^n in (cohomological) degree n . A better variant of this functor is the (cosimplicial version) of the reduced Moore functor NM^\bullet defined as

$$(NM^\bullet) = \bigcap_{i=0}^{n-1} \ker (s_i : M^i \rightarrow M^{i-1}) \subseteq M^n$$

where the s_i are the codegeneracy operators. The differential is defined as the restriction of the old differential, i.e. by the alternating sum of face operators. Thus we have that $(NM^\bullet, d) \subseteq (M^\bullet, d)$ is a subcochain complex. This inclusion is a quasi-isomorphism (in fact a chain homotopy equivalence).

Proposition 3.19 (Dold-Kan). *The functor*

$$N : \text{Fun}(\Delta, \text{Mod}_R) \rightarrow \text{Ch}_R^{\geq 0}$$

is an equivalence of categories.

Remark 3.20. *If A^\bullet is a cosimplicial algebra then the DGA structure on (A^\bullet, d) restricts to a DGA structure on $(NA^\bullet, d) \subseteq (A^\bullet, d)$. This defines a variant of the functor $\text{csCAlg}_R \rightarrow \text{DGA}_R$. Since the inclusion is a quasi-iso this does not make a real difference.*

*To verify this one has to check that for two elements $x \in A^k$ and $y \in A^l$ in the kernel of all codegeneracy maps the product $(\sigma_f)_*x \cdot (\sigma_b)_*y$ is also in the kernel of all codegeneracy maps. We first use that*

$$s_i(\sigma_f x \cdot \sigma_b y) = (s_i \circ \sigma_f)(x) \cdot (s_i \circ \sigma_b)(y).$$

For $i < k$ we have that $s_i \circ \sigma_f = \sigma_f \circ s_i : [k] \rightarrow [k+l]$. For $i \geq k$ we have that $s_i \circ \sigma_b = \sigma_b \circ s_{i-k} : [l] \rightarrow [k+l]$ as one verifies by an explicit calculation.

One can use the fact that the source of our Dold-Kan correspondence is a functor category and the fact that N is obviously colimit preserving to derive an ‘explicit’ form of the inverse. The inverse $\mathcal{K} : \text{Ch}_R^{\geq 0} \rightarrow \text{Fun}(\Delta, \text{Mod}_R)$ is given by $\mathcal{K}(C^\bullet)_n = \text{Hom}_{\text{Ch}_R}(NG_n, C^\bullet)$ where G_n is the cosimplicial R -module

$$R[\text{Hom}_\Delta([n], [0])] \xrightarrow[\partial_1]{\partial_0} R[\text{Hom}_\Delta([n], [1])] \xrightarrow[\partial_2]{\partial_0} R[\text{Hom}_\Delta([n], [2])] \cdots$$

where $R[X]$ denotes the free R -module generated by the set X . This functor is by construction right adjoint to N and to prove the Dold-Kan equivalence one makes the functor \mathcal{K} even more explicit, e.g. using the following Lemma. This we first learned from the article [Get15] of Ezra Getzler.

Lemma 3.21. *There is a canonical isomorphism $NG_n \cong (\Lambda_n)^\vee$ where Λ^n is the exterior algebra (over R) on $R^{\{0, \dots, n\}}$ where the generators $e_i \in R^{\{0, \dots, n\}}$ sit in cohomological degree -1 . The differential is defined on generators by $de_i = 1$. This isomorphism is compatible with the simplicial maps and sends the algebra structure on Λ^n to the coalgebra structure on NG_n (this coalgebra structure comes from the coalgebra structure on G_n using Proposition 3.24 below).*

As a consequence we get an induced isomorphism

$$\begin{aligned} \mathcal{K}(C^\bullet)^n &\cong \text{Hom}_{\text{Ch}_R}(NG_n, C^\bullet) \\ &\cong \text{Hom}_{\text{Ch}_R}(R[0], C^\bullet \otimes NG_n^\vee) \\ &\cong Z^0(C^\bullet \otimes \Lambda_n) \end{aligned}$$

where we have used that NG_n is dualisable. If C^\bullet is a CDGA the latter clearly carries a commutative product. More generally from this description the functor \mathcal{K} admits the structure of a lax symmetric monoidal functor.

Proposition 3.22. *This gives a well defined functor $\text{CDGA}_R \rightarrow \text{csCAlg}_R$. The composition $\text{CDGA}_R \rightarrow \text{csCAlg}_R \xrightarrow{N} \text{DGA}_R$ is equivalent to the identity.*

Proof. The first part of the claim is clear. It remains to show that the composition is equivalent to the identity... \square

Without our explicit description of the functor there is another abstract argument showing that the functor in question admits a lax symmetric monoidal structure. By abstract nonsense this is the same as an oplax symmetric monoidal structure on the inverse N . This is a well known structure called the Eilenberg-Zilber or shuffle map.

Proposition 3.23. *The functor $N : \text{Fun}(\Delta, \text{Mod}_R) \rightarrow \text{Ch}_R^{\geq 0}$ admits the structure of a oplax symmetric monoidal transformation. That is there is a natural morphism*

$$\Delta : N(C^\bullet \otimes D^\bullet) \rightarrow NC^\bullet \otimes ND^\bullet$$

which is symmetric and associative and an isomorphism $N(R^{\text{const}}) \cong R[0]$. Moreover Δ is a quasi-iso for all C^\bullet and D^\bullet .

Proof. We define the map Δ in degree n

$$\Delta : N(C^\bullet \otimes D^\bullet)^n \subseteq C^n \otimes D^n \rightarrow (NC^\bullet \otimes ND^\bullet)^n \subseteq \bigoplus_{p+q=n} C^p \otimes C^q$$

and in the (p, q) factor as

$$\Delta(x \otimes y) = \sum_{\sigma} \text{sign}(\sigma) \cdot f_{\sigma}(x) \otimes b_{\sigma}(y)$$

where the sum runs over all shuffle permutations $\sigma \in \Sigma_{p+q}$ with $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(p+q)$. For any such σ the maps $f_{\sigma} : [p+q] \rightarrow [p]$ and $b_{\sigma} : [p+q] \rightarrow [q]$ are given by

$$f_{\sigma} = s_{\sigma(p+1)} \cdots s_{\sigma(p+q)} \quad \text{and} \quad b_{\sigma} = s_{\sigma(1)} \cdots s_{\sigma(p)}$$

(HERE IS A MISTAKE WITH THESE MAPS) Now there are a couple of things to check. First symmetry is clear since interchanging x and y exactly results in a sign as needed. Associativity is seen similarly by using a 3-dimensional shuffle description. \square

Proposition 3.24. *The ∞ -category csCAlg_R^{∞} is presentable and the canonical forgetful functor $\text{csCAlg}_R^{\infty} \rightarrow \mathcal{D}(R)$ preserves limits (and reflects equivalences).*

Proof. There is a well known combinatorial model structure on cosimplicial commutative rings such that the forgetful functor $\text{csCAlg}_R \rightarrow \text{Ch}_R$ is right Quillen. This will be discussed in more detail later. \square

Corollary 3.25. *The functors $\mathcal{S}^{op} \xrightarrow{\mathcal{A}_{PL}} \text{CDGA}_{\mathbb{Q}}^{\infty} \rightarrow \text{csCAlg}_{\mathbb{Q}}^{\infty}$ and $\mathcal{S}^{op} \xrightarrow{C^*(-, \mathbb{Q})} \text{csCAlg}_{\mathbb{Q}}^{\infty}$ are equivalent.*

Proof. Both functors preserve limits (as the underlying $\mathcal{D}(R)$ -valued functors do). Thus it suffices to compare them on the point which is easy. \square

The next corollary has essentially the same proof as the last. It will be used later to prove that there can not be a variant of \mathcal{A}_{PL} in non-rational situations (see Proposition 1.11 in the introduction).

Corollary 3.26. *Assume we are given a functor $\mathcal{A}_R : \text{Top}^{op} \rightarrow \text{CDGA}_R$ whose composite $\text{NTop}^{op} \rightarrow \text{NCDGA}_R \rightarrow \mathcal{D}(R)$ is equivalent to $C^*(-, R)$. Then the functor $\text{NTop}^{op} \rightarrow \text{NCDGA}_R \rightarrow \text{csCAlg}_R^{\infty}$ is equivalent to the cochains functor. In particular for every space X there is an zig-zag of quasi-isomorphisms of cosimplicial rings $C^*(X, R) \simeq \mathcal{A}_R(X)$.*

4 Coalgebras and Goerss' theorem

In the last section we have seen that the chains $C^*(X, R)$ have the extra structure of a cosimplicial commutative R -algebra or in the rational case (up to equivalence) the structure of a CDGA. There is a coalgebra structure on singular chains $C_*(X, R)$ which dualises to this structure. Thus the coalgebra structure should be considered the more fundamental object. More precisely for every simplicial set X the chains $C_*(X, R)$ which are given in degree n by $R[X_n]$ (i.e. the R -module generated by X_n). This carries a cocommutative coalgebra structure induced from the diagonal $X_n \rightarrow X_n \times X_n$ since $R[X_n \times X_n] \cong R[X_n] \otimes_R R[X_n]$.

Definition 4.1. Let coCAlg_R denote the category of cocommutative coalgebras over a commutative ring R . By scoCAlg_R we denote the category of simplicial objects in coCAlg_R and refer to it as simplicial (cocommutative) coalgebras. For convenience we will suppress the adjective cocommutative and take the convention that all coalgebras are cocommutative.

By coCAlg_R^∞ we denote the ∞ -category obtained from scoCAlg_R by inverting the underlying quasi-isomorphisms.

With this notation we have a functor $C_*(-, R) : \text{sSet} \rightarrow \text{scoCAlg}_R$ which refines to a functor $C_*(-, R) : \mathcal{S} \rightarrow \text{scoCAlg}_R^\infty$. Recall that we denote by \mathcal{S}_R the Bousfield HR -local spaces. Then we clearly get an induced functor $\mathcal{S}_R \rightarrow \text{scoCAlg}_R^\infty$ by restriction (which is the same as factoring using the fact that \mathcal{S}_R is a DK-localization at the R -homology equivalences). The result that we want to prove now is

Theorem 4.2 (Goerss). *The functor $C_*(-, k) : \mathcal{S}_k \rightarrow \text{scoCAlg}_k^\infty$ is fully faithful for an algebraically closed field k .*

Let us note that the target category is a presentable ∞ -category as a consequence of model structures that we will discuss later. The idea of the proof is to construct an explicit right adjoint with unit and equivalence. We first start by an analogous fact in ordinary land.

Lemma 4.3. *For every ring R the functor*

$$R[-] : \text{Set} \rightarrow \text{coCAlg}_R$$

admits a right adjoint given by

$$(-)^{\text{gp}} : \text{coCAlg}_R \rightarrow \text{Set} \quad C \mapsto \text{Hom}_{\text{coCAlg}_R}(R, C)$$

Proof. Clear. □

First we will denote the comultiplication for a coalgebra over R by $\Delta : C \rightarrow C \otimes_R C$ and the counit by $\epsilon : C \rightarrow R$. For example for the coalgebra R (over R) the comultiplication is given by $\Delta(r) = r \cdot (1 \otimes 1) \in R \otimes_R R$ and the counit by the identity.

Using this the right adjoint $(-)^{\text{gp}}$ can be made more explicit. Therefore note that a morphism $R \rightarrow C$ of coalgebras is given by sending $1 \in R$ to an element $c \in C$ which has the property that $\Delta(c) = c \otimes c$ and $\epsilon(c) = 1$. Such elements are called *grouplike*. Vice versa every grouplike element in C determines a unique morphism of coalgebras $R \rightarrow C$. Thus the right adjoint is given by sending a coalgebra C to the subset $C^{\text{gp}} \subseteq C$.

Lemma 4.4. *Let R be an integral domain. Then the unit $X \rightarrow R[X]^{\text{gp}}$ of the adjunction is a bijection. In particular $R[-] : \text{Set} \rightarrow \text{coCAlg}_R$ is fully faithful.*

Proof. The comultiplication $\Delta : R[X] \rightarrow R[X] \otimes R[X]$ is given by $\Delta(x) = x \otimes x$ where $x \in R[X]$ denotes the basis element x corresponding to $x \in X$. The counit $\epsilon : R[X] \rightarrow R$ is given by $\epsilon(x) = 1$. Thus clearly all basis elements are grouplike. Let $y = \sum_{x \in X} y_x \cdot x \in R[X]$ be a grouplike element of $R[X]$. Then we get that $\sum_{x \in X} y_x = 1$ and that

$$\sum_{x \in X} y_x \cdot x \otimes x = \Delta(y) = y \otimes y = \sum_{x, x' \in X} y_x \cdot y_{x'} \cdot x \otimes x'.$$

We obtain that $y_x \cdot y_{x'} = \delta_{x, x'} y_x$. By the first condition there has to be a non-zero coefficient, say y_{x_0} . Then since $y_{x_0}^2 = y_{x_0}$ by the second property we find that $y_{x_0} = 1$ and by $y_{x_0} \cdot y_x = 0$ for $x \neq x_0$ we find that all other coefficients are zero. \square

Using this observation and assuming that R is a domain we find that the functor $C_*(-, R) : \mathbf{sSet} \rightarrow \mathbf{scoAlg}_R$ is fully faithful and in fact has a left inverse given by $(-)_\Delta^{\text{gp}} : \mathbf{scoAlg}_R \rightarrow \mathbf{sSet}$. All we need to do to prove Goerss' theorem is to show that this adjoint functor also descends to a functor $\mathbf{scoAlg}_R^\infty \rightarrow \mathcal{S}_R$ to prove that $C_*(-, R)$ is fully faithful (since then this induces an adjunction on the level of ∞ -categories). Thus we have to investigate when the functor $(-)_\Delta^{\text{gp}}$ sends quasi-isomorphisms to R -homology equivalences of spaces. Since R -homology equivalences are detected by applying $C_*(-, R)$ this comes down to trying to understand the composition functor

$$\mathbf{scoAlg}_R \rightarrow \mathbf{scoAlg}_R \quad C_\bullet \mapsto R[C_\bullet^{\text{gp}}]$$

and when it preserves quasi-isomorphisms (a priori it could also be that we have to derive the functor but this will not be the case here...). It will turn out that this is the case for $R = k$ an algebraically closed field. In fact we will show that the counit map $k[C_\bullet^{\text{gp}}] \rightarrow C_\bullet$ is naturally split injective in this case.

4.1 Structure theory for coalgebras

A good source for the basic theory of coalgebras is [Swe69]. The presentation here follows Goerss' paper [Goe95] with some additions. Recall that all coalgebras are cocommutative.

Proposition 4.5 (Fundamental theorem of coalgebras). *Let C be a coalgebra over a field k and $x \in C$. Then there exists a finite dimensional subcoalgebra $D \subseteq C$ with $x \in D$.*

Proof. Write $\Delta(x) = \sum_i x_i \otimes c_i$ and

$$(\Delta \otimes \text{id})(\Delta(x)) = \sum_i \Delta(x_i) \otimes c_i = \sum_{i, j} a_j \otimes b_{ij} \otimes c_i.$$

We can assume that the $(c_i)_{i \in I}$ are linearly independent and the $(a_j)_{j \in J}$ are also linearly independent (to see that the form given above and the independence is possible assume for example that the c_i and a_j all lie in a prefixed basis). Let $D \subseteq C$ be the subspace spanned by the b_{ij} . Now by counitality we have $x = \sum_{i, j} \epsilon(a_j) \cdot b_{ij} \cdot \epsilon(c_i) \in D$. We want to show that D is a subcoalgebra, i.e., that $\Delta(D) \subseteq D \otimes D$ which then finishes the proof since D is by definition finite dimensional.

First note that we have

$$\sum_{i, j} \Delta(a_j) \otimes b_{ij} \otimes c_i = \sum_{i, j} a_j \otimes \Delta(b_{ij}) \otimes c_i$$

by coassociativity. Since the c_i are linearly independent this implies that for every i we have $\sum_j \Delta(a_j) \otimes b_{ij} = \sum_j a_j \otimes \Delta(b_{ij})$. Thus

$$\sum_j a_j \otimes \Delta(b_{ij}) \in C \otimes C \otimes D$$

and since the $(a_j)_{j \in J}$ are linearly independent this implies that $\Delta(b_{ij}) \in C \otimes D$. A similar argument shows $\Delta(b_{ij}) \in D \otimes C$ and thus we have $\Delta(b_{ij}) \in C \otimes D \cap D \otimes C = D \otimes D$. \square

It easily follows that every finite sequence of elements x_1, \dots, x_n or every finite dimensional subspace is contained in a finite dimensional subcoalgebra (exercise!).

Corollary 4.6. *Every coalgebra C is the filtered colimit over its finite dimensional subcoalgebras.*

Proof. This follows immediately from the last result. The only non-trivial thing is to show that the vector space colimit C of a diagram C_i of coalgebras is again a coalgebra (and in the fact the colimit in the category of coalgebras). This is straightforward. \square

Remark 4.7. *Note that the 'dual' statement for algebras is totally wrong. Not every algebra is a filtered limit of finite dimensional algebras. For example the polynomial ring $k[x]$ is certainly not. In fact the limit over the finite dimensional quotients is the power series ring $k[[x]]$.*

Corollary 4.8. *The category coAlg_k for a field k is presentable and the forgetful functor $\text{coAlg}_k \rightarrow \text{Vect}_k$ has a right adjoint (the cofree coalgebra).*

Proof. First coAlg_k admits all colimits (which are formed underlying). Now we consider the full subcategory of finite dimensional coalgebras. This is essentially small and every coalgebra is a filtered colimit of finite dimensional coalgebras. Thus it only remains to show that finite dimensional coalgebras are compact objects in coAlg_k . This is clear since every morphism into a filtered colimit has to factor through a finite stage. Thus we have shown that coAlg_k is presentable. Then the adjoint functor theorem implies that the forgetful functor $\text{coAlg}_k \rightarrow \text{Vect}_k$ has a right adjoint since it preserves all colimits. Show that it admits colimits and then use adjoint functor theorem. In fact its a nice exercise to get an explicit formula for the cofree coalgebra on a vector space V (it is a bit tricky). \square

A coalgebra C over k is called simple if it has no non-trivial subcoalgebras (i.e. besides 0 and C). Let K/k be a finite field extension. Then $K^\vee = \text{Hom}_k(K, k)$ is a finite dimensional coalgebra over k and is simple, because subcoalgebras of K^\vee corresponds to quotients of K which do not exist.

Proposition 4.9. *All simple coalgebras over k are up to isomorphism of the form K^\vee for a finite field extension K over k .*

Proof. The fundamental theorem of coalgebras implies that every non-finite dimensional coalgebra over k has a non-trivial subcoalgebra. Thus every simple coalgebra C is automatically finite dimensional. Then C^\vee is a finite dimensional, commutative algebra over k which has no non-trivial quotients. This implies C^\vee has only the trivial ideals (zero and C^\vee) which implies that it is a field. Since everything was finite dimensional we get that $C \cong C^{\vee\vee}$. \square

The latter in particular implies that if k is algebraically closed there are no simple coalgebras besides k itself.

Definition 4.10. Let C be a coalgebra over k . Then the étale part EC is defined to be the direct sum $\bigoplus_{C_\alpha \subseteq C} C_\alpha$ where C_α runs through all simple subcoalgebras of C (counted several times if the same isomorphism class occurs several times).

Lemma 4.11. Let $C = \sum_{i \in I} C_i$ be a (not necessarily direct) sum of subcoalgebras $C_i \subseteq C$. Then every simple subcoalgebra of C is a subcoalgebra of one of the summands.

Proof. Let $D \subseteq C$ be the simple subcoalgebra in question. Since D is finite dimensional it lies in finitely many summands. Inductively we can assume that D lies in two summands, i.e. it suffices to prove that if $D \subseteq C_1 + C_2$ then D lies in one of the summands. If D is not contained in C_1 then we have $D \cap C_1 = 0$ since this intersection is a subcoalgebra of D . We choose a linear map $f : C \rightarrow k$ with $f|_D = \epsilon_D$ and $f|_{C_1} = 0$. Then we find for $d \in D$:

$$(f \otimes \text{id})\Delta(d) = (\epsilon_D \otimes \text{id})\Delta(d) = d.$$

But $\Delta(D) \subseteq \Delta(C_1) + \Delta(C_2) = C_1 \otimes C_1 + C_2 \otimes C_2$. Since $f|_{C_1} = 0$ we find for every $d \in D$ that $d = (f \otimes \text{id})\Delta(d) \in C_2$. \square

Lemma 4.12. The canonical morphism $EC \rightarrow C$ is an injective morphism of coalgebras. In fact the assignment $C \mapsto EC$ refines to an endofunctor $E : \text{coAlg}_k \rightarrow \text{coAlg}_k$ such that the inclusion $EC \rightarrow C$ is natural in C .

Proof. We want to show that the sum of simple coalgebras $\sum C_\alpha \subseteq C$ is direct. Thus we have to show that $C_{\alpha_0} \cap \sum_{\alpha \neq \alpha_0} C_\alpha = 0$ for every α_0 . Since C_{α_0} is simple it follows that if $C_{\alpha_0} \cap \sum_{\alpha \neq \alpha_0} C_\alpha \neq 0$ then $C_{\alpha_0} \subseteq \sum_{\alpha \neq \alpha_0} C_\alpha$. Thus by Lemma 4.11 this implies that $C_{\alpha_0} = C_{\alpha_1}$, a contradiction.

The second claim of functoriality follows if we know that for a morphism of coalgebras $f : C \rightarrow D$ the image $f(C_\alpha) \subseteq D$ is simple for every simple subcoalgebra $C_\alpha \subseteq C$ (note that the image of a coalgebra map is a subcoalgebra). Since the image is a quotient of C_α this will follow if we show that all quotients of simple coalgebras are again simple. Using 4.9 this is equivalent to saying that subalgebras of finite field extensions are again field extensions which is obvious (the inverse of every element is polynomial in the elements since it is an algebraic extension). \square

Example 4.13. Let X be a set and consider $k[X] = \bigoplus_X k$ as a coalgebra as before. We find that $E(k[X]) = k[X]$ since every point $x \in X$ defines an inclusion of coalgebras $k \rightarrow k[X]$.

Proposition 4.14. Let k be an algebraically closed field and C be a coalgebra. Then the adjunction unit $k[C^{\text{sp}}] \rightarrow C$ factors through the inclusion $EC \subseteq C$ and induces a natural equivalence

$$k[C^{\text{sp}}] \cong EC$$

Proof. Since the coalgebra $k[C^{\text{sp}}]$ is equal to its étale part it follows by the functoriality of $E(-)$ that the coalgebra morphism $k[C^{\text{sp}}] \rightarrow C$ factors through the étale part. Now a simple subcoalgebra of C is given by k , thus the étale part is given by the direct sum over all homomorphisms $k \rightarrow C$ (which is automatically injective) but this is exactly the description of C^{sp} . \square

Finally the most important result in this section is the following:

Theorem 4.15. *Let k be a perfect field. Then for every coalgebra C the inclusion $EC \rightarrow C$ has a unique and natural split which is a map of coalgebras.*

To prove this theorem we will need a number of preparation steps. We will call a coalgebra irreducible if it has a unique simple subcoalgebra, i.e. if the étale part consists of a single summand. A subcoalgebra $D \subseteq C$ is called irreducible component if it is a maximal irreducible subcoalgebra of C .

Lemma 4.16. *Every coalgebra over a field k (not necessarily perfect) is the direct sum over its irreducible components. More precisely for every simple subcoalgebra $C_\alpha \subseteq C$ there is a unique irreducible component $\overline{C_\alpha} \subseteq C$ that contains C_α and the canonical morphism*

$$\bigoplus_{\alpha} \overline{C_\alpha} \rightarrow C$$

is an isomorphism of coalgebras where the sum is indexed over the simple subcoalgebras of C .

Proof. First we note that the sum of all irreducible subcoalgebras of C which contain C_α is again an irreducible subcoalgebra. To see this note that if another simple C_β is a subcoalgebra of this sum then it has to factor through one of the summands (as shown in Lemma 4.12) which can't be. By construction this sum contains C_α and is maximal thus an irreducible component. Also by construction it is unique.

To see that the irreducible components are disjoint we assume that they are not, i.e. there is a non-trivial intersection $\overline{C_{\alpha_0}} \cap \sum_{\alpha \neq \alpha_0} \overline{C_\alpha}$ for some α_0 . Then this intersection contains a simple subcoalgebra which has to be C_{α_0} since it is a subcoalgebra of $\overline{C_{\alpha_0}}$. An application of Lemma 4.11 shows that it also has to lie in one $\overline{C_\alpha}$ for $\alpha \neq \alpha_0$ which is a contradiction.

Finally we need to see that every element $c \in C$ lies in a sum of irreducible components. For this it suffices to show that it lies in a finite sum of irreducible subcoalgebras since every subcoalgebra lies in an irreducible component as shown above. Let $\overline{\{c\}}$ be the subcoalgebra generated by c , i.e. the intersection of all subcoalgebras containing c . This is by the fundamental theorem of coalgebras finite dimensional. Replacing C by $\overline{\{c\}}$ we can thus without loss of generality assume that C is finite dimensional (think about it for a second why this reduction is allowed). Then by the structure theory of artinian algebras we have that $A \cong A_1 \oplus \dots \oplus A_n$ for local artinian subalgebras. Then $C = A_1^\vee \oplus \dots \oplus A_n^\vee$ and thus it suffices to show that each A_i^\vee is irreducible. This is clear by definition, since local algebras have only a single field quotient. \square

Lemma 4.17. *Let $f : C \rightarrow D$ be a morphism of coalgebras. Then f restricts to a morphism of irreducible components $\overline{C_\alpha} \rightarrow \overline{f(C_\alpha)}$ for every simple $C_\alpha \subseteq C$.*

Proof. First let's assume that f is surjective and that C is irreducible. We claim that then automatically D is irreducible as well. The unique simple subcoalgebra of D is then automatically given by $f(C_0)$ for $C_0 \subseteq C$ the simple subcoalgebra (recall that this image is always simple). To see that D is irreducible we first write $C = \varinjlim C_i$ as a filtered colimit of finite dimensional algebras. We get that $D = \varinjlim f(C_i)$. If we can show that $f(C_i)$ is irreducible, then D is irreducible as well. Thus we can without loss of generality assume that C and D are finite dimensional.

By dualizing we get an inclusion of algebras $A \subseteq B$ with $B = C^\vee$ a local k -algebra and $A \cong D^\vee$. We have to show that A is local as well. It suffices to show that the subset of all elements A which are not a unit form an ideal. This follows since an element $d \in A$ is a unit precisely if it is a unit in B : one direction is clear for the other we use that if $d \in B$ is a unit the inverse d^{-1} is polynomial in d , i.e. $d^{-1} = p(d)$ for $p \in k[x]$ by finite dimensionality.

More in detail: there is map $k[x] \rightarrow B$ sending x to b . Then this map factors as $k[x]/q \rightarrow B$ for some polynomial $0 \neq q \in k[x]$. But $q(0) \neq 0$ since otherwise x would be a zero divisor in $k[x]/q$ and thus b a zero divisor in B which cannot be for a unit. But then we can assume that $q(0) = 1$ and $p := (1 - q)/x$ has the property that $p(b) \cdot b = 1 - q(b) = 1$.

Finally lets get back to the general situation of a morphism $f : C \rightarrow D$ of coalgebras. By the first part we know that the image $f(\overline{C_\alpha})$ is irreducible. Thus it is contained in an irreducible component which contains $f(C_\alpha)$, thus in $\overline{f(C_\alpha)}$. \square

Lemma 4.18. *Let C be an irreducible coalgebra over a perfect field k . Then there is a unique retract of the inclusion $EC \subseteq C$ which is a map of coalgebras. This retract is natural in C .*

Proof. First note that $EC = K^\vee$ is the dual of a finite field extension K of k . Thus we can use the fundamenal theorem of coalgebras to write C as the filtered colimit over finite dimensional subcoalgebras $C_i \subseteq C$ which all contain EC and are still all irreducible. If we prove the result for all C_i then it follows for C . Thus we can assume without loss of generality that C is finite dimensional over k and dualise the whole situation. Then the statement becomes the following:

Given a finite dimensional, local algebra A over a perfect field k . Denote the unique maximal ideal $\mathfrak{m} \subseteq A$. Then there is a unique subfield $K \subseteq A$ such that $A = K \oplus \mathfrak{m}$ (as vector spaces). Note that then automatically $K \cong A/\mathfrak{m}$.

Since k is perfect the field extension A/\mathfrak{m} is seperable over k . Thus by the primitive element theorem it is generated by a single element i.e. of the form $A/\mathfrak{m} = k(\alpha)$ for some $\alpha \in A/\mathfrak{m}$. Let $p \in k[x]$ be the minimal polynomial of α which is a separable polynomial over k . Thus $p'(\alpha) \neq 0$. Now note that A is complete with respect to the \mathfrak{m} -adic topology since we have $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n which by Nakayama implies $\mathfrak{m}^n = 0$. Thus it follows from Hensel's lemma that there is a unique element $x \in \alpha \subset A$ such that $p(x) = 0$ in A . Then we define $K = k(x) \subseteq A$. This has the required property since the composition $K \rightarrow A \rightarrow A/\mathfrak{m}$ is an isomorphism by construction.

For naturality we have to show that if $C \rightarrow D$ is a morphism of irreducible coalgebras over k then for the commutative diagram

$$\begin{array}{ccc} EC & \longrightarrow & ED \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

also the diagram of retracts $C \rightarrow EC$ and $D \rightarrow ED$ commutes. If the coalgebra map is injective we have that $EC \xrightarrow{\sim} ED$ is an isomorphism. Then the commutativity follows from the uniqueness. Since a general coalgebra map can be factored into a surjective map followed by an injective map it remains to treat the surjective case $C \rightarrow D$. Writing C as a filtered colimit of finite dimensional algebras we can again reduced to a finite dimensional situation, i.e. we have an inclusion $A \subseteq B$ of local algebras. Then we have to show that

diagram

$$\begin{array}{ccc} A/\mathfrak{n} & \longrightarrow & B/\mathfrak{m} \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

commutes. Since $\mathfrak{n} = \mathfrak{m} \cap A$ one can use a similar argument as above to deduce the uniqueness of the complement and thus one gets that $A = B/\mathfrak{m} \cap A \oplus \mathfrak{m} \cap A = k \oplus A$ which implies the claim. \square

Note that in the case of an algebraically closed field k (which is the most relevant case for now) the last lemma is much easier to prove since then $EC = k$ and the retract is given by the counit of the coalgebra.

Proof of Theorem 4.15. We need to show that for every coalgebra C over a perfect field k there is a natural and unique split $C \rightarrow EC$. By Lemma 4.16 we have that $C = \bigoplus \overline{C_\alpha}$ and $EC = \bigoplus C_\alpha$. There can not be a split mixing the components since it has to fix C_α thus we have to take for every α the split $\overline{C_\alpha} \rightarrow C_\alpha$ of Lemma 4.18. It is natural in C by Lemma 4.17. \square

Corollary 4.19. *Let k be an algebraically closed field. Then the counit of the adjunction $k[-] : \text{Set} \rightleftarrows \text{coCAlg}_k : (-)^{\text{sp}}$ given by $k[C^{\text{sp}}] \rightarrow C$ has a natural retract.*

Proof. Combine Proposition 4.14 with Theorem 4.15. \square

Finally we can give a proof of Goerss theorem, stated as 4.2 above:

Theorem (Goerss). *The functor $C_*(-, k) : \mathcal{S}_k \rightarrow \text{scoCAlg}_R^\infty$ is fully faithful for an algebraically closed field k . For every space X the canonical map*

$$X \rightarrow \text{Map}_{\text{scoCAlg}_R^\infty}(k, C_*(X, k))$$

exhibits the target as the k -localization of X .

Proof. The adjunction between $k[-] : \text{Set} \rightarrow \text{coCAlg}_k$ and $(-)^{\text{sp}} : \text{coCAlg}_k \rightarrow \text{Set}$ induces an adjunction $\text{sSet} \leftrightarrow \text{coCAlg}_k$. We equip the left hand side with the k -local weak equivalences (recall that this only depends on the characteristic of k) and the right hand side with the underlying quasi-isomorphisms. Then both functors preserve weak equivalences, the first by definition and for the second by Corollary 4.19. Thus we get an induced adjunction on the level of ∞ -categories. Since the initial adjunction has an isomorphism as counit it follows that the induced adjunction has an equivalence as counit. This shows fully faithfulness.

For the second part it is clear by the general theory of localization that the right adjoint factors through the local objects and the counit is a localization (see Remarks 2.26 and 2.27). Thus the only remaining thing is to show that the right adjoint has the claimed form (despite being the induced map from the 1-categorical right adjoint). But this follows from the universal property of spaces as stated in Proposition 3.1 and below. \square

4.2 The case of \mathbb{F}_p

We now want to give a description in the case $k = \mathbb{F}_p$. This is inspired by the work of Kriz [?]. We first have to introduce some terminology.

Definition 4.20. A commutative \mathbb{F}_p -algebra A is called Boolean (or just a p -ring following McCoy and Montgomery) if for every $x \in A$ we have $x^p = x$. In other words if the Frobenius is the identity.

Example 4.21. The ring \mathbb{F}_p is Boolean and also the rings $\prod_X \mathbb{F}_p \cong \text{Map}(X, \mathbb{F}_p)$ for every set X . The field \mathbb{F}_{p^n} is not Boolean. More generally the field \mathbb{F}_p is the only p -Boolean domain.

Example 4.22. For a ring R the relation $x^2 = x$ already implies that $2 = 0$ since $0 = 2^2 - 2 = 2$. But $x^p = x$ does not imply characteristic p . For example in $\mathbb{Z}/2$ we have $x^3 = x$ for every element but its not a 3-Boolean ring.

Remark 4.23. The name Boolean ring arises as follows: for $p = 2$ it turns out that 2-Boolean rings (usually only called Boolean rings) are equivalent to what is classically called Boolean algebras. The latter are sets B together with operations $\vee : B \times B \rightarrow B$ (the join/disjunction/OR) and $\wedge : B \times B \rightarrow B$ (the meet/conjunction/AND) and $\neg : B \rightarrow B$ (negation/NOT) satisfying a list of axioms akin to classical logic rules. Given a 2-Boolean ring we construct a Boolean algebra by letting $B = R$ the same underlying set and $a \wedge b := a \cdot b$ and $a \vee b := a + b + ab$ and $\neg x := 1 + x$. This in fact defines an isomorphism of categories.

As an example to remember the relation consider the set $P(X) \cong \text{Map}(X, \mathbb{F}_2)$ for a set X . Then it becomes a classical Boolean algebra with $\vee = \cup$, $\wedge = \cap$ and $\neg(A) = X \setminus A$. On the other hand with pointwise operations it becomes a 2-Boolean algebra as defined above. Under the identification as above we have $A + B = A \cup B \setminus A \cap B$ thus $A \cup B = A + B + A \cap B$ and $X \setminus A = X - A = 1 + A$.

Now we want to formulate a dual version of this statement. To this end we need to introduce the Frobenius endomorphism of a coalgebra in characteristic p .

Definition 4.24. Let k be a field of characteristic p and C be a k -coalgebra. Then the Frobenius $\varphi_p : C \rightarrow C$ is the unique coalgebra morphism which has the following properties:

- It is natural in coalgebra maps
- the dual $\varphi_p^\vee : C^\vee \rightarrow C^\vee$ is given by the usual Frobenius $x \mapsto x^p$.

This uniquely fixes φ_p since it does for finite dimensional coalgebras and thus for all by the fundamental theorem and naturality. The only little thing to check is that the morphism constructed this way really has the property that the morphism φ_p is the pre-dual of the Frobenius morphism for infinite dimensional coalgebras. But this again follows by naturality of the usual Frobenius and the fact that $C^\vee = \varprojlim(C_i^\vee)$ for finite dimensional subcoalgebras $C_i \subseteq C$.

Definition 4.25. A coalgebra C over \mathbb{F}_p is called Boolean (or more precisely p -Boolean) if $\varphi_p(x) = x$ for every $x \in C$. We denote the category of cocommutative Boolean coalgebras over \mathbb{F}_p by $\text{coCAlg}_{\mathbb{F}_p}^{\text{Bool}}$.

We now want to give a different and more conceptual description of the Frobenius of a coalgebra. This will be the description that we generalize later on to ringspectra. To this end we have to consider the algebraic Tate-construction. Let M be an abelian group with an action by a finite group G . Then we denote by M^{tG} the quotient of the G -fixed points M^G modulo the norms, i.e. elements of the form $N(m) = \sum_{g \in G} gm$ for some $m \in M$. For example if the G -action on M is trivial we have that $M^{tG} = M/|G|$. Since every G -module is a module over the G -module with trivial action this implies that $|G|$ is invertible in all the Tate groups.

Lemma 4.26. *For every abelian group M and every prime p the map*

$$\Delta_p : M \rightarrow (M \otimes \dots \otimes M)^{tC_p} \quad m \mapsto [m \otimes \dots \otimes m]$$

is additive. If M is p -torsion then it is an isomorphism.

Proof. For additivity we compute

$$\begin{aligned} \Delta_p(m_0 + m_1) &= \left[\sum_{(i_0, \dots, i_p) \in \{0,1\}^p} m_{i_0} \otimes m_{i_1} \otimes \dots \otimes m_{i_p} \right] \\ &= \Delta_p(m_0) + \Delta_p(m_1) + \left[\sum_{[i_0, \dots, i_p]} N(m_{i_0} \otimes m_{i_1} \otimes \dots \otimes m_{i_p}) \right] = \Delta_p(m_0) + \Delta_p(m_1) \end{aligned}$$

where in the second sum $[i_0, \dots, i_p]$ runs through a set of representatives of orbits of the cyclic C_p -action on the set $S = \{0, 1\}^p \setminus \Delta$ which are all isomorphic to C_p .

To see that Δ_p is an isomorphism if M is an \mathbb{F}_p -vector space we claim that the endofunctor

$$\text{Ab} \rightarrow \text{Ab} \quad M \mapsto (M \otimes \dots \otimes M)^{tC_p}$$

commutes with arbitrary sums. The proof is a categorification of the proof above. For a sum of two vector spaces $V_0 \oplus V_1$ we get the same decomposition as above and use that for C_p -modules induced up from the trivial group the Tate-construction vanishes. Then arbitrary sums follow from the more general fact that the functor commutes with arbitrary filtered colimit. To see this we use that

$$(\text{colim} M_i \otimes \dots \otimes \text{colim} M_i)^{tC_p} \cong (\text{colim}(M_i \otimes \dots \otimes M_i))^{tC_p} \cong \text{colim}(M_i \otimes \dots \otimes M_i)^{tC_p}$$

where the latter equivalence follows since taking fixed points commutes with filtered colimits and taking the quotient of course does. Finally this reduces the claim to the case of \mathbb{F}_p in which it is obvious. \square

Before we go on, we want to give a flavour of what the Tate construction actually does in characteristic p . In the category $\text{Vect}_{\mathbb{F}_p}^{C_p}$ there exist exactly the indecomposables V_d for $1 \leq d \leq p$ which are d -dimensional and where the generator of C_p acts in a basis by the Jordan matrix

$$A = \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \dots & \\ & & & 1 \end{pmatrix}.$$

We get that $V_d^{C_p} \cong \mathbb{F}_p$ spanned by the first basis vector (using that these are the Eigenspaces of 1) and for $d = p$ we get that

$$N(e_d) = e_d + Ae_d + A^2e_d + \dots + A^{p-1}e_d.$$

In this sum the only occurrence of e_1 is in the very last summand, since the matrices A^k for $k < p - 1$ all have a zero on the upper right but A^{p-1} has a one. Thus we get that $N(e_d) = e_d$. We conclude that $(V_p)^{tC_p} = 0$ for $d = p$. For the other representations one checks easily that the norm is zero, thus we get that $V_d^{tC_p} \cong \mathbb{F}_p$ for $d < p$. Thus taking the Tate-construction corresponds to giving a one dimensional space for all occurrences of the non-maximal dimensional indecomposable.

Lemma 4.27. *The functor $(-)^{tC_p} : \text{Vect}_{\mathbb{F}_p}^{C_p} \rightarrow \text{Vect}_{\mathbb{F}_p}$ admits a lax symmetric monoidal structure.*

Proof. To construct a lax symmetric monoidal structure on the functor $(-)^{tC_p}$. To this end we have to come up with a map

$$V^{tC_p} \otimes W^{tC_p} \rightarrow (V \otimes W)^{tC_p}$$

for every pair of C_p -representations V, W . We have an obvious inclusion $V^{C_p} \otimes W^{C_p} \subseteq (V \otimes W)^{C_p}$, in particular the fixed points functor admits a lax symmetric monoidal structure. We claim that this descends to a functor $V^{tC_p} \otimes W^{tC_p} \rightarrow (V \otimes W)^{tC_p}$. To see this we have to check that for elements $v \in V$ and $w \in W^{C_p}$ the element $N(v) \otimes w$ vanishes in $(V \otimes W)^{tC_p}$ and vice versa. But this follows since

$$N(v) \otimes w = \left(\sum_{g \in C_p} gv \right) \otimes w = \sum_{g \in C_p} (gv \otimes gw) = N(v \otimes w).$$

For the unit we get that $\mathbb{F}_p^{tC_p} \cong \mathbb{F}_p$. □

Remark 4.28. *One can check that the lax symmetric monoidal structure $\Phi_{V,W}$ on the Tate functor $\text{Vect}_{\mathbb{F}_p}^{C_p} \rightarrow \text{Vect}_{\mathbb{F}_p}$ is strong exactly in the case $p = 2$. To see this consider the matrix*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

This defines the two dimensional indecomposable C_p -representation V_2 for every p . In characteristic 2 this is isomorphic to the regular representation $\mathbb{F}_2[C_2]$ thus we get that $V_2 \otimes V_2 \cong \mathbb{F}_2[C_2 \times C_2] \cong V_2 \oplus V_2$ but in characteristic $p > 2$ we get that $V_2 \otimes V_2 \cong V_1 \oplus V_3$ as a straightforward computation of the Jordan normal form of the Kronecker product of the matrix A with itself shows. Thus we get that

$$(V_2 \otimes V_2)^{tC_p} \cong (V_1 \oplus V_3)^{tC_p} = V_1^{tC_p} \oplus V_3^{tC_p} \cong \mathbb{F}_p \oplus \mathbb{F}_p.$$

For the last equation we need that $p > 3$. In this case we get that $V_2^{tC_2} \otimes V_2^{tC_2}$ is one dimensional but $(V_2 \otimes V_2)^{tC_p}$ is two dimensional. It might still be (as I had initially claimed) that the map is injective....

Corollary 4.29. *The lax symmetric monoidal structure on the functor*

$$\mathrm{Vect}_{\mathbb{F}_p} \rightarrow \mathrm{Vect}_{\mathbb{F}_p} \quad V \mapsto (V \otimes \dots \otimes V)^{tC_p}$$

inherited from the one of $(-)^{tC_p}$ is strong symmetric monoidal and the map $V \rightarrow (V \otimes \dots \otimes V)^{tC_p}$ is a symmetric monoidal transformation. In fact the latter exhibits an equivalence of this functor to the identity as lax symmetric monoidal functors.

Proof. Use the universal property or a direct argument to show that the Tate-diagonal transformation from the identity functor is compatible with the lax symmetric monoidal structure. \square

Lemma 4.30. *Assume C is a coalgebra over \mathbb{F}_p . Then $(C \otimes \dots \otimes C)^{tC_p}$ admits a coalgebra structure such that the Tate diagonal Δ_p is a map of coalgebras. Also the map*

$$C \xrightarrow{\Delta^p} (C \otimes \dots \otimes C)^{C_p} \xrightarrow{\mathrm{can}} (C \otimes \dots \otimes C)^{tC_p}$$

is a morphism of coalgebras. Here Δ^p denotes the p -fold iteration of the coproduct defined inductively as $\Delta^2 := \Delta$ and $\Delta^n := (\Delta \otimes \mathrm{id}) \circ \Delta^{n-1} = (\mathrm{id} \otimes \Delta) \circ \Delta^{n-1}$.

Proof. The first and second part are clear by the claim before. For the third part we consider the following commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{\Delta^p} & (C \otimes \dots \otimes C)^{C_p} & \xrightarrow{\mathrm{can}} & (C \otimes \dots \otimes C)^{tC_p} \\ \downarrow \Delta & & \downarrow (\Delta \otimes \dots \otimes \Delta)^{C_p} & & (\Delta \otimes \dots \otimes \Delta)^{tC_p} \downarrow \\ C \otimes C & \xrightarrow{\Delta^p \otimes \Delta^p} & ((C \otimes \dots \otimes C) \otimes (C \otimes \dots \otimes C))^{C_p} & \xrightarrow{\mathrm{can}} & ((C \otimes \dots \otimes C) \otimes (C \otimes \dots \otimes C))^{tC_p} \\ \parallel & & \uparrow & & \sim \uparrow \\ C \otimes C & \xrightarrow{\Delta^p \otimes \Delta^p} & (C \otimes \dots \otimes C)^{C_p} \otimes (C \otimes \dots \otimes C)^{C_p} & \xrightarrow{\mathrm{can} \otimes \mathrm{can}} & (C \otimes \dots \otimes C)^{tC_p} \otimes (C \otimes \dots \otimes C)^{tC_p} \end{array}$$

and conclude that the outer two compositions agree. This shows that the upper horizontal map is a map of coalgebras. \square

Proposition 4.31. *The Frobenius of a coalgebra C over \mathbb{F}_p is given by the composition*

$$C \xrightarrow{\Delta^p} (C \otimes \dots \otimes C)^{C_p} \xrightarrow{\mathrm{can}} (C \otimes \dots \otimes C)^{tC_p} \xrightarrow{\Delta_p^{-1}} C.$$

Proof. By the last lemma it follows that the map above is a natural morphism of the identity functor on the category of coalgebras over \mathbb{F}_p . That is also sometimes called the center of the category of coalgebras. Since the category of coalgebras is the Ind-completion of the category of algebras this is the same as the center of the category of finite dimensional algebras over \mathbb{F}_p . We claim that the center of the category of finite dimensional \mathbb{F}_p -algebras is given by \mathbb{N} generated by the Frobenius endomorphism. We leave this as an exercise (more generally the reader should try to prove that the center of the category of R -modules for a ring R is given by the center of R).

We conclude that there has to a natural number n such that the given composition is equal to n -th iteration of the Frobenius operator. We now consider the coalgebra $C =$

$\mathbb{F}_p\{1, t, t^2, t^3, \dots\}$ with comultiplication given by $\Delta(t^n) = \sum_{i+j=n} t^i \otimes t^j$. This is the predual of the power series ring $\mathbb{F}_p[[t]]$. The Frobenius operator of C is given by

$$t^n \mapsto \begin{cases} t^{n/p} & \text{if } n \text{ is divisible by } p \\ 0 & \text{else} \end{cases}$$

On the other hand the map

$$C \xrightarrow{\Delta^p} (C \otimes \dots \otimes C)^{C_p} \xrightarrow{\text{can}} (C \otimes \dots \otimes C)^{tC_p} \xrightarrow{\Delta_p^{-1}} C$$

is given by

$$t^n \mapsto \Delta_p^{-1} \left[\sum_{i_1 + \dots + i_p = n} t^{i_1} \otimes \dots \otimes t^{i_p} \right] = \Delta_p^{-1} [t^{n/p} \otimes \dots \otimes t^{n/p}] = t^{n/p}$$

if p divides n and zero otherwise. Thus the transformation has to be the Frobenius. \square

Corollary 4.32. *A coalgebra C over \mathbb{F}_p is Boolean precisely if the following diagram is commutative:*

$$\begin{array}{ccc} & (C \otimes \dots \otimes C)^{C_p} & \\ \Delta^p \nearrow & & \downarrow \text{can} \\ C & \xrightarrow{\Delta_p} & (C \otimes \dots \otimes C)^{tC_p} \end{array}$$

In other words $[\Delta^p(c)] = \Delta_p(c)$ for all $c \in C$.

Theorem 4.33 (coStone Duality). *The functor $\mathbb{F}_p[-] : \text{Set} \rightarrow \text{coCAlg}_{\mathbb{F}_p}$ factors through the subcategory $\text{coCAlg}_{\mathbb{F}_p}^{\text{Bool}} \subseteq \text{coCAlg}_{\mathbb{F}_p}$ and induces an equivalence $\text{Set} \simeq \text{coCAlg}_{\mathbb{F}_p}^{\text{Bool}}$.*

Proof. Since $\mathbb{F}_p[X]$ for a set X is given by $\bigoplus \mathbb{F}_p$ we immediately obtain that the Frobenius is trivial.

Conversely we claim that for a coalgebra with trivial Frobenius the inclusion $EC \rightarrow C$ is an isomorphism and the simple subcoalgebras are all of the form \mathbb{F}_p^\vee . The second claim is clear since the only finite field extension of \mathbb{F}_p with trivial Frobenius is the identity.

For the second claim we have to show that every irreducible coalgebra over \mathbb{F}_p with trivial Frobenius agrees with \mathbb{F}_p . After reducing with the fundamental theorem to the finite dimensional case this means that every finite dimensional, local \mathbb{F}_p -algebra A with trivial Frobenius is isomorphic to \mathbb{F}_p . To see this we note that as earlier we get that $\mathfrak{m}^n = 0$ for some n using the Nakayama Lemma. We can assume that p -divides n . But then $x^n = x \in \mathfrak{m}$ for all $x \in \mathfrak{m}$ this can only happen if $\mathfrak{m} = 0$. \square

Corollary 4.34. *For a general coalgebra C over \mathbb{F}_p we can describe the counit $\mathbb{F}_p[C^{\text{gp}}] \subseteq C$ as the fixed points of the Frobenius $\varphi_p : C \rightarrow C$.*

Corollary 4.35. *The category of simplicial sets is equivalent to the category of simplicial Boolean coalgebras over \mathbb{F}_p . Under this equivalence the quasi-isomorphisms of coalgebras correspond to \mathbb{F}_p -equivalences of spaces.*

Definition 4.36. The ∞ -category of simplicial Boolean algebras over \mathbb{F}_p is the ∞ -category obtained by formally inverting the underlying quasi-isomorphisms in the category $\text{scoCAlg}_{\mathbb{F}_p}^{\text{Bool}}$. In formulas we have

$$(\text{scoCAlg}_{\mathbb{F}_p}^{\text{Bool}})^\infty := \text{NscCAlg}_{\mathbb{F}_p}^{\text{Bool}}[q^{-1}].$$

Remark 4.37. *We will see soon, that this is also the same as the ∞ -category of simplicial coalgebras equipped with a homotopy between the Frobenius and the identity.*

Corollary 4.38. *The canonical functor*

$$C_*(-, \mathbb{F}_p) : \mathcal{S}_{\mathbb{F}_p} \rightarrow (\text{scoCAlg}_{\mathbb{F}_p}^{\text{Bool}})^{\infty}$$

is an equivalence of ∞ -categories.

4.3 Non algebraically closed fields

Assume now that we are given a perfect field k with algebraic closure \bar{k} . Then for a coalgebra C over k we try to compare C^{gp} with $(C \otimes \bar{k})^{\text{gp}}$. There is an obvious map $C^{\text{gp}} \rightarrow (C \otimes \bar{k})^{\text{gp}}$. We denote the Galois group of \bar{k} over k by G . There is an obvious action of G on $(C \otimes \bar{k})^{\text{gp}}$.

Lemma 4.39. *For every coalgebra C over k the canonical map induces an isomorphism*

$$C^{\text{gp}} \rightarrow ((C \otimes \bar{k})^{\text{gp}})^G.$$

Proof. Let us describe the canonical map $C^{\text{gp}} \rightarrow (C \otimes \bar{k})^{\text{gp}}$. It takes a grouplike element $c \in C$ (i.e. $\epsilon(c) = 1$ and $\Delta(c) = c \otimes c$) to the element $c \otimes 1 \in C \otimes \bar{k}$. It is not hard to check that this is also grouplike. This map is obviously injective since $C \rightarrow C \otimes_k \bar{k}$.

Moreover it clearly lands in the fixed points for the G action since 1 is fixed. Since we have that $(C \otimes_k \bar{k})^G \cong C \otimes \bar{k}^G \cong C$ we get that a fixed point in $C \otimes_k \bar{k}$ is fixed under G precisely if it lies in the image of the canonical map. \square

Note that the Galois group G is among other things a profinite group. We will have to deal with that soon. But let's first try to understand the case of finite absolute Galois group as a toy example. Think for example about the case \mathbb{C} over \mathbb{R} . In fact it is a consequence of a theorem of Artin and Schreier that C_2 is the only non-trivial finite group that can occur as the absolute Galois group, namely iff the field is real closed. In this case it is automatically of characteristic 0. But we are in any case more interested in the methods than the exact group G . It will be our standing assumption that the absolute Galois group of k is finite until stated otherwise.

We consider the ∞ -category $\mathcal{S}^G = \text{Fun}(BG, \mathcal{S})$ which is given by 'naive' G -spaces. One explicit model for $\text{Fun}(BG, \mathcal{S})$ is given by $\text{N}(\text{sSet}^G)[W^{-1}]$ where sSet^G denotes the category of simplicial sets equipped with an action by G and W denotes the underlying weak equivalences. Thus we will first try to understand the pure algebra.

Proposition 4.40. *There is a left adjoint functor $\text{Set}^G \rightarrow \text{coCAlg}_k$ which sends a G -set X to the coalgebra*

$$\left(\bar{k}^{\vee} [X] \right)_G =: k^{\sigma} [X]$$

obtained as the coinvariants of the G -action obtained by acting on \bar{k} and X (recall that colimits of coalgebras are formed underlyingly). The right adjoint $\text{coCAlg}_k \rightarrow \text{Set}^G$ is given by sending a coalgebra C to the set

$$\text{Hom}_{\text{coCAlg}_k} \left(\bar{k}^{\vee}, C \right) \cong \text{Hom}_{\text{coCAlg}_{\bar{k}}} \left(\bar{k}, C \otimes \bar{k} \right) \cong (C \otimes \bar{k})^{\text{gp}}$$

of grouplike elements over \bar{k} equipped with its canonical G -action.

Proof. The first half of the statement is clear since the given functor obviously preserves colimits. Then also the first given form of the right adjoint is clear by adjunction since the left adjoint sends the generator G to \bar{k}^\vee . The only statement that requires proof is the last isomorphism. At the level of vector spaces this is clear since

$$\mathrm{Hom}_k(\bar{k}^\vee, C) \cong \mathrm{Hom}_k(k, C \otimes \bar{k}) \cong \mathrm{Hom}_{\bar{k}}(\bar{k}, C \otimes \bar{k})$$

which follows from finite dimensionality of k . Then it remains to check that under this isomorphisms maps of coalgebras correspond to maps of coalgebras which is left as an exercise. \square

Example 4.41. *The G -set G with left multiplication gets mapped to \bar{k}^\vee . Since this is a generator of Set^G this already determines the whole adjunction. On the other hand the trivial G -set pt gets mapped to k as a k -coalgebra. The transitive cosets G/H get mapped to all intermediate field extension $(\bar{k}^H)^\vee$ as a result of the Galois correspondence.*

Corollary 4.42. *The functor $k^\sigma[-] : \mathrm{Set}^G \rightarrow \mathrm{coCAlg}_k$ is fully faithful.*

Proof. We have to understand the unit of the adjunction which is the morphism

$$X \rightarrow \mathrm{Hom}_{\mathrm{coCAlg}_{\bar{k}}}(\bar{k}, k^\sigma[X] \otimes_k \bar{k})$$

We get an isomorphism

$$\left(\bar{k}^\vee[X]/G\right) \otimes_k \bar{k} \cong \left(\bar{k}^\vee[X] \otimes_k \bar{k}\right) / G \cong \left((\bar{k} \otimes_k \bar{k})^\vee[X]\right) / G \cong \left(\bigoplus_G \bar{k}[X]\right) / G \cong \bar{k}[X]$$

under which the counit corresponds to the counit for the field \bar{k} . We have seen earlier that this is an isomorphism. Note that we have seen in reality that the composite functor $\mathrm{Set}^G \rightarrow \mathrm{coCAlg}_k \rightarrow \mathrm{coCAlg}_{\bar{k}}$ is equivalent to the functor $\mathrm{Set}^G \rightarrow \mathrm{Set} \rightarrow \mathrm{coCAlg}_{\bar{k}}$. \square

Now we can give a slightly more conceptual proof of Lemma 4.39 as follows: there is a factorization

$$k[-] : \mathrm{Set} \xrightarrow{\mathrm{triv}} \mathrm{Set}^G \xrightarrow{k^\sigma[-]} \mathrm{coCAlg}_k$$

where the left hand functor is given by considering a set as a G -set with trivial action. The right adjoint to the functor triv is the fixed point functor, thus the claim follows by composition of right adjoints, namely that $C^{\mathrm{gp}} \cong \mathrm{Hom}_{\mathrm{coCAlg}_k}(\bar{k}, C \otimes_k \bar{k})^G$.

There is an obvious simplicial version

$$C_*^\sigma(-, k) : \mathrm{sSet}^G \rightarrow \mathrm{scoCAlg}_k$$

obtained by applying $k^\sigma[-]$ levelwise.

Remark 4.43. *There is a dual version of this functor for cohomology which might be a bit more suggestive. It is given by*

$$C_\sigma^*(X, k) = C_*^\sigma(X, k)^\vee = C^*(X, \bar{k})^G$$

In order to study the homotopical properties of the functor $C_*^\sigma(-, k) : \mathrm{sSet}^G \rightarrow \mathrm{scoCAlg}_k$ we say that a morphism of G -spaces (resp. G -simplicial sets) $X \rightarrow Y$ is a k -equivalence if the underlying morphism is a k -equivalence.

Proposition 4.44. *The notion of k -equivalence defines a Bousfield localization of \mathcal{S}^G . For a space X with G -action the Bousfield localization $X \rightarrow X_k$ in \mathcal{S}^G is given by taking the Bousfield k -localization of the underlying spaces $X \rightarrow X_k$ and equipping it with the induced G -action.*

Proof. This follows from the more general fact that for a functor category into a Bousfield localization $L : \mathcal{C} \rightarrow \mathcal{C}^0 \rightarrow \mathcal{C}$ there is an induced Bousfield localization $\text{Fun}(I, \mathcal{C}) \rightarrow \text{Fun}(I, \mathcal{C}^0)$ so that the local equivalences are the pointwise equivalences. The easiest way to see this is the note that the endofunctor $L_* : \text{Fun}(I, \mathcal{C}) \rightarrow \text{Fun}(I, \mathcal{C})$ is clearly idempotent since L is. Thus it corresponds to a localization. The equivalence as well as the local objects are pointwise. \square

Lemma 4.45. *The functor $C_*^\sigma : \text{sSet}^G \rightarrow \text{scoCAlg}_k$ sends k -equivalences to underlying quasi isomorphisms of simplicial coalgebras. More generally the underlying chain complex of $C_*^\sigma(X, k)$ is naturally isomorphic to $C_*(X, k)$.*

Proof. We have that as a G -module $\bar{k}^\vee \cong k[G]$ and thus $\bar{k}^\vee[X] \cong k[G \times X]$ as a G -module. Thus we have a natural isomorphism $k^\sigma[X] \cong k[G \times X]/G \cong k[X]$ as a k -vector space. This implies the second claim and the first is an immediate consequence. \square

Note that this statement really shows that we could also have taken homotopy orbits as opposed to actual orbits. That is how we should think about it: $C_*^\sigma(X, k) \simeq C_*(X, \bar{k}^\vee)_{hG}$. Also note that while $C_*^\sigma(X, k)$ as a vector space does not depend on the G -action on X at all the coalgebra structure does since e.g. $C_*^\sigma(G_t, k) \cong \bar{k}^\vee$ for G as a left G -set but $C_*^\sigma(G_t, k) \cong C_*^\sigma(G_t, k) = k[G]$ for the trivial action of G on G .

Theorem 4.46. *The functor*

$$C_*^\sigma : \mathcal{S}_k^G \rightarrow \text{coCAlg}_k^\infty$$

is fully faithful. For a space X with G -action the morphism $X \rightarrow \text{Map}_{\text{coCAlg}_k^\infty}(\bar{k}^\vee, C_^\sigma(X, k))$ exhibits the target as the k -localization of X (in the category \mathcal{S}^G).*

Proof. The proof is literally the same as the proof of Goerss theorem. We claim that the 1-categorical adjunction descends to an ∞ -categorical adjunction which is then automatically also fully faithful. We have just seen that the left adjoint preserves weak equivalences. The right adjoint is as described above given by sending C_\bullet to $(C_\bullet \otimes_k \bar{k})^{\text{gp}}$ which preserves weak equivalences since base changing from k to \bar{k} does and since taking group like elements does send quasi-isomorphism to \bar{k} -local equivalences. But these are the same as the k -local equivalences. \square

Corollary 4.47. *For a space X (without G -action which we consider as trivial G -action) the map $X \rightarrow \text{Map}_{\text{coCAlg}_k^\infty}(k, C_*(X, k))$ is equivalent to the map $X \rightarrow \text{Map}_{\mathcal{S}}(BG, X_k)$ where X_k is the k -localization of X .*

Proof. By Theorem 4.46 we get that

$$\text{Map}_{\text{coCAlg}_k^\infty}(k, C_*(X, k)) \simeq \text{Map}_{(\mathcal{S}^G)_k}(\text{pt}, X) \simeq \text{Map}_{\mathcal{S}^G}(\text{pt}, X_k) \simeq (X_k)^{hG}$$

which agrees with the given formula since the action is trivial. \square

Note that the last corollary is far from optimal since the next one always applies under our assumption of the finiteness of the absolute Galois group.

Corollary 4.48. *Let $k = \mathbb{R}$ (or in fact every real closed field). Then the functor*

$$C_*(-, \mathbb{R}) : \mathcal{S}_{\mathbb{Q}} \rightarrow \text{coCAlg}_{\mathbb{R}}^{\infty}$$

is fully faithful.

Proof. The counit of the adjunction is given by the map $X \rightarrow \text{Map}(BG, X_{\mathbb{Q}}) \simeq X_{\mathbb{Q}}$. \square

Now we want to turn to the usual case where G is not finite. Then G is among other things a profinite group, i.e. a formal inverse limit of finite groups. Moreover \bar{k} is a discrete G -module which can be expressed in a couple of equivalent ways. More generally for a G -set X the following are equivalent:

- When we equip X with the discrete topology and G with the profinite topology then the action is continuous
- All orbits are finite with closed stabilizer group.
- All stabiliser subgroups are open.
- We have that $X = \varinjlim_{G_{\alpha}} X^{G_{\alpha}}$ where the colimit is taken over open subgroups $G_{\alpha} \subseteq G$ (necessarily of finite index) ordered by the superset relation.

The equivalence between these conditions is easily seen. A key fact to use is that a subgroup $H \subseteq G$ is open iff it is closed and of finite index. This is a consequence of the profinite topology. Similar descriptions apply to modules, rings etc. with group action.

There is an obvious category of discrete G -sets. Let us give some equivalent descriptions of this category.

1. Let Fin^G the category of finite, discrete G -sets. Call a family $\{S_i \rightarrow S\}_{i \in I}$ of such a cover if it is jointly surjective. Then consider $\text{Shv}_{\text{Set}}(\text{Fin}^G)$.
2. Consider $\text{Orb}_G^{\text{fin}} := \text{Fin}_{\text{trans}}^G \subseteq \text{Fin}^G$ and endow it with the Grothendieck topology consisting of all maps (clearly all maps are surjective). Then consider $\text{Shv}_{\text{Set}}(\text{Orb}_G^{\text{fin}})$.
3. Consider the (finite) étale site $\text{Spec}(k)_{\text{ét}}$. Then consider $\text{Shv}_{\text{Set}}(\text{Spec}(k)_{\text{ét}})$. Note that every profinite group arises as a Galois group of a field but not necessarily as the absolute Galois group.
4. Write $G \cong \varprojlim G_i$ as a cofiltered limit of finite groups (which is possible since we have a profinite group). One can always take $G \cong \varprojlim_{U \subseteq G \text{ open, normal}} G/U$. Then consider the limit $\varprojlim \text{Set}^{G_i}$ of categories where the transition morphisms $\text{Set}^{G_i} \rightarrow \text{Set}^{G_j}$ for $f : G_i \rightarrow G_j$ are given by the right adjoint $f_* : \text{Set}^{G_i} \rightarrow \text{Set}^{G_j}$. This right adjoint is given by taking the K_{ij} -fixed points where $K_{ij} \subseteq G_i$ is the kernel of f (at least for surjective f and we can without loss of generality assume that all maps are surjective).

Proposition 4.49. *All these categories are equivalent to the category of discrete G -sets.*

Proof. Let us first prove the equivalence between the categories (1)-(4). For (1) and (3) we claim that the sites are actually equivalent. To see this note that every scheme étale over $\text{Spec}(k)$ is of the form $\coprod \text{Spec}(k_i)$ for finite field extensions k_i of k . Really we should say finite, separable field extensions, but this is automatic since we assumed that k is perfect. Now we define a coproduct preserving functor $\text{Fin}^G \rightarrow X_{\text{ét}}$ which sends G/H to \overline{k}^H . This is essentially surjective by what we have just said and clearly fully faithful. It is also easy to check that covers exactly correspond to jointly surjective maps. The equivalence between (1) and (2) is clear since the descent property implies that coproducts of orbits are sent to products. Then the only descent that is left is ‘Galois descent’. Now finally to see that (2) and (4) are equivalent we just use that the limit in categories is formed by taking sequences of objects together with isomorphisms upon applying the transition functors (THE WAY IT IS DONE HERE ONLY WORKS FOR ABELIAN GROUPS OTHERWISE ONE HAS TO DEAL WITH THE FACT THAT NOT EVERY SUBGROUP IS NORMAL).

It remains to understand the equivalence to the category of discrete G -sets. Therefore we first have to describe a functor $\text{Shv}(\text{Spec}(k)_{\text{ét}}) \rightarrow \text{Set}^G$. This functor is given by taking the ‘stalk’ at $\text{Spec}(\overline{k})$. The concrete formula is

$$F_{\text{Spec}(\overline{k})} = \text{colim}_{K \subseteq \overline{k}, \text{ finite over } k} F(\text{Spec}(K))$$

We can also directly describe the functor $\text{Shv}_{\text{Set}}(\text{Orb}_G^{\text{fin}}) \rightarrow \text{Set}^G$ as $F \mapsto \text{colim}_{U \subseteq G \text{ open}} F(G/U)$. Then it is immediate that it carries a G -action and that it is discrete (using the last of our equivalent conditions). Then the inverse functor $\text{Set}^G \rightarrow \text{Shv}_{\text{Set}}(\text{Orb}_G^{\text{fin}})$ is given by sending a discrete G -set X to the functor which sends $G/U \rightarrow X^U$. It is straightforward to check that these two functors are inverse to each other. \square

Note that the proof of the equivalence between (1) to (4) actually works for sheaves with values in any ∞ -category in place of the category of sets (if one is careful in defining the functors).

Remark 4.50. *The functor*

$$\text{Shv}_{\text{Set}}(\text{Orb}_G^{\text{fin}}) \rightarrow \text{Set} \quad F \mapsto F_{G/\text{pt}} = \varinjlim_{U \subseteq G \text{ open}} F(G/U)$$

has the following properties:

- *It detects equivalences. This is a consequence of the fact that it factors as $\text{Shv}_{\text{Set}}(\text{Orb}_G^{\text{fin}}) \xrightarrow{\sim} \text{Set}^G \xrightarrow{\text{forget}} \text{Set}$. Here the second functor does as is easily checked.*
- *It is a topos point, i.e. it preserves finite limits and all colimits. For the colimits part we have to verify that the functor $\text{PSh}_{\text{Set}}(\text{Orb}_G^{\text{fin}}) \rightarrow \text{Set}$ descends to the Bousfield localization, i.e. that it sends covers of G/U to colimits. But we find that $(G/U)_G \cong G/U$ which makes this clear. Alternatively one can also easily see that the forgetful functor $\text{Set}^G \rightarrow \text{Set}$ commutes with colimits. The finite limits part follows since the indexing category of the stalk is filtered or again by looking at the forgetful functor. Note that it does not commute with arbitrary limits, e.g. $G = \varprojlim G/N$ is not discrete!*

Together this shows that the topos has ‘enough points’.

Now we want to pass to other settings like non-concrete categories or ∞ -categories where it does a priori not make sense to talk about ‘discrete’ G -sets. The idea is to use the equivalent descriptions of a G -action given in Proposition 4.49 which also make sense for sheaves with values in an arbitrary category or ∞ -category. However there is a catch that does not allow to do this completely naively. This has to do with sheaves versus hypersheaves.

4.3.1 Digression: Hypercompletion

Let \mathcal{X} be an ∞ -topos. Then there is a truncation functor $\tau_{\leq 0} : \mathcal{X} \rightarrow \tau_{\leq 0}\mathcal{X}$. We can also consider the morphism $X^{S^n} \rightarrow X$ induced by evaluation at the basepoint $\text{pt} \rightarrow S^n$. This makes \mathcal{X}^{S^n} an object of the slice topos \mathcal{X}/X and we define the n -th homotopy sheaf of X to be

$$\pi_n(X) := \tau_{\leq n}(X^{S^n}) \in \tau_{\leq 0}(\mathcal{X}/X).$$

Example 4.51. Let $\mathcal{X} = \mathcal{S}$ be the ∞ -topos of spaces. For a space X we have that

$$\tau_{\leq 0}(\mathcal{S}/X) \simeq \tau_{\leq 0}(\text{Fun}(X, \mathcal{S})) \simeq \text{NFun}(\tau_{\leq 1}X, \text{Set}).$$

Under the equivalence $\mathcal{S}/X \simeq \text{Fun}(X, \mathcal{S})$ the object $X^{S^n} \rightarrow X$ corresponds to the functor which sends $x \in X$ to $\text{Map}_*((S^n, \text{pt}), (X, x))$. Thus we get that

$$\pi_n(X) : \tau_{\leq 1}X \rightarrow \text{Set} \quad x \mapsto \pi_n(X, x).$$

Note that π_0 is in this example and in general a constant functor. One can also easily show that the higher homotopy groups are group objects using that $\tau_{\leq 0}$ commutes with finite products.

Example 4.52. Let $\mathcal{X} = \text{Shv}(\text{NC})$ be a sheaf ∞ -topos for \mathcal{C} a Grothendieck site \mathcal{C} . Then the internal homotopy groups can be described as follows: for a sheaf $F : \text{NC}^{\text{op}} \rightarrow \mathcal{S}$ we have that $\pi_0(F) \in \text{Shv}_{\text{Set}}(\mathcal{C})$ (or rather its pullback to the terminal object) is the sheafification of the presheaf $\text{NC}^{\text{op}} \rightarrow \mathcal{S} \xrightarrow{\pi_0} \text{NSet}$. For the higher homotopy groups consider the objects $c \in \mathcal{C}$ as basepoints. By this we mean that we study the pullback of $\pi_n(F)$ along the morphism $\text{Shv}(\text{NC}/c) \simeq \mathcal{X}/c \rightarrow \mathcal{X}/X$ for every morphism $c \rightarrow X$. We get that $\pi_n(F, c)$ is the sheafification of the presheaf $(\text{NC}/c)^{\text{op}} \xrightarrow{F} \mathcal{S}_* \xrightarrow{\pi_n} \text{Set}$.

A morphism $f : F \rightarrow G$ in \mathcal{X} is called ∞ -connected if it induces isomorphisms on all homotopy sheaves. By this we mean that it induces an isomorphism between the homotopy sheaves of X and the pullbacks of the homotopy sheaves of Y (QUESTION: Do we need to assume that it is additionally an effective epimorphism?). The following is an equivalent description: a morphism is ∞ -connected if all the truncations $\tau_{\leq n}X \rightarrow \tau_{\leq n}Y$ are equivalences. Since Postnikov towers do not converge in a general ∞ -topos this does not necessarily imply that it is an equivalence.

Definition 4.53. Let \mathcal{X} be an ∞ -topos. An object $F \in \mathcal{X}$ is called hypercomplete if it local with respect to the ∞ -connected morphisms. We say that \mathcal{X} is hypercomplete if every object is hypercomplete or equivalently every ∞ -connected morphism is an equivalence, i.e. Whitehead's theorem holds.

Example 4.54. The ∞ -topos \mathcal{S} is hypercomplete.

The following is essentially shown in [Lur09, Section 6.5.3]. Recall that an augmented simplicial object $U^\bullet \rightarrow M$ in a site \mathcal{C} is called hypercovering if for each $n \geq 0$ the morphisms

$$U_n \rightarrow \text{cosk}_{n-1}(U_\bullet)$$

is a covering. Here cosk is taken over M , i.e. in the slice category \mathcal{C}/M . For example for a hypercover we get that $U_0 \rightarrow M$ is a covering, $U_1 \rightarrow U_0 \times_M U_0$ is a covering etc. For convenience we have assumed that disjoint unions and pullbacks along covers in the site exist. Otherwise one has to use sieves or directly work with effective epimorphisms in the associated topos.

Proposition 4.55. *A sheaf F is hypercomplete precisely if it is a hypersheaf in the sense that for every hypercovering U^\bullet of M we have a limit*

$$F(M) \simeq \lim_{\Delta} F(U^\bullet)$$

Note that for ordinary sheaves it is equivalent to have hyperdescent or descent. It only differs in the ∞ -categorical setting.

Example 4.56. *Assume that \mathcal{C} is the nerve of an ordinary site which has enough points or more generally the ∞ -topos associated to a 1-topos with enough points.. Then the stalks of the homotopy sheaves are exactly the homotopy groups of the stalks (this follows since geometric morphisms commute with truncations and finite limits). Thus a morphism is ∞ -connected iff it induces an equivalence on all stalks. In particular the ∞ -topos has enough points. Vice versa if an ∞ -topos has enough points then it is automatically hypercomplete since \mathcal{S} is.*

Example 4.57. *Consider the profinite group $G = \mathbb{Z}^\wedge \cong \text{Gal}(\overline{\mathbb{F}}_p, \mathbb{F}_p)$. Then we claim that the ∞ -topos $\text{Shv}(\text{NFin}^G) \simeq \text{Shv}(\text{NOrb}_G^{\text{fin}}) \simeq \text{Shv}(\text{NSpec}(\mathbb{F}_p)_{\text{ét}})$ is not hypercomplete.*

To see this consider a finite, connected nontrivial CW complex M all of whose homotopy groups are torsion. For example take a Moore space $M(\mathbb{Z}/n\mathbb{Z}, k)$ for some $n > 1$ and $k > 0$. Consider the functor $F : \text{NOrb}_G^{\text{fin}} \rightarrow \mathcal{S}$ which is constant with value M . We claim that F is a sheaf, i.e. belongs to $\text{Shv}(\text{NOrb}_G^{\text{fin}})$. To see this first note that for every finite group H acting trivial on M we get that the diagonal inclusion $M \rightarrow M^{hH} = \text{Map}(\text{BH}, M)$ is a homotopy equivalence by the Sullivan conjecture (as proven by Miller). This immediately implies that the sheaf condition is satisfied.

On the other hand consider the functor $F' : \text{NOrb}_G^{\text{fin}} \rightarrow \mathcal{S}$ given by

$$\mathbb{Z}/n\mathbb{Z} \mapsto \text{Map}(\mathbb{R}/n\mathbb{Z}, M) \simeq LM$$

Then F' is also a sheaf since the functor $\text{Orb}_G^{\text{fin}} \mapsto \mathcal{S}$ given by $\mathbb{Z}/n\mathbb{Z} \mapsto \mathbb{R}/n\mathbb{Z} \simeq S^1$ sends quotients to quotients. In other words the homotopy fixed points for the action of $\mathbb{Z}/k\mathbb{Z}$ on the free loop space LM are equivalent to the free loop space again.

Now there is an obvious map $F \rightarrow F'$ which is the inclusion of constant loops. We claim that this morphism is ∞ -connected. By what we have said before we can test this at the stalk at G , i.e. it suffices to check that the morphism

$$M \simeq \varinjlim_U F(G/U) \rightarrow F'(G/U) = \varinjlim \text{Map}(\mathbb{R}/n\mathbb{Z}, M) \simeq \varinjlim LM$$

The claim that this is a weak equivalence. To see this we identify the homotopy groups of the target, i.e. of LM with $\pi_*(X) \oplus \pi_{*+1}(M)$ such that the map $M \rightarrow LM$ acts as the inclusion $\pi_*(M) \rightarrow \pi_*(M) \oplus \pi_*(LM)$ (this can be done using the natural split inclusion). Then we get that

$$\pi_*(\varinjlim \text{Map}(\mathbb{R}/n\mathbb{Z}, M)) \cong \pi_*(M) \oplus \varinjlim \pi_{*+1}(LM)$$

where the colimit is taken over the divisibility poset of \mathbb{N} . The transition maps in this poset act by multiplication with any natural number. Thus since all homotopy groups of M are torsion the colimit vanishes. This shows that the map $F \rightarrow F'$ is ∞ -connected. But it is certainly not an equivalence, since the Loop space LM is not weakly equivalent to M unless M is discrete which was excluded.

This example is a variant of [Lur09, Warning 7.2.2.31] and is attributed there to Wieland.

The following statement is proven in [Lur09, Section 6.5.2].

Proposition 4.58. *For every ∞ -topos \mathcal{X} the full subcategory of hypercomplete objects $\mathcal{X}_{\text{hyper}}^\wedge \subseteq \mathcal{X}$ is a subcategory of local objects for a left exact, accessible localization of \mathcal{X} . In particular it is itself an ∞ -topos.*

Definition 4.59. Let G be a profinite group. We define the ∞ -category \mathcal{S}^G to be the hypercomplete ∞ -topos $\text{HyShv}(\text{NOrb}_G^{\text{fin}}) \simeq \text{HyShv}(\text{NFin}^G)$.

Clearly as before, if G is the absolute Galois group of a field k then this is equivalent to $\text{HyShv}(\text{NSpec}(k)_{\text{ét}})$.

Example 4.60. *If G is finite then this is equivalent to $\mathcal{S}^G \simeq \text{Fun}(\text{BG}, \mathcal{S})$.*

Also from the general theory we see that there is a geometric morphism $\mathcal{S}^G \rightarrow \mathcal{S}$ given by taking the stalk at G . This geometric morphism is conservative. In fact it maps to the naive category of G -spaces given by $\text{Fun}(\text{BG}, \mathcal{S})$. Thus we can think of \mathcal{S}^G as some sort of special G -spaces. QUESTION: Is the functor $\mathcal{S}^G \rightarrow \text{Fun}(\text{BG}, \mathcal{S})$ fully faithful, most probably not?

Proposition 4.61. *There is an equivalence*

$$\text{NsSet}^G[W^{-1}] \simeq \mathcal{S}^G$$

where the left hand site denotes simplicial objects in discrete G -sets and W denotes the underlying weak equivalence, i.e. after forgetting the G -action.

Proof. The category sSet^G is equivalent to simplicial sheaves on $\text{Orb}_G^{\text{fin}}$. The underlying weak equivalences correspond under this equivalence to the stalkwise equivalences. This is the Joyal model structure. The claim thus is equivalent to stating that the Joyal model structure models the hypercomplete ∞ -topos which is well known. \square

Now we want to discuss Bousfield-localizations of the category of hypersheaves.

Definition 4.62. Consider the ∞ -category $\text{Shv}(\text{NOrb}_G^{\text{fin}})$ and a spectrum E . We call a morphism $F \rightarrow G$ an E -local equivalence if the morphism on stalks $F_G \rightarrow G_G$ is a E -local equivalence.

Proposition 4.63. *The notion of E -local equivalence defines a Bousfield localization of $\text{PSh}(\text{NOrb}_G^{\text{fin}})$ (and also of $\text{Shv}(\text{NOrb}_G^{\text{fin}})$ and $\text{HyShv}(\text{NOrb}_G^{\text{fin}}) \rightarrow \mathcal{S}$).*

Proof. The stalk functor $\text{PSh}(\text{NOrb}_G^{\text{fin}}) \rightarrow \text{Shv}(\text{NOrb}_G^{\text{fin}}) \rightarrow \text{HyShv}(\text{NOrb}_G^{\text{fin}}) \rightarrow \mathcal{S}$ is a left adjoint functor. Thus this follows from the left adjoint condition in Theorem 2.25. \square

Now we try to understand the E -local replacement.

Proposition 4.64. *The ∞ -category of E -local objects in \mathcal{S}^G is equivalent to the localization of the ∞ -category $(\mathcal{S}_E)^G$ at the stalkwise equivalences, i.e. under the stalk functor*

$$(\mathcal{S}_E)^G \rightarrow \mathcal{S}_E.$$

In particular a presheaf $F : (\text{NOrb}_G^{\text{fin}})^{\text{op}} \rightarrow \mathcal{S}$ that is E -local is levelwise k -local and a hypersheaf.

Proof. We consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{HyShv}_{\mathcal{S}_E}(\mathrm{NOrb}_G^{\mathrm{fin}}) & \xleftarrow{L} & \mathrm{HyShv}_{\mathcal{S}}(\mathrm{NOrb}_G^{\mathrm{fin}}) \\ \downarrow (-)_G & & \downarrow (-)_G \\ \mathcal{S}_E & \xleftarrow{L} & \mathcal{S} \end{array}$$

which commutes by the way colimits are computed in \mathcal{S}_E . Thus it follows that E -local equivalences can be tested on the associated stalk of the localization. This implies the claim. \square

Lemma 4.65. *Assume that \mathcal{D} is a compactly generated ∞ -category (i.e. presentable and ω -accessible). Then the stalk functor*

$$\mathrm{HyShv}_{\mathcal{D}}(\mathrm{NOrb}_G^{\mathrm{fin}}) \rightarrow \mathcal{D}$$

is conservative.

Proof. Let $(d_i)_{i \in I}$ be a set of compact generators. Then the family of functors $\mathrm{Map}(d_i, -) : \mathcal{D} \rightarrow \mathcal{S}$ is conservative. Thus we consider the induced diagrams

$$\begin{array}{ccc} \mathrm{HyShv}_{\mathcal{D}}(\mathrm{NOrb}_G^{\mathrm{fin}}) & \xrightarrow{\mathrm{Map}(d_i, -)_*} & \mathrm{HyShv}_{\mathcal{S}}(\mathrm{NOrb}_G^{\mathrm{fin}}) \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow{\mathrm{Map}(d_i, -)} & \mathcal{S} \end{array}$$

which commutes by the fact that $\mathrm{Map}(d_i, -)$ commutes with filtered colimits, thus with the stalk-functor. Now we use that the stalk functor for the category of spaces is conservative. It follows that the stalk functor of \mathcal{D} is conservative. \square

The last lemma makes sure that the ‘hypercompletion’ is already local with respect to stalkwise equivalences. We would like to apply this to the ∞ -category $\mathcal{S}_{H\mathbb{Q}}$ of rational spaces. Unfortunately we do not know that this is compactly generated.

Question 4.66. Is the ∞ -category of rational spaces compactly generated? If this is the case then probably the point is a compact generator. Thus it would be implied if the following is true: is a filtered colimit of rational spaces again rational?

Now let me give some indication why the answer could be ‘yes’: first a simply connected (or more generally nilpotent) space is rational if and only if its homotopy groups are uniquely divisible. This property is obviously closed under filtered colimits. Thus a filtered colimit of simply connected, rational spaces is again rational. Bousfield characterizes rationality for arbitrary spaces as a property of homotopy groups, but this property is also homotopy theoretic in nature so that I am unable to decide if it is preserved by filtered colimits.

The second indication is that one can study the stable analogue of this question: is a filtered colimit of rational spectra again rational. The answer is yes. This is in fact equivalent to the fact that rationalization is a smashing localization. This observation also shows that the class of \mathbb{S}/p -local spectra for a prime p cannot be closed under filtered colimits since p -completion is not smashing. A consequence is that \mathbb{F}_p -local spaces (a.k.a. $\mathbb{S}/p = M(\mathbb{Z}/p)$ -local spaces) are not closed under filtered colimits (recall that the functor Ω^∞ preserves filtered colimit as well as \mathbb{S}/p -localizations).

Proposition 4.67. *Let X be a nilpotent connected, \mathbb{Q} -local space and $G = \text{Gal}(\bar{\mathbb{Q}}, \mathbb{Q})$. Then the constant functor*

$$\mathbf{N}(\text{Orb}_G^{\text{fin}})^{\text{op}} \rightarrow \mathcal{S} \quad G/U \mapsto X$$

is \mathbb{Q} -local.

Proof. It is clear that the constant functor is a sheaf (since rationally homotopy fixed points for a trivial action of a finite group are trivial) and that it is levelwise \mathbb{Q} -local. Thus it only remains to show that it is a hypersheaf. To see this we write X as the limit $\tau_{\leq n} X$. Then all the truncations are also \mathbb{Q} -local (as we have seen before). Then the constant sheaf on X is the limit of the constant sheaves on $\tau_{\leq n} X$. Thus it follows that the constant sheaf on X is a limit of truncated sheaves. The truncated sheaves are all hypercomplete, thus also X is. Need finally an argument about how it can be written as a limit. \square

Question: What about the fixed points in $(\mathcal{S}^{\mathbb{Z}^\wedge})^{\text{BZ}}$

4.3.2 Back to Coalgebras

Note that it is important to note that by definition the ∞ -category \mathcal{S}^G is generated by finite G -orbits which can equivalently be considered as $\text{Spec}(K)$ for a finite field extension K over k . The same applies of course to the category Set^G of discrete G -sets. Now we can start to generalize Theorem 4.46 to the setting of non-finite Galois group. Thus let's recall that G is the profinite absolute Galois group of k with fixed algebraic closure \bar{k} .

Proposition 4.68. *There is a left adjoint functor $k^\sigma[-]: \text{Set}^G \rightarrow \text{coCAlg}_k$ which sends a finite G -orbit G/U (for $U \subseteq G$ open) to the coalgebra $(\bar{k}^U)^\vee$. The right adjoint $\text{coCAlg}_k \rightarrow \text{Set}^G$ is given by sending a coalgebra C to the set*

$$\varinjlim_U \text{Hom}_{\text{coCAlg}_k} \left((\bar{k}^U)^\vee, C \right) \cong \text{Hom}_{\text{coCAlg}_{\bar{k}}} (\bar{k}, C \otimes \bar{k}) \cong (C \otimes \bar{k})^{\text{gp}}$$

of grouplike elements over \bar{k} equipped with its canonical discrete G -action.

Proof. By the presentation $\text{Set}^G \simeq \text{Shv}(\text{NOrb}_G^{\text{fin}})$ to construct the left adjoint it suffices to provide a functor

$$\text{NOrb}_G^{\text{fin}} \rightarrow \text{coCAlg}_k$$

and check that it satisfies 'codescent'. Such a functor is given by the assignment $G/U \mapsto (\bar{k}^U)^\vee$ and it satisfies codescent since $G/U \mapsto \bar{k}^U$ satisfies descent and everything is finite dimensional.

The right adjoint RC for a coalgebra C is now as a sheaf given by

$$G/U \mapsto \text{Hom}_{\text{coCAlg}_k} \left((\bar{k}^U)^\vee, C \right)$$

which in turn is (by an analysis as in Proposition 4.40) isomorphic to

$$\text{Hom}_{\text{coCAlg}_{\bar{k}^U}} \left(\bar{k}^U, C \otimes_k \bar{k}^U \right) \cong \left(C \otimes \bar{k}^U \right)^{\text{gp}}$$

Now passing to the colimit we get the claimed result. \square

Lemma 4.69. *The underlying vector space of $k^\sigma[X]$ for a discrete G -set X is naturally isomorphic to the underlying vector space of $k[X]$.*

Proof. The more precise claim is that we have a commutative diagram of colimit preserving functors

$$\begin{array}{ccc} \text{Set}^G & \longrightarrow & \text{coCAlg}_k \\ \downarrow & & \downarrow \\ \text{Set} & \xrightarrow{k[-]} & \text{Vect}_k \end{array}$$

This can be checked on generators where we have a natural equivalence

$$k^\sigma[G/U] \cong (\bar{k}^U)^\vee \cong ((k[G])^U)^\vee \cong k[G/U].$$

□

Lemma 4.70. *The composite functor*

$$\text{Set}^G \xrightarrow{k^\sigma[-]} \text{coCAlg}_k \xrightarrow{\otimes_{\bar{k}}} \text{coCAlg}_{\bar{k}}$$

is equivalent to the functor

$$\text{Set}^G \rightarrow \text{Set} \xrightarrow{\bar{k}[-]} \text{coCAlg}_{\bar{k}}.$$

Proof. Check again on generators. This is similar to Corollary 4.42. □

Lemma 4.71. *The functor $k^\sigma[-]: \text{Set}^G \rightarrow \text{coCAlg}_k$ is fully faithful.*

Proof. We have to understand the unit of the adjunction

$$X \rightarrow (k^\sigma[X] \otimes_k \bar{k})^{\text{gp}}$$

By the last lemma the right hand side is given by $\bar{k}[X]^{\text{gp}}$ in which case we have seen earlier that the canonical morphism is an isomorphism. □

Finally we again get a twisted chains functor

$$C_*^\sigma(-, k) : \text{sSet}^G \rightarrow \text{scoCAlg}_k$$

Remark 4.72. *Again there is a slightly easier description of the dual functor*

$$(C_*^\sigma(X, k))^\vee \cong C^*(X, \bar{k})^G =: C_\sigma^*(X, k)$$

which can be seen by observing that both sides send colimits of discrete G -spaces to limits. Therefore we can reduce it to finite orbits in which case it is clear by definition.

Theorem 4.73. *The functor $C_*^\sigma : \text{sSet}^G \rightarrow \text{scoCAlg}_k$ sends k -equivalences to quasi isomorphisms of simplicial coalgebras and thus induces a functor*

$$\mathcal{S}_k^G \rightarrow \text{scoCAlg}_k^\infty.$$

This functor is fully faithful.

Proof. The first part immediately follows from Lemma 4.69 since the equivalence only depend on the chains of the underlying simplicial set. For the second part we want to invoke Lemma 4.71. Therefore it suffices to show that the right adjoint functor sends quasi-isos to k -local equivalences. Since this is the same as \bar{k} -local equivalences and follows as before. □

Corollary 4.74. *The functor*

$$C_* : \mathcal{S}_{\mathbb{Q}} \rightarrow \text{scoCAlg}_{\mathbb{Q}}^\infty$$

is fully faithful when restricted to nilpotent rational spaces.

Proof. Study the counit of the adjunction. □

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