GROUP THEORY FOR HOMOTOPY THEORISTS

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Abstract. We demonstrate how to effectively work with the theory of groups using Quillen model structures avoiding the overly abstract definition of a group as a set with a binary operation. Our approach is highly inspired by the modern point set approaches to the category of spectra. One major advantage is that in our approach it is easy to write down examples of free groups and colimits of groups. We also use it to define the tensor product of abelian groups.

1. The category of groups

Definition 1.1. *A group is given by a pair consisting of a set S (called the 'generators') and a set R of words in $S \amalg S^{-1}$ (called the 'relations'). Here S^{-1} is a set which is in bijection to S but whose elements we write as s^{-1} for $s \in S$. We will also write the group as $\langle S|R\rangle$.*

Example 1.2. We have the *free group* $\mathrm{Fr}(S)$ for any set S given by the pair $\langle S \mid \emptyset\rangle$.

Example 1.3. The torus group is given by

$$\langle a, b \mid aba^{-1}b^{-1}\rangle$$

i.e. it has a set of 2 generators denoted a and b and one relation.

Definition 1.4. *A morphism of groups $\langle S|R\rangle \to \langle S'|R'\rangle$ is a map $f : S \to S'$ such that f sends relations in R to relations in R'. We denote the category of groups by* Grp*.*

We will later establish a model structure on the category Grp, and eventually we it will be the resulting homotopy theory that we are interested in (the 'group theory'). But it will be of central importance for use to have a good model for groups in order to perform computations.

There is an evident forgetful functor Grp $\to$ Set which forgets the set of relations, i.e. it sends $\langle S|R\rangle \mapsto S$. This functor has both adjoints given by equipping a set S with the empty set of relations and the set of all relations.

Proposition 1.5. *The category* Grp *has all limits and colimits.*

Proof. Form the colimit/limit of underlying sets and equip it with the initial, resp. terminal set of relations that makes the structure maps morphisms of groups. □

2. The homotopy theory of groups

In this section we want to equip the category of groups with a Quillen model structure.

Definition 2.1. *Every morphism $\langle S|R\rangle \to \langle S'|R'\rangle$ in* Grp *is a cofibration. Accordingly a map is called a trivial fibration if it is an isomorphism in* Grp.

Clearly every morphism in Grp can be factored into a cofibration followed by a trivial fibration. We now want to define generating trivial cofibrations. To this end we consider the set of morphisms

$$\begin{aligned}\langle a,b \mid ab^{-1}\rangle &\to \langle c \mid cc^{-1}\rangle \\ \langle S \mid \emptyset\rangle &\to \langle S \amalg \{a\} \mid aw\rangle \\ \langle S \mid vw\rangle &\to \langle S \mid vw, wv\rangle \\ \langle S \mid v,w\rangle &\to \langle S \mid v,w,vw\rangle \\ \langle S \mid v,vw\rangle &\to \langle S \mid v,vw,w\rangle\end{aligned} \tag{1}$$

Here S always refers to an arbitrary finite set, and v, w to arbitrary words in $S \amalg S^{-1}$.

Definition 2.2. *A map $\langle S|R\rangle \to \langle S'|R'\rangle$ in* Grp *is called a fibration if it admits the right lifting property against the generating trivial cofibrations, i.e. the maps in* (1)*. It is called a weak equivalence if it is in the strongly saturated class generated by the maps* (1)*.* [1]

Theorem 2.3. *The choices of cofibrations, fibrations and weak equivalences determine a closed model category structure on* Grp*. A map $\langle S|R\rangle \to \langle S'|R'\rangle$ between fibrant groups is a weak equivalence if and only if the underlying map $S \to S'$ is an isomorphism.*

Proof. By the small object argument, which works since the generating cofibrations have compact soruce and target, every morphism can be factored into a trivial cofibration (which is the same as a weak equivalence since every map is a cofibration) followed by a fibration. The rest of the axioms follows straightforwardly. □

We will refer to the model structure of the last theorem as the standard model structure on Grp. The homotopy category of this model structure Ho(Grp) is equivalent to the classical category of groups. We will not get into this comparison here since we want to develop group theory from scratch without having to use the old one.

Remark 2.4. There is another model structure on the category Grp in which every morphism is a cofibration and a fibration. The weak equivalences are the isomorphisms in Grp. We will refer to this model structure as the *genuine* model structure. The identity functor Grp $\to$ Grp is a left Quillen functor with respect to the genuine model structure on the source and the standard model structure on the target. In fact the standard model structure is a left Bousfield localization of the genuine model structure.

[1]Strongly saturated here means the smallest class of maps containing the maps in (1) and being closed under pushouts, retract and transfinite compositions. In this case, the class is in fact closed under all colimits.

3. Some examples

In the following example we explain how the model structure on Grp can be used to effectively compute maps between groups. T

Example 3.1. We want to compute homotopy classes, i.e. maps in the homotopy category Ho(Grp), from the free group $F = \langle x \mid \emptyset \rangle$ to itself. Clearly we have that

$$\mathrm{Hom}_{\mathrm{Grp}}(F, F) = \{\mathrm{id}_F\}$$

consists of a singleton. But there will be more maps in the homtopy category. We first have to construct a fibrant replacement $F \to F'$. We set

$$F' = \langle ..., x_{-2}, x_{-1}, x_0, x_1, x_2, ... \mid R\rangle$$

where R consists of all words in the letters x_i and x_i^{-1} whose *multiplicity* is zero. For a word

$$x_{i_1}^{j_1} \dots x_{i_k}^{j_k}$$

the multiplicity is defined to be the weighted sum $i_1 \cdot j_1 + ... + i_k \cdot j_k$. Then the claim is that the map $F \to F'$ which sends x to x_1 is an equivalence and F' is fibrant. Both of these claims are elementary using pushouts respectively lifts aginst the generating trivial cofibrations (1) and are left to the reader.

As a result we get that

$$\mathrm{Hom}_{\mathrm{Ho(Grp)}}(F, F) = \mathrm{Hom}_{\mathrm{Grp}}(F, F')/\sim$$

is the quotient of the countable set $\mathrm{Hom}_{\mathrm{Grp}}(F, F')$. The equivalence relation is given by homotopy. But the factorization $F \amalg F \xrightarrow{\nabla} F \to F$ exhibits F as a good cylinder object for F. Therefore the equivalence relation is trivial and we get that $\mathrm{Hom}_{\mathrm{Ho(Grp)}}(F, F) = \mathrm{Hom}_{\mathrm{Grp}}(F, F')$ is a countable set.

Unfortunately we are currently not able to express the composition of morphisms in $\mathrm{Hom}_{\mathrm{Ho(Grp)}}(F, F)$ since this composition makes it into a monoid and we have not yet revisited the theory of monoids in modern language (i.e. similar to how we treat groups here). We plan to address this point in future work.

Example 3.2. We want to demonstrate how to see that the groups $\langle a, b \mid \emptyset\rangle$ and $\langle a, b \mid aba^{-1}b^{-1}\rangle$ are not weakly equivalent. We do this by exhibiting a fibrant group G such that

$$\mathrm{Hom}_{\mathrm{Ho(Grp)}}(\langle a, b \mid \emptyset\rangle, G) \not\cong \mathrm{Hom}_{\mathrm{Ho(Grp)}}(\langle a, b \mid aba^{-1}b^{-1}\rangle, G).$$

To construct G, consider the set $S = \{x_\emptyset, x_{12}, x_{23}, x_{13}, x_{123}, x_{132}\}$. To each of those elements (and the ones of S^{-1}) we assign a map $\{1, 2, 3\} \to \{1, 2, 3\}$ according to the following table:

	1	2	3
$x_\emptyset, x_\emptyset^{-1}$	1	2	3
x_{12}, x_{12}^{-1}	2	1	3
x_{23}, x_{23}^{-1}	1	3	2
x_{13}, x_{13}^{-1}	3	2	1
x_{123}, x_{132}^{-1}	2	3	1
x_{132}, x_{123}^{-1}	3	1	2

More generally, to any word w in those symbols we assign a map $\rho_w : \{1,2,3\} \to \{1,2,3\}$ by composition. We let

$$G := \langle x_\emptyset, x_{12}, x_{23}, x_{13}, x_{123}, x_{132} \mid \text{all } w \text{ with } \rho_w = \mathrm{id}\rangle.$$

It is not hard to check explicitly that this is fibrant, by checking the right lifting property against the generating cofibrations 1. The details are left to the reader.

We now compute $\mathrm{Hom}_{\mathrm{Ho(Grp)}}(\langle a, b \mid \emptyset\rangle, G)$ and $\mathrm{Hom}_{\mathrm{Ho(Grp)}}(\langle a, b \mid aba^{-1}b^{-1}\rangle, G)$. By the cyclinder object argument from Example 3.1, we can identify those with $\mathrm{Hom}_{\mathrm{Grp}}(\langle a, b \mid \emptyset\rangle, G)$ and $\mathrm{Hom}_{\mathrm{Grp}}(\langle a, b \mid \emptyset\rangle, G)$. But we have:

$$\begin{aligned} &\# \mathrm{Hom}_{\mathrm{Grp}}(\langle a, b \mid \emptyset\rangle, G) \\ = &\# \mathrm{Hom}_{\mathrm{Set}}(\{a, b\}, \{x_\emptyset, x_{12}, x_{23}, x_{13}, x_{123}, x_{132}\}) = 36, \end{aligned}$$

while

$$\# \mathrm{Hom}_{\mathrm{Grp}}(\langle a, b \mid aba^{-1}b^{-1}\rangle, G) < 36,$$

since for example $x_{12}x_{13}x_{12}^{-1}x_{13}^{-1}$ is not among the relations.

Note that it is crucial for these computations to work in the category of all groups (instead of just the fibrant ones), as computing maps out of an explicit fibrant replacement of $\langle x \mid \emptyset\rangle$ or $\langle a, b \mid \emptyset\rangle$ would be quite unwieldy.

4. Homotopy limits and colimits

Since the category Grp admits a model structure it automatically comes with homotopy limits and colimits. Homotopy colimits are easy to understand since every map is a cofibration. This implies that every diagram $I \to \mathrm{Grp}$ is cofibrant in the projective model structure on the functor category and therefore the homotopy colimit agrees with the colimit in the category Grp.

For limits the situation turns out to be more complicated as the following example shows.

Example 4.1. We have the pullback squares

$$\begin{array}{ccc} \langle \emptyset \mid \emptyset\rangle & \longrightarrow & \langle a \mid \emptyset\rangle \\ \downarrow & & \downarrow \\ \langle b \mid \emptyset\rangle & \longrightarrow & \langle a, b \mid ab\rangle \end{array}$$

in Grp. The right vertical map as well as the lower horizontal map are weak equivalences, but the upper horizontal and left vertical maps are not. Thus the pullback is not invariant under weak equivalence of groups.

Fortunately, the model structure on Grp gives us a way of deriving the pullback and we get the homotopy pullback or right derived functor of pullback modelling the 'correct' pullback of groups. Also the model structure gives us a way of computing these pullbacks. For Example 4.1, we find that the homotopy pullback of the lower right corner is weakly equivalent to either of the three lower right terms.

Another construction that needs to be derived is the kernel of a group homomorphism. The derived kernel of a group homomorphism $G \to H$ is defined as the homotopy pullback of the following diagram:

$$\begin{array}{ccc} & & G \\ & & \downarrow \\ \langle \emptyset \mid \emptyset \rangle & \longrightarrow & H \end{array}$$

Observe that the nonderived kernel is always $\langle \emptyset \mid \emptyset \rangle$. But of course the fact that there is an interesting derived kernel is a very important fact in group theory.

5. The tensor product of groups

In this section we will construct an operation that models the tensor product of abelian groups (if the reader is familiar with this concept from the classical approach to groups).

Definition 5.1. *We define the smash product*[2] *of two groups as*

$$\langle S \mid R \rangle \wedge \langle S' \mid R' \rangle := \langle S \times S' \mid R \times S' \cup S \times R' \rangle$$

where $R \times S'$ denotes the set of words obtained from the words in R as follows: for every word $s_1^{\pm} s_2^{\pm} \dots s_n^{\pm} \in R$ with $s_i \in S$ and every element $t \in S'$ we add the word

$$(s_1^{\pm}, t)(s_2^{\pm}, t) \dots (s_n^{\pm}, t)$$

to $R \times S'$. In other words we have

$$R \times S' = \bigcup_{t \in S'} i_t(R)$$

for $i_t : S \to S \times S'$ sending s to (s, t). The set of words $S \times R'$ is defined dually.

We also have a unit for the smash product, given by the group $\langle x \mid \emptyset \rangle$.

Proposition 5.2. *The smash product equips the category* Grp *with a symmetric monoidal structure.*

Proof. Clear. □

Unfortunately, it turns out that the standard model structure on groups is not quite compatible with the smash product of groups, as the following example shows.

Example 5.3. Consider the weak equivalence $\langle x \mid \emptyset \rangle \to \langle x, x' \mid x'x^{-2} \rangle$ and smash it with the group $\langle a, b, c \mid abc^{-1} \rangle$. We get the map

$$\langle a, b, c \mid abc^{-1} \rangle \to \left\langle a, b, c, a', b', c' \mid a'a^{-2}, b'b^{-2}, c'c^{-2}, abc^{-1}, a'b'(c')^{-1} \right\rangle$$

[2] This is sometimes called tensor product in the classical literature. We prefer $\wedge$ because the operation is very similar to the smash product of pointed sets. In particular there is a functor $\mathrm{Set}_* \to \mathrm{Grp}$ which takes a pointed set (S, s) to the group $\langle S \mid s \rangle$. This functor takes the smash product of pointed sets to a group weakly equivalent to the smash product of groups.

and we claim that this is not an equivalence (which it would have to be if the model structure were compatible since every object is cofibrant). To see that it is not an equivalence note that we have weak equivalences

$$\langle a, b \mid \emptyset \rangle \simeq \langle a, b, c \mid abc^{-1} \rangle$$

and

$$\langle a, b \mid aba^{-1}b^{-1} \rangle \simeq \left\langle a, b, c, a', b', c' \mid a'a^{-2}, b'b^{-2}, c'c^{-2}, abc^{-1}, a'b'(c')^{-1} \right\rangle$$

which follow easily by factoring into generating trivial cofibrations, and are left as an exercise. But this is a problem since the groups $\langle a, b \mid \emptyset \rangle$ and $\langle a, b \mid aba^{-1}b^{-1} \rangle$ are not weakly equivalent.

One can fortunately solve this problem by passing to a Bousfield localization of our model structure.

Definition 5.4. *The abelian model structure on* Grp *is the left Bousfield localization of the standard model structure at the morphism*

$$\langle a, b \mid \emptyset \rangle \to \langle a, b \mid aba^{-1}b^{-1} \rangle$$

In particular an abelian group in our perspective is the same as a group which allows to uniformize a lot of arguments. Only the non-existence of the smash products for (non-abelian) groups remains a mystery. However for abelian groups we have:

Theorem 5.5. *The abelian model structure is symmetric monoidal with respect to the smash product of groups.*

Proof. We have to verify the pushout-product axiom for generating trivial cofibrations (1) against all maps, which is left as a (tedious) exercise to the reader. □

6. Epilogue

This document is of course not entirely serious and we hope that no one takes offense. It grew out of an attempt to illustrate our perspective on the following points.

(1) A spectrum is a sequence of pointed spaces $(X_n)_{n\in\mathbb{N}}$ together with homotopy equivalences $\varphi_n : X_n \simeq \Omega X_{n+1}$. We like to call the more general object where one only has maps $\varphi_n : X_n \to \Omega X_{n+1}$ (or equivalently $\Sigma X_n \to X_{n+1}$), but they are not required to be homotopy equivalences, a *prespectrum*. Of course, every prespectrum $X = (X_n, \varphi_n)$ gives rise to a spectrum by replacing it with $X' = (X'_n, \varphi'_n)$ with

$$X'_n = \underrightarrow{\operatorname{colim}}_{k\to\infty} \Omega^k X_{n+k}.$$

which is a fibrant replacement in the (stable) Bousfield-Friedlander model structure. We think of X as being a 'generators and relations description' of X'. Of course a lot of examples of spectra will arise through generators and relations, i.e. suspension spectra or Thom spectra. Also computing (homotopy) colimits is much easier through prespectra. But the same is already true for groups, as shown in this document. The real advantage of spectra over prespectra is that maps between spectra are much easier to understand, namely a map of spectra $X \to Y$ is given by a sequence of maps $f_n : X_n \to Y_n$ for every n together with homotopies between the two compositions in the square

$$\begin{array}{ccc} X_n & \xrightarrow[\varphi_n]{\simeq} & \Omega X_{n+1} \\ \big\downarrow{\scriptstyle f_n} & & \big\downarrow{\scriptstyle \Omega f_{n+1}} \\ Y_n & \xrightarrow[\psi_n]{\simeq} & \Omega Y_{n+1} \end{array} .$$

The homotopies are not just required to exist, but are part of the datum of a map.

(2) It is sometimes really confusing to use model categories since the 'objects' of the category are always implicitly a presentation of their cofibrant-fibrant replacements. This is especially confusing if there are several model structures around and therefore the same object presents different replacements.

(3) We prefer the notation $\otimes$ for the smash/tensor product of spectra over $\wedge$, since it pretty much behaves like the tensor product of abelian groups, even though in the model it starts to look like the smash product of pointed spaces and the suspension spectrum functor (like the free abelian group) sends smash products of pointed spaces to smash products of spectra. Similarly we like to write $\oplus$ for the wedge of spectra, which is a coproduct but also equivalent to the product and thus a biproduct (in the higher categorical sense or in the homotopy category).