The group completion theorem via localizations of ring spectra

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Abstract

In this short expository note we give a proof of the group completion theorem (see [MS76]) using localizations of ring spectra. We also relate the group completion of a topological monoid to the plus construction. Most of the results are well-known but we hope that our treatment can shed some new light on some aspects.

Let R be a multiplicative generalized homology theory represented by an $\mathbb{E}_1$-ring spectrum (also called R) and M an $\mathbb{E}_1$-space.[1] We consider the multiplicatively closed subset $\pi_R \subseteq R_0(M)$ which is the image of the Hurewicz homomorphism $\pi_0(M) \to R_0(M)$.

Theorem 1. *Assume that π_R satisfies the left Ore condition[2] in the graded ring $R_*(M)$ e.g. if it lies in the center.*

1. *The canonical map $M \to \Omega BM$ induces a ring isomorphism*

$$\pi_R^{-1} R_*(M) \to R_*(\Omega BM).$$

2. *If E is a left R-module spectrum then we have an isomorphism*

$$\pi_R^{-1} E_*(M) \to E_*(\Omega BM)$$

of $R_*(M)$-modules.

Example 2. *Assume that $R = H\mathbb{Z}$ is ordinary homology and $\pi = \pi_{H\mathbb{Z}} = \pi_0(M)$ is central in $H_*(M, \mathbb{Z})$. Then our first result reduces to the classical group completion theorem of Segal-McDuff, namely that we have an isomorphism of rings*

$$\pi^{-1} H_*(M, \mathbb{Z}) \cong H_*(\Omega BM, \mathbb{Z}) \ .$$

Example 3. *Assume that M is homotopy commutative, so that $\pi = \pi_{\mathbb{S}} = \pi_0(M)$ is central in $\mathbb{S}_*(M)$. Then for every generalized cohomology theory E we have an isomorphism $\pi_{\mathbb{S}}^{-1} E_*(M) \to E_*(\Omega BM)$.*

[1] We will work entirely in an ∞-categorical setting following [Lur09, Lur16].

[2] See Definition 14 below for a review of the left Ore condition and for left fractions

Proof of Theorem 1. The space ΩBM is an $\mathbb{E}_1$-space and the map $i : M \to \Omega BM$ is a map of $\mathbb{E}_1$-spaces. Moreover every element $x \in \pi_0(M)$ is mapped to a unit in ΩBM since ΩBM is grouplike as is every loopspace. We claim that ΩBM is initial with respect to these properties in the ∞-categorical sense, that is for every other $\mathbb{E}_1$-space N the map

$$\mathrm{Map}_{\mathbb{E}_1}(\Omega BM, N) \xrightarrow{i^*} \mathrm{Map}^u_{\mathbb{E}_1}(M, N)$$

is an equivalence of spaces, where $\mathrm{Map}^u_{\mathbb{E}_1}$ denotes the space of those $\mathbb{E}_1$-maps $M \to N$ for which $\pi_0(M)$ is send to a unit in $\pi_0(N)$. The space $\mathrm{Map}^u_{\mathbb{E}_1}(M, N)$ is equivalent to the space of maps $\mathrm{Map}_{\mathbb{E}_1}(M, N^\times)$ where $N^\times \subseteq N$ is the inclusion of the connected components which are units in $\pi_0(N)$. Since every $\mathbb{E}_1$-map $\Omega BM \to N$ also factors through $N^\times$ it suffices to prove that

$$\mathrm{Map}_{\mathbb{E}_1}(\Omega BM, N^\times) \xrightarrow{i^*} \mathrm{Map}_{\mathbb{E}_1}(M, N^\times)$$

is an equivalence. In other words we can assume that N is grouplike. Therefore we are claiming that ΩBM is the group completion of M in the ∞-categorical sense. This is well-known and follows from the following pair of observations:

1. For every $\mathbb{E}_1$-space M the $\mathbb{E}_1$-space ΩBM is grouplike
2. For every grouplike $\mathbb{E}_1$-space M the map $M \to \Omega BM$ is an equivalence of $\mathbb{E}_1$-spaces.

Thus the endofunctor $\Omega B : \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S}) \to \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S})$ is a localization (see [Lur09, Section 5.2.7]) with local objects the grouplike $\mathbb{E}_1$-spaces.

Now we consider the map of ringspectra

$$R[M] \to R[\Omega BM]$$

where $R[M]$ is the spherical group ring defined as $R \otimes \Sigma^\infty_+ M$. It follows from the above universal property of $M \to \Omega BM$ and the fact that the functor

$$R[-] : \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S}) \to \mathrm{Alg}_{\mathbb{E}_1}(R)$$

is left adjoint to the functor Ω^∞ (equipped with the multiplicative $\mathbb{E}_1$-structure) that $R[\Omega BM]$ is the universal localization of the ring spectrum $R[M]$ at the class π_R, i.e. initial among all maps of $\mathbb{E}_1$-ringspectra with source $R[M]$ which invert π_R. If π_R satisfies the left Ore condition in $\pi_* R[M] = R_*(M)$ then this universal ring spectrum has homotopy groups $\pi_R^{-1} R_*(M)$ which implies the result. This fact about Ore localization of ring spectra has been shown by Jacob Lurie, see [Lur16, Section 7.2.3] and will be reviewed in Appendix A. Note that in the most relevant case that R and M are both commutative this formula for the homotopy groups of the localization is quite classical, see e.g. [EKMM97, Chapter VIII.2].

Similarly it follows for every R-module E that $E[\Omega BM]$ is as an $R[M]$-module the localization of $E[M]$ at π_R and thus also has homtopy group $\pi_R^{-1} E_*(M)$ which is also due to Lurie and will also be explained in Appendix A. $\square$

1 The commutative case and the plus construction

In this section we want to relate the group completion ΩBM to Quillen's plus construction. Therefore assume for the whole section that M is an $\mathbb{E}_\infty$-space[3]. In this case the Ore

[3] A lot of what we are going to say also works for homotopy commutative $\mathbb{E}_1$-spaces. But since in the application to K-theory everything is $\mathbb{E}_\infty$ we will not need this extra generality

condition for $\mathbb{S}_*(M)$ and $H_*(M, \mathbb{Z})$ is automatic so that we can descibe the homology or stable homotopy of ΩBM very explicitly as a telescope. We want to give a space level interpretation of this localization. This is very much along the lines of the original description of Segal-McDuff [MS76] and almost all the results in this section can also be found in [RW13].

Construction 4. *Let $m \in M$ be an element. Then we consider the colimit of spaces*

$$M_m := \varinjlim (M \xrightarrow{m} M \xrightarrow{m} M \xrightarrow{m} \dots)$$

where the maps are given by left multiplication with the element m. Since M is $\mathbb{E}_\infty$ all the maps in this diagram are canonically M-linear, i.e. left multiplication is a left M-module map using the symmetry. Concretely this means in particular that for an element $n \in M$ we use left multiplication with the symmetry path $\tau = \tau_{(n,m)}$ in M to fill the diagram

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "n"'] & M \arrow[d, "n"] \arrow[dl, Rightarrow, "\tau"'] \\ M \arrow[r, "m"'] & M \end{tikzcd}$$

which (together with other coherences that we will not need here) exhibits left multiplication with m as being M-linear. As a result of the M-linearity the colimit M_m inherits a left action by M as it can be interpreted as the colimit in spaces with a left M-action. Note that the colimit defining M_m makes more generally sense for every space with a left action of M in place of M. Thus for an ordered finite set of elements $m_1, ..., m_n$ of M we can define inductively

$$M_{\{m_1,..,m_n\}} := (M_{\{m_1,...,m_{n-1}\}})_{m_n} .$$

Now we choose a set $(m_i)_{i \in I}$ of generators of $\pi_0(M)$ and a well-ordering on I. Then we define a left M-space M_∞ by the colimit

$$M_\infty := \varinjlim_{S \subseteq I} M_S$$

indexed over all finite subsets S of I. We will assume for the rest of the section that such an ordered generating set has been chosen once and for all.

The space M_∞ comes with two canonical maps of left M-spaces

$$M \to M_\infty \to \Omega BM .$$

The first map is the structure map of the colimit and the second map is obtained from the map $M \to \Omega BM$ by applying the construction $(-)_\infty$ (which can also be applied to any space with a a left action of M) and noting that ΩBM is equivalent to $(\Omega BM)_\infty$ since multiplication with every element is an equivalence.

In an ideal world the second map $M_\infty \to \Omega BM$ would be an equivalence giving us a very concrete description of the group completion ΩBM as a telescope. Note that this map is automatically a stable equivalence by the results of the first section, but the following example shows that it is not always an equivalence.

Example 5. *Consider the groupoid of finite sets and isomorphisms. It is symmetric monoidal with respect to disjoint union. The nerve is an $\mathbb{E}_\infty$-space (recall that spaces are Kan complexes for us). Concretely it is equivalent to $M = \coprod_{\mathbb{N}} B\Sigma_n$. In fact this is the free $\mathbb{E}_\infty$-space* *on the generator $m_1 \in \pi_0(B\Sigma_1) \subseteq \pi_0(\coprod B\Sigma_n)$. If we form M_∞ using the generator 1 in $\pi_0(M) = \mathbb{N}$ we obtain*

$$M_\infty \simeq \coprod_{\mathbb{Z}} B\Sigma_\infty \ .$$

The group completion ΩBM is given by free grouplike $\mathbb{E}_\infty$-space on a single generator, since M is the free $\mathbb{E}_\infty$-space on one generator. Thus we have $\Omega BM \simeq QS^0$ (this fact is called the Barrat-Priddy-Quillen theorem). The maps are given by

$$\coprod_{\mathbb{N}} B\Sigma_n \to \coprod_{\mathbb{Z}} B\Sigma_\infty \to QS^0$$

neither of which are equivalences. Note that this also means that m does not act invertibly on $\coprod_{\mathbb{Z}} B\Sigma_\infty$. In fact one can check that left multiplication by m on $\coprod_{\mathbb{Z}} B\Sigma_\infty$ shifts the component by one and inserts the constant unit in the first coordinate of the permutation.

There are however conditions which make sure that M_∞ is already the desired group completion. For the next proposition we note that there is for every element $m \in M$ in an $\mathbb{E}_\infty$-monoid and every $n \in \mathbb{N}$ a map

$$B\Sigma_n \xrightarrow{(m,\ldots,m)} (M^n)_{h\Sigma_n} \to M$$

where the second map is the coherent multiplication. This induces a map $\Sigma_n \to \pi_1(M, m^n)$.

Proposition 6. *For the left M-space M_∞ the following are equivalent:*

1. *M_∞ is $\pi_0(M)$-local, that is $\pi_0(M)$ acts invertibly on M_∞;*
2. *The map $M \to M_\infty$ exhibits M_∞ as the universal $\pi_0(M)$-local space;*
3. *The canonical map $M_\infty \to \Omega BM$ is an equivalence;*
4. *The fundamental groups of all components of M_∞ are abelian;*
5. *The fundamental groups of all components of M_∞ are hypoabelian[4];*
6. *For every m_i the map*
   $$\Sigma_3 \to \pi_1(M, m_i^3) \to \pi_1(M_\infty, m_i^3)$$
   has (123) in its kernel. [5]
7. *For every m_i there is an $n \geq 2$ such that the map*
   $$\Sigma_n \to \pi_1(M, m_i^n) \to \pi_1(M_\infty, m_i^n)$$
   has the permutation $(123...n)$ in its kernel.

[4] A group is called hypoabelian if it has no nontrivial perfect subgroups.

[5] This is the cyclic invariance condition that has been introduced by Veovodsky and plays a crucial role in 'stabilizations' of categories with respect to objects, see e.g. [**?**].

Proof. (1)⇒(2): We have to show that the map $M \to M_\infty$ is a local equivalence, that is mapping it into any space X with a left action on which $\pi_0(M)$-acts invertibly is an equivalence. But this is clear, since the maps in the colimit all are local equivalences.
(2)⇒(3): We note that the localization as a left module is automatically the localization as a $\mathbb{E}_1$-monoid which can be seen as in Corollary 13 Appendix A (see also [BNT15, Proposition C.5]).
(3) ⇒ (4) ⇒ (5): Clear.
(4) ⇒ (6): The map $\Sigma_3 \to \pi_1(M_\infty, m_i^3)$ factors through the abelianization of Σ_3 if the target is abelian. But the abelianization of Σ_3 is given by C_2 and has (123) in its kernel.
(5) ⇒ (7): The map $\Sigma_n \to \pi_1(M_\infty, m_i^n)$ factors through the hypoabelianization of Σ_n if the target is hypoabelian. But the hypoabelianization of Σ_n is given by C_2 for $n \geq 5$, thus for $n = 5$ it has (12345) in its kernel.
(6) ⇒ (7): Clear
(7) ⇒ (1): It is obviously enough to show that m_i acts invertibly on M_∞ for every $i \in I$. Let us first assume that the set of generators $\{m_i\}_{i\in I}$ consists of a single element m. The following argument appears essentially verbatim in [Rob12, Proposition 4.21], see also the discussion in [BNT15, Appendix C].

The left multiplication with m as a map $M_\infty \to M_\infty$ is induced by the diagram

$$\begin{tikzcd}
M \arrow[r, "m"] \arrow[d, "m"'] & M \arrow[r, "m"] \arrow[d, "m"'] \arrow[dl, phantom, "\overset{\tau}{\Swarrow}"] & M \arrow[r, "m"] \arrow[d, "m"'] \arrow[dl, phantom, "\overset{\tau}{\Swarrow}"] & \ldots \arrow[dl, phantom, "\overset{\tau}{\Swarrow}"] \\
M \arrow[r, "m"'] & M \arrow[r, "m"'] & M \arrow[r, "m"'] & \ldots
\end{tikzcd}$$

as one sees directly from the definition. Note that we only draw the 0,1 and 2-cells in this diagram and ignore the higher cells in the discussion because there are no coherence issues.[6] In order to show that this induced map is an equivalence we want to exhibit an inverse. There is an obvious candidate for an inverse $M_\infty \to M_\infty$, namely the map induced by the diagram

$$\begin{tikzcd}
M \arrow[r, "m"] \arrow[dr, "\mathrm{id}"] & M \arrow[r, "m"] \arrow[d, equal, "\overset{\mathrm{id}}{\Downarrow}"] \arrow[dr, "\mathrm{id}"] & M \arrow[r, "m"] \arrow[d, equal, "\overset{\mathrm{id}}{\Downarrow}"] \arrow[dr, "\mathrm{id}"] & \ldots \arrow[d, equal, "\overset{\mathrm{id}}{\Downarrow}"] \\
M \arrow[r, "m"'] & M \arrow[r, "m"'] & M \arrow[r, "m"'] & M \arrow[r] & \ldots
\end{tikzcd}$$

The composition in both directions is given by the map induced from the diagram

$$\begin{tikzcd}
M \arrow[r, "m"] \arrow[dr, "m"] & M \arrow[r, "m"] \arrow[d, equal, "\overset{\tau}{\Downarrow}"] \arrow[dr, "m"] & M \arrow[r, "m"] \arrow[d, equal, "\overset{\tau}{\Downarrow}"] \arrow[dr, "m"] & \ldots \arrow[d, equal, "\overset{\tau}{\Downarrow}"] \\
M \arrow[r, "m"'] & M \arrow[r, "m"'] & M \arrow[r, "m"'] & M \arrow[r] & \ldots
\end{tikzcd} \tag{1}$$

If the 2-cells τ in this diagram were equivalent to identities a cofinality argument would show that this map $M_\infty \to M_\infty$ is equivalent to the identity. The cell τ being the identity path means that the permutation (12) lies in the kernel of the map $\Sigma_2 \to \pi_1(M, m^2)$. In fact less is sufficient:

[6] More precisely the ∞-category obtained from the poset of natural numbers $(\mathbb{N}, \leq)$ is the pushout

$$\Delta^1 \cup_{\Delta^0} \Delta^1 \cup_{\Delta^0} \Delta^1 \cup_{\Delta^0} \ldots$$

in $\mathcal{C}\mathrm{at}_\infty$ and thus functors and transformations out of it are entirely determined on maps between adjacent natural numbers.

- We do not need that the 2-cells in diagram (1) itself are equivalent to the identity but only that they are mapped to the identity in M_∞. That means it is sufficient to assume that the permutation (12) is in the kernel of the composition $\Sigma_2 \to \pi_1(M, m^2) \to \pi_1(M_\infty, m^2)$. This is the case $n = 2$ of our assumption.

- We do not need to consider the 2-cells in diagram (1) itself but we can consider the $(n-1)$-fold horizontal composition of the two cells with itself by a further cofinality argument. But the $(n-1)$-fold horizontal composition in the diagram is given by left multiplication with the symmetry path $\tau_{1,2} \circ \tau_{2,3} \circ ... \tau_{n-1,n}$ where $\tau_{i,i+1}$ is the path from m^n to itself that applies the symmetry to the factors i and $i+1$. This corresponds to the permutation $(12...n)$. Thus it is sufficient that $(12...n)$ is in kernel of the map $\Sigma_2 \to \pi_1(M, m^n) \to \pi_1(M_\infty, m^n)$.

Together these arguments show that the assertion (7)⇒(1) holds for a single generator $[m]$ of $\pi_0(M)$. For a set of generators the construction M_∞ is an iteration of the construction for a single generator (applied to the left M-module obtained by 'inverting' the earlier elements) and thus follows by the same argument as the case of a single element. □

Corollary 7. *Assume that the fundamental groups of all components of M are hypoabelian, then the map $M_\infty \to B\Omega M$ is an equivalence. In particular the fundamental groups of all components of M_∞ are abelian.*

Proof. We need to show that the map $\Sigma_n \to \pi_1(M, m_i^n) \to \pi_1(M_\infty, m_i^n)$ sends the cycle $(123..n)$ to zero. But if M has hypoabelian fundamental groups the first map $\Sigma_n \to \pi_1(M, m_i^n)$ factors through the hypoabelianization of Σ_n. For $n \geq 5$ the group Σ_n has hypoabelianization C_2 and the cycle $(123...n)$ lies in the kernel for n odd. □

Example 8. *Consider the E_∞-monoid $M = \coprod BU(n)$ which is given as the realization of the topological groupoid of complex vector spaces with direct sum as addition. All components have abelian fundamental groups, thus we get that*

$$\mathbb{Z} \times BU \simeq \coprod_{\mathbb{Z}} BU \simeq M_\infty \simeq \Omega BM.$$

We see that the obstruction for the map $M_\infty \to \Omega BM$ to be an equivalence is the fact that the fundamental groups of M_∞ are not hypoabelian. But there is a universal way of forcing a space to have hypoabelian fundamental group: Quillen's plus construction. More precisely if X is a space then the plus construction is another space X^+ with a map $X \to X^+$ such that X^+ has hypoabelian fundamental groups for all components and that X^+ is initial with respect to this property (in the ∞-category of spaces). More specifically we apply the plus construction to every connected component of $[x] \in \pi_0(X)$ separately and we always mean the plus construction with respect to the maximal perfect subgroup PG of $G = \pi_1(X, x)$. Then $G/PG = \pi_1(X^+, x)$ is the hypoabelianization of G

This universal property is not quite the way the plus construction is usually presented but it can easily seen to be equivalent by the following pair of observations (see also [Ber99] for a proof and a nice discussion):

- The usual plus construction can be constructed functorialy, that is we get a functor $(-)^+ : \mathcal{S} \to \mathcal{S}$ which lands in the full subcategory $\mathcal{S}^{\text{hypo}} \subseteq \mathcal{S}$ of spaces with hypoabelian fundamental groups.

- it comes with a map $X \to X^+$ natural in X (in the ∞-categorical sense) which is an equivalence for X hypoabelian.

Thus the plus construction is a localization with local objects the hypoabelian spaces. It is an interesting excercise for the reader to only use the universal property of the plus construction to deduce that $X \to X^+$ is a homology equivalence. It is also an elementary excercise to check that the plus construction

$$(-)^+ : \mathcal{S} \to \mathcal{S}$$

preserves products. In particular it preserves monoid structures.

Theorem 9. *There are equivalences*

$$(M_\infty)^+ \simeq (M^+)_\infty \simeq \Omega BM \ .$$

More precisely the canonical maps $M_\infty \to \Omega BM$ and $M_\infty \to (M^+)_\infty$ are plus constructions.

The equivalence $M_\infty^+ \simeq \Omega BM$ in this generality seems to be well-known for a while and is implicitly claimed by several people. The first written proof that we have been able to find is due to Randal-Williams [RW13]. The other equivalence seems to be new.

Note that from the theorem one can conclude that for every connected component $[x] \in \pi_0(M_\infty)$ the fundamental group $G = \pi_1(M_\infty, x)$ has perfect commutator subgroup $[G, G]$ by observing that its hypoabelianization G/PG is already abelian. Such groups G are called quasi-perfect. See also [RW13, Section 3] for a direct proof of this fact.

Proof. We first prove that $(M^+)_\infty \simeq (M_\infty)^+$ or more precisely that the map

$$M_\infty \to (M^+)_\infty$$

is a plus construction. First of all, from Corollary 7 we see that $(M^+)_\infty$ has (hypo)abelian fundamental group. Thus in order to check the universal property it suffices to check that the map induces an equivalence on mapping spaces as we map into a hypoabelian space Y. We again assume for simplicity that the set of generators consists of a single element $m \in M$ and the general case is an iteration of this case. Then we find:

$$\begin{aligned}
\mathrm{Map}_{\mathcal{S}}((M^+)_\infty, Y) &\simeq \varprojlim\big(\ldots \xrightarrow{m^*} \mathrm{Map}_{\mathcal{S}}(M^+, Y) \xrightarrow{m^*} \mathrm{Map}_{\mathcal{S}}(M^+, Y) \xrightarrow{m^*} \mathrm{Map}_{\mathcal{S}}(M^+, Y)\big) \\
&\simeq \varprojlim\big(\ldots \xrightarrow{m^*} \mathrm{Map}_{\mathcal{S}}(M, Y) \xrightarrow{m^*} \mathrm{Map}_{\mathcal{S}}(M, Y) \xrightarrow{m^*} \mathrm{Map}_{\mathcal{S}}(M, Y)\big) \\
&\simeq \mathrm{Map}_{\mathcal{S}}(M_\infty, Y) \ .
\end{aligned}$$

Now we look at the commutative diagram

$$\begin{tikzcd}
(M_\infty)^+ \arrow[r] \arrow[d] & \Omega BM \arrow[d] \\
(M^+)_\infty \arrow[r] & \Omega B(M^+) \ .
\end{tikzcd}$$

The left vertical map is an equivalence as we have just shown, the lower horizontal map is an equivalence by Corollary 7 thus it suffices to show that the right hand map is an equivalence. Both spaces are simple, thus it suffices to check that it it is a homology isomorphism which follows from the group completion theorem. [7] □

[7] One can also see this directly using the universal properties.

Remark 10. *Note that the space M_∞ as constructed above has as π_0 automatically the group completion $K_0(M)$ of the monoid $\pi_0(M)$ and is always of the form*

$$M_\infty \simeq K_0(M) \times \mathrm{BGL}_\infty M$$

where $\mathrm{GL}_\infty M$ is just a name for the $\mathbb{E}_1$-space $\Omega(M_\infty)$. This follows from a cofinality argument and is a decomposition of spaces not compatible with any extra structure. The obstruction for this to admit an $\mathbb{E}_\infty$-structuture is by our results above exactly the fact that $\pi_0\mathrm{GL}_\infty M = \pi_1\mathrm{BGL}_\infty M$ might be non-abelian. Then $(M_\infty)^+$ is of the form $\mathrm{K}_0(M) \times (\mathrm{BGL}_\infty M)^+$ as the plus construction is applied component-wise which then admits a grouplike $\mathbb{E}_\infty$-structure (but the splitting is not a splitting of $\mathbb{E}_\infty$-spaces).

A Review of Ore localization of ring spectra

In this appendix we review Ore localizations of ring spectra. All the material is due to Lurie [Lur16, Section 7.2.3] and fairly standard. We mostly include it for the convenience of the reader.

Let A be an $\mathbb{E}_1$-ring spectrum. We want to invert a set $S \subseteq \pi_* A$. That means we want to find a map of algebras $i : A \to A'$ such that

1. S acts invertibly on A', that is all elements $i(S) \subseteq \pi_*(A')$ are units.
2. The ring spectrum A' is initial among all ring spectra with property (1).

It turns out that it is more convenient to construct the localization A' not only for the ring spectrum A itself but for every A-module spectrum M. To this end we make the following definition:

Definition 11. Let M be a left A-module spectrum. We say that S-acts invertibly on M or that M is S-local if every $s \in S$ acts invertibly on $\pi_*(M)$. We let $\mathrm{Mod}_A^{\mathrm{Loc}(S)} \subseteq \mathrm{Mod}_A$ be the full subcategory consisting of all S-local A-modules (here Mod_A denotes the ∞-category of left R-modules).

Proposition 12. *The inclusion $\mathrm{Mod}_A^{\mathrm{Loc}(S)} \subseteq \mathrm{Mod}_R$ has a left adjoint L which can be described as*

$$L(M) = A' \otimes_A M$$

where A' is a uniquely determined A–A-bimodule with a map $A \to A'$ of bimodules.

Proof. We want to invoke the theory of localizations of ∞-categories to show the existence of the left adjoint. First of all, by [Lur16, Corollary 4.2.3.7] the ∞-category Mod_A is presentable. We consider the set S' of morphisms in Mod_A defined as

$$S' := \left\{ \Sigma^{-n} A \xrightarrow{\Sigma^{-n} R_s} \Sigma^{-n} A \mid s \in S, n \in \mathbb{N} \right\}$$

where R_s denotes right multiplication with $s \in S$. The object $\Sigma^{-n}A$ is a left A-module. Now we claim that an A-module M is S-local in the sense of Definition 11 precisely if it is S'-local in the sense of Bousfield localizations. The latter means by definition that the induced maps

$$\mathrm{Map}_A(\Sigma^{-n}A, M) \xrightarrow{R_s^*} \mathrm{Map}_A(\Sigma^{-n}A, M)$$

are equivalences of spaces for each $s \in S$ and $n \in \mathbb{N}$. But this map is equivalent to

$$\Omega^\infty \Sigma^n M \xrightarrow{\Sigma^n L_s} \Omega^\infty \Sigma^n M$$

where $L_s : M \to M$ is left multiplication with s. This is an equivalence for all n iff $L_s : M \to M$ is an equivalence which shows our claim.

It follows from the general theory of Bosufield localizations of presentable ∞-categories [Lur09, Section 5.5.4] which is essentially due to J. Smith in a slightly different language, that there is a left adjoint L to the inclusion $\mathrm{Mod}_A^{\mathrm{Loc}(S)} \subseteq \mathrm{Mod}_A$ which preserves κ-compact objects for some regular cardinal κ and that $\mathrm{Mod}_A^{\mathrm{Loc}(S)}$ is itself presentable. In this case we can even say more, namely that the left adjoint preserves ω-compact objects. This is equivalent to the assertion that the inclusion $\mathrm{Mod}_A^{\mathrm{Loc}(S)} \subseteq \mathrm{Mod}_A$ preserves ω-filtered colimits. But it is obvious that the subcategory of S-local modules is closed under all colimits. As a result the endofunctor $L : \mathrm{Mod}_A \to \mathrm{Mod}_A$ given by the localization (composed with the inclusion which we supress in the notation) preserves all colimits. By standard Morita theory it follows that it is given by tensoring with a bimodule $A' := L(A)$ which reveives a map $A \to A'$ induced by the map $\mathrm{id} \to L$ of endofunctors. □

Corollary 13. *The bimodule A' admits the unique[8] structure of an algebra such that $A \to A'$ is a map of algebras (compatible with the bimodule structure). This structure makes it a localization of A at S in the sense defined at the beginning of the section. The ∞-category $\mathrm{Mod}_A^{\mathrm{Loc}(S)}$ can be identified with $\mathrm{Mod}_{A'}$.*

Proof. The bimodule A' is an idempotent in the category ${}_A\mathrm{Mod}_A$ (since L is a localization). That means the two canonical maps $A' \to A' \otimes_A A'$ are both homotopic and equivalences. It follows that A' admits the unique structure of an associative algebra in the ∞-category of A-A-bimodules and that the category of local objects is equivalent to $\mathrm{Mod}_{A'}$. □

While the discussion so far is very pleasing from a theoretical point of view the localization $A \to A'$ is unfortunately not very explicit. In the situation of ordinary rings similar arguments show that there is an abstract localization at any set of elements but since it is produced by abstract methods it does not have a useful description besides its universal property. Fortunately one can sometimes describe the localization in terms of 'fractions', similar to the commutative case. First we review the classical Ore localization from algebra.

Definition 14. Let A_* be a graded ring. We say that a multiplicatively closed subset $S \subseteq A_*$ satisfies the left Ore condition if the following holds:

1. For every pair of elements $s \in S$ and $a \in A_*$ there exist elements $s' \in S$ and $a' \in A_*$ such that $as' = sa'$ in A_*.
2. For every pair of elements $a \in A_*$ and $t \in S$ such that $at = 0$ there exists $s \in S$ such that $sa = 0$.

In this case we define for every graded A_* module M_* the graded module of left fractions $S^{-1}M_*$ is the quotient of $S \times M_*$ by the equivalence relation

$$(s, m) \sim (s', m') \quad :\Leftrightarrow \quad \exists a, a' \in A_* \colon as = a's' \in S \text{ and } am = a'm' \ .$$

[8]Unique in this setting always means unique up to contractible choices

We denote the equivalence class of a pair (s, m) by $s^{-1}m$. In particular we get a map $M_* \to S^{-1}M_*$ sending m to $[1, m] = 1^{-1}m$. Now $S^{-1}M_*$ is a graded abelian group in an obvious sense and it admits the structure of an A_*-module such that the map $M_* \to S^{-1}M_*$ is a map of A_*-modules. For the case $M_* = A_*$ there is a unique structure of a graded ring on $S^{-1}A_*$ compatible with the ring structure of A_* (this is somewhat lengthy but straightforward to check).

Note that in the commutative case the Ore condition is automatic and the left fractions are given by the usual localization of commutative rings.

Definition 15. We say that a map $f : M_* \to M'_*$ of A_*-modules is an S-nil equivalence if following pair of conditions is satisfied:

1. For every element $m \in M_*$ with $f(m) = 0$ in M'_* there exists $s \in S$ such that $sm = 0$.
2. for every $m' \in M'_*$ there exists $s \in S$ such that sm' is in the image of $f : M_* \to M'_*$.

The proof of the following Lemma is straightforward and left to the reader as an exercise (or see [Lur16, insert]).

Lemma 16. *1. S satisfies the left Ore condition precisely if for each $s \in S$ the map $A_* \xrightarrow{R_s} A_*$ given by right multiplication with s is a S-nil equivalence.*

2. A map $M_* \to M'_*$ of graded A_*-modules is isomorphic to the map $M_* \to S^{-1}M_*$ precisely if it is an S-nil-equivalence and the target M'_* is S-local (meaning that S-acts invertibly).

Now let us switch back to the setting of ring spectra. We say that $S \in \pi_*(A)$ satisfies the left Ore condition if it does as a subset of the graded ring $A_* = \pi_*(A)$. Similarly a map of A-modules $M \to M'$ is called S-nil equivalence if the induced map $M_* \to M'_*$ is a S-nil equivalence. This is equivalent to the assertion that the fibre is S-nilpotent. The key fact now is that the class of S-nil equivalences of A-module spectra is closed under pushouts, retracts, transfinite compositions and shifts as one easily checks from the long exact sequences.

Theorem 17 (Lurie). *Let A be an $\mathbb{E}_1$-ringspectrum. If $S \subseteq \pi_*(A)$ satisfies the left Ore condition then for every A module spectrum M the universal localization $M \to M'$ (which inverts S) is an S-nil equivalence. In particular the homotopy groups of M' are given by the left fractions*

$$\pi_*(M') \cong S^{-1}\pi_*(M) \ .$$

Proof. From the proof of Proposition 12 we see that the morphism $M \to M'$ lies in the smallest saturated class of morphisms containing the maps $R_S : A \to A$ and is closed under shifts. By Lemma 16(1) and the fact that S-nil equivalences are closed under saturation this implies that the map $M \to M'$ is a S-nil equivalence which shows the first claim. The second claim is a consequence of the first by Lemma 16(2). □

Remark 18. *One might wonder why an analogous statement to Theorem 17 is not true in the ∞-category of spaces, i.e. for the group completion of an $\mathbb{E}_\infty$-monoid. Example 5 clearly shows that this does not work. On a technical level the answer is that if one tries to follow the same proof strategy defining S-nilpotent maps then these will not be a saturated class. In fact we exactly see from Proposition 6 and Theorem 9 that the fundamental group of spaces (more precisely the fact that it is not abelian) is the crucial difference between the stable and the unstable setting.*

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