

Cartier modules and cyclotomic spectra

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Abstract

We construct and study a t -structure on p -typical cyclotomic spectra and explain how to recover crystalline cohomology of smooth schemes over perfect fields using this t -structure. Our main tool is a new approach to p -typical cyclotomic spectra via objects we call p -typical topological Cartier modules. Using these, we prove that the heart of the cyclotomic t -structure is the full subcategory of derived V -complete objects in the abelian category of p -typical Cartier modules.

Key Words. Topological Hochschild homology, cyclotomic spectra, Cartier modules, Dieudonné modules, de Rham–Witt complexes.

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1 Introduction

The notion of a cyclotomic spectrum was introduced in [BHM93]. The importance of cyclotomic spectra in arithmetic contexts was understood in the work of Hesselholt and his collaborators (see for example [Hes96, GH99, HM03, Hes06, Hes18]). The homotopy theory of cyclotomic spectra is more recent and was developed by Kaledin [Kal10, Kal13], Blumberg–Mandell [BM15], and Barwick–Glasman [BG16a]. Using cyclotomic spectra, Bhatt, Morrow, and Scholze give in [BMS19] a topological construction of (completed) prismatic cohomology theories, which generalize crystalline and $A\Omega$ cohomology. Their work relies on work of the second author with P. Scholze [NS18], which provides a simple description of bounded below cyclotomic spectra. In this paper, we give another way to understand cyclotomic spectra, which is better suited to answering the question: what are the building blocks of a cyclotomic spectrum?

1.1 Statement of results

Fix a prime number p . A p -typical Cartier module is an abelian group M equipped with endomorphisms V and F such that $FV = p$. The building blocks of p -typical cyclotomic spectra are, in a precise sense, certain p -typical Cartier modules.

Theorem 1 (see Theorems 2.1 and 3.26). *The ∞ -category $(\mathbf{CycSp}_p)_{\geq 0}$ of connective p -typical cyclotomic spectra is the connective part of a t -structure on \mathbf{CycSp}_p , the ∞ -category of p -typical cyclotomic spectra. The heart $\mathbf{CycSp}_p^\heartsuit$ is equivalent to the abelian category of derived V -complete Cartier modules.*

We say that a p -typical Cartier module M is derived V -complete if the natural map $M \rightarrow \lim_n M/V^n$ is an equivalence in $\mathcal{D}(\mathbb{Z})$.¹

The existence and uniqueness of such a t -structure is a formal consequence of the fact that $(\mathbf{CycSp}_p)_{\geq 0}$ is presentable and is closed under colimits and extensions in \mathbf{CycSp}_p . The difficult part of the theorem is the identification of the heart.

Recall Bökstedt’s theorem, which says that $\pi_*\mathrm{THH}(\mathbb{F}_p) = \mathbb{F}_p[b]$, a polynomial ring on a degree 2 generator. More generally, using the vanishing of the cotangent complex, one deduces that $\pi_*\mathrm{THH}(k) = k[b]$ for any perfect ring k . Our interest in the cyclotomic t -structure was piqued by the discovery of the next result.

¹Unless otherwise specified, all quotients M/V^n are computed in the derived sense (and hence are given as the cofiber of $M \xrightarrow{V^n} M$ in the derived category) as are all limits.

Theorem 2 (see Theorem 6.14). *If k is a perfect ring of characteristic p , then $\mathrm{THH}(k) \in \mathbf{CycSp}_p^\heartsuit$.*

Despite the higher homotopy groups, $\mathrm{THH}(k)$ is discrete as a cyclotomic spectrum. On the Cartier module side of the story, when k is a perfect ring of characteristic p , $\mathrm{THH}(k)$ corresponds to $W(k)$, the ring of p -typical Witt vectors over k , with its Witt vector Verschiebung and Frobenius operations. The fact that $\mathrm{THH}(k)$ is in $\mathbf{CycSp}_p^\heartsuit$ is consistent with the fact, due to Hesselholt–Madsen [HM97, Theorem B] for perfect fields of characteristic p , that $\pi_i \mathrm{TC}(k) = 0$ for $i > 0$. However, the theorem is much stronger. It says that for any cyclotomic spectrum X with $\pi_i X = 0$ for $i < 0$ one has $\mathrm{Hom}_{\mathbf{CycSp}_p}(X[i], \mathrm{THH}(k)) = 0$ for $i > 0$. To reconcile the fact that $\mathrm{THH}(k)$ is not at all discrete as a spectrum or even as a spectrum with S^1 -action with the fact that it is discrete as a p -typical cyclotomic spectrum, observe that the S^1 -equivariant map $b: \mathrm{THH}(k)[2] \rightarrow \mathrm{THH}(k)$ is not a map of cyclotomic spectra.

For any p -typical cyclotomic spectrum X , the homotopy groups with respect to the t -structure of Theorem 1 are denoted by $\pi_i^{\mathrm{cyc}} X$. These are objects of $\mathbf{CycSp}_p^\heartsuit \subseteq \mathbf{CycSp}_p$. Thus, they can be considered either as p -typical cyclotomic spectra, with underlying spectrum with S^1 -action and Frobenius $\varphi: \pi_i^{\mathrm{cyc}} X \rightarrow (\pi_i^{\mathrm{cyc}} X)^{tC_p}$, or as derived V -complete p -typical Cartier modules under the equivalence of Theorem 1. We will typically not distinguish notationally between these two points of view. Write $\mathbf{CycSp}_{[m,n]} = (\mathbf{CycSp}_p)_{\geq m} \cap (\mathbf{CycSp}_p)_{\leq n}$. The objects of $\mathbf{CycSp}_{[m,n]}$ are p -typical cyclotomic spectra X , which are bounded with respect to the cyclotomic t -structure, and such that $\pi_i^{\mathrm{cyc}} X \simeq 0$ for $i \notin [m, n]$.

Now, we discuss some of the consequences for schemes, especially in characteristic p . Recall the classical Hochschild–Kostant–Rosenberg theorem [HKR62], which states that when k is a commutative ring and R is a smooth k -algebra, there is a natural isomorphism $\mathrm{HH}_i(R/k) \cong \Omega_{R/k}^i$. Say that a commutative k -algebra R is ind-smooth if it is a filtered colimit of smooth k -algebras. If R is ind-smooth, then $L_{R/k}$ is a flat R -algebra and the HKR theorem continues to hold. The next result is a reinterpretation of a theorem of Hesselholt [Hes96, Theorem C] in the context of the cyclotomic t -structure.

Theorem 3 (see Theorem 6.18). *Let k be a perfect ring of characteristic p . If R is an ind-smooth commutative k -algebra, then for each i there is a natural isomorphism $\pi_i^{\mathrm{cyc}} \mathrm{THH}(R) \cong W\Omega_R^i$ of Cartier modules, where $W\Omega_R^i$ is the i th term in the de Rham–Witt complex of R .*

In particular, if R is smooth and has relative dimension d over a perfect ring of characteristic p , then $W\Omega_R^i \cong 0$ for $i > d$ and hence $\mathrm{THH}(R) \in \mathbf{CycSp}_{[0,d]}$.

Let k be a commutative ring. Then, $\mathbf{CycSp}_{\mathrm{THH}(k)} = \mathrm{Mod}_{\mathrm{THH}(k)}(\mathbf{CycSp}_p)$ admits a t -structure with $(\mathbf{CycSp}_{\mathrm{THH}(k)})_{\geq 0} \simeq \mathrm{Mod}_{\mathrm{THH}(k)}((\mathbf{CycSp}_p)_{\geq 0})$. We let $\mathbf{CycSp}_{\mathrm{THH}(k)}^\heartsuit$ denote the heart, which we identify in Corollary 12.

Corollary 4 (see Proposition 6.12 and Theorem 6.18). *If X is smooth and quasi-compact over a perfect field k of characteristic p , then there is a convergent spectral sequence*

$$E_2^{s,t} \cong H^{-s}(X, W\Omega_{\mathcal{O}_X}^t) \Rightarrow \pi_{s+t}^{\mathrm{cyc}} \mathrm{THH}(X)$$

in the abelian category $\mathbf{CycSp}_{\mathrm{THH}(k)}^\heartsuit$.

Corollary 5. *Let \mathcal{C} be a smooth and proper dg category over a commutative ring R which itself is smooth of relative dimension d over a perfect ring k of characteristic p . Then, $\mathrm{THH}(\mathcal{C})$ is bounded. Specifically, if $\mathrm{THH}(\mathcal{C}) \in (\mathbf{CycSp}_p)_{\geq -e}$, then it is in $\mathbf{CycSp}_{[-e, d+e]}$.*

Proof. Note that if \mathcal{C} is smooth and proper over R , then $\mathrm{THH}(\mathcal{C})$ is a perfect $\mathrm{THH}(R)$ -module spectrum and hence bounded below as a spectrum. It suffices to show that if $X \in (\mathbf{CycSp}_p)_{\geq d+e+1}$, then the mapping space $\mathrm{Map}_{\mathbf{CycSp}_p}(X, \mathrm{THH}(\mathcal{C})) \simeq 0$. However, this is equivalent to $\mathrm{Map}_{\mathbf{CycSp}_{\mathrm{THH}(R)}}(X \otimes \mathrm{THH}(R), \mathrm{THH}(\mathcal{C}))$ and hence to $\mathrm{Map}_{\mathbf{CycSp}_{\mathrm{THH}(R)}}(X \otimes \mathrm{THH}(\mathcal{C}), \mathrm{THH}(R))$ using that $\mathrm{THH}(\mathcal{C})$ is self dual over $\mathrm{THH}(R)$. Now, $X \otimes \mathrm{THH}(\mathcal{C})$ is contained in $(\mathbf{CycSp}_p)_{\geq d+1}$, so the result follows from the fact that $\mathrm{THH}(R) \in (\mathbf{CycSp}_p)_{\leq d}$. \square

Let R be a quasisyntomic ring in the sense of [BMS19, Definition 4.9], meaning that R is p -complete with bounded p^∞ -torsion and the cotangent complex L_{R/\mathbb{Z}_p} has p -complete Tor-amplitude contained in $[0, 1]$. Note that a characteristic p ring R is quasisyntomic if and only if L_{R/\mathbb{F}_p} has Tor-amplitude contained in $[0, 1]$.

For R quasisyntomic, Bhatt–Morrow–Scholze [BMS19] use syntomic descent to construct a filtration $F_{\text{BMS}}^* \text{TP}(R)$ on $\text{TP}(R)$ whose graded pieces are given by an absolute prismatic cohomology theory, suitably completed. In the special case where R is smooth over a perfect field of characteristic p , the graded pieces are given by crystalline cohomology $\text{R}\Gamma_{\text{crys}}(R/W)$ by [BMS19, Theorem 1.10].

Again in the case of an ind-smooth algebra R over a perfect field k of characteristic p , we can directly construct a filtration on $\text{TP}(R)$ whose graded pieces are given canonically by the de Rham–Witt complex of de Rham and Illusie (see [Ill79]) in the smooth affine case.

Corollary 6 (see Theorem 6.24). *Let k be a perfect field of characteristic p and let R be an ind-smooth k -algebra and let $\text{TP}(\tau_{\geq *}\text{THH}(R))$ be the filtration on $\text{TP}(R)$ induced by the cyclotomic Whitehead tower. The induced Whitehead tower with respect to the Beilinson t -structure on filtered spectra defines a natural complete exhaustive multiplicative decreasing \mathbb{Z} -indexed filtration $F_{\text{B}}^* \text{TP}(R)$ in filtered spectra with graded pieces given by*

$$\text{gr}_{\text{B}}^i \text{TP}(R) \simeq \text{W}\Omega_R^*[2i]$$

for all i . Moreover, this filtration agrees with the BMS filtration on $\text{TP}(R)$ after forgetting the secondary filtration.

We call the filtration here the Beilinson filtration as it is constructed using the Beilinson t -structure following an idea of [Ant19]. Both the BMS filtration $F_{\text{BMS}}^* \text{TP}(R)$ and the Beilinson filtration $F_{\text{B}}^* \text{TP}(R)$ are equipped with secondary filtrations. For the BMS filtration, one obtains the Nygaard filtration $\mathcal{N}^{\geq *}\text{R}\Gamma(R/W)$ as the residual filtration on the graded pieces, while for the Beilinson filtration, one obtains the Hodge filtration $\text{W}\Omega_R^{\geq *}$ on the graded pieces. Forgetting these secondary filtrations, the Beilinson and BMS filtrations agree on $\text{TP}(R)$. By remembering the secondary filtration in $F_{\text{B}}^* \text{TP}(R)$, we recover a specific complex, the de Rham–Witt complex, which computes crystalline cohomology.

Example 7. Let X be a K3 surface over a perfect field k of characteristic p , meaning a smooth and proper surface over k such that the canonical bundle $\Omega_{X/k}^2$ is trivial and $H^1(X, \mathcal{O}_X) = 0$. The formal Brauer group of X is, by the work of Artin and Mazur [AM77], a commutative formal Lie group. The nature of the formal Brauer group stratifies K3 surfaces over k into two types: the **Hodge–Witt** K3 surfaces, for which $\widehat{\text{Br}}_X$ is a formal p -divisible group of height h where $1 \leq h \leq 10$, and the **supersingular** K3 surfaces, where $\widehat{\text{Br}}_X$ is unipotent (and even isomorphic to $\widehat{\mathbb{G}}_a$, the formal completion of the additive group). When X is supersingular, the p -typical Cartier module $H^2(X, \text{W}\Omega_X^0)$ associated to $\widehat{\text{Br}}_X$ is isomorphic to $\mathbb{F}_p[[x]]$, the p -typical Cartier module with $F = 0$, $Vx^n = x^{n+1}$, and $ax = xa^p$ (see [Ill79, 7.2]). The spectral sequence of Corollary 4 implies that $\pi_{-2}^{\text{cyc}} \text{THH}(X) \cong H^2(X, \text{W}\Omega_X^0)$ since there can be no differentials in or out. Now, the spectrum underlying $\pi_{-2}^{\text{cyc}}(\text{THH}(X))$ has homotopy groups k in degree 0 and $k[[x]]$ in degrees ≥ 2 by Figure 1 on page 6. In particular, it is not compact and hence not dualizable as a $\text{THH}(k)$ -module spectrum. Since the forgetful functor $\mathbf{CycSp}_{\text{THH}(k)} \rightarrow \mathcal{D}(\text{THH}(k))$ is symmetric monoidal, it follows that $\pi_{-2}^{\text{cyc}} \text{THH}(X)$ is not dualizable as a cyclotomic spectrum when X is a supersingular K3 surface. Thus, we see that the t -structure on $\mathbf{CycSp}_{\text{THH}(k)}$ does not restrict to a t -structure on $\mathbf{CycSp}_{\text{THH}(k)}^{\text{dual}}$, the full subcategory of dualizable objects in $\mathbf{CycSp}_{\text{THH}(k)}$. Moreover, as this example shows, $\text{THH}(X)$ cannot be perfect in $\mathbf{CycSp}_{\text{THH}(k)}$. Indeed, if it were perfect, then each cyclotomic homotopy group $\pi_i^{\text{cyc}} \text{THH}(X)$ would be a finitely presented $W(k)$ -module, which this example shows is not the case. This recovers the counterexample of [AMN18].

In general, despite the coincidence of the filtrations on $\text{TP}(R)$ when R is a smooth ring over a perfect field k of characteristic p , the BMS filtration and the cyclotomic Whitehead tower do not agree on $\text{THH}(R)$ when R is quasisyntomic. In fact, even when $R = k$ is a perfect field they differ. Indeed, the BMS filtration

is not a filtration by cyclotomic spectra. We have seen by Theorem 2 that the cyclotomic Whitehead tower for $\mathrm{THH}(k)$ is concentrated in a single degree. But, the BMS filtration is given by $\tau_{\geq 2\star}\mathrm{THH}(k)$, i.e., the classical Whitehead tower. The maps $\tau_{\geq 2\star}\mathrm{THH}(k) \rightarrow \mathrm{THH}(k)$ cannot be given the structure of cyclotomic maps. Nevertheless, we have the following result, which Scholze suggested to us.

Theorem 8 (see Theorem 6.23). *The BMS filtration and the t -structure filtration on $\mathrm{TC}(X)$ agree when X is a smooth quasi-compact scheme over a perfect field k of characteristic p .*

1.2 Idea of proofs: topological Cartier modules

To prove our theorems, we introduce the stable ∞ -category \mathbf{TCart}_p of p -typical topological Cartier modules.² These are spectra M equipped with an S^1 -action together with an S^1 -equivariant factorization

$$M_{hC_p} \xrightarrow{V} M \xrightarrow{F} M^{hC_p}$$

of the C_p -norm

$$M_{hC_p} \xrightarrow{\mathrm{Nm}_{C_p}} M^{hC_p}.$$

Maps are defined as usual to be maps of spectra with S^1 -action which commute with the V and F operations and the homotopy witnessing the factorization.

There is a t -structure on \mathbf{TCart}_p where an object M is in $(\mathbf{TCart}_p)_{\geq 0}$ (resp. $(\mathbf{TCart}_p)_{\leq 0}$) if and only if $\pi_i M = 0$ for $i < 0$ (resp. $\pi_i M = 0$ for $i > 0$). The heart of this t -structure is the abelian category of p -typical Cartier modules introduced above. Given a p -typical topological Cartier module M , the cofiber of V , which we write as M/V , naturally admits the structure of a p -typical cyclotomic spectrum. On bounded below objects with respect to this t -structure on \mathbf{TCart}_p and the cyclotomic t -structure, we obtain the following theorem.

Theorem 9 (see Theorem 3.21). *The functor $(-)/V: \mathbf{TCart}_p^- \rightarrow \mathbf{CycSp}_p^-$ admits a fully faithful t -exact right adjoint given by TR . The essential image of TR is the full subcategory of bounded below p -typical topological Cartier modules M such that $\pi_i M$ is derived V -complete for all i .*

From Theorem 9, we can read off two things: first, that the heart of the t -structure on cyclotomic spectra is given by derived V -complete p -typical topological Cartier modules; second, that for a cyclotomic spectrum X , the cyclotomic homotopy groups $\pi_i^{\mathrm{cyc}} X$ are given by $\pi_i \mathrm{TR}(X)$ equipped with the canonical V and F operations induced from transfer and inclusion of fixed points. This gives the proof of Theorem 1. Previous calculations in TR of commutative rings of Hesselholt–Madsen [HM97, Theorem 5.5] and Hesselholt [Hes96, Theorem C] then suffice to establish Theorems 2 and 3 for smooth algebras over perfect fields, which we show is enough to prove the general case of each theorem. The next three corollaries are immediate consequences.

Corollary 10. *If X is a bounded below p -typical cyclotomic spectrum, then the natural S^1 -equivariant map $\mathrm{TR}(X) \rightarrow X$ induces a p -adic equivalence $\mathrm{TR}(X)^{tS^1} \rightarrow X^{tS^1}$.*

Proof. By Theorem 9, the counit map $\mathrm{TR}(X)/\mathrm{TR}(X)_{hC_p} \rightarrow X$ is an equivalence. In particular, we have a cofiber sequence

$$\mathrm{TR}(X)_{hC_p} \rightarrow \mathrm{TR}(X) \rightarrow X.$$

Applying $(-)^{tS^1}$, we obtain a cofiber sequence p -adically equivalent to

$$((\mathrm{TR}(X)_{hC_p})^{tC_p})^{hS^1} \rightarrow (\mathrm{TR}(X)^{tC_p})^{hS^1} \rightarrow (X^{tC_p})^{hS^1}$$

by [NS18, Lemma II.4.2]. But, $(\mathrm{TR}(X)_{hC_p})^{tC_p} \simeq 0$ by the Tate orbit lemma [NS18, Lemma I.2.1]. \square

²In early talks on this project, we called these topological Dieudonné modules.

Corollary 11. *If X is a bounded below p -typical cyclotomic spectrum such that $\mathrm{TR}(X) \simeq 0$, then $X \simeq 0$.*

Proof. Indeed, since $\mathrm{TR}: \mathbf{CycSp}_p^- \rightarrow \mathbf{TCart}_p^-$ is fully faithful, the adjoint $\mathrm{TR}(X)/V \rightarrow X$ is an equivalence, so $X \simeq 0$. \square

By [HM97, Theorem F], which says that $\pi_0 \mathrm{TR}(k) \cong W(k)$ when k is a commutative ring, and using Theorem 9, we obtain the following corollary.

Corollary 12 (see Theorem 6.1). *For any commutative ring k , $\pi_0^{\mathrm{cyc}} \mathrm{THH}(k) \simeq W(k)$. Moreover, $\mathbf{CycSp}_{\mathrm{THH}(k)}^\heartsuit$ is equivalent to the abelian category of derived V -complete $W(k)$ -modules in p -typical Cartier modules.*

Example 13. The cyclotomic homotopy groups $\pi_i^{\mathrm{cyc}} X$ of a cyclotomic spectrum X are given by derived V -complete p -typical Cartier modules. In particular, any derived V -complete p -typical Cartier module M has an underlying cyclotomic spectrum M/V . Its homotopy groups are computed using the cofiber sequence $M_{hC_p} \xrightarrow{V} M \rightarrow M/V$. They are given in Figure 1.

$$\pi_i M/V \cong \begin{cases} \mathrm{coker}(M \xrightarrow{V} M) & \text{if } i = 0, \\ \mathrm{ker}(M \xrightarrow{V} M) & \text{if } i = 1, \\ \mathrm{coker}(M \xrightarrow{p} M) & \text{if } i \geq 2 \text{ is even, and} \\ \mathrm{ker}(M \xrightarrow{p} M) & \text{if } i \geq 2 \text{ is odd.} \end{cases}$$

Figure 1: This table gives the homotopy groups of the cyclotomic spectrum M/V if M is a p -typical Cartier module (i.e., an object of $\mathbf{TCart}_p^\heartsuit$).

Example 14. The theory of p -typical Cartier modules arises in the study of commutative formal groups over commutative $\mathbb{Z}_{(p)}$ -algebras (see [Zin84]). We describe briefly the connection in characteristic p . Let k be a perfect field of characteristic p . Because $VF = p$ on $W(k)$, one finds that $VF = p$ on any $W(k)$ -module in $\mathbf{TCart}_p^\heartsuit$ (see Remark 6.16). A Dieudonné module is an abelian group M with endomorphisms F and V such that $FV = VF = p$. Let G be a finite flat group scheme and let $M(G)$ denote the covariant Dieudonné module (the dual of the construction in [Dem72, Chapter III]). We find that if G is infinitesimal (like α_p) or multiplicative (like μ_p), then V is nilpotent and thus $M(G)$ is derived V -complete. Similarly, if G is a formal p -divisible group, then $M(G)$ is a derived V -complete Dieudonné module. In particular, we obtain a functor

$$\{\text{formal } p\text{-divisible groups over } k\} \rightarrow \mathbf{CycSp}_{\mathrm{THH}(k)}^\heartsuit.$$

We will see in [AN] that this functor induces an equivalence between the bounded derived ∞ -category of isogeny classes of p -divisible groups and isogeny classes of dualizable cyclotomic spectra over perfect fields.

The t -structures on \mathbf{TCart}_p and \mathbf{CycSp}_p are compatible with the natural symmetric monoidal structures, so we obtain induced symmetric monoidal structures on the abelian categories of p -typical Cartier modules and derived V -complete p -typical Cartier modules, respectively. When k is a perfect field of characteristic p , Goerss [Goe99] had previously constructed a symmetric monoidal structure on $W(k)$ -modules in p -typical Cartier modules. We prove in Section 4 that our symmetric monoidal structure agrees with his in this case. In future work we will study these symmetric monoidal structures more closely and deduce generalizations of the HKR theorem (Theorem 3) to non-perfect rings.

Remark 15. In Section 5, we show that p -typical topological Cartier modules are to genuine fixed points as genuine cyclotomic spectra are to geometric fixed points. Let $\mathbb{TSp}_p^{\text{gen}}$ denote the ∞ -category of genuine S^1 -spectra with respect to the finite p -subgroups of S^1 . The ∞ -category \mathbf{TCart}_p is equivalent to $\text{Fix}_{(-)^{C_p}}(\mathbb{TSp}_p^{\text{gen}})$, the ∞ -category of fixed points for the endofunctor $(-)^{C_p}$ of $\mathbb{TSp}_p^{\text{gen}}$. In particular, an equivalent way of defining an object of \mathbf{TCart}_p is to give a genuine S^1 -spectrum $M \in \mathbb{TSp}_p^{\text{gen}}$ together with an equivalence $M^{C_p} \simeq M$ of genuine S^1 -spectra.

Outline. In Section 2, we prove the existence of the t -structure of Theorem 1. We prove some basic, but important, properties of the cyclotomic t -structure. Section 3 introduces p -typical topological Cartier modules and establishes Theorem 9. In Section 4, we compare the natural symmetric monoidal structure on $\mathbf{CycSp}_p^\heartsuit$ with the monoidal structure previously constructed by Goerss [Goe99] on $W(k)$ -modules in p -typical Cartier modules when k is a perfect field of characteristic p . We study some genuine equivariant homotopy-theoretic aspects of the story in Section 5. Finally, Section 6 contains our applications to THH of rings and schemes. Appendix A gives some background on t -structures.

Conventions. We will freely use the theory of ∞ -categories developed by Lurie in [Lur09, Lur17, Lur18]. Unless otherwise mentioned, we work with cyclotomic spectra as studied in [NS18]. We will make one important deviation from the notation in *op. cit.* Namely, a p -typical cyclotomic spectrum will be a spectrum $X \in \text{Sp}^{BS^1}$ with S^1 -action equipped with an S^1 -equivariant map $\varphi_p: X \rightarrow X^{tC_p}$, where X^{tC_p} carries the residual $S^1/C_p \cong S^1$ -action. See Remark 2.5 for more about this choice. We write \mathbf{CycSp}_p for the stable ∞ -category of p -typical cyclotomic spectra.

Notation. Let \mathcal{C} be an ∞ -category with objects $x, y \in \mathcal{C}$. We will write $\text{Map}_{\mathcal{C}}(x, y)$ for the ∞ -groupoid (space) of maps from x to y in \mathcal{C} . If \mathcal{C} is stable, we will write $\mathbf{Map}_{\mathcal{C}}(x, y)$ for the mapping *spectrum* from x to y . Given an \mathbb{E}_∞ -ring spectrum R , let $\mathcal{D}(R) \simeq \text{Mod}_R(\text{Sp})$ denote the stable ∞ -category of R -module spectra. If R is connective, then we equip $\mathcal{D}(R)$ with the canonical Postnikov t -structure, where $\mathcal{D}(R)^\heartsuit \simeq \text{Mod}_{\pi_0 R}$, the abelian category of $\pi_0 R$ -modules. Unless specified otherwise, all limits, colimits, and tensor products are computed in $\mathcal{D}(R)$. This convention holds even for discrete rings; thus, limits and colimits of abelian groups are computed in $\mathcal{D}(\mathbb{Z})$ as opposed to $\text{Mod}_{\mathbb{Z}}$: quotients are given by cofibers, limits by derived limits, and the tensor product by the derived tensor product. For example, if M is an abelian group, then M/p is the cofiber of $M \xrightarrow{p} M$. Hence, M/p is an object of $\mathcal{D}(\mathbb{Z})$ with $\pi_i M/p = 0$ for $i \neq 0, 1$, $\pi_1 M/p \cong \ker(M \xrightarrow{p} M) \cong \text{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/p)$, and $\pi_0 M/p \cong \text{coker}(M \xrightarrow{p} M) \cong \text{Tor}_0^{\mathbb{Z}}(M, \mathbb{Z}/p)$. Note that the objects of $\mathcal{D}(\mathbb{Z})$ can be modeled by either chain complexes or \mathbb{Z} -module spectra. We will typically write $\pi_* M$ for the homotopy groups of M viewed as a spectrum; these are isomorphic to the homology groups $H_* M$ when M is viewed as a chain complex.

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2 The cyclotomic t -structure

In this section we define the **cyclotomic t -structure** for integral and p -typical cyclotomic spectra in their genuine and non-genuine flavors. With some difficulty, one can prove some basic facts about truncations in the cyclotomic t -structure, including for example the fact that $\mathrm{THH}(\mathbb{F}_p)$ is in the heart. By working instead with topological Cartier modules, introduced in Section 3, such computations are more transparent. Hence, in this section, we restrict ourselves to discussing formal properties of the cyclotomic t -structure.

For the necessary background on t -structures, see Appendix A.

2.1 The cyclotomic t -structure

There is a t -structure on Sp^{BS^1} , the ∞ -category of spectra with S^1 -action, where the connective objects are the *connective* spectra with S^1 -action. This is reviewed in Proposition A.17. The heart is the abelian category of abelian groups since BS^1 is simply connected.

We let \mathbf{CycSp}_p be the lax equalizer

$$\mathrm{LEq}(\mathrm{id}, (-)^{tC_p} : \mathrm{Sp}^{BS^1} \rightrightarrows \mathrm{Sp}^{BS^1}).$$

(For background on lax equalizers, see [NS18, Section II.1].) An object of \mathbf{CycSp}_p is a spectrum X with S^1 -action and an S^1 -equivariant map $\varphi : X \rightarrow X^{tC_p}$, where X^{tC_p} carries the residual $S^1 \cong S^1/C_p$ -action.

Let $(\mathbf{CycSp}_p)_{\geq 0} \subseteq \mathbf{CycSp}_p$ denote the full subcategory of p -typical cyclotomic spectra X such that the underlying spectrum is connective. In this case, the cyclotomic structure map $\varphi : X \rightarrow X^{tC_p}$ factors canonically through the connective cover $\tau_{\geq 0}(X^{tC_p})$.

Recall that a t -structure on a stable ∞ -category \mathcal{C} is **compatible with a symmetric monoidal structure \mathcal{C}^{\otimes} on \mathcal{C}** if $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ is closed under tensor products and the unit object of \mathcal{C} is in $\mathcal{C}_{\geq 0}$. The purpose of this section is to prove the following theorem.

Theorem 2.1. *The ∞ -category $(\mathbf{CycSp}_p)_{\geq 0}$ forms the connective part of an accessible, left complete t -structure on \mathbf{CycSp}_p , which is compatible with the symmetric monoidal structure on \mathbf{CycSp}_p .*

Example 2.2. If X is a connective spectrum, then $X^{\mathrm{triv}} \in (\mathbf{CycSp}_p)_{\geq 0}$. Recall from [NS18, Example II.1.2 and Section IV.4] that X^{triv} is the cyclotomic spectrum with trivial S^1 -action and cyclotomic Frobenius given by the composition $X \rightarrow X^{hC_p} \rightarrow X^{tC_p}$. The cyclotomic sphere spectrum $\mathbb{S}^{\mathrm{triv}}$ is the unit object for the natural symmetric monoidal structure on \mathbf{CycSp}_p (see [NS18, Section IV.2]).

Example 2.3. If R is a connective \mathbb{E}_1 -ring spectrum, then $\mathrm{THH}(R) \in (\mathbf{CycSp}_p)_{\geq 0}$.

By construction, the forgetful functor $\mathbf{CycSp}_p \rightarrow \mathrm{Sp}^{BS^1}$ is right t -exact. It is not left t -exact or even left bounded: as we will see $\mathrm{THH}(\mathbb{F}_p)$ is in the heart in \mathbf{CycSp}_p but is not bounded above when viewed as an object of Sp^{BS^1} .

VARIANT 2.4. The same arguments that we give to prove Theorem 2.1 will work for \mathbf{CycSp} , the ∞ -category of global cyclotomic spectra, for $\mathbf{CycSp}_R = \mathrm{Mod}_R(\mathbf{CycSp})$ where R is a connective \mathbb{E}_{∞} -algebra in cyclotomic spectra, and for $\mathrm{Mod}_R(\mathbf{CycSp}_p)$ when R is a connective \mathbb{E}_{∞} -algebra in p -typical cyclotomic spectra. In all of these cases the connective part of the t -structure consists of the full subcategory of cyclotomic (or p -typical cyclotomic) R -modules X for which the underlying spectrum with S^1 -action is connective.

Remark 2.5. In this paper we shall concentrate on the p -typical aspects of the theory. The integral case will be pursued in future work. For this reason, we write $\mathrm{TC}(X)$ and $\mathrm{TR}(X)$ for p -typical versions of TC and TR . Since our definition of p -typical cyclotomic spectra differs slightly from that of [NS18], note that for us,

$\mathrm{TC}(-)$ is the theory representable by $\mathbb{S}^{\mathrm{triv}}$ in \mathbf{CycSp}_p ; hence, $\mathrm{TC}(X)$ is the equalizer of $X^{hS^1} \rightrightarrows (X^{tC_p})^{hS^1}$, where the two maps are given by the canonical map $X^{hS^1} \rightarrow X^{tS^1} \rightarrow (X^{tC_p})^{hS^1}$ and by φ^{hS^1} .

We let $\mathrm{TR}^{n+1}(X)$ be defined as the iterated pullback

$$\mathrm{TR}^{n+1}(X) = X^{hC_{p^n}} \times_{(X^{tC_p})^{hC_{p^{n-1}}}} \cdots \times_{(X^{tC_p})^{hC_p}} X^{hC_p} \times_{X^{tC_p}} X$$

where the maps to the left are induced by φ and the maps to the right are the canonical maps. When X is bounded below, the methods of [NS18, Chapter II] endow X with the structure of an S^1 -spectrum which is genuine with respect to the subgroups C_{p^n} for $n \geq 0$ and $\mathrm{TR}^{n+1}(X) \simeq X^{C_{p^n}}$. There are natural maps $R: \mathrm{TR}^{n+1}(X) \rightarrow \mathrm{TR}^n(X)$ given by forgetting the first factor in the above pullback and $\mathrm{TR}(X) \simeq \lim \mathrm{TR}^{n+1}(X)$. In general, there is a map $\mathrm{TC}(X) \rightarrow \mathrm{TR}(X)$ which induces a p -adic equivalence $\mathrm{TC}(X) \rightarrow \mathrm{fib}(\mathrm{TR}(X) \xrightarrow{1-F} \mathrm{TR}(X))$.

Warning 2.6. Note that $\mathrm{TC}(X, p)$ is typically defined as $\mathrm{fib}(\mathrm{TR}(X) \xrightarrow{1-F} \mathrm{TR}(X))$, so our definition agrees with $\mathrm{TC}(X, p)$ only after p -completion. Rather, $\mathrm{fib}(\mathrm{TR}(X) \xrightarrow{1-F} \mathrm{TR}(X))$ is equivalent to the equalizer of the canonical and Frobenius maps $X^{hC_{p^\infty}} \rightrightarrows (X^{tC_p})^{hC_{p^\infty}}$.

Definition 2.7. For an object $X \in \mathbf{CycSp}_p$, we will write $\pi_i^{\mathrm{cyc}} X \in \mathbf{CycSp}_p^\heartsuit$ for the i th homotopy object of X . Note that this is a cyclotomic spectrum and hence has an underlying spectrum with S^1 -action. The homotopy groups of this underlying spectrum are $\pi_*(\pi_i^{\mathrm{cyc}} X)$.

Example 2.8. If $X \in \mathbf{CycSp}_p$, a necessary condition for $X \in (\mathbf{CycSp}_p)_{\leq 0}$ is that $\pi_i \mathrm{TC}(X) = 0$ for $i > 0$ since $\mathbb{S}^{\mathrm{triv}} \in (\mathbf{CycSp}_p)_{\geq 0}$ and $\mathrm{TC}(X)$ is the mapping spectrum from $\mathbb{S}^{\mathrm{triv}}$ to X . Thus, $\mathbb{Z}^{\mathrm{triv}}$ is not in $\mathbf{CycSp}_p^\heartsuit$, because $\pi_i \mathrm{TC}(\mathbb{Z}^{\mathrm{triv}}) \cong \mathbb{Z}_p$ for all positive odd i .

Warning 2.9. This t -structure is not compatible with filtered colimits. In particular, the heart $\mathbf{CycSp}_p^\heartsuit$ is not closed under filtered colimits in $(\mathbf{CycSp}_p)_{\geq 0}$. See Example 3.27 for details.

Lemma 2.10. *The following diagram*

$$\begin{array}{ccc} (\mathrm{Sp}^{BS^1})_{\geq 0} & \xrightleftharpoons[\tau_{\geq 0}(-)^{tC_p}]{\mathrm{id}} & (\mathrm{Sp}^{BS^1})_{\geq 0} \\ \downarrow & & \downarrow \\ \mathrm{Sp}^{BS^1} & \xrightleftharpoons[(-)^{tC_p}]{\mathrm{id}} & \mathrm{Sp}^{BS^1} \end{array}$$

induces via the natural transformations of functors $\tau_{\geq 0}(-)^{tC_p} \rightarrow (-)^{tC_p}$ an equivalence

$$\mathrm{LEq}\left(\mathrm{id}, \tau_{\geq 0}(-)^{tC_p}: (\mathrm{Sp}^{BS^1})_{\geq 0} \rightrightarrows (\mathrm{Sp}^{BS^1})_{\geq 0}\right) \simeq (\mathbf{CycSp}_p)_{\geq 0}.$$

In particular, $(\mathbf{CycSp}_p)_{\geq 0}$ is presentable and the inclusion functor $(\mathbf{CycSp}_p)_{\geq 0} \rightarrow \mathbf{CycSp}_p$ preserves colimits.

Proof. The equivalence is clear given that for a connective cyclotomic spectrum X the cyclotomic structure map determines and is determined by its factorization through the connective cover. We can appeal to [NS18, II.1.5(3)] which gives presentability (since $\tau_{\geq 0}(-)^{tC_p}$ is accessible). Now, both forgetful functors $(\mathbf{CycSp}_p)_{\geq 0} \rightarrow (\mathrm{Sp}^{BS^1})_{\geq 0}$ and $\mathbf{CycSp}_p \rightarrow \mathrm{Sp}^{BS^1}$ preserve and detect colimits. Since $(\mathrm{Sp}^{BS^1})_{\geq 0} \rightarrow \mathrm{Sp}^{BS^1}$ is closed under colimits, the claim about preservation of colimits follows. \square

Proof of Theorem 2.1. By definition, $(\mathbf{CycSp}_p)_{\geq 0}$ is closed under extensions in \mathbf{CycSp}_p . Combined with Lemma 2.10, it follows from [Lur17, 1.4.4.11] that there exists a unique t -structure $((\mathbf{CycSp}_p)_{\geq 0}, (\mathbf{CycSp}_p)_{\leq 0})$ on \mathbf{CycSp}_p . The cyclotomic t -structure is accessible because $(\mathbf{CycSp}_p)_{\geq 0}$ is presentable by Lemma 2.10.

It is clear that \mathbf{CycSp}_p is left separated: an object in

$$\bigcap_{n \in \mathbb{Z}} (\mathbf{CycSp}_p)_{\geq n}$$

has contractible underlying S^1 -spectrum. We check that it is left complete by showing that $(\mathbf{CycSp}_p)_{\geq 0}$ is closed under countable products in \mathbf{CycSp}_p and applying [Lur17, 1.2.1.19], which says that under this hypothesis left completeness is equivalent to left separatedness. In fact, we will show that $(\mathbf{CycSp}_p)_{\geq 0}$ is closed under all products in \mathbf{CycSp}_p .

We already know that $(\mathrm{Sp}^{BS^1})_{\geq 0}$ is closed under products in Sp^{BS^1} by Proposition A.17. Thus, to conclude, it is enough to show that the functor $\mathbf{CycSp} \rightarrow \mathrm{Sp}^{BS^1}$ commutes with products of connective objects. By [NS18, II.1.5(v)], it is enough to see that for a product $\prod_{i \in I} X_i$ of connective spectra with S^1 -action the natural map

$$\left(\prod_i X_i \right)^{tC_p} \rightarrow \prod_i X_i^{tC_p}$$

is an equivalence. This follows from Lemma 2.11 below.

To see that the cyclotomic t -structure is compatible with the symmetric monoidal structure on \mathbf{CycSp}_p we simply note that $\mathbb{S}^{\mathrm{triv}} \in (\mathbf{CycSp}_p)_{\geq 0}$ and that $(\mathbf{CycSp}_p)_{\geq 0}$ is closed under the tensor product, which follows from the fact that the tensor product of two connective spectra with S^1 -action is again connective. This completes the proof. \square

We used the following lemma in the proof.

Lemma 2.11. *Suppose that $F: I \rightarrow \mathrm{Sp}^{BS^1}$ is an I -diagram in spectra with S^1 -action with limit $X = \lim_i F(i)$.*

- (a) *If $I = \mathbb{Z}^{\mathrm{op}}$, so that X is the limit of the tower $\cdots \rightarrow F(i+1) \rightarrow F(i) \rightarrow \cdots$, and if the fiber of $X \rightarrow F(i)$ is n_i -connective where $n_i \rightarrow \infty$ as $i \rightarrow \infty$, then $X_{hC_p} \simeq \lim_i (F(i)_{hC_p})$ and $X^{tC_p} \simeq \lim_i (F(i)^{tC_p})$.*
- (b) *If there exists d such that $\lim_I: \mathrm{Sp}^I \rightarrow \mathrm{Sp}$ sends $\mathrm{Sp}_{\geq 0}^I$ to $\mathrm{Sp}_{\geq -d}$ and there exists N such that each $F(i)$ is N -connective, then $X_{hC_p} \simeq \lim_i (F(i)_{hC_p})$ and $X^{tC_p} \simeq \lim_i (F(i)^{tC_p})$.*

Example 2.12. If $I = \mathbb{Z}^{\mathrm{op}}$ so that X is a sequential limit, then (b) applies since $d = 1$. If I is discrete, so that X is a product, then (b) applies since $d = 0$.

Proof of Lemma 2.11. Each statement for the Tate construction follows from the corresponding statement for homotopy orbits using the fiber sequence $X_{hC_p} \rightarrow X^{hC_p} \rightarrow X^{tC_p}$ and the fact that X^{hC_p} commutes with all limits of spectra with C_p -action.

The proof of (a) is the same as the proof of [NS18, Lemma I.2.6]. It is easy enough to repeat here: the fiber of $X_{hC_p} \rightarrow F(i)_{hC_p}$ is n_i -connective since taking homotopy orbits is right t -exact. Thus, the limit of the fibers vanishes.

For (b), consider for each n the map $\tau_{\leq n} X_{hC_p} \rightarrow \lim_I \tau_{\leq n} F(i)_{hC_p}$. The functor $\tau_{\leq n}(-)_{hC_p}$ is computed as a colimit over the skeleton of BC_p on uniformly bounded below objects, so it commutes with I -limits of uniformly bounded below objects. Hence, $\lim_I \tau_{\leq n} F(i)_{hC_p} \simeq (\tau_{\leq n}(\lim_I F(i)))_{hC_p} \simeq \lim \tau_{\leq n} X_{hC_p}$. Taking the limit over n on both sides, we obtain the desired equivalence. \square

Remark 2.13. The heart $\mathbf{CycSp}_p^\heartsuit$ is by definition $(\mathbf{CycSp}_p)_{\geq 0} \cap (\mathbf{CycSp}_p)_{\leq 0}$ and it is an abelian category (by [BBD82, Théorème 1.3.6]). In fact, $\mathbf{CycSp}_p^\heartsuit} \subseteq (\mathbf{CycSp}_p)_{\geq 0}$ is the full subcategory of 0-truncated objects (see [Lur17, 1.2.1.9]), so it is presentable because $(\mathbf{CycSp}_p)_{\geq 0}$ is presentable. Truncation $\tau_{\leq 0}: \mathcal{C}_{\geq 0} \rightarrow \mathcal{C}^\heartsuit$ gives a left adjoint to the inclusion. See [Lur09, 5.5.6.21] for details.

Because $(\mathbf{CycSp}_p)_{\geq 0}$ is closed under the tensor product in \mathbf{CycSp}_p , and since the unit \mathbb{S}^{triv} of \mathbf{CycSp}_p is connective, $(\mathbf{CycSp}_p)_{\geq 0}$ naturally inherits a symmetric monoidal structure such that the inclusion $(\mathbf{CycSp}_p)_{\geq 0} \rightarrow \mathbf{CycSp}_p^\heartsuit$ is symmetric monoidal. It is easy to see that $\mathbf{CycSp}_p^\heartsuit$ inherits a symmetric monoidal structure from $(\mathbf{CycSp}_p)_{\geq 0}$. For details, see A.2.

Corollary 2.14. *The abelian category $\mathbf{CycSp}_p^\heartsuit$ inherits a symmetric monoidal structure \otimes^\heartsuit from $(\mathbf{CycSp}_p)_{\geq 0}$ such that the localization functor $\pi_0^{\text{cyc}}: (\mathbf{CycSp}_p)_{\geq 0} \rightarrow \mathbf{CycSp}_p^\heartsuit$ is symmetric monoidal. Moreover, \otimes^\heartsuit is compatible with colimits in each variable.*

Proof. The first claim follows from Lemma A.12. To prove the second, we use that colimits in $\mathbf{CycSp}_p^\heartsuit$ are obtained by computing colimits in $(\mathbf{CycSp}_p)_{\geq 0}$ and applying π_0^{cyc} . Since the tensor product of connective cyclotomic spectra commutes with colimits in each variable and since $\pi_0^{\text{cyc}}: (\mathbf{CycSp}_p)_{\geq 0} \rightarrow (\mathbf{CycSp}_p)^\heartsuit$ commutes with colimits (being a left adjoint), the claim follows. \square

Remark 2.15. Concretely, if $A, B \in \mathbf{CycSp}_p^\heartsuit$, then $A \otimes^\heartsuit B \simeq \pi_0^{\text{cyc}}(A \otimes B)$, the 0-truncation of the tensor product of A and B when viewed as objects of $(\mathbf{CycSp}_p)_{\geq 0}$.

Example 2.16. The fiber F of $\mathbb{S}^{\text{triv}} \rightarrow \mathbb{Z}^{\text{triv}}$ in cyclotomic spectra is connected. Hence, $\pi_0^{\text{cyc}} F \simeq 0$ and it follows that $\pi_0^{\text{cyc}} \mathbb{S}^{\text{triv}} \simeq \pi_0^{\text{cyc}} \mathbb{Z}^{\text{triv}}$. In particular, every object of $\mathbf{CycSp}_p^\heartsuit$ is canonically a \mathbb{Z}^{triv} -module in \mathbf{CycSp}_p .

2.2 The genuine cyclotomic t -structure

In this section we introduce the t -structure on genuine cyclotomic spectra after introducing the Mackey t -structure on genuine S^1 -spectra. This will only become relevant for Section 5 and the reader who does not want to get involved with the intricacies of genuine homotopy theory can safely skip this section.

We will use genuine equivariant homotopy theory as a black box, but remind the reader that there is a stable presentable ∞ -category

$$\mathbb{TSp}_{\mathcal{F}}^{\text{gen}}$$

of genuine S^1 -spectra with respect to a family \mathcal{F} of subgroups of S^1 . Here we write \mathbb{T} for S^1 to distinguish it from the homotopical circle. A genuine spectrum $X \in \mathbb{TSp}_{\mathcal{F}}^{\text{gen}}$ has fixed points X^H for all closed subgroup $H \subseteq \mathbb{T}$ that lie in \mathcal{F} . If \mathcal{F} is the family just consisting of the trivial group $1 \subseteq \mathbb{T}$ then $\mathbb{TSp}_{\mathcal{F}}^{\text{gen}} \simeq \text{Sp}^{BS^1}$. We will mostly be concerned with the family consisting of finite p -subgroups of \mathbb{T} . In this case we write $\mathbb{TSp}_p^{\text{gen}}$ for this ∞ -category:

$$\mathbb{TSp}_p^{\text{gen}} := \mathbb{TSp}_{\{1, C_p, C_{p^2}, \dots\}}^{\text{gen}}.$$

There are several equivalent ways of describing \mathbb{TSp}_p . A treatment using equivariant orthogonal spectra is reviewed in [NS18] based on lecture notes by Schwede [Sch18]. An elegant ∞ -categorical model using spectral Mackey functors is due to Barwick [Bar17] and Barwick–Glasman [BG16a] based on a model of Guillou–May [GM11]. We assume the reader is familiar with basic constructions such as fixed points, geometric fixed points, classifying spaces for families, the tom Dieck splitting and the isotropy separation sequence. For the remainder of the section we shall work with genuine spectra $X \in \mathbb{TSp}_p^{\text{gen}}$ and for simplicity just refer to them as genuine S^1 -spectra or even genuine spectra.

Let X be a genuine S^1 -spectrum. We say that X is connective if for each $n \geq 0$ the fixed points spectrum $X^{C_{p^n}}$ is connective. Similar we say that X is coconnective if for each $n \geq 0$ the spectrum $X^{C_{p^n}}$ is coconnective. This defines two full subcategories

$$(\mathbb{TSp}_p^{\text{gen}})_{\geq 0} \quad \text{and} \quad (\mathbb{TSp}_p^{\text{gen}})_{\leq 0}$$

of $\mathbb{TSp}_p^{\text{gen}}$.

Proposition 2.17. *The pair $((\mathbb{TSp}_p^{\text{gen}})_{\geq 0}, (\mathbb{TSp}_p^{\text{gen}})_{\leq 0})$ defines a left and right complete, accessible t -structure on $\mathbb{TSp}_p^{\text{gen}}$ which is compatible with filtered colimits.*

Note that by construction the fixed point functor

$$(-)^{C_{p^n}} : \mathbb{TSp}_p^{\text{gen}} \rightarrow \text{Sp}$$

is t -exact for each $n \geq 0$.

Definition 2.18. We call this t -structure on $\mathbb{TSp}_p^{\text{gen}}$ the **Mackey t -structure**.

The terminology is motivated by the fact that the heart $(\mathbb{TSp}_p^{\text{gen}})^{\heartsuit}$ is equivalent to the abelian category of Mackey functors. The notion of a Mackey functor might not be entirely standard in this setting (since \mathbb{T} is not a finite group) but Mackey t -structures exist for categories of genuine G -spectra where G is finite, in which case the heart is equivalent to the abelian category of classical Mackey functors on G . The construction and proof is the same as in Proposition 2.17.

Remark 2.19. One equivalent description of $\mathbb{TSp}_p^{\text{gen}}$ following Barwick [Bar17] is as the ∞ -category of product-preserving functors from the effective Burnside category of the orbit category $\{S^1/C_{p^n}\}_{n \geq 0} \subseteq \mathcal{S}^{BS^1}$ to spectra. In this language, $(\mathbb{TSp}_p^{\text{gen}})_{\geq 0}$ is equivalent to the ∞ -category of product-preserving functors from the Burnside category to $\text{Sp}_{\geq 0}$ and similarly for the ∞ -category of coconnective objects. From this description, the existence of the t -structure in Proposition 2.17 is clear. We give, however, a presentation-independent proof.

Proof of Proposition 2.17. Consider the compact generators $\Sigma_+^\infty \mathbb{T}/C_{p^n} \in \mathbb{TSp}_p^{\text{gen}}$. By [Lur17, Proposition 1.4.4.11] the smallest subcategory $\mathcal{C}_{\geq 0} \subseteq \mathbb{TSp}_p^{\text{gen}}$ that contains $\Sigma_+^\infty \mathbb{T}/C_{p^n}$ and is closed under colimits and extensions is the ∞ -category of connective objects for a t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ on $\mathbb{TSp}_p^{\text{gen}}$. An object $X \in \mathbb{TSp}_p^{\text{gen}}$ is then (-1) -truncated (i.e. in $\mathcal{C}_{\leq -1}$) if and only if the mapping spectrum

$$\mathbf{Map}_{\mathbb{TSp}_p^{\text{gen}}}(\Sigma_+^\infty \mathbb{T}/C_{p^n}, X) \simeq X^{C_{p^n}}$$

is (-1) -truncated. We claim that an object X is connective precisely if for every n the spectrum $X^{C_{p^n}}$ is connective. If X is connective, then it follows that $X^{C_{p^n}}$ is connective since this is true for the generators. Conversely, assume that $X^{C_{p^n}}$ is connective for each n . We consider the truncation $X \rightarrow \tau_{\leq -1} X$. The fibre of this map is given by $\tau_{\geq 0} X \rightarrow X$. We get a cofiber sequence

$$(\tau_{\geq 0} X)^{C_{p^n}} \rightarrow X^{C_{p^n}} \rightarrow (\tau_{\leq -1} X)^{C_{p^n}}$$

for each $n \geq 0$. The first two terms are connective spectra, thus so is the third. But since $\tau_{\leq -1} X$ is (-1) -truncated the last term is also (-1) -truncated as it is a mapping spectrum from a connective object. Therefore it has to be zero. Since the orbits are generators of $\mathbb{TSp}_p^{\text{gen}}$, it follows that $\tau_{\leq -1} X \simeq 0$. Thus, X is connective.

This establishes the existence of t -structure. The other claims now immediately follow using the conservativity of family $\{(-)^{C_{p^n}} : n \geq 0\}$ of functors $\mathbb{TSp}_p^{\text{gen}} \rightarrow \text{Sp}$ and the fact that fixed points preserve limits and colimits. \square

Lemma 2.20. *A genuine S^1 -spectrum $X \in \mathbb{TSp}_p^{\text{gen}}$ is connective in the Mackey t -structure precisely if all geometric fixed points $X^{\Phi_{C_{p^n}}} \in \mathbb{Sp}$ are connective.*

Proof. To see this we argue by induction over n . Assume that all spectra X^{C_k} and $X^{\Phi_{C_{p^k}}}$ for $k = 0, \dots, n-1$ are connective. Consider the isotropy separation sequence

$$((E_{\mathcal{P}_{\text{prop}}} C_{p^n})_+ \otimes X)^{C_{p^n}} \rightarrow X^{C_{p^n}} \rightarrow X^{\Phi_{C_{p^n}}}$$

where the left hand side is a colimit of a diagram only involving fixed points of proper subgroups of C_{p^n} .³ Thus it is a connective spectrum. It follows that $X^{C_{p^n}}$ is connective if and only if $X^{\Phi_{C_{p^n}}}$ is connective. \square

We recall the definition of genuine cyclotomic spectra.

Definition 2.21. The ∞ -category $\mathbf{CycSp}_p^{\text{gen}}$ of **genuine p -typical cyclotomic spectra** is defined to be fixed points for the endofunctor $(-)^{\Phi_{C_p}}$:

$$\mathbf{CycSp}_p^{\text{gen}} := \text{Fix}_{(-)^{\Phi_{C_p}}}(\mathbb{TSp}_p^{\text{gen}}).$$

In other words, a genuine p -typical cyclotomic spectrum is a genuine S^1 -spectrum X equipped with an equivalence $X^{\Phi_{C_p}} \simeq X$ of genuine S^1 -spectra.

By Lemma 2.20, the functor $(-)^{\Phi_{C_p}}$ is right t -exact. Using that accessible t -structures are closed under limits of right t -exact left adjoint functors, we obtain the following corollary.

Corollary 2.22. *There is an accessible t -structure on $\mathbf{CycSp}_p^{\text{gen}}$ where*

$$(\mathbf{CycSp}_p^{\text{gen}})_{\geq 0} \simeq \text{Fix}_{(-)^{\Phi_{C_p}}}((\mathbb{TSp}_p^{\text{gen}})_{\geq 0});$$

the forgetful functor to $\mathbb{TSp}_p^{\text{gen}}$ is right t -exact.

We call the induced t -structure on p -typical genuine cyclotomic spectra the **genuine cyclotomic t -structure**.

Remark 2.23. With the same arguments one also gets a Mackey t -structure on $\mathbb{TSp}_{\mathcal{F}}^{\text{gen}}$ where \mathcal{F} is the family of finite subgroups. This then induces also a t -structure on the ∞ -category of global genuine cyclotomic spectra $\mathbf{CycSp}^{\text{gen}}$. The latter is defined as the ∞ -category of homotopy fixed points

$$\mathbf{CycSp}^{\text{gen}} := (\mathbb{TSp}_{\mathcal{F}}^{\text{gen}})^{h\mathbb{N}_{>0}},$$

where the multiplicative monoid $\mathbb{N}_{>0}$ acts on $\mathbb{TSp}_{\mathcal{F}}^{\text{gen}}$ via $n \mapsto (-)^{\Phi_{C_n}}$. See [NS18, Section 2.3].

The next result says that the t -structure of Corollary 2.22 reduces to the cyclotomic t -structure of Section 2.1 when restricted to bounded below objects.

Theorem 2.24. *The natural functors $\mathbf{CycSp}^{\text{gen}} \rightarrow \mathbf{CycSp}$ and $\mathbf{CycSp}_p^{\text{gen}} \rightarrow \mathbf{CycSp}_p$ are right t -exact and restrict to equivalences $\mathbf{CycSp}_{\geq 0}^{\text{gen}} \simeq \mathbf{CycSp}_{\geq 0}$ and $(\mathbf{CycSp}_p^{\text{gen}})_{\geq 0} \simeq (\mathbf{CycSp}_p)_{\geq 0}$.*

Proof. This is simply a restatement of Theorem [NS18, Theorem II.3.8], once we note that an object X in $\mathbf{CycSp}^{\text{gen}}$ or $\mathbf{CycSp}_p^{\text{gen}}$ is bounded below in the cyclotomic t -structure if and only if the underlying spectrum of X is bounded below. This follows from Lemma 2.20 and the fact that $\Phi_{C_{p^n}} X \simeq X$ for all n . \square

In particular, we see that $\mathbf{CycSp}_p^{\text{gen}, \heartsuit} \simeq \mathbf{CycSp}_p^{\heartsuit}$ and similarly $\mathbf{CycSp}^{\text{gen}, \heartsuit} \simeq \mathbf{CycSp}^{\heartsuit}$.

³In fact this can even be simplified in this case since the subgroup lattice of C_{p^n} is very simple but we prefer to write the proof in a form that also works for more complicated families.

2.3 The cyclotomic t -structure and the generalized Segal conjecture

Recall that the Segal conjecture for C_p (a theorem of Lin [Lin80] and Gunawardena [Gun80]) says that the Tate diagonal $\mathbb{S} \rightarrow (\mathbb{S}^{\otimes p})^{tC_p} \simeq \mathbb{S}^{tC_p}$ is p -completion. This map is also equivalent to the trivial map that factors through homotopy fixed points \mathbb{S}^{hC_p} . We prove that every bounded above object in the cyclotomic t -structure satisfies an analogue of the Segal conjecture.

Proposition 2.25. *Suppose that $M \in \mathbf{CycSp}_p$. If $M \in (\mathbf{CycSp}_p)_{\leq d}$, then the cyclotomic Frobenius map $M \xrightarrow{\varphi} M^{tC_p}$ is d -truncated.⁴*

Proof. Fix a point $x \in BS^1$. We endow $x_1\mathbb{S}[d+1]$ with the structure of a p -typical cyclotomic spectrum by letting $x_1\mathbb{S}[d+1] \rightarrow (x_1\mathbb{S}[d+1])^{tC_p}$ be the zero map. The mapping spectrum $\mathbf{Map}_{\mathbf{CycSp}_p}(x_1\mathbb{S}[d+1], M)$ is the equalizer of

$$\mathbf{Map}_{S^1}(x_1\mathbb{S}[d+1], M) \rightrightarrows \mathbf{Map}_{S^1}(x_1\mathbb{S}[d+1], M^{tC_p}),$$

where the two maps are given by

$$(x_1\mathbb{S}[d+1] \xrightarrow{f} M) \mapsto (x_1\mathbb{S}[d+1] \xrightarrow{f} M \xrightarrow{\varphi} M^{tC_p})$$

and

$$f \mapsto x_1\mathbb{S}[d+1] \xrightarrow{0} (x_1\mathbb{S}[d+1])^{tC_p} \xrightarrow{f^{tC_p}} M^{tC_p}.$$

Since $\mathbf{Map}_{S^1}(x_1\mathbb{S}[d+1], M) \simeq M[-d-1]$ and $\mathbf{Map}_{S^1}(x_1\mathbb{S}[d+1], M^{tC_p}) \simeq M^{tC_p}[-d-1]$, we find that $\mathbf{Map}_{\mathbf{CycSp}_p}(x_1\mathbb{S}[d+1], M)$ is the fiber of $M[-d-1] \xrightarrow{\varphi[-d-1]} M^{tC_p}[-d-1]$. Since $M \in (\mathbf{CycSp}_p)_{\leq d}$, it follows that

$$\pi_i \mathbf{Map}_{\mathbf{CycSp}_p}(x_1\mathbb{S}[d+1], M) = 0$$

for $i \geq 0$. Hence, $\pi_i M \xrightarrow{\pi_i(\varphi)} \pi_i M^{tC_p}$ is an injection for $i = d+1$ and an isomorphism for $i \geq d+2$. In other words, φ is d -truncated. \square

Remark 2.26. The proposition gives a necessary but certainly not sufficient condition for an object to be bounded above. For example, $\mathbb{S}_p^{\text{triv}}$ (meaning the p -complete sphere with the trivial cyclotomic structure) satisfies the conclusion of Proposition 2.25 by the Segal conjecture but not the hypothesis since $\text{TC}(\mathbb{S}_p^{\text{triv}})$ contains \mathbb{S}_p as a summand.

We now give an example of a class of bounded above objects in the cyclotomic t -structure. Many more will appear later, in Section 6, once we have access to the p -typical topological Cartier module machinery developed in Section 3.

Example 2.27. Let $M \in \text{Sp}_{\leq d}$ be a bounded above spectrum and let x_1M denote the induced S^1 -spectrum $M \otimes S_+^1$. Now, $(x_1M)^{tC_p} \simeq 0$. To see this, we can reduce to the case where M is concentrated in a single degree using the easy generalization of [NS18, I.2.6(ii)] for weak Postnikov towers and the fact that x_1 preserves colimits. We make x_1M into a p -typical cyclotomic spectrum in the only way we can: we let the cyclotomic Frobenius $x_1M \rightarrow (x_1M)^{tC_p} \simeq 0$ be the zero map. Now, we show that $x_1M \in (\mathbf{CycSp}_p)_{\leq d+1}$. To see this, fix another cyclotomic spectrum X . We see that the mapping spectrum $\mathbf{Map}_{\mathbf{CycSp}_p}(X, x_1M) \simeq \mathbf{Map}_{S^1}(X, x_1M)$ using the equalizer formula, since $(x_1M)^{tC_p} \simeq 0$. Of course, $x_1M \in (\text{Sp}^{BS^1})_{\leq d+1}$. Hence, if $X \in (\mathbf{CycSp}_p)_{\geq d+2}$, the mapping space $\text{Map}_{\mathbf{CycSp}_p}(X, x_1M)$ vanishes.

⁴Recall that a map of spectra $X \rightarrow Y$ is d -truncated if it induces isomorphisms $\pi_n X \cong \pi_n Y$ for $n \geq d+2$ and an injection $\pi_{d+1} X \rightarrow \pi_{d+1} Y$. In other words, the fiber of $X \rightarrow Y$ is d -truncated.

3 Topological Cartier modules

In this section we give a new description of p -typical cyclotomic spectra based on a topological version of p -typical Cartier modules. Using this description, we find that the cyclotomic homotopy groups $\pi_i^{\text{cyc}} X$ of a p -typical cyclotomic spectrum X “are” the homotopy groups of $\text{TR}(X) = \lim_{n,R} X^{C_{p^n}}$ equipped with operations V and F such that $FV = p$.

As for cyclotomic spectra, p -typical topological Cartier modules admit two flavors: a genuine flavor and a simplistic flavor. Unlike cyclotomic spectra, where one needs a boundedness assumption to show that the simple version agrees with the genuine version, these flavors are unconditionally the same for p -typical topological Cartier modules, as we prove in Section 5.

3.1 The ∞ -category of topological Cartier modules

Classically, a p -typical Cartier module is an abelian group M equipped with endomorphisms V and F such that $FV = p$.⁵

Definition 3.1. A p -typical topological Cartier module is a spectrum M with an S^1 -action together with an S^1 -equivariant factorization of the C_p -norm

$$M_{hC_p} \xrightarrow{V} M \xrightarrow{F} M^{hC_p},$$

where M_{hC_p} and M^{hC_p} carry the residual $S^1 = S^1/C_p$ actions and the norm $M_{hC_p} \rightarrow M^{hC_p}$ is equivariant for this action. We will refer to V as the **Verschiebung** and F as the **Frobenius**.

Here are some examples and constructions with p -typical topological Cartier modules.

Example 3.2. Let M be an abelian group considered as an Eilenberg–MacLane spectrum with (necessarily) trivial S^1 -action. A p -typical topological Cartier module structure on M is equivalent to a p -typical Cartier module structure on M since the maps V and F necessarily have to factor through the truncations $M_{hC_p} \rightarrow \tau_{\leq 0} M_{hC_p} \simeq M$ and $M \simeq \tau_{\geq 0} M^{hC_p} \rightarrow M^{hC_p}$ and the norm factors as $M_{hC_p} \rightarrow M \xrightarrow{p} M \rightarrow M^{hC_p}$.

Example 3.3. For every p -typical topological Cartier module M the homotopy groups $\pi_n M$ are p -typical Cartier modules: the maps $V, F : \pi_n M \rightarrow \pi_n M$ are induced by π_n of the compositions

$$M \rightarrow M_{hC_p} \xrightarrow{V} M \quad \text{and} \quad M \xrightarrow{F} M^{hC_p} \rightarrow M.$$

To see that the composition FV is p , note that the composition is equivalent to $M \rightarrow M_{hC_p} \xrightarrow{\text{Nm}_{C_p}} M^{hC_p} \rightarrow M$ and we can reduce to the case where M is discrete, where the claim follows from the description of the norm from group homology to group cohomology. For more about the structure of $\pi_* M$ when M is a p -typical topological Cartier module, see Section 3.5.

Example 3.4. Let X be a p -typical cyclotomic spectrum. The S^1 -spectrum

$$\begin{aligned} \text{TR}(X) &\simeq \left(\dots \times_{(X^{tC_p})^{hC_{p^2}}} X^{hC_{p^2}} \times_{(X^{tC_p})^{hC_p}} X^{hC_p} \times_{X^{tC_p}} X \right) \\ &\simeq \varprojlim_{n,R} \left(X^{hC_{p^n}} \times_{(X^{tC_p})^{hC_{p^{n-1}}}} \dots \times_{(X^{tC_p})^{hC_p}} X^{hC_p} \times_{X^{tC_p}} X \right) \\ &\simeq \varprojlim_{n,R} (\text{TR}^{n+1}(X)) \end{aligned}$$

⁵In contrast to the case of Dieudonné modules we do not include the condition that $VF = p$. This is only appropriate when working over Witt vectors of a perfect ring of characteristic p , see Example 4.18 and Section 6.2.

canonically carries the structure of a topological Cartier module. The map $V: \mathrm{TR}(X)_{hC_p} \rightarrow \mathrm{TR}(X)$ is given by the canonical map

$$\mathrm{TR}(X)_{hC_p} \simeq \left(\varprojlim_{n,R} \mathrm{TR}^n(X) \right)_{hC_p} \rightarrow \varprojlim_{n,R} (\mathrm{TR}^n(X)_{hC_p}) \quad (1)$$

followed by the levelwise norm

$$\begin{array}{ccc} \dots (X^{hC_p})_{hC_p} \times_{(X^{tC_p})_{hC_p}} X_{hC_p} \times_0 & 0 & \\ & \downarrow V & \\ \dots X^{hC_{p^2}} \times_{(X^{tC_p})_{hC_p}} X^{hC_p} \times_{X^{tC_p}} X. & & \end{array} \quad (2)$$

The map $F: \mathrm{TR}(X) \rightarrow \mathrm{TR}(X)^{hC_p}$ is given by the projection

$$\begin{array}{ccc} \dots \times_{(X^{tC_p})^{hC_{p^2}}} X^{hC_{p^2}} \times_{(X^{tC_p})^{hC_p}} X^{hC_p} \times_{X^{tC_p}} X & & \\ & \downarrow F & \\ \dots \times_{(X^{tC_p})^{hC_{p^2}}} X^{hC_{p^2}} \times_{(X^{tC_p})^{hC_p}} X^{hC_p}. & & \end{array} \quad (3)$$

The composition is evidently the norm; this will also follow from a more detailed analysis in Construction 3.18.

Example 3.5. Let M be a p -typical topological Cartier module. We consider the S^1 -spectrum

$$X = M/V = \mathrm{cofib} \left(M_{hC_p} \xrightarrow{V} M \right)$$

together with the ‘quotient’ map $\rho: M \rightarrow X$, which is also S^1 -equivariant. The S^1 -spectrum X admits a canonical S^1 -equivariant map $X \rightarrow M^{tC_p}$ induced from the commutative square

$$\begin{array}{ccc} M_{hC_p} & \xrightarrow{\mathrm{id}} & M_{hC_p} \\ \downarrow V & & \downarrow \mathrm{Nm}_{C_p} \\ M & \xrightarrow{F} & M^{hC_p} \end{array}$$

by taking vertical cofibers. The composition $X \rightarrow M^{tC_p} \xrightarrow{\rho^{tC_p}} X^{tC_p}$ endows X with the structure of a cyclotomic spectrum. Note that the map $M^{tC_p} \rightarrow X^{tC_p}$ is an equivalence if M is bounded below, since the fiber is given by $(M_{hC_p})^{tC_p}$ which vanishes by the Tate orbit lemma [NS18, Lemma I.2.1]. When M is a p -typical Cartier module, viewed as a p -typical topological Cartier module via Example 3.2, the homotopy groups of M/V are given in Figure 1 on page 6.

Now, we give a rigorous construction of the ∞ -category of p -typical topological Cartier modules.

Definition 3.6. The ∞ -category \mathbf{TCart}_p of p -typical topological Cartier modules is the pullback

$$\begin{array}{ccc} \mathbf{TCart}_p & \longrightarrow & (\mathrm{Sp}^{BS^1})^{\Delta^2} \\ \downarrow & & \downarrow (\mathrm{ev}_1, \partial^1) \\ \mathrm{Sp}^{BS^1} & \xrightarrow{(\mathrm{id}, \mathrm{Nm}_{C_p})} & \mathrm{Sp}^{BS^1} \times (\mathrm{Sp}^{BS^1})^{\Delta^1}, \end{array} \quad (4)$$

where the bottom arrow sends an object M to the pair $(M, M_{hC_p} \xrightarrow{\text{Nm}_{C_p}} M^{hC_p})$ and the right-hand vertical arrow sends a 2-simplex

$$\begin{array}{ccc} & M_1 & \\ v \nearrow & & \searrow f \\ M_0 & \xrightarrow{n} & M_2, \end{array}$$

expressing that $n \simeq f \circ v$, to the pair $(M_1, M_0 \xrightarrow{n} M_2)$. We will write $M = (M, V_M, F_M, \sigma_M)$ for an object of \mathbf{TCart}_p , where M is a spectrum with S^1 -action, $V_M : M_{hC_p} \rightarrow M$, $F_M : M \rightarrow M^{hC_p}$, and σ_M is a 2-simplex expressing an equivalence $\text{Nm}_{C_p} \simeq F_M \circ V_M$.

For the rest of the section we will establish some facts about the ∞ -category \mathbf{TCart}_p including the fact that it is presentable and a formula for the mapping spaces. Since this is a bit technical the reader might want to skip the rest of this section on a first reading.

We recall the ∞ -category $\mathbf{CycSp}_p^{\text{Fr}}$ of cyclotomic spectra with Frobenius lifts. The objects are spectra X with S^1 -action equipped with an S^1 -equivariant map $\psi_p : X \rightarrow X^{hC_p}$. As an ∞ -category $\mathbf{CycSp}_p^{\text{Fr}}$ is defined as the pullback

$$\begin{array}{ccc} \mathbf{CycSp}_p^{\text{Fr}} & \longrightarrow & (\text{Sp}^{BS^1})^{\Delta^1} \\ \downarrow & & \downarrow (\text{ev}_0, \text{ev}_1) \\ \text{Sp}^{BS^1} & \xrightarrow{(\text{id}, (-)^{hC_p})} & \text{Sp}^{BS^1} \times \text{Sp}^{BS^1}, \end{array}$$

or as $\text{LEq}(\text{id}, (-)^{hC_p})$ in the language of [NS18].

Lemma 3.7. *There is a pullback square of stable ∞ -categories*

$$\begin{array}{ccc} \mathbf{TCart}_p & \longrightarrow & \text{Sp}^{BS^1} \times \text{Sp}^{BS^1} \\ \downarrow & & \downarrow \\ \mathbf{CycSp}_p^{\text{Fr}} & \longrightarrow & (\text{Sp}^{BS^1})^{\Delta^1} \end{array}$$

where

- (1) the left vertical functor sends (M, V, F, σ) to (M, F) ;
- (2) the upper horizontal map sends (M, V, F, σ) to the pair $(M_{hC_p}, M^{hC_p}/F)$, where M^{hC_p}/F denotes the cofiber of $M \xrightarrow{F} M^{hC_p}$;
- (3) the lower horizontal functor sends (M, F) to the composition $M_{hC_p} \xrightarrow{\text{Nm}_{C_p}} M^{hC_p} \rightarrow M^{hC_p}/F$;
- (4) the right vertical map sends (X, Y) to the zero map $X \xrightarrow{0} Y$.

Proof. We consider the diagram of stable ∞ -categories

$$\begin{array}{ccccc}
\mathbf{TCart}_p & \longrightarrow & (\mathrm{Sp}^{BS^1})^{\Delta^2} & & (5) \\
\downarrow & & \downarrow i & & \\
& & (\mathrm{Sp}^{BS^1})^{\Lambda_2^2} & \xrightarrow{\partial_0} & (\mathrm{Sp}^{BS^1})^{\Delta^1} \\
& & \downarrow (ev_1, \partial_1) & & \downarrow (ev_0, ev_1) \\
\mathrm{Sp}^{BS^1} & \xrightarrow{(id, Nm_{C_p})} & \mathrm{Sp}^{BS^1} \times (\mathrm{Sp}^{BS^1})^{\Delta^1} & \xrightarrow{id \times ev_1} & \mathrm{Sp}^{BS^1} \times \mathrm{Sp}^{BS^1}
\end{array}$$

where the left hand square is the defining pullback for \mathbf{TCart}_p as in Definition 3.6 and the right hand side is induced from the diagram of simplicial sets⁶

$$\begin{array}{ccccc}
& & \Delta^2 & & \\
& & \uparrow i & & \\
& & \Lambda_2^2 & \xleftarrow{\partial^0} & \Delta^1 \\
& & \uparrow \{1\} \sqcup \partial^1 & & \uparrow \partial^1 \sqcup \partial^0 \\
\Delta^0 \sqcup \Delta^1 & \xleftarrow{id \sqcup \partial^0} & \Delta^0 \sqcup \Delta^0 & &
\end{array}$$

with $i: \Lambda_2^2 \rightarrow \Delta^2$ the ‘inclusion’ and $\{1\}$ denotes the map that hits the object $\{1\} \in \Lambda_2^2$. The square in this diagram of simplicial sets is a pushout of simplicial sets and the lower horizontal map is a monomorphism. Therefore it is a pushout in Cat_∞ and thus the right hand square in (5) is a pullback of stable ∞ -categories. We can insert a further pullback and obtain a diagram

$$\begin{array}{ccccc}
\mathbf{TCart}_p & \longrightarrow & (\mathrm{Sp}^{BS^1})^{\Delta^2} & & (6) \\
\downarrow & & \downarrow i & & \\
\mathbf{CycSp}_p^{\mathrm{Fr}} & \longrightarrow & (\mathrm{Sp}^{BS^1})^{\Lambda_2^2} & \xrightarrow{\partial_0} & (\mathrm{Sp}^{BS^1})^{\Delta^1} \\
\downarrow & & \downarrow (ev_1, \partial_1) & & \downarrow (ev_0, ev_1) \\
\mathrm{Sp}^{BS^1} & \xrightarrow{(id, Nm_{C_p})} & \mathrm{Sp}^{BS^1} \times (\mathrm{Sp}^{BS^1})^{\Delta^1} & \xrightarrow{id \times ev_1} & \mathrm{Sp}^{BS^1} \times \mathrm{Sp}^{BS^1}
\end{array}$$

where the new term is equivalent to $\mathbf{CycSp}_p^{\mathrm{Fr}}$ by pasting of pullback squares and the left upper square is a pullback for the same reason. Finally we use that for every stable ∞ -category \mathcal{D} there is a pullback square of

⁶Note that we use ev_i for evaluation at the vertex i in Δ^n . For example for Δ^1 we have that ev_0 corresponds to ∂^1 and ev_1 to ∂^0 .

the form

$$\begin{array}{ccc} \mathcal{D}^{\Delta^2} & \xrightarrow{(\text{ev}_0, \text{cofib}(\partial_1))} & \mathcal{D} \times \mathcal{D} \\ \downarrow i & & \downarrow \\ \mathcal{D}^{A_2^2} & \longrightarrow & \mathcal{D}^{\Delta^1}, \end{array} \quad (7)$$

where the lower horizontal map sends a diagram

$$\begin{array}{ccc} & B & \\ & \searrow & \\ A & \longrightarrow & C \end{array} \quad (8)$$

to the composition $A \rightarrow C \rightarrow C/B$ and the right hand vertical map sends (X, Y) to the zero morphism $X \xrightarrow{0} Y$. This pullback is just a manifestation of the fact that filling a diagram as (8) is equivalent to choosing a nullhomotopy of the composition $A \rightarrow B \rightarrow C/B$.

To finish the proof we paste together the upper square in diagram (6) with the square (7) for $\mathcal{D} = \text{Sp}^{BS^1}$. \square

Now, we can give the desired formula for the mapping spectra in \mathbf{TCart}_p . Let $M = (M, V_M, F_M, \sigma_M)$ and $N = (N, V_N, F_N, \sigma_N)$ be topological Cartier modules. There is a map

$$\vartheta : \mathbf{Map}_{\mathbf{CycSp}_p^{\text{Fr}}}(M, N) \rightarrow \mathbf{Map}_{\text{Sp}^{BS^1}}(M_{hC_p}, \text{fib}(F_N))$$

which sends a map $g : M \rightarrow N$ in $\mathbf{CycSp}_p^{\text{Fr}}$ to the factorization of the map

$$gV_M - V_N g_{hC_p} : M_{hC_p} \rightarrow N$$

through the map $\text{fib}(F_N) \rightarrow N$ induced from the canonical nullhomotopy

$$F_N(gV_M - V_n g_{hC_p}) \simeq gF_M V_M - F_N V_N g_{hC_p} \simeq 0,$$

arising from the natural transformation $(-)_{hC_p} \xrightarrow{\text{Nm}_{C_p}} (-)_{hC_p}$.

Proposition 3.8. *For every pair of topological Cartier modules M and N there is a fiber sequence*

$$\mathbf{Map}_{\mathbf{TCart}_p}(M, N) \rightarrow \mathbf{Map}_{\mathbf{CycSp}_p^{\text{Fr}}}(M, N) \xrightarrow{\vartheta} \mathbf{Map}_{\text{Sp}^{BS^1}}(M_{hC_p}, \text{fib}(F_N))$$

of spectra.

Proof. From the pullback square of Lemma 3.7 we get an induced pullback square on mapping spectra of the form

$$\begin{array}{ccc} \mathbf{Map}_{\mathbf{TCart}_p}(M, N) & \longrightarrow & \mathbf{Map}_{\text{Sp}^{BS^1}}(M_{hC_p}, N_{hC_p}) \times \mathbf{Map}_{\text{Sp}^{BS^1}}(M^{hC_p}/F_M, N^{hC_p}/F_N) \\ \downarrow & & \downarrow \\ \mathbf{Map}_{\mathbf{CycSp}_p^{\text{Fr}}}(M, N) & \longrightarrow & \mathbf{Map}_{(\text{Sp}^{BS^1})^{\Delta^1}}(M_{hC_p} \rightarrow M^{hC_p}/F_M, N_{hC_p} \rightarrow N^{hC_p}/F_N). \end{array}$$

Thus, to establish the fiber sequence in question, we have to identify the cofiber of the right hand map. The right hand map has an obvious section which is induced from the functor $(\mathrm{Sp}^{BS^1})^{\Delta^1} \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} \mathrm{Sp}^{BS^1} \times \mathrm{Sp}^{BS^1}$. The fiber of this section is equivalent to the mapping spectrum

$$\mathbf{Map}_{\mathrm{Sp}^{BS^1}}(M_{hC_p}, \Omega(N^{hC_p}/F_N)).$$

Since $\Omega(N^{hC_p}/F_N) \simeq \mathrm{fib}(F_N)$ this shows that the cofiber of the map $\mathbf{Map}_{\mathbf{TCart}_p}(M, N) \rightarrow \mathbf{Map}_{\mathbf{CycSp}^{\mathrm{Fr}}}(M, N)$ has the claimed homotopy type. Tracing through the identification lets us identify the maps as stated. \square

Remark 3.9. For every S^1 -spectrum M we can consider M_{hC_p} as a cyclotomic spectrum with Frobenius lift, where we choose the Frobenius lift to be the zero map $M_{hC_p} \rightarrow (M_{hC_p})^{hC_p}$. For every cyclotomic spectrum N with Frobenius lift $F_N : N \rightarrow N^{hC_p}$ we get an equivalence

$$\mathbf{Map}_{\mathrm{Sp}^{BS^1}}(M_{hC_p}, \mathrm{fib}(F_N)) \simeq \mathbf{Map}_{\mathbf{CycSp}^{\mathrm{Fr}}}(M_{hC_p}, N)$$

as one can directly verify from the description of mapping spectra in $\mathbf{CycSp}_p^{\mathrm{Fr}}$ using [NS18, Proposition II.1.5]. Using this equivalence we can rewrite the fiber sequence of Proposition 3.8 as

$$\mathbf{Map}_{\mathbf{TCart}_p}(M, N) \rightarrow \mathbf{Map}_{\mathbf{CycSp}^{\mathrm{Fr}}}(M, N) \rightarrow \mathbf{Map}_{\mathbf{CycSp}_p^{\mathrm{Fr}}}(M_{hC_p}, N),$$

in which the right hand map can be described as the map sending $g : M \rightarrow N$ to the map $gV_M - V_n f_{hC_p} : M_{hC_p} \rightarrow N$ in Sp^{BS^1} which canonically refines to a map in $\mathbf{CycSp}_p^{\mathrm{Fr}}$. This fact will be useful in Section 3.3.

There is also a ‘dual’ version of Proposition 3.8 which we record to use later. We consider the ∞ -category

$$\mathrm{Alg}_{(-)_{hC_p}}(\mathrm{Sp}^{BS^1}) := \mathrm{LEq}\left(\mathrm{Sp}^{BS^1} \begin{array}{c} \xrightarrow{(-)_{hC_p}} \\ \xrightarrow{\mathrm{id}} \end{array} \mathrm{Sp}^{BS^1}\right)$$

of $(-)_{hC_p}$ -algebras. We will also abbreviate this ∞ -category as $\mathrm{Alg}_{(-)_{hC_p}}$. For every topological Cartier module (M, F, V, σ) we get an object $(M, V) \in \mathrm{Alg}_{(-)_{hC_p}}$.

Proposition 3.10. *For every pair of topological Cartier modules M, N there is a fiber sequence*

$$\mathbf{Map}_{\mathbf{TCart}_p}(M, N) \rightarrow \mathbf{Map}_{\mathrm{Alg}_{(-)_{hC_p}}}(M, N) \rightarrow \mathbf{Map}_{\mathrm{Sp}^{BS^1}}(\mathrm{cofib}(V_M), N^{hC_p})$$

where the right hand map admits a dual description to the of Proposition 3.8.

Proof. The proof is entirely dual to the one of Proposition 3.8: we first get a pullback square of stable ∞ -categories

$$\begin{array}{ccc} \mathbf{TCart}_p & \longrightarrow & \mathrm{Sp}^{BS^1} \times \mathrm{Sp}^{BS^1} \\ \downarrow & & \downarrow \\ \mathrm{Alg}_{(-)_{hC_p}} & \longrightarrow & (\mathrm{Sp}^{BS^1})^{\Delta^1} \end{array}$$

similar to the one of Lemma 3.7. Here the upper horizontal map sends (M, V, F, σ) to $(\mathrm{fib}(V), M^{hC_p})$, the lower horizontal map sends (M, V) to the map $(\mathrm{fib}(V) \rightarrow M_{hC_p} \xrightarrow{\mathrm{Nm}} M^{hC_p})$ and the right vertical map sends a pair (X, Y) to the zero morphism from X to Y . Then as in the proof of Proposition 3.8 we compute the mapping space in this pullback to get the result. \square

Proposition 3.11. *The ∞ -category \mathbf{TCart}_p is stable and presentable and the forgetful functor $\mathbf{TCart}_p \rightarrow \mathrm{Sp}^{BS^1}$ preserves limits and colimits.*

Proof. We follow the proof of [NS18, II.1.5]. The ∞ -category \mathbf{TCart}_p is stable because it is the pullback of stable ∞ -categories along exact functors (see [Lur17, 1.1.4.2]). Since \mathbf{TCart}_p is the pullback of accessible ∞ -categories along accessible functors, it is accessible itself by [Lur09, 5.4.6.6]. To see that \mathbf{TCart}_p admits colimits and that $\mathbf{TCart}_p \rightarrow \mathrm{Sp}^{BS^1}$ preserves them, first suppose that $K \rightarrow \mathbf{TCart}_p$ is a diagram which admits an extension to $K^\triangleright \rightarrow \mathbf{TCart}_p$ such that the induced map $K^\triangleright \rightarrow \mathrm{Sp}^{BS^1}$ is a colimit diagram. Since $(-)_hC_p$ preserves colimits and since the forgetful functor $\mathbf{CycSp}_p^{\mathrm{Fr}} \rightarrow \mathrm{Sp}^{BS^1}$ preserves colimits (see [NS18, Proposition II.1.5]), it follows from Proposition 3.8 that $K^\triangleright \rightarrow \mathbf{TCart}_p$ is a colimit too. Now, let $K \rightarrow \mathbf{TCart}_p$ be an arbitrary diagram where K is a small simplicial set. The composition $K \rightarrow \mathbf{TCart}_p \rightarrow \mathrm{Sp}^{BS^1}$ admits a colimit because Sp^{BS^1} is presentable. Let

$$(\mathrm{colim}_{k \in K} M(k))_{hC_p} \simeq \mathrm{colim}_{k \in K} (M(k)_{hC_p}) \xrightarrow{\mathrm{colim} V_{M(k)}} \mathrm{colim}_{k \in K} M(k)$$

be the Verschiebung. Similarly, let

$$\mathrm{colim}_{k \in K} M(k) \xrightarrow{\mathrm{colim}_{k \in K} F_{M(k)}} \mathrm{colim}_{k \in K} M(k)^{hC_p} \rightarrow (\mathrm{colim}_{k \in K} M(k))^{hC_p}$$

define the Frobenius map. It is enough to show that the composition is equivalent to Nm_{C_p} . But, the composition is equivalent to

$$\mathrm{colim}_{k \in K} (M(k)_{hC_p}) \xrightarrow{\mathrm{colim}_{k \in K} \mathrm{Nm}_{C_p}|_{M(k)}} \mathrm{colim}_{k \in K} (M(k)^{hC_p}) \rightarrow (\mathrm{colim}_{k \in K} M(k))^{hC_p}. \quad (9)$$

Since Nm_{C_p} is a natural transformation, there is a canonical commutative diagram

$$\begin{array}{ccc} \mathrm{colim}_{k \in K} M(k)_{hC_p} & \longrightarrow & (\mathrm{colim}_{k \in K} M(k))_{hC_p} \\ \downarrow \mathrm{colim} \mathrm{Nm}_{C_p}|_{M(k)} & & \downarrow \mathrm{Nm}_{C_p}|_{\mathrm{colim} M(k)} \\ \mathrm{colim}_{k \in K} M(k)^{hC_p} & \longrightarrow & (\mathrm{colim}_{k \in K} M(k))^{hC_p}. \end{array}$$

The top arrow is an equivalence, and we see that (9) is the desired norm. The proof that $\mathbf{TCart}_p \rightarrow \mathrm{Sp}^{BS^1}$ preserves limits is the same, using Proposition 3.8, the fact that the forgetful functor $\mathbf{CycSp}_p^{\mathrm{Fr}} \rightarrow \mathrm{Sp}^{BS^1}$ preserves limits by [NS18, Proposition II.1.5], and the fact that $(-)^{hC_p}$ preserves limits. \square

Remark 3.12. The ∞ -category \mathbf{TCart}_p is equivalent to the ∞ -category of pullback squares in Sp^{BS^1} of the form

$$\begin{array}{ccc} M & \longrightarrow & X \\ \downarrow & & \downarrow \\ M^{hC_p} & \longrightarrow & M^{tC_p} \end{array}$$

(where $X = M/V$). This description is akin to Tate squares and shows that a topological Cartier module M gives rise to a genuine C_p -spectrum⁷ whose categorical fixed points are equivalent to the underlying spectrum.

⁷The C_p -spectrum M admits an S^1 -action which is compatible with the C_p -equivariant structure, so it is really an S^1 -spectrum that is genuine for the family consisting only of C_p and the trivial group.

We will make this analogy rigorous in Section 5.1. This category of pullback squares has an obvious symmetric monoidal structure (algebras are such that all objects are algebras and all maps are algebra maps), which will be discussed in more detail in future work (see also Section 4).

Remark 3.13. Proposition 3.11 formally implies that \mathbf{TCart}_p has a single compact generator $K \in \mathbf{TCart}_p$: we denote the left adjoint to the forgetful functor $\mathbf{TCart}_p \rightarrow \mathbf{Sp}$ by L and set $K = L(\mathbb{S})$. Then K is compact since if $M \simeq \operatorname{colim}_i M_i$ is a filtered colimit in \mathbf{TCart}_p , then

$$\mathbf{Map}_{\mathbf{TCart}_p}(K, \operatorname{colim}_i M_i) \simeq \mathbf{Map}_{\mathbf{Sp}}(\mathbb{S}, \operatorname{colim}_i M_i) \simeq \operatorname{colim}_i \mathbf{Map}_{\mathbf{Sp}}(\mathbb{S}, M_i) \simeq \operatorname{colim}_i \mathbf{Map}_{\mathbf{TCart}_p}(K, M_i).$$

It is a generator because $\mathbf{Map}_{\mathbf{TCart}_p}(K, M) \simeq 0$ implies by adjunction that $M \simeq 0$. It thus follows that \mathbf{TCart}_p is equivalent to the ∞ -category of module spectra over

$$\mathbf{Map}_{\mathbf{TCart}_p}(K, K) = \mathbf{Map}_{\mathbf{Sp}}(\mathbb{S}, K) \simeq K.$$

This endomorphism ring spectrum is a spectral version of the Raynaud–Cartier ring which controls the operations on the de Rham–Witt complex. We will analyze this ring spectrum in future work.

3.2 The t -structure on \mathbf{TCart}_p

We introduce a t -structure on \mathbf{TCart}_p which will turn out to be compatible via the constructions in Examples 3.4 and 3.5 with the t -structure introduced in Section 2 on cyclotomic spectra.

Definition 3.14. Let $(\mathbf{TCart}_p)_{\geq 0}$ denote the full subcategory of topological Cartier modules whose underlying spectrum is connective and let $(\mathbf{TCart}_p)_{\leq 0}$ denote the full subcategory whose underlying spectrum is coconnective. We will call the objects of these ∞ -categories the connective and coconnective p -typical topological Cartier modules, respectively.

Proposition 3.15. *The pair $((\mathbf{TCart}_p)_{\geq 0}, (\mathbf{TCart}_p)_{\leq 0})$ defines an accessible t -structure on \mathbf{TCart}_p with the following properties:*

- (i) *the forgetful functor $\mathbf{TCart}_p \rightarrow \mathbf{Sp}^{BS^1}$ is t -exact;*
- (ii) *the functor $(-)/V: \mathbf{TCart}_p \rightarrow \mathbf{CycSp}_p$ constructed in Example 3.5 is right t -exact;*
- (iii) *the t -structure is compatible with filtered colimits;*
- (iv) *the t -structure is left and right complete;*
- (v) *the heart $\mathbf{TCart}_p^\heartsuit$ is equivalent to the abelian category of abelian groups equipped with endomorphisms V and F such that $FV = p$.*

Proof. Because $(\mathbf{TCart}_p)_{\geq 0}$ is presentable and closed under colimits and extensions in \mathbf{TCart}_p , by [Lur17, 1.4.4.11], there is some t -structure $((\mathbf{TCart}_p)_{\geq 0}, \mathcal{C})$ on \mathbf{TCart}_p with connective part $(\mathbf{TCart}_p)_{\geq 0}$ and coconnective part some full subcategory $\mathcal{C} \subseteq \mathbf{TCart}_p$. Note that by Proposition 3.8, for every $M \in (\mathbf{TCart}_p)_{\geq 0}$ and every $N \in (\mathbf{TCart}_p)_{\leq -1}$, the mapping space $\mathbf{Map}_{\mathbf{TCart}_p}(M, N)$ is contractible. It follows that $(\mathbf{TCart}_p)_{\leq 0} \subseteq \mathcal{C}$. To prove that the inclusion is an equivalence, we will prove that every topological Cartier module structure on a spectrum M with S^1 -action extends to a structure on $\tau_{\geq 0}M$ and $\tau_{\leq -1}M$. In fact, it will be enough to do this for $\tau_{\geq 0}M$ since it follows then for $\tau_{\leq -1}M$ by taking cofibers.

Consider the composition $(\tau_{\geq 0}M)_{hC_p} \rightarrow M_{hC_p} \xrightarrow{V} M$. Since $(\tau_{\geq 0}M)_{hC_p}$ is connective, this factors through the connective cover $\tau_{\geq 0}M \rightarrow M$, giving a commutative diagram

$$\begin{array}{ccc} (\tau_{\geq 0}M)_{hC_p} & \longrightarrow & \tau_{\geq 0}M \\ \downarrow & & \downarrow \\ M_{hC_p} & \xrightarrow{V} & M. \end{array}$$

Similarly, the composition $\tau_{\geq 0}M \rightarrow M \xrightarrow{F} M^{hC_p}$ factors through $\tau_{\geq 0}(M^{hC_p}) \simeq \tau_{\geq 0}((\tau_{\geq 0}M)^{hC_p})$, where the equivalence follows because the right adjoint functor $\tau_{\geq 0}$ commutes with limits. Hence, we get a commutative diagram

$$\begin{array}{ccc} \tau_{\geq 0}M & \longrightarrow & (\tau_{\geq 0}M)^{hC_p} \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & M^{hC_p}. \end{array}$$

Consider the prism $\Delta^2 \times \Delta^1$. We would like to construct a new object of $(\mathrm{Sp}^{BS^1})^{\Delta^2}$ defining a topological Cartier module structure on $\tau_{\geq 0}M$ where the F and V maps are the top horizontal arrows in the two commutative diagrams above. Moreover, we need to have a map of topological Cartier modules $\tau_{\geq 0}M \rightarrow M$. This will be provided by a map $\Delta^1 \rightarrow (\mathrm{Sp}^{BS^1})^{\Delta^2}$ or, by adjunction, by a map $\Delta^2 \times \Delta^1 \rightarrow \mathrm{Sp}^{BS^1}$.

The commutative diagrams constructed above as well as the natural transformation Nm_{C_p} from $(-)_hC_p$ to $(-)^{hC_p}$ provide a map from $(\partial\Delta^2) \times \Delta^1$ to Sp^{BS^1} . Moreover, the ‘bottom’ of this triangular cylinder can be filled in with the 2-simplex σ associated to the topological Cartier module structure on M . Decomposing $\Delta^2 \times \Delta^1$ into three 3-simplices (tetrahedra), we can use the fact that we can fill inner horns (since Sp^{BS^1} is an ∞ -category), to inductively fill in $\Delta^2 \times \Delta^1$, thus obtaining the desired map.

Now, the cofiber of $\tau_{\geq 0}M \rightarrow M$ is a \mathbf{TCart}_p -structure on $\tau_{\leq -1}M$. Since $\tau_{\geq 0}M \in (\mathbf{TCart}_p)_{\geq 0}$ and $\tau_{\leq -1}M \in (\mathbf{TCart}_p)_{\leq -1} \subseteq \mathcal{C}[-1]$, it follows that in fact this is the truncation sequence associated to M in the t -structure $((\mathbf{TCart}_p)_{\geq 0}, \mathcal{C})$. In particular, we see that $(\mathbf{TCart}_p)_{\leq 0} \simeq \mathcal{C}$, as desired.

Accessibility of the t -structure follows because $(\mathbf{TCart}_p)_{\geq 0}$ is presentable. The t -exactness of $\mathbf{TCart}_p \rightarrow \mathrm{Sp}^{BS^1}$ follows by construction. The right t -exactness of $\mathbf{TCart}_p \rightarrow \mathbf{CycSp}_p$ follows because if $M \in (\mathbf{TCart}_p)_{\geq 0}$, then M_{hC_p} is also connective and hence so is the cofiber M/V . Compatibility with filtered colimits follows because filtered colimits are preserved by the forgetful functor $\mathbf{TCart}_p \rightarrow \mathrm{Sp}^{BS^1}$ by Proposition 3.11 and because the t -structure on Sp^{BS^1} is compatible with filtered colimits by Proposition A.17. Right completeness follows from the right complete version of [Lur17, Proposition 1.2.1.19] since \mathbf{TCart}_p is right separated (by the conservativity of the forgetful functor $\mathbf{TCart}_p \rightarrow \mathrm{Sp}^{BS^1}$) and because $(\mathbf{TCart}_p)_{\leq 0}$ is closed under filtered colimits and hence in particular countable coproducts inside of \mathbf{TCart}_p . For left completeness, use that \mathbf{TCart}_p is left separated and recall that $\mathbf{TCart}_p \rightarrow \mathrm{Sp}^{BS^1}$ preserves limits by Proposition 3.11. It follows, as usual by conservativity, that $(\mathbf{TCart}_p)_{\geq 0}$ is closed under countable products. We conclude by [Lur17, Proposition 1.2.1.19]. Finally, we see that the heart consists precisely of topological Cartier modules whose underlying spectrum is discrete. Since the S^1 -action is automatically trivial, we are reduced to the objects of Example 3.2. \square

Remark 3.16. Alternatively, to prove the existence, accessibility, and completeness of the t -structure on \mathbf{TCart}_p , one can show that the ring spectrum K of Remark 3.13 is connective and identify the t -structure above on \mathbf{TCart}_p with the Postnikov t -structure on K -module spectra, which has the desired properties by [Lur17, 7.1.1.13].

3.3 Topological Cartier modules and cyclotomic spectra

The functor

$$(-)/V : \mathbf{TCart}_p \rightarrow \mathbf{CycSp}_p$$

of Example 3.5 preserves all colimits and thus is a left adjoint by the adjoint functor theorem. The first result of this section identifies the right adjoint.

Proposition 3.17. *The functor $\mathrm{TR} : \mathbf{CycSp}_p \rightarrow \mathbf{TCart}_p$ is right adjoint to $(-)/V$.*

Construction 3.18. Let us review our candidate functor TR first, which was sketched in Example 3.4. By construction, for every cyclotomic spectrum X (no boundedness assumptions) the spectrum

$$\mathrm{TR}(X) = \left(\dots \times_{(X^{tC_p})^{hC_{p^2}}} X^{hC_{p^2}} \times_{(X^{tC_p})^{hC_p}} X^{hC_p} \times_{X^{tC_p}} X \right)$$

comes equipped with a canonical S^1 -action and an S^1 -equivariant map $\pi : \mathrm{TR}(X) \rightarrow X$. Moreover there is an evident S^1 -equivariant equivalence

$$\Phi : \mathrm{TR}(X) \xrightarrow{\cong} X \times_{X^{tC_p}} \mathrm{TR}(X)^{hC_p}, \quad (10)$$

where the pullback involves the cyclotomic Frobenius $X \rightarrow X^{tC_p}$ and the map $\mathrm{TR}(X)^{hC_p} \rightarrow X^{tC_p}$ is given by one of the two equivalent compositions in the square

$$\begin{array}{ccc} \mathrm{TR}(X)^{hC_p} & \xrightarrow{\pi^{hC_p}} & X^{hC_p} \\ \downarrow \text{can} & & \downarrow \text{can} \\ \mathrm{TR}(X)^{tC_p} & \xrightarrow{\pi^{tC_p}} & X^{tC_p}. \end{array}$$

Under Φ , the map π corresponds to projection onto the first factor. Now let us construct the structure of a topological Cartier module $\mathrm{TR}(X)$ from Example 3.4 a little bit more carefully. The map $F : \mathrm{TR}(X) \rightarrow \mathrm{TR}(X)^{hC_p}$ is obtained as the composite

$$F : \mathrm{TR}(X) \xrightarrow{\Phi} X \times_{X^{tC_p}} \mathrm{TR}(X)^{hC_p} \xrightarrow{\mathrm{Pr}_2} \mathrm{TR}(X)^{hC_p}$$

and the map V as the composite

$$V : \mathrm{TR}(X)_{hC_p} \xrightarrow{(0, \mathrm{Nm})} X \times_{X^{tC_p}} \mathrm{TR}(X)^{hC_p} \xrightarrow{\Phi^{-1}} \mathrm{TR}(X)$$

where the first map is zero onto the first factor and the norm into the second factor, using the canonical nullhomotopy of $\mathrm{can} \circ \mathrm{Nm}$. Now the composition $F \circ V$ comes by definition with a homotopy to the norm.

Moreover by construction of V the composite $\mathrm{TR}(X)_{hC_p} \xrightarrow{V} \mathrm{TR}(X) \xrightarrow{\pi} X$ is canonically nullhomotopic and thus gives rise to a map $\mathrm{TR}(X)/V \rightarrow X$. This map admits a refinement to a map of cyclotomic spectra induced by the compatible nullhomotopies of the horizontal maps in the commutative diagram

$$\begin{array}{ccccc} \mathrm{TR}(X)_{hC_p} & \xrightarrow{V} & \mathrm{TR}(X) & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow F & & \downarrow \\ \mathrm{TR}(X)_{hC_p} & \xrightarrow{\mathrm{Nm}} & \mathrm{TR}(X)^{hC_p} & \longrightarrow & X^{tC_p}. \end{array}$$

Unwinding the definitions, we see that the map $\mathrm{TR}(X)/V \rightarrow X$ is an equivalence precisely if the map

$$\pi^{tC_p}: \mathrm{TR}(X)^{tC_p} \rightarrow X^{tC_p} \quad (11)$$

is an equivalence. Indeed, the fiber F of $\mathrm{TR}(X) \rightarrow X$ is naturally equivalent to the fiber of $\mathrm{TR}(X)^{hC_p} \rightarrow X^{tC_p}$ and there is a natural map from $\mathrm{TR}(X)_{hC_p}$ to this fiber. We have $\mathrm{TR}(X)_{hC_p} \simeq F$ (equivalently, $\mathrm{TR}(X)/V \simeq X$) if and only if X^{tC_p} is the cofiber of the norm map $\mathrm{TR}(X)_{hC_p} \rightarrow \mathrm{TR}(X)^{hC_p}$.

In general, the map $\mathrm{TR}(X)/V \rightarrow X$ induces a natural transformation of functors $\mathrm{TR} \rightarrow R$, where R is the right adjoint to $(-)/V$.

Proof of Proposition 3.17. We have to verify that TR satisfies the universal property of the right adjoint, i.e., that for every pair of a topological Cartier module M and a cyclotomic spectrum X the induced map

$$\mathbf{Map}_{\mathbf{TCart}_p}(M, \mathrm{TR}(X)) \rightarrow \mathbf{Map}_{\mathbf{CycSp}_p}(M/V, X)$$

is an equivalence of spectra. We use the fiber sequence for the mapping space in \mathbf{TCart}_p in the form established in Remark 3.9:

$$\mathbf{Map}_{\mathbf{TCart}_p}(M, \mathrm{TR}(X)) \rightarrow \mathbf{Map}_{\mathbf{CycSp}_p^{\mathrm{Fr}}}(M, \mathrm{TR}(X)) \rightarrow \mathbf{Map}_{\mathbf{CycSp}_p^{\mathrm{Fr}}}(M_{hC_p}, \mathrm{TR}(X)) .$$

Recall that the objects of $\mathbf{CycSp}_p^{\mathrm{Fr}}$ are spectra M with S^1 -action equipped with an S^1 -equivariant map $\psi_p: M \rightarrow M^{hC_p}$. Composing with the canonical map $M^{hC_p} \rightarrow M^{tC_p}$ defines a p -typical cyclotomic spectrum, giving a colimit preserving functor

$$\mathbf{CycSp}_p^{\mathrm{Fr}} \rightarrow \mathbf{CycSp}_p .$$

A result of A. Krause and the second author [KN18, Proposition 10.3] says that the right adjoint is given by TR . Using this we can retranslate the fiber sequence above into

$$\mathbf{Map}_{\mathbf{TCart}_p}(M, \mathrm{TR}(X)) \rightarrow \mathbf{Map}_{\mathbf{CycSp}_p}(M, X) \rightarrow \mathbf{Map}_{\mathbf{CycSp}_p}(M_{hC_p}, X) .$$

where the cyclotomic spectrum M_{hC_p} has the trivial Frobenius $M_{hC_p} \xrightarrow{0} (M_{hC_p})^{tC_p}$. Using the description of the maps we see that the right hand map in this fiber sequence is the map induced from the map $V: M_{hC_p} \rightarrow M$ of cyclotomic spectra.⁸ Now the claim follows from the fact that the cyclotomic spectrum M/V is by construction the cofiber of the map $M_{hC_p} \rightarrow M$. \square

Lemma 3.19. *The natural map $\mathrm{TR}(X)/V \rightarrow X$ is an equivalence if X is bounded below.*

Proof. We first recall that $(-)_{hC_p}$ and $(-)^{tC_p}$ preserve sequential limits of uniformly bounded below spectra with S^1 -action by Lemma 2.11. Now, as noted above, $\mathrm{TR}(X)/V \rightarrow X$ is an equivalence if and only if the arrow of (11) is an equivalence. There are cofiber sequences $\mathrm{TR}^n(X)_{hC_p} \rightarrow \mathrm{TR}^{n+1}(X) \rightarrow X$ for each $n \geq 0$. Applying $(-)^{tC_p}$ and using the Tate orbit lemma, which holds as X is bounded below, we see that $\mathrm{TR}^{n+1}(X)^{tC_p} \simeq X^{tC_p}$ for all n . Since $(-)^{tC_p}$ commutes with limits of towers of uniformly bounded below spectra by the discussion at the beginning of the proof, $\mathrm{TR}(X)^{tC_p} \simeq X^{tC_p}$. \square

Let \mathbf{TCart}_p^- and \mathbf{CycSp}_p^- denote the full subcategories of \mathbf{TCart}_p and \mathbf{CycSp}_p on the bounded below objects.

⁸Note that this is not a map of cyclotomic spectra with Frobenius lifts, even though source and target have Frobenius lifts.

Definition 3.20. Say that a bounded below $M \in \mathbf{TCart}_p^-$ is V -complete if the limit of the tower

$$\cdots \rightarrow M_{hC_{p^2}} \xrightarrow{V_{hC_p}} M_{hC_p} \xrightarrow{V} M$$

vanishes. Write V^n for the composition $M_{hC_{p^n}} \xrightarrow{V_{hC_{p^{n-1}}}} M_{hC_{p^{n-1}}} \rightarrow \cdots \rightarrow M_{hC_p} \xrightarrow{V} M$. Let $\widehat{\mathbf{TCart}}_p^- \subseteq \mathbf{TCart}_p^-$ be the full subcategory of V -complete bounded below p -typical topological Cartier modules. We will say that a map $M \rightarrow \hat{M}$ in \mathbf{TCart}_p^- is a V -completion if \hat{M} is V -complete and if the natural forgetful map $\mathbf{Map}_{\mathbf{TCart}_p^-}(\hat{M}, N) \rightarrow \mathbf{Map}_{\mathbf{TCart}_p^-}(M, N)$ is an equivalence for every N in $\widehat{\mathbf{TCart}}_p^-$.

Theorem 3.21. *The functor $\mathrm{TR}: \mathbf{CycSp}_p^- \rightarrow \mathbf{TCart}_p^-$ is fully faithful and t -exact with left adjoint $M \mapsto M/V$. The essential image is the full subcategory of V -complete bounded below p -typical topological Cartier modules.*

Proof. The first claim follows immediately from the fact that $M \mapsto M/V$ is right t -exact, that TR is t -exact, and that the counit map $\mathrm{TR}(X)/V \rightarrow X$ is an equivalence for $X \in \mathbf{CycSp}_p^-$ bounded below, which was proved in Lemma 3.19. The second claim follows from the next proposition. \square

Proposition 3.22. *If $M \in \mathbf{TCart}_p^-$ is bounded below, then $\lim_n M/V^n$ admits the structure of a p -typical Cartier module and there is a natural map $M \rightarrow \lim_n M/V^n$ exhibiting $\lim_n M/V^n$ as the V -completion of M .*

Proof. By the first part of Theorem 3.21, there are two claims to check: first that $\mathrm{TR}(X)$ is V -complete for any bounded below p -typical cyclotomic spectrum X (apply this to $X = M/V$); second that $\mathrm{TR}(M/V) \simeq \lim_n M/V^n$ for a bounded below p -typical topological Cartier module. Indeed, the p -typical Cartier module structure on $\lim_n M/V^n$ comes from that on $\mathrm{TR}(M/V)$ via Construction 3.18. The natural map is then the unit map $M \rightarrow \mathrm{TR}(M/V)$ of the adjunction. We have that $\mathrm{TR}(N/V)$ is V -complete and hence $N \simeq \mathrm{TR}(N/V)$ for any V -complete M . In particular, $\mathbf{Map}_{\mathbf{TCart}_p^-}(\mathrm{TR}(M/V), N) \simeq \mathbf{Map}_{\mathbf{TCart}_p^-}(\mathrm{TR}(M/V), \mathrm{TR}(N/V)) \simeq \mathbf{Map}_{\mathbf{CycSp}_p^-}(M/V, N/V) \simeq \mathbf{Map}_{\mathbf{TCart}_p^-}(M, N)$ for $N \in \widehat{\mathbf{TCart}}_p^-$ by Proposition 3.17, which shows that $\mathrm{TR}(M/V)$ is a V -completion of M .

For the first claim, note that for each n the cofiber sequence

$$\mathrm{TR}(X)_{hC_{p^n}} / \mathrm{TR}(X)_{hC_{p^{n+1}}} \rightarrow \mathrm{TR}(X) / \mathrm{TR}(X)_{hC_{p^{n+1}}} \rightarrow \mathrm{TR}(X) / \mathrm{TR}(X)_{hC_{p^n}}$$

is equivalent to the cofiber sequence

$$X_{hC_{p^n}} \rightarrow X^{C_{p^n}} \xrightarrow{R} X^{C_{p^{n-1}}},$$

from which it follows immediately that $\mathrm{TR}(X) \simeq \lim_n \mathrm{TR}(X) / \mathrm{TR}(X)_{hC_{p^{n+1}}}$, or equivalently that $\lim_n \mathrm{TR}(X)_{hC_{p^{n+1}}} \simeq 0$. Similarly, the cofiber sequence

$$(M/V)_{hC_{p^n}} \rightarrow (M/V)^{C_{p^n}} \xrightarrow{R} (M/V)^{C_{p^{n-1}}}$$

is inductively equivalent to

$$M_{hC_{p^n}} / M_{hC_{p^{n+1}}} \rightarrow M / M_{hC_{p^{n+1}}} \rightarrow M / M_{hC_{p^n}},$$

which is what we wanted to show. \square

Remark 3.23. We have defined what it means to be V -complete for a bounded below topological Cartier module. In general, one can consider the Bousfield localization of \mathbf{TCart}_p at the mod V -equivalences, i.e. all the maps in \mathbf{TCart}_p that become equivalences after applying the functor $\mathbf{TCart}_p \rightarrow \mathbf{CycSp}_p$. This defines a Bousfield localization of the ∞ -category \mathbf{TCart}_p by the results of [Lur09, Section 5.5.4]. The local objects are those topological Cartier modules N for which the induced map

$$\mathbf{Map}_{\mathbf{TCart}_p}(M, N) = 0$$

whenever $M/V = 0$. It follows from the adjunction of Proposition 3.17 that $\mathrm{TR}(X)$ is mod V -local for every cyclotomic spectrum X . In general, the map $M \rightarrow \mathrm{TR}(M/V)$ is not a mod V -equivalence, but it is if M is bounded below. Therefore, a bounded below topological Cartier module is mod V -local precisely if it is V -complete in the sense of Definition 3.20. For a bounded below topological Cartier module the map $M \rightarrow \mathrm{TR}(M/V) \simeq \lim M/V^n$ is the mod V -localization. When M is not bounded below, this localization is mysterious and we do not know how to describe it or how to understand the mod V -local objects. See also Proposition 3.32 below.

3.4 The heart

From Theorem 3.21, we see that $\mathbf{CycSp}_p^\heartsuit$ is the full subcategory of $\mathbf{TCart}_p^\heartsuit$ consisting of V -complete objects. This notion of V -complete is in the ambient stable ∞ -category \mathbf{TCart}_p^- . The goal of this section is to describe a more intrinsic notion, making reference only to the abelian category of p -typical Cartier modules, that agrees with V -completeness for objects of $\mathbf{TCart}_p^\heartsuit$.

Definition 3.24. If M is a p -typical Cartier module, we say that M is **derived V -complete** if the limit $\lim(\cdots \rightarrow M \xrightarrow{V} M \xrightarrow{V} M)$ vanishes (in the derived category $\mathcal{D}(\mathbb{Z})$), or equivalently, if the canonical map $M \rightarrow \lim_n \mathrm{cofib}(M \xrightarrow{V^n} M)$ is an equivalence. We denote the category of p -typical Cartier modules by \mathbf{Cart}_p and the full subcategory of derived V -complete Cartier modules by $\widehat{\mathbf{Cart}}_p$.

Lemma 3.25. *Let M be a p -typical topological Cartier module concentrated in a single degree as in Example 3.2. Then M is V -complete as a p -typical topological Cartier module if and only if it is derived V -complete as a p -typical Cartier module.*

Proof. First, assume that M is derived V -complete. We have to show that the limit of the diagram

$$\cdots \xrightarrow{V_{hC_{p^2}}} M_{hC_{p^2}} \xrightarrow{V_{hC_p}} M_{hC_p} \xrightarrow{V} M \quad (12)$$

of spectra vanishes as well. This diagram can be written as the diagonal of the diagram

$$\begin{array}{ccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow V \otimes BC_{p^2} & & \downarrow V \otimes BC_p & & \downarrow V \\ \cdots & \xrightarrow{\mathrm{id} \otimes t_2} & M \otimes BC_{p^2} & \xrightarrow{\mathrm{id} \otimes t_1} & M \otimes BC_p & \xrightarrow{\mathrm{id} \otimes t_0} & M \\ & & \downarrow V \otimes BC_{p^2} & & \downarrow V \otimes BC_p & & \downarrow V \\ \cdots & \xrightarrow{\mathrm{id} \otimes t_2} & M \otimes BC_{p^2} & \xrightarrow{\mathrm{id} \otimes t_1} & M \otimes BC_p & \xrightarrow{\mathrm{id} \otimes t_0} & M \end{array}$$

where $t_n: BC_{p^{n+1}} \rightarrow BC_{p^n}$ is the canonical projection. Then the diagonal limit can also be computed by first computing the vertical limits followed by the horizontal one. We claim that all the vertical limits are

already trivial so that also the limit of (12) vanishes. For the rightmost column this is by assumption true. For the other columns we invoke part (b) of Lemma 2.11 to see that they also vanish.

The converse statement follows since for any diagram of spectra $\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0$ with vanishing limit the induced diagram $\cdots \rightarrow \pi_0(X_2) \rightarrow \pi_0(X_1) \rightarrow \pi_0(X_0)$ as a diagram of abelian groups has vanishing derived limit. This can be seen by writing the inverse limit as the fiber of a map

$$\prod X_i \rightarrow \prod X_i$$

which is then an equivalence since the limit vanishes. Thus it induces a bijection on π_0 which implies the claim since the homotopy groups of the derived limit of the π_0 -diagram are the kernel (\lim) and cokernel (\lim^1) of this map. \square

Theorem 3.21 and Lemma 3.25 imply the following theorem, which completes the proof of Theorem 1.

Theorem 3.26. *The heart $\mathbf{CycSp}_p^\heartsuit \subseteq \mathbf{TCart}_p^\heartsuit$ is the full subcategory of derived V -complete p -typical Cartier modules.*

Proof. Since $\mathrm{TR}: \mathbf{CycSp}_p^- \rightarrow \mathbf{TCart}_p^-$ is fully faithful and t -exact, it embeds the heart $\mathbf{CycSp}_p^\heartsuit$ fully faithfully into $\mathbf{TCart}_p^\heartsuit$. An object M of $\mathbf{TCart}_p^\heartsuit$ is moreover in $\mathbf{CycSp}_p^\heartsuit$ if and only if $M \rightarrow \mathrm{TR}(M/V)$ is an equivalence. Indeed, if $M \in \mathbf{TCart}_p^\heartsuit$ is equivalent to $\mathrm{TR}(N)$ for some $N \in \mathbf{CycSp}_p^\heartsuit$, then M is V -complete by Theorem 3.21 and $M \simeq \mathrm{TR}(M/V)$. On the other hand, if $M \simeq \mathrm{TR}(M/V)$, then $\pi_i^{\mathrm{cyc}} M/V = 0$ for $i > 0$ since TR is fully faithful on bounded below objects. Since the cyclotomic t -structure is left separated, this implies $M/V \in \mathbf{CycSp}_p^\heartsuit$. Now, $M \simeq \mathrm{TR}(M/V)$ if and only if M is V -complete as a p -typical topological Cartier module by Theorem 3.21. This happens if and only if M is derived V -complete by Lemma 3.25. \square

Using the theorem, we can prove that the t -structure on \mathbf{CycSp}_p is not compatible with filtered colimits.

Example 3.27. For the purposes of this example, we consider every abelian group M as a p -typical Cartier module with $V = p$ and $F = \mathrm{id}$. Then, M is derived V -complete if and only if M is derived p -complete. Now we have the filtered colimit

$$\mathrm{colim}_n \mathbb{Z}/p^n = \mathbb{Q}_p/\mathbb{Z}_p,$$

which we consider as a filtered colimit of p -typical Cartier modules. The p -typical Cartier modules \mathbb{Z}/p^n are derived V -complete. The module $\mathbb{Q}_p/\mathbb{Z}_p$ is not derived V -complete and the derived V -completion is given as the cofiber of the derived V -completions of \mathbb{Z}_p and \mathbb{Q}_p . The first is already derived V -complete and the derived V -completion of \mathbb{Q}_p is trivial. Hence, the derived V -completion of $\mathbb{Q}_p/\mathbb{Z}_p$ is $\mathbb{Z}_p[1]$. Thus the heart $\mathbf{CycSp}_p^\heartsuit$ is not closed under filtered colimits in $(\mathbf{CycSp}_p)_{\geq 0}$.

Theorem 3.26 generalizes by induction to give a description of all V -complete bounded below p -typical topological Cartier modules. The reader should compare this to the fact that a complex $M \in \mathcal{D}(\mathbb{Z})$ is p -complete if and only if each homology group $H_n(M)$ is derived p -complete.

Proposition 3.28. *A bounded below p -typical topological Cartier module M is V -complete precisely if all its homotopy groups are derived V -complete when considered as p -typical Cartier modules.*

Proof. Assume that M is connective and V -complete. Then, as in the second part of the proof of Lemma 3.25, we see that $\pi_0 M$ is derived V -complete as a p -typical Cartier module. We consider the fiber sequence $\tau_{\geq 1} M \rightarrow M \rightarrow \pi_0 M$. By Lemma 3.25, $\pi_0 M$ is V -complete and by assumption M is. It follows that $\tau_{\geq 1} M$ is as well. By induction we get that all homotopy groups are derived V -complete when considered as classical Cartier modules.

For the converse assume that all homotopy groups are derived V -complete as p -typical Cartier modules. We first write M as the limit of its Postnikov tower. Since completion commutes with this limit (the construction uses homotopy orbits, cofibers, and a sequential limit, all of which preserve limits of uniformly bounded below objects by Lemma 2.11), we can assume that M is bounded above. Then, we can reduce to a single homotopy group by using iterated extensions as in the first part of the proof. Finally, we can apply Lemma 3.25. \square

We now describe the algebraic ‘completion’ functor for algebraic Cartier modules to get a better understanding of the heart. This will be relevant in the next section.

Lemma 3.29. *The inclusion of derived V -complete p -typical Cartier modules into p -typical Cartier modules is reflective. The left adjoint LM can be described as π_0 of the V -completion of M as a topological Cartier module or as $\pi_0 \text{TR}(M/V)$. It is also equivalent to H_0 of the derived inverse limit of the tower*

$$\cdots \rightarrow \text{cofib}(M \xrightarrow{V^2} M) \rightarrow \text{cofib}(M \xrightarrow{V} M)$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. By the previous results we have a commutative diagram

$$\begin{array}{ccc} \widehat{\mathbf{Cart}}_p & \longrightarrow & \mathbf{Cart}_p \\ \downarrow & & \downarrow \\ (\widehat{\mathbf{TCart}}_p)_{\geq 0} & \longrightarrow & (\mathbf{TCart}_p)_{\geq 0}, \end{array}$$

where $(\widehat{\mathbf{TCart}}_p)_{\geq 0} = (\mathbf{TCart}_p)_{\geq 0} \cap \widehat{\mathbf{TCart}}_p^-$. Passing to left adjoints and applying to discrete topological Cartier modules we get the first claim. Now observe that π_0 of the inverse limit

$$M_V^\wedge = \lim(\cdots \rightarrow M/V^3 \rightarrow M/V^2 \rightarrow M/V)$$

(recall that M/V^n means the cofiber of the map $V^n : M_{hC_{p^n}} \rightarrow M$) is equivalent to π_0 of the inverse limit

$$\lim(\cdots \rightarrow \tau_{\leq 1}(M/V^3) \rightarrow \tau_{\leq 1}(M/V^2) \rightarrow \tau_{\leq 1}(M/V))$$

and that the canonical map $\tau_{\leq 1}(M/V^n)$ to $\text{cofib}(M \xrightarrow{V^n} M)$ is an equivalence which follows immediately from the long exact sequences. \square

We warn the reader that L is a slightly non-standard operation and we will refer to it as the algebraic derived V -completion. The crucial property is that $L : \mathbf{Cart}_p \rightarrow \widehat{\mathbf{Cart}}_p$ is a left adjoint (and hence is right exact) and that for V -torsion free modules it is just given by the usual inverse limit $\lim_n M/V^n$.

Lemma 3.30. *If M is V -torsion free, then so is LM .*

Proof. Since M is V -torsion free, the kernel of $V : \lim_n M/V^n \rightarrow \lim_n M/V^n$ is given by the inverse limit of the kernels of the maps

$$V : \text{coker}(M \xrightarrow{V^n} M) \rightarrow \text{coker}(M \xrightarrow{V^n} M).$$

But these kernels are isomorphic to $\text{coker}(M \xrightarrow{V} M)$ through the map $\text{coker}(M \xrightarrow{V} M) \rightarrow \ker(\text{coker}(M \xrightarrow{V^n} M) \rightarrow \text{coker}(M \xrightarrow{V^n} M))$ which sends m in M to $V^{n-1}(m)$. Under this equivalence the diagram over which we have to take the limit is the diagram

$$\cdots \xrightarrow{0} \text{coker}(M \xrightarrow{V} M) \xrightarrow{0} \text{coker}(M \xrightarrow{V} M) \xrightarrow{0} \text{coker}(M \xrightarrow{V} M)$$

whose limit is 0. This shows that LM is V -torsion free. \square

Lemma 3.31. *For a map $M \rightarrow N$ between V -torsion free Cartier modules, the induced map $LM \rightarrow LN$ on algebraic V -completions is an isomorphism precisely if the induced map $\operatorname{coker}(M \xrightarrow{V} M) \rightarrow \operatorname{coker}(N \xrightarrow{V} N)$ is an isomorphism.*

Proof. By the derived V -completion we shall mean the object of the derived category given on underlying abelian groups by the (derived) inverse limit

$$\lim_n \operatorname{cofib}(M \xrightarrow{V^n} M).$$

The V -operator on this limit can be described in the obvious way coming from a map of diagrams. We claim that a map $M \rightarrow N$ between not necessarily V -torsion free p -typical Cartier modules induces an equivalence on derived V -completions precisely if it induces an equivalence $\operatorname{cofib}(M \xrightarrow{V} M) \rightarrow \operatorname{cofib}(N \xrightarrow{V} N)$. To see this note that $\operatorname{cofib}(\lim_n \operatorname{cofib}(M \xrightarrow{V^n} M) \xrightarrow{V} \lim_n \operatorname{cofib}(M \xrightarrow{V^n} M)) \simeq \operatorname{cofib}(M \xrightarrow{V} M)$ which implies that a map which is an equivalence after derived V -completion is also an equivalence after mod V reduction. Conversely, if $\operatorname{cofib}(M \xrightarrow{V} M) \rightarrow \operatorname{cofib}(N \xrightarrow{V} N)$ is an equivalence, then one shows inductively using the fiber sequence

$$\operatorname{cofib}(M \xrightarrow{V} M) \rightarrow \operatorname{cofib}(M \xrightarrow{V^n} M) \rightarrow \operatorname{cofib}(M \xrightarrow{V^{n-1}} M)$$

and the analogous one for N to show that that $\operatorname{cofib}(M \xrightarrow{V^n} M) \rightarrow \operatorname{cofib}(N \xrightarrow{V^n} N)$ is an equivalence for every n which implies that the map on the limit is an equivalence. \square

To conclude, we note that the right t -completion of the ∞ -category of cyclotomic spectra is a localization of \mathbf{TCart}_p . This will not be needed in the rest of the paper.

Proposition 3.32. *The functor $(-)/V: \mathbf{TCart}_p \rightarrow \mathbf{CycSp}_p$ factors through the right completion*

$$\operatorname{Sp}((\mathbf{CycSp}_p)_{\geq 0}) \rightarrow \mathbf{CycSp}_p$$

of p -typical cyclotomic spectra with respect to the cyclotomic t -structure. The induced map $\mathbf{TCart}_p \rightarrow \operatorname{Sp}((\mathbf{CycSp}_p)_{\geq 0})$ exhibits the right completion as a localization of \mathbf{TCart}_p . The local objects are those topological Cartier modules all of whose homotopy groups are derived V -complete.⁹

Proof. Because TR is t -exact, we have a commutative diagram

$$\begin{array}{ccccc} \cdots & \longrightarrow & (\mathbf{CycSp}_p)_{\geq n} & \xrightarrow{\tau_{\geq n+1}} & (\mathbf{CycSp}_p)_{\geq n+1} & \longrightarrow & \cdots \\ & & \downarrow \operatorname{TR} & & \downarrow \operatorname{TR} & & \\ \cdots & \longrightarrow & (\mathbf{TCart}_p)_{\geq n} & \xrightarrow{\tau_{\geq n+1}} & (\mathbf{TCart}_p)_{\geq n+1} & \longrightarrow & \cdots \end{array}$$

of ∞ -categories, where the vertical maps are fully faithful with essential image consisting of n -connective topological Cartier modules that are V -complete. By Proposition 3.28 these are equivalently those n -connective topological Cartier modules whose homotopy groups are derived V -complete. Taking the limit, as $n \rightarrow -\infty$, we obtain a fully faithful embedding

$$\operatorname{Sp}((\mathbf{CycSp}_p)_{\geq 0}) \rightarrow \mathbf{TCart}_p,$$

where $\operatorname{Sp}((\mathbf{CycSp}_p)_{\geq 0})$ is the right completion of \mathbf{CycSp}_p with respect to the cyclotomic t -structure. (Recall from Proposition 3.15 that \mathbf{TCart}_p is already right complete.) This inclusion functor preserves limits and is accessible, and thus admits a left adjoint. Moreover the essential image consists of those topological Cartier modules all of whose homotopy groups are derived V -complete. \square

⁹We would like to thank the anonymous referee for suggesting this explicit characterization of the local objects.

3.5 p -typical Cartier complexes

Let M be a p -typical topological Cartier module. We have already seen in Section 3.3 that the homotopy groups π_*M admit the structure of p -typical Cartier modules, i.e. for every $k \in \mathbb{Z}$ the induced maps

$$V : \pi_k M \rightarrow \pi_k M \quad \text{and} \quad F : \pi_k M \rightarrow \pi_k M$$

satisfy $FV = p$. Moreover from the S^1 -action on M we get a ‘Connes operator’

$$d : \pi_k M \rightarrow \pi_{k+1} M$$

which satisfies $d^2 = \eta d = d\eta$ where η here denotes the map

$$\eta : \pi_k M \rightarrow \pi_{k+1} M$$

given by acting with the Hopf element $\eta \in \pi_1(\mathbb{S})$. In particular the map η is 2-torsion and $\eta^4 = 0$. To see this we note that the map d is defined by acting with an element d in π_1 of the spherical group ring $\mathbb{S}[S^1]$ which is given by the fundamental class in $\pi_1(S^1)$ shifted to the basepoint 0 in $\mathbb{S}[S^1]$. Then the claim follows from the fact that

$$\pi_*(\mathbb{S}[S^1]) = (\pi_*\mathbb{S})[d]/(d^2 = \eta d)$$

which is implied by the determination of stable homotopy class of the multiplication map $S^1 \times S^1 \rightarrow S^1$. This map is after a single suspension given by the map

$$S^2 \vee S^2 \vee S^3 \rightarrow S^2$$

which is the inclusions on the first two summands and the Hopf map on the last. See also [HM04] for a discussion.

Lemma 3.33. *For a topological Cartier module M we have on π_*M the relations*

$$Vd = pdV, \quad dF = pFd, \quad FdV = \begin{cases} d & \text{for } p > 2, \\ d + \eta & \text{for } p = 2 \end{cases}$$

and the maps F, V commute with η .

Proof. The S^1 -equivariant map $V : M_{hC_p} \rightarrow M$ can equivalently be considered as an S^1 -equivariant map $M \rightarrow \text{res}_p M$ where $\text{res}_p M$ has the S^1 -action given by restricting the S^1 -action on M along the p -fold cover map $S^1 \rightarrow S^1$ given by $z \mapsto z^p$. Thus we get that on homotopy groups $V(d(x)) = d'V(x)$ where $d' : \pi_k M \rightarrow \pi_{k+1} M$ is the Connes operator associated with the S^1 -action on $\text{res}_p M$. But $d' = pd$ since the p -fold cover map is of degree p . Hence, we see that

$$Vd = d'V = pdV.$$

The Frobenius can dually be considered as a map $\text{res}_p M \rightarrow M$ and we therefore find

$$dF = Fd' = Fpd = pFd.$$

To identify $FdV : \pi_*M \rightarrow \pi_{*+1}M$ we note that we can write this as a composite

$$\pi_*M \xrightarrow{\iota} \pi_*(M_{hC_p}) \xrightarrow{V_*} \pi_*M \xrightarrow{d} \pi_{*+1}M \xrightarrow{F_*} \pi_{*+1}(M^{hC_p}) \xrightarrow{\kappa} \pi_{*+1}M,$$

where ι and κ are the structure maps and the maps V_* and F_* are the maps induced by the spectral maps V and F on homotopy groups. The operator called abusively V above was defined as the composite $V_* \circ \iota$ and similar for F . Since V is S^1 -equivariant we find that the composite of the maps is equal to

$$\kappa \circ d' \circ F_* \circ V_* \circ \iota = \kappa \circ d' \circ \text{Nm}_{C_p} \circ \iota,$$

where d' is the Connes operator for the residual S^1 -action on M_{hC_p} .¹⁰ Now we claim that for every S^1 -spectrum M (not necessarily a p -typical topological Cartier module) the composite $\kappa \circ d' \circ \text{Nm}_{C_p} \circ \iota$ is given by $d : \pi_* M \rightarrow \pi_{*+1} M$ for p odd and by $d + \eta : \pi_* M \rightarrow \pi_{*+1} M$ for $p = 2$. It is enough to show this claim for M the free S^1 -spectrum on a generator in degree 0, which is to say for $M = \Sigma_+^\infty S^1$ and the class $x \in \pi_0(M)$ induced by the basepoint $1 \in S^1$. We have identifications

$$M \simeq M_{hC_p} \simeq M^{hC_p} \simeq \mathbb{S} \oplus \Sigma \mathbb{S}$$

such that under these identifications the map $\iota : M \rightarrow M_{hC_p}$ is given by the map $\text{id} \oplus p : \mathbb{S} \oplus \Sigma \mathbb{S} \rightarrow \mathbb{S} \oplus \Sigma \mathbb{S}$, the norm is given by the identity and the map $\kappa : M^{hC_p} \rightarrow M$ is given by the map $p \oplus \text{id} : \mathbb{S} \rightarrow \mathbb{S}$ for p odd and by the map

$$\begin{pmatrix} p & \eta \\ 0 & \text{id} \end{pmatrix} : \mathbb{S} \oplus \Sigma \mathbb{S} \rightarrow \mathbb{S} \oplus \Sigma \mathbb{S}$$

for $p = 2$ (see the proof of Lemma [Hes96, Lemma 1.5.1] for an argument). Under these identifications the operator d' takes $1 \in \mathbb{S} \oplus \Sigma \mathbb{S} = M_{hC_p}$ to the unit element in π_1 of $\Sigma \mathbb{S}$. Then the claim follows by a straightforward computation. The commutativity of the maps F and V with η is clear since these are stable maps. \square

Now, we make this structure into a definition which is inspired by the definition of a Dieudonné complex in [BLM18].

Definition 3.34. A p -typical Cartier complex is a \mathbb{Z} -graded abelian group C^* together with operators

$$V, F : C^* \rightarrow C^* \quad \text{and} \quad \eta, d : C^* \rightarrow C^{*+1}$$

satisfying

$$\begin{aligned} FV &= p, & d^2 &= \eta d = d\eta, & 2\eta &= \eta^4 = 0, \\ Vd &= pdV, & dF &= pFd, & FdV &= \begin{cases} d & \text{for } p > 2, \\ d + \eta & \text{for } p = 2. \end{cases} \end{aligned}$$

Remark 3.35. Hesselholt–Madsen [HM04] and Hesselholt [Hes15, Definition 4.1] introduced the notion of a Witt complex. The universal example is the absolute de Rham–Witt complex. These Witt complexes give examples of p -typical Cartier complexes in the sense of Definition 3.34 with η given by multiplication with $d \log[-1] = [-1] \cdot d[-1]$. In fact Witt complexes should be considered as a multiplicative version of Cartier complexes (with some additional structure like a map from the Witt vectors). We will study the precise relation in future work.

Lemma 3.33 shows that the homotopy groups $C^* := \pi_* M$ of a p -typical topological Cartier module M naturally form a p -typical Cartier complex. The statement of Proposition 3.28 is that for bounded below M the V -completeness of M is equivalent to the derived V -completeness of the terms $\pi_* M$. In particular,

¹⁰This map is also equal to $\kappa \circ \text{Nm}_{C_p} \circ d' \circ \iota$ for d' corresponding to the residual S^1 -action on M^{hC_p} but we shall not need this fact.

the completeness of M can be entirely expressed in terms of the associated p -typical Cartier complex. More generally one can ask that all the terms C^i be derived V -complete for a general Cartier complex C^* . It turns out in practice that this condition is somewhat hard to verify since the V -adic filtration does not take the differential d into account. For example, in the case of the de Rham–Witt complex which will be treated later, the quotients by iterations of V will not be the truncated de Rham–Witt complexes. We will now give a completeness condition for a p -typical Cartier complex that is equivalent to degreewise derived V -completeness but closer to the notions of completeness that arise for the de Rham–Witt complex.

Construction 3.36. Let C^* be a p -typical Cartier complex. For every $i \in \mathbb{Z}$ and $r \geq 0$ we define a derived quotient $C^i/(V^r + dV^r)$ in $\mathcal{D}(\mathbb{Z})$ as the total cofiber of the commutative square

$$\begin{array}{ccc} C^{i-1} & \xrightarrow{p^r} & C^{i-1} \\ \downarrow d & & \downarrow dV^r \\ C^i & \xrightarrow{V^r} & C^i. \end{array}$$

This total cofiber is by definition the cofiber of the map $V^r + dV^r : C^i \oplus_{C^{i-1}} C^{i-1} \rightarrow C^i$ (where the source is the derived pushout) and this justifies the notation $C^i/(V^r + dV^r)$.

There is a natural diagram

$$\dots \rightarrow C^i/(V^3 + dV^3) \rightarrow C^i/(V^2 + dV^2) \rightarrow C^i/(V + dV)$$

where the map $C^i/(V^{r+1} + dV^{r+1}) \rightarrow C^i/(V^r + dV^r)$ is induced from the map of squares given by the identity on the lower right term, by pV on the upper left term and by V on the other two terms. We will denote the (derived) limit of this diagram as $(C^i)_{V+dV}^\wedge \in \mathcal{D}(\mathbb{Z})$. There is a canonical map

$$C^i \rightarrow (C^i)_{V+dV}^\wedge$$

induced from the structure map out of lower right corner of the defining square for $C^i/(V^r + dV^r)$.

Definition 3.37. A Cartier complex C^* is called derived $(V + dV)$ -**complete** if for every i the map

$$C^i \rightarrow (C^i)_{V+dV}^\wedge$$

from Construction 3.36 is an equivalence in $\mathcal{D}(\mathbb{Z})$.

Now we show that for a bounded below Cartier complex this notion of $(V + dV)$ -completeness is equivalent to the naive notion of degreewise derived V -completeness. The latter means that for every i the Cartier module C^i is derived complete in the sense of Definition 3.24. Bounded below means that there exists i_0 such that $C^i = 0$ for $i < i_0$ (we hope that the cohomological notation does not lead to confusion).

Proposition 3.38. *A bounded below Cartier module C^* is derived $(V + dV)$ -complete precisely if it is degreewise derived V -complete.*

Proof. We can assume by shifting that $C^i = 0$ for $i < 0$. For C^0 we then obtain that $C^0/(V^r + dV^r) = C^0/V^r$ and that $(C^0)_{V+dV}^\wedge = (C^0)_V^\wedge$ where the latter is the derived V -completion. It follows that C^0 is derived $(V + dV)$ -complete precisely if it is derived V -complete.

Now we proceed by induction over i . Assume that C^{i-1} is derived V and $(V + dV)$ -complete. We will show that C^i is derived $(V + dV)$ -complete precisely if it is derived V -complete. By definition $(C^i)_{V+dV}^\wedge$ is the limit of total cofibers of squares

$$\begin{array}{ccc} C^{i-1} & \xrightarrow{p^r} & C^{i-1} \\ \downarrow d & & \downarrow dV^r \\ C^i & \xrightarrow{V^r} & C^i. \end{array}$$

The limit is taken over maps of total cofibers obtained from maps of squares. Thus we can equivalently described $(C^i)_{V+dV}^\wedge$ as the total cofiber of the square obtained as the limit of these maps of squares. This limit square is

$$\begin{array}{ccc} \lim(\dots \xrightarrow{pV} C^{i-1} \xrightarrow{pV} C^{i-1} \xrightarrow{pV} C^{i-1}) & \longrightarrow & \lim(\dots \xrightarrow{V} C^{i-1} \xrightarrow{V} C^{i-1} \xrightarrow{V} C^{i-1}) \\ \downarrow & & \downarrow \\ \lim(\dots \xrightarrow{V} C^i \xrightarrow{V} C^i \xrightarrow{V} C^i) & \longrightarrow & C^i. \end{array}$$

The upper right term vanishes by the assumption that C^{i-1} is derived V -complete; the upper left term vanishes since the inverse limit can be written equivalently as the limit of

$$\dots \xrightarrow{p} \lim_{i,V} C^{i-1} \xrightarrow{p} \lim_{i,V} C^{i-1} \xrightarrow{p} \lim_{i,V} C^{i-1}$$

where $\lim_{i,V} C^{i-1}$ is the term in the upper right corner, which we have already noted is zero. In other words the limit square takes the form

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \lim_{i,V} C^i & \longrightarrow & C^i. \end{array}$$

From this description it follows that the map from C^i to the total cofiber of this square is an equivalence precisely if the lower left corner vanishes, i.e. if C^i is derived V -complete. This finishes the proof. \square

4 The symmetric monoidal structure

According to Corollary 2.14 we get an induced symmetric monoidal structure on $\mathbf{CycSp}_p^\heartsuit$ from the tensor product of cyclotomic spectra. In this section we shall give an explicit formula for this symmetric monoidal structure and explore some consequences. We will come back to this in later work.

4.1 The tensor product of topological Cartier modules

In this section we shall describe the symmetric monoidal structure on $\widehat{\mathbf{TCart}}_p^-$ induced through the equivalence $\widehat{\mathbf{TCart}}_p^- \simeq \mathbf{CycSp}_p^-$ from the one on cyclotomic spectra. We denote the tensor product corresponding to the symmetric monoidal structure on $\widehat{\mathbf{TCart}}_p^-$ by $\widehat{\boxtimes}$. More precisely we have that

$$M\widehat{\boxtimes}N := \mathrm{TR}(M/V \otimes N/V),$$

where \otimes is the tensor product of cyclotomic spectra whose underlying spectrum is just the tensor product of spectra. In this section we will follow the convention that \otimes shall always refer to symmetric monoidal structures that are taken ‘underlying’ in this sense. Our first task is to give a more explicit formula for $M\widehat{\boxtimes}N$ and then we will identify the induced symmetric monoidal structure on the heart explicitly. In future work we will show that this symmetric monoidal structure on $(\mathbf{TCart}_p^-)_{V+dV}^\wedge$ is induced by a natural symmetric monoidal structure on \mathbf{TCart}_p^- which will be denoted by \boxtimes and use this to understand the relation to de Rham–Witt complexes that will be explained in Section 6 over general bases.

Recall the ∞ -category $\mathbf{CycSp}_p^{\text{Fr}}$ of cyclotomic spectra with Frobenius lift. There are natural forgetful functors

$$\mathbf{TCart}_p \rightarrow \mathbf{CycSp}_p^{\text{Fr}} \rightarrow \mathbf{CycSp}_p$$

where the first functor just forgets the V operator and the second sends (X, ψ_p) to the cyclotomic spectrum X equipped with the Frobenius $X \rightarrow X^{hC_p} \xrightarrow{\text{can}} X^{tC_p}$. As proven in [KN18, Proposition 10.3], the second functor admits a right adjoint given by TR , which, as we have proved above, factors canonically through the first category. The first functor $\mathbf{TCart}_p \rightarrow \mathbf{CycSp}_p^{\text{Fr}}$ admits both adjoints by the adjoint functor theorem, but we will only be concerned with the left adjoint here.

Lemma 4.1. *The forgetful functor*

$$\mathbf{TCart}_p \rightarrow \mathbf{CycSp}_p^{\text{Fr}}$$

is monadic. The underlying S^1 -spectrum of the left adjoint applied to M in $\mathbf{CycSp}_p^{\text{Fr}}$ is given by $M[V] := \bigoplus_{n \geq 0} M_{hC_{p^n}}$ with the structure of a topological Cartier module that will be described in the proof.

Proof. The forgetful functor reflects equivalences and preserves all colimits and limits. Thus, it follows from Lurie’s version of the monadicity theorem that it is monadic (see [Lur17, 4.7.0.3]). To understand the left adjoint functor we give a construction of an object $M[V] \in \mathbf{TCart}_p$. As a spectrum with S^1 -action we set

$$M[V] := \bigoplus_{n \geq 0} M_{hC_{p^n}}$$

where $M_{hC_{p^n}}$ carries the residual $S^1/C_{p^n} \cong S^1$ -action. We now want to equip $M[V]$ with the structure of a p -typical topological Cartier module. To this end, we define the V -operator as the inclusion

$$M[V]_{hC_p} \simeq \bigoplus_{n \geq 1} M_{hC_{p^n}} \rightarrow \bigoplus_{n \geq 0} M_{hC_{p^n}} = M[V]$$

and the F -operator as the composition

$$\bigoplus_{n \geq 0} M_{hC_{p^n}} \rightarrow \bigoplus_{n \geq 0} (M_{hC_{p^n}})^{hC_p} \rightarrow \left(\bigoplus_{n \geq 0} M_{hC_{p^n}} \right)^{hC_p} = M[V]^{hC_p}$$

where the second map is the canonical interchange map and the first is given on the $(n = 0)$ -summand by F and on the n th summand by the C_p -norm map

$$M_{hC_{p^n}} \simeq (M_{hC_{p^{n-1}}})_{hC_p} \rightarrow (M_{hC_{p^{n-1}}})^{hC_p}$$

followed by the inclusion into the direct sum. There is a canonical equivalence between the composition $F \circ V$ and the norm map, as the latter also factors as the composition

$$M[V]_{hC_p} \simeq \bigoplus_{n \geq 1} M_{hC_{p^n}} \xrightarrow{\bigoplus \text{Nm}_{C_p}} \bigoplus_{n \geq 1} (M_{hC_{p^{n-1}}})^{hC_p} \rightarrow \left(\bigoplus_{n \geq 1} M_{hC_{p^{n-1}}} \right)^{hC_p}.$$

Now we compute the mapping spectrum $\mathbf{Map}_{\mathbf{TCart}_p}(M[V], N)$ and show that the canonical map to $\mathbf{Map}_{\mathbf{CycSp}_p^{\text{Fr}}}(M, N)$, induced by the forgetful functor $\mathbf{TCart}_p \rightarrow \mathbf{CycSp}_p^{\text{Fr}}$ and the map $M \rightarrow M[V]$ of cyclotomic spectra with Frobenius lifts, is an equivalence. Here we abusively denote the ‘underlying’

cyclotomic spectrum with Frobenius lift of N also by N . To this end we use the formula for the mapping space in \mathbf{TCart}_p given in Proposition 3.10 and get a fibre sequence

$$\mathbf{Map}_{\mathbf{TCart}_p}(M[V], N) \rightarrow \mathbf{Map}_{\mathbf{Alg}_{(-)hC_p}}(M[V], N) \xrightarrow{\vartheta} \mathbf{Map}_{\mathbf{Sp}^{BS^1}}(\mathrm{cofib}(V_M), N^{hC_p}) . \quad (13)$$

Note that $\mathrm{cofib}(V_M) = M$. Moreover, it is straightforward to check that $\mathbf{Map}_{\mathbf{Alg}_{(-)hC_p}}(M[V], N) \simeq \mathbf{Map}_{\mathbf{Sp}^{BS^1}}(M, N)$. In fact $M[V]$ is by construction free as a $(-)_{hC_p}$ -algebra if we neglect the F -operators. Under these identifications the fibre sequence (13) takes the form

$$\mathbf{Map}_{\mathbf{TCart}_p}(M[V], N) \rightarrow \mathbf{Map}_{\mathbf{Sp}^{BS^1}}(M, N) \xrightarrow{\vartheta} \mathbf{Map}_{\mathbf{Sp}^{BS^1}}(M, N^{hC_p})$$

with the right hand map ϑ is given by sending g to $F_N \circ g - g^{hC_p} \circ F_M$. But this is also the mapping spectrum in $\mathbf{CycSp}^{\mathrm{Fr}}$. \square

Remark 4.2. For any spectrum with S^1 -action M the spectrum $M[V] = \bigoplus_{n \geq 0} M_{hC_p^n}$ of the last lemma is free as a spectrum with V operator. The lemma shows that if M admits an F -operator then $M[V]$ is a topological Cartier module and is also free as a topological Cartier module. One can also express this by saying that the commutative square of forgetful functors

$$\begin{array}{ccc} \mathbf{TCart}_p & \longrightarrow & \mathbf{Alg}_{(-)hC_p} \\ \downarrow & & \downarrow \\ \mathbf{Alg}_{(-)hC_p} & \longrightarrow & \mathbf{Sp}^{BS^1} \end{array}$$

remains commutative after passing to left adjoints of the horizontal maps, i.e., the square is left adjointable.

As usual we denote by $(\mathbf{CycSp}_p^{\mathrm{Fr}})^- \subseteq \mathbf{CycSp}_p^{\mathrm{Fr}}$ the full subcategory of bounded below objects. The adjunction of Lemma 4.1 induces an adjunction

$$(\mathbf{CycSp}_p^{\mathrm{Fr}})^- \xrightleftharpoons{\quad} \widehat{\mathbf{TCart}}_p^- \quad (14)$$

whose left adjoint sends M to the V -completion of $\bigoplus_{n \geq 0} M_{hC_p^n}$ which is $\prod_{n \geq 0} M_{hC_p^n}$ and will be denoted by $M[[V]]$. We note that this is also equivalent to $\mathrm{TR}(M)$ where M is considered as a cyclotomic spectrum with the Frobenius $M \xrightarrow{F} M^{hC_p} \xrightarrow{\mathrm{can}} M^{tC_p}$ (since this is the mod V reduction).

We now use that the ∞ -category $\mathbf{CycSp}_p^{\mathrm{Fr}}$ has a symmetric monoidal structure given by the ‘underlying’ tensor product. Formally this symmetric monoidal structure is constructed exactly as the one on \mathbf{CycSp}_p in [NS18, Construction IV.2.1] using that the functor $(-)_{hC_p} : \mathbf{Sp}^{BS^1} \rightarrow \mathbf{Sp}^{BS^1}$ admits a canonical lax symmetric monoidal structure. The symmetric monoidal structure on $\mathbf{CycSp}_p^{\mathrm{Fr}}$ restricts to one on the full subcategory $(\mathbf{CycSp}_p^{\mathrm{Fr}})^- \subseteq \mathbf{CycSp}_p^{\mathrm{Fr}}$ since $X \otimes Y$ is bounded below for X and Y bounded below.

Proposition 4.3. *The left adjoint $-[[V]] : (\mathbf{CycSp}_p^{\mathrm{Fr}})^- \rightarrow \widehat{\mathbf{TCart}}_p^-$ admits a canonical refinement to a symmetric monoidal functor.*

Proof. By definition, the functor $(-)/V : \widehat{\mathbf{TCart}}_p^- \rightarrow \mathbf{CycSp}_p^-$ is a symmetric monoidal equivalence. Thus, it suffices to equip the composition

$$(\mathbf{CycSp}_p^{\mathrm{Fr}})^- \xrightarrow{-[[V]]} \widehat{\mathbf{TCart}}_p^- \xrightarrow{(-)/V} \mathbf{CycSp}_p^-$$

with a symmetric monoidal structure. We claim that this composite is equivalent to the functor which takes $X \in (\mathbf{CycSp}_p^{\text{Fr}})^-$ to the ‘underlying’ cyclotomic spectrum of X , i.e. X equipped with the composition $X \rightarrow X^{hC_p} \xrightarrow{\text{can}} X^{tC_p}$ as Frobenius. To see this we have to compute the cofiber of the map

$$V : \left(\prod_{n \geq 0} M_{hC_{p^n}} \right)_{hC_p} \rightarrow \prod_{n \geq 0} M_{hC_{p^n}} .$$

Since the product is uniformly bounded below it commutes with the orbits and the claim follows from the description of the V and F operators given in the proof of Lemma 4.1 above. Finally the functor $\mathbf{CycSp}_p^{\text{Fr}} \rightarrow \mathbf{CycSp}_p$ admits by construction of the symmetric monoidal structures a symmetric monoidal refinement since the transformation $(-)^{hC_p} \rightarrow (-)^{tC_p}$ is a symmetric monoidal transformation. \square

We now observe that for every p -typical topological Cartier module there is a natural cofiber sequence

$$(M_{hC_p})[V] \rightarrow M[V] \rightarrow M \tag{15}$$

of p -typical topological Cartier modules. Here the S^1 -spectrum M_{hC_p} is considered as a cyclotomic spectrum with Frobenius lift, where the Frobenius lift is given by the zero map. Then the map $(M_{hC_p})[V] \rightarrow M[V]$ is induced from the map $M_{hC_p} \rightarrow M[V]$ that is given as the composition

$$M_{hC_p} \xrightarrow{(\text{id}, -V)} M_{hC_p} \oplus M \xrightarrow{i} \bigoplus_{n \geq 0} M_{hC_{p^n}} = M[V]$$

where i is the summand inclusion. This is a map in $\mathbf{CycSp}^{\text{Fr}}$ and therefore gives rise to a map $(M_{hC_p})[V] \rightarrow M[V]$. Concretely this map is given by $i - V$ where i is the inclusion $(M_{hC_p})[V] \rightarrow M[V]$ and $V = \bigoplus_{n \geq 0} V_{hC_{p^n}}$ is just applied levelwise. It is easy to see that the composite to M comes with a preferred nullhomotopy (as we also only have to check that on M_{hC_p}) and that on underlying spectra this is a cofiber sequence. This implies that it is a cofiber sequence of p -typical topological Cartier modules.¹¹

If M is bounded below and V -complete, we also get a cofiber sequence

$$(M_{hC_p})[[V]] \rightarrow M[[V]] \rightarrow M \tag{16}$$

in $\widehat{\mathbf{TCart}}_p^-$ by completion of (15).

Corollary 4.4. *For every pair of V -complete bounded below p -typical topological Cartier modules M and N the tensor product $M \widehat{\boxtimes} N \in \widehat{\mathbf{TCart}}_p^-$ is equivalent to the total cofiber of a square*

$$\begin{array}{ccc} (M_{hC_p} \otimes N_{hC_p})[[V]] & \longrightarrow & (M \otimes N_{hC_p})[[V]] \\ \downarrow & & \downarrow \\ (M_{hC_p} \otimes N)[[V]] & \longrightarrow & (M \otimes N)[[V]]. \end{array}$$

¹¹Note that this gives rise to our standard cofiber sequence $M_{hC_p} \rightarrow M \rightarrow M/V$ of cyclotomic spectra upon taking the mod V reductions (again M_{hC_p} has the trivial Frobenius). But while the first map is a map of cyclotomic spectra between cyclotomic spectra that admit Frobenius lifts, it is not a map of cyclotomic spectra with Frobenius lifts. That is important to keep in mind in identifying some of the maps later.

Proof. We use that the symmetric monoidal structure $\widehat{\boxtimes}$ on $\widehat{\mathbf{TCart}}_p^-$ commutes with colimits and the cofiber sequence (16) for M and N to deduce that the tensor product is the total cofiber of a square

$$\begin{array}{ccc} M_{hC_p}[[V]] \widehat{\boxtimes} N_{hC_p}[[V]] & \longrightarrow & M[[V]] \widehat{\boxtimes} N_{hC_p}[[V]] \\ \downarrow & & \downarrow \\ M_{hC_p}[[V]] \widehat{\boxtimes} N[[V]] & \longrightarrow & M[[V]] \widehat{\boxtimes} N[[V]]. \end{array}$$

Now we use that $-[[V]]$ is symmetric monoidal as shown in Proposition 4.3 to get the result. (Note that we do not need to complete the total cofiber as it is already V -complete since this is a finite colimit.) \square

Remark 4.5. One can also work out the maps in the diagram of Corollary (4.4) by using the fact that this is a diagram in TR for the corresponding diagram

$$\begin{array}{ccc} (M_{hC_p} \otimes N_{hC_p}) & \xrightarrow{V \otimes \text{id}} & (M \otimes N_{hC_p}) \\ \downarrow \text{id} \otimes V & & \downarrow \text{id} \otimes V \\ (M_{hC_p} \otimes N) & \xrightarrow{V \otimes \text{id}} & (M \otimes N) \end{array}$$

of cyclotomic spectra. We again issue the warning that this is not a diagram with Frobenius lifts. Therefore one has to use the identification of TR with $-[[V]]$ in each term but the maps are not compatible with that identification and we get additional terms coming from the Frobenius operators.

Now assume that M and N are connective. We want to determine $\pi_0(M \widehat{\boxtimes} N)$ in terms of $\pi_0(M)$ and $\pi_0(N)$. For M connective we have that $\pi_0(M[[V]]) = \bigoplus_{n \geq 0} \pi_0(M)$. Thus, on π_0 , the effect of $-[[V]]$ is given by adjoining V freely as a module. We will write this formula as $\pi_0(M[[V]]) = (\pi_0 M)[V]$, thus use the notation $A[V]$ for an abelian group with a map $F : A \rightarrow A$ for the p -typical Cartier module $\bigoplus_{n \geq 0} A$ formed analogously to the topological case. We shall write a typical element in $A[V]$ as a polynomial in V , i.e. as $\sum_{i=1}^n a_i V^i$. Similarly, we have that $\pi_0(M[[V]]) = \pi_0(M)[V]$.

Now, observe that in the formula of Corollary 4.4 all terms are connective if M and N are. As a result we find that $\pi_0(M \widehat{\boxtimes} N)$ is the total cokernel of a square

$$\begin{array}{ccc} (\pi_0 M \otimes \pi_0 N)[[V]] & \longrightarrow & (\pi_0 M \otimes \pi_0 N)[[V]] \\ \downarrow & & \downarrow \\ (\pi_0 M \otimes \pi_0 N)[[V]] & \longrightarrow & (\pi_0 M \otimes \pi_0 N)[[V]]. \end{array} \tag{17}$$

Of course the total cokernel does not depend on the upper left term (this is why this is not a concept one ever hears of), so it is just the quotient of $(\pi_0 M \otimes \pi_0 N)[[V]]$ by the image of the two maps into it.

Corollary 4.6. *For any pair of V -complete connective p -typical topological Cartier modules M and N we can describe $\pi_0(M \widehat{\boxtimes} N)$ as π_0 of the algebraic derived V -completion (see Lemma 3.29 and the following discussion for this notion) of a p -typical Cartier module*

$$(\pi_0 M \otimes \pi_0 N)[V] / \sim$$

where the equivalence relation is generated additively by

$$(m \otimes Vn)V^k \sim (Fm \otimes n)V^{k+1} \quad (Vm \otimes n)V^k \sim (m \otimes Fn)V^{k+1}$$

for all $m \in \pi_0 M$, $n \in \pi_0 N$ and $k \in \mathbb{N}$.

Proof. It is clear that the square (17) is the V -completion of a square with terms $(\pi_0 M \otimes \pi_0 N)[V]$ instead of $(\pi_0 M \otimes \pi_0 N)[V]$. The rest follows from the description of the maps in the square (17) which can be done as explained in Corollary 4.5 or just using that they have to be compatible with F (where F is zero on the terms with orbits). Concretely one gets the following description of the maps:

1. the upper horizontal map sends $(m \otimes n)V^k$ to $(Vm \otimes n)V^k$;
2. the left vertical map sends $(m \otimes n)V^k$ to $(m \otimes Vn)V^k$;
3. the lower horizontal map sends $(m \otimes n)V^k$ to $(Vm \otimes n)V^k - (m \otimes Fn)V^{k+1}$;
4. the right vertical map sends $(m \otimes n)V^k$ to $(m \otimes Vn)V^k - (Fm \otimes n)V^{k+1}$.

This implies the claim. □

Corollary 4.6 identifies the tensor product on the category of derived V -complete p -typical Cartier modules induced from the symmetric monoidal structure on \mathbf{CycSp}_p . We shall describe this tensor product in the next section purely algebraically.

4.2 The tensor product of classical Cartier modules

Inspired by the results of the last section, we will construct in this section a symmetric monoidal structure \boxtimes on the abelian category \mathbf{Cart}_p of p -typical Cartier modules. Our guiding principle is that if R is a commutative ring, then the ring of p -typical Witt vectors $W(R)$, with its usual Frobenius and Verschiebung operations F and V defines a commutative algebra object with respect to \boxtimes . In particular we will get induced symmetric monoidal structures on modules over $W(R)$ (internal to the category \mathbf{Cart}_p). We will explicitly identify this category and in the case $R = \mathbb{F}_p$ we will see that the category is equivalent to classical Dieudonné modules.

The induced symmetric monoidal structure $\boxtimes_{W(\mathbb{F}_p)}$ on Dieudonné modules was first defined by Goerss in [Goe99] and later studied by [BL07]. Under the relation to genuine spectra and Mackey functors that will be explained in Section 5 our tensor product can be understood using the tensor product of Mackey functors.

We thank Achim Krause for helpful discussion around this section and for the crucial idea in the proof of Proposition 4.14.

Definition 4.7. Let M, N, Q be p -typical Cartier modules. A bilinear map of abelian groups $(-, -) : M \times N \rightarrow Q$ will be called (V, F) -bilinear if it satisfies the relations

$$\begin{aligned} F(x, y) &= (F(x), F(y)) \\ V(x, F(y)) &= (V(x), y) \\ V(F(x), y) &= (x, V(y)) \end{aligned}$$

for any $x \in M$ and $y \in N$. We let $\mathrm{Hom}_{(V, F)}(M \times N, Q)$ denote the group of (V, F) -bilinear maps $M \times N \rightarrow Q$.

Example 4.8. For a commutative ring R the multiplication $W(R) \times W(R) \rightarrow W(R)$ is (V, F) -bilinear.

For an abelian group M with a map $F : M \rightarrow M$ we will denote by $M[V]$ the p -typical Cartier module $\bigoplus_{n \in \mathbb{N}} M$ with the V and F operators as in the last section. We will again adopt the notation that we write an element as a polynomial in V , i.e. in the form $\sum a_i V^i$.

Lemma 4.9. *Given p -typical Cartier modules M and N , the functor $\mathrm{Hom}_{(V,F)}(M \times N, -)$ from p -typical Cartier modules to abelian groups is corepresentable by a p -typical Cartier module $M \boxtimes N$. Explicitly we have that*

$$M \boxtimes N = (M \otimes N)[V] / \sim$$

where the equivalence relation is generated additively by

$$(m \otimes Vn)V^k \sim (Fm \otimes n)V^{k+1} \quad (Vm \otimes n)V^k \sim (m \otimes Fn)V^{k+1}$$

for all $m \in M$, $n \in N$ and $k \in \mathbb{N}$ with the V and F operators induced from the ones on $(M \otimes N)[V]$.

Proof. We want to define $M \boxtimes N$ by the above formula. First we have to show that this is even a p -typical Cartier module, i.e. that the V and F operators descend to the quotient. For V this is obvious, since

$$V((m \otimes Vn)V^k) = (m \otimes Vn)V^{k+1}.$$

The operator F is given on additive generators by

$$F((m \otimes n)V^k) = \begin{cases} (Fm \otimes Fn)V^0 & \text{for } k = 0 \\ p(m \otimes n)V^{k-1} & \text{for } k > 0 \end{cases}$$

Thus it follows that

$$F((m \otimes Vn)V^k) \sim F((Fm \otimes n)V^{k+1})$$

and similarly for the other relation. Finally we have to understand maps $M \boxtimes N \rightarrow Q$ in the category of p -typical Cartier modules. By construction these are just maps $\beta : (M \otimes N)[V] \rightarrow Q$ of p -typical Cartier modules which satisfy the relations

$$\beta((m \otimes Vn)V^k) = \beta((Fm \otimes n)V^{k+1}) \quad \beta((Vm \otimes n)V^k) = \beta((m \otimes Fn)V^{k+1}).$$

Being V -linear, every such map β is determined by its restriction $M \otimes N \rightarrow M \otimes N[V] \rightarrow Q$. This restriction precisely satisfies the relation of Definition 4.7. Vice versa every map satisfying the relations of Definition 4.7 can be extended to such a map β in a unique way. \square

Remark 4.10. We can also write this tensor product also the cokernel of the map of p -typical Cartier modules

$$(M \otimes N)[V] \oplus (M \otimes N)[V] \rightarrow (M \otimes N)[V]$$

given by assembling together the maps $(M \otimes N)[V] \rightarrow (M \otimes N)[V]$ given by

$$\begin{aligned} (m \otimes n)V^k &\mapsto (Vm \otimes n)V^k - (m \otimes Fn)V^{k+1}, \\ (m \otimes n)V^k &\mapsto (m \otimes Vn)V^k - (Fm \otimes n)V^{k+1}. \end{aligned}$$

Here, the summands $(M \otimes N)[V]$ in the source of the map are p -typical Cartier modules with Frobenius that is zero in V -degree 0. Note that this cokernel description is basically reversing the line of thought in the proof of Corollary 4.6.

Proposition 4.11. *The assignment $(M, N) \mapsto M \boxtimes N$ defines a symmetric monoidal structure on p -typical Cartier modules which is compatible with small colimits in each variable and with unit the p -typical Cartier module $\mathbb{Z}[V] \subseteq W(\mathbb{Z})$ which is given by*

$$\mathbb{Z}[V] = \bigoplus_{n \geq 0} \mathbb{Z} \cdot V^n(1) \subseteq \prod_{n \geq 0} \mathbb{Z} \cdot V^n(1) = \mathbb{Z}[[V]] \cong W(\mathbb{Z}),$$

a subring of $W(\mathbb{Z})$.¹²

Proof. The proof is the same as the proof that the tensor product on abelian groups defines a symmetric monoidal structure. Since the bilinear conditions are symmetric, the result is symmetric. The claim about colimits is clear because $\mathrm{Hom}_{(V,F)}(M \times N, -)$ takes colimits in N to limits in the functor category, and hence colimits of corepresentables. Now, we claim that $\mathbb{Z}[V] \boxtimes M \cong M \cong M \boxtimes \mathbb{Z}[V]$. For this, by symmetry, it is enough to construct a natural isomorphism $\mathrm{Hom}_{(V,F)}(M \times \mathbb{Z}[V], -) \cong \mathrm{Hom}_{\mathbf{Cart}_p}(M, -)$. Given a (V, F) -bilinear pairing $(-, -) : M \times \mathbb{Z}[V] \rightarrow Q$, we let $f : M \rightarrow N$ be given by $f(x) = (x, 1)$. Then,

$$f(F(x)) = (F(x), 1) = (F(x), F_W(1)) = F(x, 1) = F(f(x))$$

and

$$f(V(x)) = (V(x), 1) = V(x, F_W(1)) = V(x, 1) = V(f(x)),$$

so that f is a map of p -typical Cartier modules. Given a map $f : M \rightarrow N$ of p -typical Cartier modules we let $(x, y)_f : M \times \mathbb{Z}[V] \rightarrow N$ be defined by

$$(x, y)_f = yf(x).$$

This is clearly bilinear. We must check that it is (V, F) -bilinear. To that end, we have

$$\begin{aligned} F(x, y)_f &= F(y(f(x))) = F_W(y)F(f(x)) = F_W(y)f(F(x)) = (F(x), F_W(y))_f, \\ V(x, F_W(y))_f &= V(F_W(y)f(x)) = yV(f(x)) = yf(V(x)) = (V(x), y)_f, \\ V(F(x), y)_f &= V(yf(F(x))) = V(yF(f(x))) = V_W(y)f(x) = (x, V_W(y))_f, \end{aligned}$$

which is what we needed to show. These operations are mutually inverse. Associativity isomorphisms are constructed by observing that $(M \boxtimes N) \boxtimes P$ corepresents (V, F) -bilinear morphisms $(M \boxtimes N) \times P \rightarrow Q$ or (V, F) -multilinear morphisms $(M \times N) \times P \rightarrow Q$. These are multilinear maps $(-, -, -)$ satisfying

$$\begin{aligned} F(x, y, z) &= (F(x), F(y), F(z)), \\ V(x, F(y), F(z)) &= (V(x), y, z), \\ V(F(x), y, F(z)) &= (x, V(y), z), \\ V(F(x), F(y), z) &= (x, y, V(z)). \end{aligned}$$

But, it is easy to check that these are precisely the same relations satisfied by the bilinear maps classified by maps $M \boxtimes (N \boxtimes P) \rightarrow Q$. The unit, pentagon, and hexagon axioms will be left to the reader. \square

Lemma 4.12. *The functor $\mathrm{coker}(V) : \mathbf{Cart}_p \rightarrow \mathbf{Ab}$ admits a symmetric monoidal structure.*

Proof. The abelian group $\mathrm{coker}(M \boxtimes N \xrightarrow{V} M \boxtimes N)$ is obtained from $M \boxtimes N$ by an equivalence relation. But according to Lemma 4.9 the group $M \boxtimes N$ is itself obtained by quotienting $(M \otimes N)[V]$ by an equivalence relation. Thus we obtain that $\mathrm{coker}(M \boxtimes N \xrightarrow{V} M \boxtimes N)$ is the quotient of $(M \otimes N)[V]$ obtained by the combined equivalence relation which is generated additively by

$$(m \otimes Vn)V^k \sim (Fm \otimes n)V^{k+1} \quad (Vm \otimes n)V^k \sim (m \otimes Fn)V^{k+1} \quad (m \otimes n)V^n \sim 0 \text{ for } n \geq 1$$

In view of the third relation the first two relations are equivalent to

$$m \otimes Vn \simeq 0, \quad Vm \otimes n \simeq 0.$$

¹²There is another similar subring of $W(\mathbb{Z})$, namely the rational Witt vectors $W_{\mathrm{rat}}(\mathbb{Z})$. There are canonical inclusions $\mathbb{Z}[V] \subseteq W_{\mathrm{rat}}(\mathbb{Z}) \subseteq W(\mathbb{Z})$ both of which are proper.

Therefore we just obtain the quotient of $M \otimes N$ by these relations. But this is $\text{coker}(M \xrightarrow{V} M) \otimes \text{coker}(N \xrightarrow{V} N)$. Together with the isomorphism

$$\text{coker}(\mathbb{Z}[V] \xrightarrow{V} \mathbb{Z}[V]) \cong \mathbb{Z},$$

this gives the functor $\text{coker}(V): \mathbf{Cart}_p \rightarrow \text{Ab}$ the desired symmetric monoidal structure. \square

We shall also need a derived version of Lemma 4.12. To this end we introduce a ‘derived’ version of \boxtimes which we denote \boxtimes^L . This is just defined as the total cofiber of the diagram

$$\begin{array}{ccc} (M \otimes^L N)[V] & \longrightarrow & (M \otimes^L N)[V] \\ \downarrow & & \downarrow \\ (M \otimes^L N)[V] & \longrightarrow & (M \otimes^L N)[V] \end{array} \quad (18)$$

where $(-)[V]$ is as before just the infinite product of $M \otimes^L N$ (not homotopy orbits) and the maps are as in the proof of Corollary 4.6. This is an object in the derived category. The ordinary tensor product \boxtimes is H_0 of \boxtimes^L .¹³

Lemma 4.13. *For every pair of p -typical Cartier modules M and N we have an equivalence*

$$\text{cofib}(M \boxtimes^L N \xrightarrow{V} M \boxtimes^L N) \simeq \text{cofib}(M \xrightarrow{V} M) \otimes^L \text{cofib}(N \xrightarrow{V} N)$$

in $\mathcal{D}(\mathbb{Z})$.

Proof. We can interchange the mod V reduction with taking the total cofiber in the square (18). Then the square is just of the form

$$\begin{array}{ccc} (M \otimes^L N) & \xrightarrow{V \otimes \text{id}} & (M \otimes^L N) \\ \downarrow \text{id} \otimes V & & \downarrow \text{id} \otimes V \\ (M \otimes^L N) & \xrightarrow{V \otimes \text{id}} & (M \otimes^L N) \end{array}$$

whose total cofiber evidently has the claimed form. \square

We now show that the tensor product of p -typical Cartier modules also induces a tensor product of derived V -complete p -typical Cartier modules. We remind the reader of the algebraic derived V -completion $L: \mathbf{Cart}_p \rightarrow \widehat{\mathbf{Cart}}_p$ discussed in Lemma 3.29 and afterwards.

Proposition 4.14. *The localization $L: \mathbf{Cart}_p \rightarrow \widehat{\mathbf{Cart}}_p$ is compatible with the symmetric monoidal structure, that is we have an induced symmetric monoidal structure on $\widehat{\mathbf{Cart}}_p$, which we denote by $\widehat{\boxtimes}$ and which is given by*

$$M \widehat{\boxtimes} N := L(M \boxtimes N)$$

and has unit $W(\mathbb{Z})$. We can also describe $M \widehat{\boxtimes} N$ as the cokernel of the map

$$(M \otimes N)[[V]] \oplus (M \otimes N)[[V]] \rightarrow (M \otimes N)[[V]],$$

the completion of the map in Remark 4.10.

¹³We note that the notation \boxtimes^L is meant to indicate that \boxtimes^L is really a tensor product on the derived category. We do not claim that it is actually the derived functor of \boxtimes in any sense and hope that this notation does not lead to confusion.

Proof. First it is clear that the cokernel of the map $(M \otimes N)[[V]] \oplus (M \otimes N)[[V]] \rightarrow (M \otimes N)[[V]]$ agrees with the completion of the tensor product by the fact that the corresponding statement is clear before completion (Remark 4.10) and the completion of a cokernel is the cokernel of the completions.

By the usual criterion for symmetric monoidal localizations we have to verify that for any pair of p -typical Cartier modules M and N the map

$$L(M \boxtimes N) \rightarrow L(LM \boxtimes N)$$

is an isomorphism. We have a resolution $\ker(q) \rightarrow M[V] \xrightarrow{q} M$ of M by V -torsion free p -typical Cartier modules. Since \boxtimes and L both commute with colimits (i.e. are right exact) we can thereby reduce the question to the V -torsion free case. Thus we assume that M is V -torsion free. Then also LM is V -torsion free by Lemma 3.30 and the map $M \rightarrow LM$ is a mod V equivalence by Lemma 3.31. Since the mod V reduction is also the derived mod V reduction we get using Lemma 4.13 that

$$\begin{aligned} \text{cofib}(M \boxtimes^L N \xrightarrow{V} M \boxtimes^L N) &\simeq \text{cofib}(M \xrightarrow{V} M) \otimes^L \text{cofib}(N \xrightarrow{V} N) \\ &\simeq \text{cofib}(LM \xrightarrow{V} LM) \otimes^L \text{cofib}(N \xrightarrow{V} N) \\ &\simeq \text{cofib}(LM \boxtimes N \xrightarrow{V} LM \boxtimes N). \end{aligned}$$

Thus the induced map is a derived mod V -equivalence, thus an equivalence after derived mod V reduction (see the proof of Lemma 3.31). But then L is H_0 of the derived completion (Lemma 3.29), thus it is also an equivalence on L . \square

We recall from Theorem 2.1 that the symmetric monoidal structure on \mathbf{CycSp}_p is compatible with the t -structure. In particular it induces a symmetric monoidal structure on the heart as shown in Corollary 2.14 (also see Appendix A.2).

Theorem 4.15. *The symmetric monoidal structure on the heart $\mathbf{CycSp}_p^\heartsuit \simeq \widehat{\mathbf{Cart}}_p$ induced from the one of cyclotomic spectra is equivalent to the symmetric monoidal structure $\widehat{\boxtimes}$ constructed above.*

Proof. From Corollary 4.6 and the description above we get that the two tensor products are naturally isomorphic. By construction it is clear that these isomorphisms are also compatible with the associators and the symmetry (since the Bar construction can be iterated as well as the construction of the algebraic tensor product). The fact that we get a compatibility of units follows since $\pi_0^{\text{cyc}}(\mathbb{S}) \cong \text{TR}_0(\mathbb{S}) \simeq W(\mathbb{Z})$ which is a result of Hesselholt and Madsen that holds for any commutative ring; see the proof of Theorem 6.1 for precise references. But in this case it is also easy to see directly since the unit is also given by $\mathbb{S}[[V]]$ as a consequence of Proposition 4.3. \square

4.3 Tensor product of Witt vectors and modules

We note that an equivalent statement to Theorem 4.15 is to say that

$$\pi_0^{\text{cyc}} = \text{TR}_0 : (\mathbf{CycSp}_p)_{\geq 0} \rightarrow \widehat{\mathbf{Cart}}_p$$

is a symmetric monoidal functor, where $\widehat{\mathbf{Cart}}_p$ is equipped with the $\widehat{\boxtimes}$ symmetric monoidal functor. This functor also preserves colimits and therefore also preserves relative tensor products over an algebra object in connective p -cyclotomic spectra. For rings R and S we have that $\text{THH}(R \otimes_{\mathbb{Z}} S) = \text{THH}(R) \otimes_{\text{THH}(\mathbb{Z})} \text{THH}(S)$. It follows from the fact that $\pi_0^{\text{cyc}}(\text{THH}(R)) = W(R)$ (see Theorem 6.1) that we have

$$\begin{aligned} W(R \otimes S) &= \pi_0^{\text{cyc}}(R \otimes_{\mathbb{Z}} S) \\ &= \pi_0^{\text{cyc}} \text{THH}(R) \widehat{\boxtimes}_{\pi_0^{\text{cyc}} \text{THH}(\mathbb{Z})} \pi_0^{\text{cyc}} \text{THH}(S) \\ &= W(R) \widehat{\boxtimes} W(S), \end{aligned}$$

where the last step uses that $W(\mathbb{Z}) = \pi_0^{\text{cyc}} \text{THH}(\mathbb{Z})$ is the unit for $\widehat{\boxtimes}$. We have not found a statement of this sort in the literature and we think that this is an important property of Witt vectors, as it for example explains how to understand Witt vectors of polynomial rings in several variables. Therefore we want to give a purely algebraic proof of this fact.

Theorem 4.16. *For every pair of rings R and S there is a natural isomorphism*

$$W(R) \widehat{\boxtimes} W(S) \cong W(R \otimes S) .$$

Note that we do not require that R and S are commutative here and use non-commutative Witt vectors, but of course the statement is of most interest in the commutative case.

Proof. Our first task is to produce a natural map

$$W(R) \widehat{\boxtimes} W(S) \rightarrow W(R \otimes S) .$$

In the commutative case this is easy: $W(R)$ and $W(S)$ are commutative algebra objectives in $\widehat{\mathbf{Cart}}_p$ and therefore the tensor product is the coproduct. As a result to construct such a map we can just combine the maps $W(R) \rightarrow W(R \otimes S)$ and $W(S) \rightarrow W(R \otimes S)$ that come from the maps $R \rightarrow R \otimes S$ and $S \rightarrow R \otimes S$.

In the associative case such map is by construction of $\widehat{\boxtimes}$ the same as a bilinear map $W(R) \times W(S) \rightarrow W(R \otimes S)$ of abelian groups with a certain compatibility with Verschiebung and Frobenius. But such a map has exactly been constructed in [KN18]. By construction of the ring structure on $W(R)$ and $W(S)$ in the commutative case through this lax symmetric monoidal map the two maps agree in the commutative case.

Now that we have the map we are left to show that it is an isomorphism. We can resolve all our rings by rings of the form $\mathbb{Z}[M]$ for a monoid M and since $W(-)$ commutes with split coequalizers reduce to this case. But for $R = \mathbb{Z}[M]$ we have a natural isomorphism $W(R) \cong R[[V]]$ with the terminology from Section 4.2. We will verify this at the end of the proof, but let us assume it for the moment. Thus the statement reduces to show that the canonical map

$$R[[V]] \widehat{\boxtimes} S[[V]] \rightarrow (R \otimes S)[[V]]$$

is an isomorphism. This in turn follows from the statement that the canonical map

$$R[V] \otimes S[V] \rightarrow (R \otimes S)[V]$$

is an isomorphism in \mathbf{Cart}_p . The latter is easy to verify using the explicit formula for the tensor product in \mathbf{Cart}_p given in Lemma 4.9.

Now we want to verify that for a ring of the form $R = \mathbb{Z}[M]$ with a monoid M we have a canonical isomorphism of Cartier modules $R[[V]] \cong W(R)$.¹⁴ For simplicity we assume that M is a commutative monoid. The statement in fact holds with the same proof also of non-commutative monoids but we do not want to get involved into non-commutative Witt vectors here (see [KN18] for a discussion which shows that the arguments given here carry over directly except one has to verify that V is injective). First, we know that $W(R)$ is derived V -complete. Therefore to construct a map of Cartier modules $R[[V]] \rightarrow W(R)$ we have to construct a map $R \rightarrow W(R)$ compatible with F -operators where the F -operator on $R = \mathbb{Z}[M]$ is given by the p -th power on M . Such a map in turn is the same as a map of multiplicative monoids $M \rightarrow W(R)$ and is canonically provided by the Teichmüller. Now both sides are V -torsion free, since the Witt vectors of every commutative ring are V -torsion free and $R[[V]]$ evidently is. Thus by Lemma 3.31 it suffices to show that the map $R[[V]] \rightarrow W(R)$ is an equivalence on the cokernel of V which is obvious. \square

¹⁴This easily follows using Theorem 6.1 from the topological statement that $\text{TR}(\mathbb{S}[M]) \simeq \text{THH}(\mathbb{S}[M])[V]$ which is a formal consequence of the fact that $\text{THH}(\mathbb{S}[M])$ has a Frobenius lift. But of course the purpose of this proof is to avoid using topological arguments.

Finally we want to study categories of modules over ring objects A in \mathbf{Cart}_p . The main example we have in mind is $W(R)$ for a commutative ring R . Thus let A be a ring object in \mathbf{Cart}_p , that is A is a ring, equipped with V and F operators such that

$$FV = p, \quad F(ab) = F(a)F(b), \quad V(aF(b)) = V(a)b, \quad V(F(a)b) = aV(b).$$

By an A -module, we mean an A -module object in \mathbf{Cart}_p with respect to the symmetric monoidal structure \boxtimes .

Lemma 4.17. *A left A -module structure on an object $M \in \mathbf{Cart}_p$ is the same as a left module structure for the underlying ring A on the underlying abelian group M such that the following conditions hold for any $x \in A$ and $y \in M$:*

- (i) $V(F_A(x)y) = xV(y)$;
- (ii) $V(xF(y)) = V_A(x)y$;
- (iii) $F(xy) = F_A(x)F(y)$.

Proof. This immediately follows from the definition of bilinear maps as in Definition 4.7. \square

Example 4.18. For $A = W(k)$ where k is a perfect ring of characteristic p we get that the category $\text{Mod}_{W(k)}(\mathbf{Cart}_p)$ is the category of $W(k)$ -modules M (in the classical sense of modules) together with a Frobenius semilinear map $F : M \rightarrow M$ and a map $V : M \rightarrow M$ that is semilinear for the inverse of Frobenius on $W(k)$, such that $VF = FV = p$. This is the category of Dieudonné modules.

5 The relationship to genuine cyclotomic spectra

In this section we shall compare the ∞ -category of p -typical topological Cartier modules with the ∞ -category of genuine p -typical cyclotomic spectra. This generality will be relevant in order to understand some cases of V -completion. It will also provide a more conceptual proof of the main result of the Section 3 on the adjointness of $(-)/V$ and TR . Moreover, although we do not explore it here, we find an equivalent formulation of the theory of p -typical topological Cartier modules in Proposition 5.5, which immediately generalizes to the integral (or big) situation. The main results can be found in Section 5.3.

5.1 Genuine p -typical topological Cartier modules

The next definition parallels that of 2.21.

Definition 5.1. The ∞ -category $\mathbf{TCart}_p^{\text{gen}}$ of **genuine p -typical topological Cartier modules** is defined to be fixed points for the endofunctor $(-)^{C_p}$:

$$\mathbf{TCart}_p^{\text{gen}} := \text{Fix}_{(-)^{C_p}}(\mathbb{TSp}_p^{\text{gen}}).$$

In other words, a genuine p -typical topological Cartier module is a genuine S^1 -spectrum M equipped with an equivalence $M^{C_p} \simeq X$ of genuine S^1 -spectra.

Our primary aim in this section is to prove that $\mathbf{TCart}_p^{\text{gen}} \simeq \mathbf{TCart}_p$, up to a theorem about genuine S^1 -spectra which we will prove in the next section. In the next proposition, we will use the notion of an F -algebra for an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$. By definition, these are organized into an ∞ -category $\text{Alg}_F(\mathcal{C}) \simeq \text{LEq}(F, \text{id} : \mathcal{C} \rightarrow \mathcal{C})$. In other words, an F -algebra is an object X of \mathcal{C} equipped with a map $FX \rightarrow X$. Similarly, if G is an endofunctor of \mathcal{D} , we have the opposite notion of a G -coalgebra, namely an object $Y \in \mathcal{D}$ equipped with a map $Y \rightarrow GY$. These are objects of the ∞ -category $\text{CoAlg}_G(\mathcal{D})$. For details, see [NS18, Section II.5].

Proposition 5.2. *There is an adjunction*

$$L : \mathbf{TCart}_p^{\text{gen}} \rightleftarrows \mathbf{CycSp}_p^{\text{gen}} : \overline{\text{TR}}$$

with the functors

$$L(M) = \text{colim}(M \rightarrow M^{\Phi C_p} \rightarrow M^{\Phi C_{p^2}} \rightarrow \dots)$$

where the map $M \rightarrow M^{\Phi C_p}$ is given by the composite $M \simeq M^{C_p} \rightarrow M^{\Phi C_p}$. The right adjoint is given by

$$\overline{\text{TR}}(X) = \lim(\dots \rightarrow X^{C_{p^2}} \rightarrow X^{C_p} \rightarrow X)$$

where the maps are induced by $X^{C_p} \rightarrow X^{\Phi C_p} \simeq X$.

Proof. This is more generally true: if we have an ∞ -category \mathcal{C} and a natural transformation $F \rightarrow G$ of functors $F, G : \mathcal{C} \rightarrow \mathcal{C}$ such that F preserves sequential limits (and these exist in \mathcal{C}) and G preserves sequential colimits (also these are assumed to exist in \mathcal{C}), then we get an adjunction

$$L : \text{Fix}_F(\mathcal{C}) \rightleftarrows \text{Fix}_G(\mathcal{C}) : R$$

where the underlying object of $L(X)$ is given by the colimit

$$\text{colim}(X \rightarrow GX \rightarrow G^2X \rightarrow \dots)$$

and the fixed point structure for G comes from commuting this colimit with G . The underlying object of $R(Y)$ is given by

$$\lim(\dots \rightarrow F^2Y \rightarrow FY \rightarrow Y)$$

and the fixed point structure comes from commuting the limit with F . To see that these functors are adjoint to one another, we observe that there is a natural transformation

$$\text{id} \rightarrow RL$$

given on underlying objects by the natural map

$$X = \lim_k F^k X \rightarrow \lim_k F^k(\text{colim}_n(G^n X)).$$

It now suffices to verify the mapping space property. To this end, we claim that for every $X \simeq FX \in \text{Fix}_F(\mathcal{C})$ and every $Y \simeq GY \in \text{Fix}_G(\mathcal{C})$ one has

$$\text{Map}_{\text{Fix}_F(\mathcal{C})}(X, RY) \simeq \text{Map}_{\text{Alg}_F(\mathcal{C})}(X, Y),$$

where Y is considered as an F -algebra through the map $FY \rightarrow GY \xrightarrow{\simeq} Y$. To see this equivalence we have used that taking the limit over iterated applications of F is the right adjoint to the inclusion $\text{Fix}_F(\mathcal{C}) \subseteq \text{Alg}_F(\mathcal{C})$ see [NS18, Section II.5]. The space $\text{Map}_{\text{Alg}_F(\mathcal{C})}(X, Y)$ is equivalent to the equalizer

$$\text{Map}_{\mathcal{C}}(X, Y) \rightrightarrows \text{Map}_{\mathcal{C}}(FX, Y)$$

where the first map is given by precomposition with $FX \rightarrow X$ and the second by applying F and then postcomposition with $FY \rightarrow Y$. We can similarly describe the mapping space $\text{Map}_{\text{Fix}_G(\mathcal{C})}(LX, Y)$ via $\text{CoAlg}_G(\mathcal{C})$ as the equalizer

$$\text{Map}_{\mathcal{C}}(X, Y) \rightrightarrows \text{Map}_{\mathcal{C}}(X, GY)$$

along the analogous maps. Since $FX \simeq X$ and $GY \simeq Y$ it is clear that $\text{Map}_{\mathcal{C}}(FX, Y) \simeq \text{Map}_{\mathcal{C}}(X, Y) \simeq \text{Map}_{\mathcal{C}}(X, GY)$. Unfolding the definitions we see that under this equivalence the maps in the equalizers correspond to each other and such that they are equivalent. Moreover the equivalence we get this way agrees (at least up to homotopy) with the map induced from the transformation $\text{id} \rightarrow RL$. \square

In order to understand the functors L and $\overline{\text{TR}}$ we need the following result, which is in some sense a genuine version of the Tate orbit Lemma, but much easier.

Lemma 5.3. *For every object $M \in \mathbb{TSp}_p^{\text{gen}}$, the canonical map*

$$(M^{C_p})^{\Phi_{C_p}} \rightarrow M^{\Phi_{C_{p^2}}}$$

is an equivalence in $\mathbb{TSp}_p^{\text{gen}}$.

Proof. For this proof we will write the left hand side as $(M^{C_p})^{\Phi_{C_{p^2}/C_p}}$ and note that it carries a residual S^1/C_{p^2} -action. By definition (see for example [NS18, Definition II.2.3]), the C_{p^2}/C_p -geometric fixed points are given by

$$(M^{C_p})^{\Phi_{C_{p^2}/C_p}} \simeq \left((M^{C_p}) \otimes E(\widetilde{S^1/C_p}) \right)^{C_p^2/C_p},$$

where the pointed genuine S^1/C_p -space $E(\widetilde{S^1/C_p})$ is characterized by having fixed points

$$E(\widetilde{S^1/C_p})^{C_{p^k}/C_p} = \begin{cases} * & k = 1, \\ S^0 & k > 1 \end{cases}$$

and the obvious maps (so that it receives a map from the space S^0). By the projection formula we get an equivalence of S^1/C_p -spectra

$$(M^{C_p}) \otimes E(\widetilde{S^1/C_p}) \simeq M \otimes f^* \left(E(\widetilde{S^1/C_p}) \right)$$

where $f : S^1 \rightarrow S^1/C_p$ is the canonical projection and f^* is inflation along this projection (i.e. restriction of the action which is left adjoint to taking fixed points). The pullback $f^* \left(E(\widetilde{S^1/C_p}) \right)$ is equivalent to the genuine S^1 -space $\widetilde{E_{p^2}S^1}$ which has fixed points

$$\widetilde{E_{p^2}S^1}^{C_{p^k}} = \begin{cases} * & k = 0, 1, \\ S^0 & k > 1. \end{cases}$$

Putting everything together we conclude that

$$(M^{C_p})^{\Phi_{C_{p^2}/C_p}} \simeq \left(M \otimes \widetilde{E_{p^2}S^1} \right)^{C_{p^2}}$$

which is by definition equivalent to $M^{\Phi_{C_{p^2}}}$. □

Let M be a genuine p -typical topological Cartier module. In the next corollary, we identify the C_{p^n} fixed points of the genuine p -typical cyclotomic spectrum LM from Proposition 5.2.

Corollary 5.4. *For every genuine p -typical topological Cartier module M we have a natural equivalence*

$$(LM)^{C_{p^n}} \simeq M/V^{n+1},$$

where V^{n+1} denotes the map $M_{hC_{p^{n+1}}} \rightarrow M^{C_{p^{n+1}}} \xrightarrow{\cong} M$.

Proof. By definition we have that $L(M) = \operatorname{colim}(M \rightarrow M^{\Phi C_p} \rightarrow M^{\Phi C_{p^2}} \rightarrow \dots)$ and since (categorical) fixed points commute with colimits we have that

$$(LM)^{C_{p^n}} = \operatorname{colim}(M^{C_{p^n}} \rightarrow (M^{\Phi C_p})^{C_{p^n}} \rightarrow (M^{\Phi C_{p^2}})^{C_{p^n}} \rightarrow \dots).$$

The transition maps in this colimit are given by the maps

$$(M^{\Phi C_{p^k}})^{C_{p^n}} \xrightarrow{\simeq} ((M^{C_p})^{\Phi C_{p^k}})^{C_{p^n}} \rightarrow (M^{\Phi C_{p^{k+1}}})^{C_{p^n}}$$

which are equivalences by Lemma 5.3 for $k \geq 1$. As a result, the colimit is equivalent to the second term $(M^{\Phi C_p})^{C_{p^n}}$. This sits in an isotropy separation cofiber sequence

$$M_{hC_{p^{n+1}}} \rightarrow M^{C_{p^{n+1}}} \rightarrow (M^{\Phi C_p})^{C_{p^n}}$$

where the middle term is identified with M through the general p -typical Cartier module structure and the left map with V^{n+1} . This shows the claim. \square

Now we come to the key assertion of the section.

Proposition 5.5. *The canonical forgetful functor $\mathbf{TCart}_p^{\text{gen}} \rightarrow \mathbf{TCart}_p$ (which will be made explicit in the proof) is an equivalence of ∞ -categories.*

Proof. Fix $n \geq 0$. We consider the ∞ -category

$$\mathbb{TSp}_{C_{p^n}}^{\text{gen}}$$

of genuine S^1 -spectra for the family of subgroups of $C_{p^n} \subseteq S^1$. In particular $\mathbb{TSp}_{C_{p^0}}^{\text{gen}} = \operatorname{Sp}^{BS^1}$. Taking C_p -fixed points gives a functor

$$(-)^{C_p}: \mathbb{TSp}_{C_{p^{n+1}}}^{\text{gen}} \rightarrow \mathbb{TSp}_{C_{p^n}}^{\text{gen}}$$

and there is another functor $U_{n+1}: \mathbb{TSp}_{C_{p^{n+1}}}^{\text{gen}} \rightarrow \mathbb{TSp}_{C_{p^n}}^{\text{gen}}$ which forgets the $C_{p^{n+1}}$ -fixed points. We define a stable ∞ -category $(\mathbf{TCart}_p^{\text{gen}})_n$ as the equalizer

$$(\mathbf{TCart}_p^{\text{gen}})_n := \operatorname{Eq}\left((-)^{C_p}, U_{n+1}: \mathbb{TSp}_{C_{p^{n+1}}}^{\text{gen}} \rightrightarrows \mathbb{TSp}_{C_{p^n}}^{\text{gen}}\right)$$

of $(-)^{C_p}$ and U_{n+1} . There is a canonical map $(\mathbf{TCart}_p^{\text{gen}})_n \rightarrow (\mathbf{TCart}_p^{\text{gen}})_{n-1}$ induced from the pair of commutative diagrams¹⁵

$$\begin{array}{ccc} \mathbb{TSp}_{C_{p^{n+1}}}^{\text{gen}} & \xrightarrow[U_{n+1}]{(-)^{C_p}} & \mathbb{TSp}_{C_{p^n}}^{\text{gen}} \\ U_{n+1} \downarrow & & \downarrow U_n \\ \mathbb{TSp}_{C_{p^n}}^{\text{gen}} & \xrightarrow[U_n]{(-)^{C_p}} & \mathbb{TSp}_{C_{p^{n-1}}}^{\text{gen}} \end{array}$$

The main point of the proof is to show that $(\mathbf{TCart}_p^{\text{gen}})_n \rightarrow (\mathbf{TCart}_p^{\text{gen}})_{n-1}$ is an equivalence for every $n \geq 1$. Let us first assume that and finish the proof. Since $\mathbb{TSp}_p^{\text{gen}} \simeq \varprojlim \mathbb{TSp}_{C_{p^n}}^{\text{gen}}$ it follows that $\mathbf{TCart}_p^{\text{gen}} \simeq \varprojlim_n (\mathbf{TCart}_p^{\text{gen}})_n$, and therefore, under our assumption,

$$\mathbf{TCart}_p^{\text{gen}} \simeq (\mathbf{TCart}_p^{\text{gen}})_0.$$

¹⁵Here, we mean that the diagram with the $(-)^{C_p}$ horizontal maps is commutative as is the diagram with the U_{n+1} and U_n horizontal maps.

To identify the $n = 0$ case, we claim that there is a pullback square

$$\begin{array}{ccc} \mathbb{TSp}_{C_p}^{\text{gen}} & \longrightarrow & (\text{Sp}^{BS^1})^{\Delta^2} \\ U_1 \downarrow & & \downarrow \partial^1 \\ \text{Sp}^{BS^1} & \xrightarrow{\text{Nm}_{C_p}} & (\text{Sp}^{BS^1})^{\Delta^1} \end{array} \quad (19)$$

of ∞ -categories. This fact is equivalent to the fact that the ‘Tate square’ determines $\mathbb{TSp}_{C_p}^{\text{gen}}$ which is well-known, see e.g. [MNN17, Theorem 6.24]. The translation between these two facts is given by the observation that a pullback square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in any stable ∞ -category (here Sp^{BS^1}) is equivalently determined by the triangle

$$\begin{array}{ccc} F & \longrightarrow & A \\ & \searrow & \downarrow \\ & & D \end{array}$$

where F is the horizontal fiber.

The two maps $\mathbb{TSp}_{C_p}^{\text{gen}} \rightarrow \mathbb{TSp}_{C_{p^0}}^{\text{gen}} = \text{Sp}^{BS^1}$ that feature in the definition of $(\mathbf{TCart}_p^{\text{gen}})_0$ are in the pullback description (19) given by the two maps that are given by the composites

$$(-)^{C_p} : \mathbb{TSp}_{C_p}^{\text{gen}} \rightarrow (\text{Sp}^{BS^1})^{\Delta^2} \xrightarrow{\text{ev}_1 = \partial^0 \partial^2} \text{Sp}^{BS^1}$$

and

$$U_1 : \mathbb{TSp}_{C_p}^{\text{gen}} \rightarrow \text{Sp}^{BS^1} .$$

It thus follows that the equalizer defining $(\mathbf{TCart}_p^{\text{gen}})_0$ is equivalent to the pullback defining \mathbf{TCart}_p .

Now in order to show that the functor $(\mathbf{TCart}_p^{\text{gen}})_n \rightarrow (\mathbf{TCart}_p^{\text{gen}})_{n-1}$ is an equivalence we will use that for $n \geq 1$ the square

$$\begin{array}{ccc} \mathbb{TSp}_{C_{p^{n+1}}}^{\text{gen}} & \xrightarrow{(-)^{C_p}} & \mathbb{TSp}_{C_{p^n}}^{\text{gen}} \\ U_{n+1} \downarrow & & \downarrow U_n \\ \mathbb{TSp}_{C_{p^n}}^{\text{gen}} & \xrightarrow{(-)^{C_p}} & \mathbb{TSp}_{C_{p^{n-1}}}^{\text{gen}} \end{array} \quad (20)$$

is a pullback square of stable ∞ -categories which we will prove in Theorem 5.11 below. Given this, the claim follows from the fact that for a general pair of maps $f, g : A \rightarrow B$ in an ∞ -category with finite limits (here the ∞ -category of stable ∞ -categories), we have a canonical equivalence

$$\text{Eq} \left(A \begin{array}{c} \xrightarrow{f} \\ \times_g \\ \xrightarrow{A} \end{array} \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \\ \xrightarrow{\text{pr}_2} \end{array} A \right) \simeq \text{Eq} \left(A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \\ \xrightarrow{B} \end{array} \right) .$$

This follows by taking the ‘total equalizer’ in the commutative diagram

$$\begin{array}{ccc} A \times A & \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{array} & A \\ \begin{array}{c} \downarrow \text{fpr}_1 \\ \downarrow \text{gpr}_2 \end{array} & & \downarrow \\ B & \xrightarrow{\quad} & * \end{array}$$

in two different ways (first horizontally and then vertically or vice versa). Then we get the diagram

$$\begin{array}{ccccc} E & \longrightarrow & A \times_B A & \rightrightarrows & A \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A \times A & \rightrightarrows & A \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & B & \rightrightarrows & * \end{array}$$

in which the two ‘outer forks’ are equalizers. □

5.2 A pullback square

Now, to complete the proof of Proposition 5.5, we have to establish that for $n \geq 1$, the square (20) is a pullback. To this end we shall introduce some terminology and establish an abstract criterion for pullbacks of stable ∞ -categories.

Definition 5.6. Let $U : \mathcal{C} \rightarrow \mathcal{D}$ be a functor of stable ∞ -categories. We say that U exhibits a recollement if it admits a fully faithful left adjoint and a fully faithful right adjoint, i.e. U is a localization and a colocalization.

For the next example we follow the notation of Proposition 5.5.

Example 5.7. The functor $U_{n+1} : \mathbb{TSp}_{C_{p^{n+1}}}^{\text{gen}} \rightarrow \mathbb{TSp}_{C_{p^n}}^{\text{gen}}$ exhibits a recollement. The left adjoint is given by forming the ‘free’ $C_{p^{n+1}}$ -spectrum $\text{Fr}(X)$ on an C_{p^n} -spectrum X . For the spectrum $\text{Fr}(X)$ we have that

$$\text{Fr}(X)_{C_{p^k}} = \begin{cases} X^{C_{p^k}} & \text{for } k = 0, \dots, n, \\ (X^{C_{p^n}})_{hC_p} & \text{for } k = n + 1. \end{cases}$$

The right adjoint is given by the ‘Borel complete’ spectrum BX with

$$BX^{C_{p^k}} = \begin{cases} X^{C_{p^k}} & \text{for } k = 0, \dots, n, \\ (X^{C_{p^n}})_{hC_p} & \text{for } k = n + 1. \end{cases}$$

The canonical maps $UB \rightarrow \text{id} \rightarrow U\text{Fr}$ are equivalences. This shows that B and Fr are fully faithful.

Lemma 5.8. *If $U : \mathcal{C} \rightarrow \mathcal{D}$ exhibits a recollement, then $\ker(U) \rightarrow \mathcal{C} \rightarrow \mathcal{D}$ is a Verdier sequence. Moreover the functor U is a Cartesian and coCartesian fibration.¹⁶*

¹⁶Here we mean the invariant concept, i.e. that every replacement by a categorical fibration is a Cartesian and coCartesian fibration.

Proof. The first statement is clear since U is a localization (or colocalization) and thus \mathcal{D} is the Verdier quotient by the acyclics. Now we want to show that U is a Cartesian fibration in the sense of [Lur09, Section 2.4]; the coCartesian case follows by passing to opposite categories. We claim that a morphism $f : c \rightarrow c'$ in \mathcal{C} is U -Cartesian precisely if the square

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \\ \downarrow & & \downarrow \\ RUc & \xrightarrow{RUf} & RUc' \end{array} \quad (21)$$

is a pullback in \mathcal{C} where $R : \mathcal{D} \rightarrow \mathcal{C}$ is the right adjoint to U and the vertical maps in the diagram are the unit maps. By definition, the morphism $f : c \rightarrow c'$ is Cartesian precisely if the square

$$\begin{array}{ccc} \mathcal{C}/c & \xrightarrow{f_*} & \mathcal{C}/c' \\ \downarrow U & & \downarrow U \\ \mathcal{D}/Uc & \xrightarrow{Uf_*} & \mathcal{D}/Uc' \end{array}$$

of ∞ -categories is a pullback. We consider the larger commutative diagram

$$\begin{array}{ccc} \mathcal{C}/c & \xrightarrow{f_*} & \mathcal{C}/c' \\ \downarrow U & & \downarrow U \\ \mathcal{D}/Uc & \xrightarrow{Uf_*} & \mathcal{D}/Uc' \\ \downarrow R & & \downarrow R \\ \mathcal{C}/RUc & \xrightarrow{RUf_*} & \mathcal{C}/RUc' \end{array} \quad (22)$$

We will show that the lower square is a pullback. Then the upper square is a pullback if the outer square is a pullback which is evidently the case if and only if the square (21) is a pullback. To see that the lower square in (22) is a pullback we first note that the vertical functors R are fully faithful since the functor $R : \mathcal{D} \rightarrow \mathcal{C}$ is. The left hand vertical functor identifies \mathcal{D}/Uc with the full subcategory of \mathcal{C}/RUc consisting of morphisms $c \rightarrow RUc$ such that c is in the essential image of R . A similar description holds for the essential image of \mathcal{D}/Uc' in \mathcal{C}/RUc' . From this description it is obvious that the square is a pullback if one uses that pullbacks of fully faithful subcategory inclusions are fully faithful and given by the obvious preimage.

We now want to argue that $U : \mathcal{C} \rightarrow \mathcal{D}$ is Cartesian. Thus assume that we have $c' \in \mathcal{C}$ and a morphism $f' : d \rightarrow Uc'$ in \mathcal{D} . We need to find a Cartesian morphism $f : c \rightarrow c'$ in \mathcal{C} with $Uc \simeq d$ and $Uf \simeq f'$. We extend the diagram

$$\begin{array}{ccc} & & c' \\ & & \downarrow \\ Rd & \xrightarrow{Rf'} & RUc' \end{array}$$

to a pullback

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \\ \downarrow & & \downarrow \\ Rd & \xrightarrow{Rf'} & RUc' \end{array}$$

and let f be the upper horizontal morphism. First we claim that the morphism $RUc \rightarrow d$ induced from the left vertical map is an equivalence. This follows by applying the exact functor RU to the whole diagram and noting that $RUc \rightarrow d$ is an equivalence. This shows that the whole diagram is equivalent to a diagram

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \\ \downarrow & & \downarrow \\ RUc & \xrightarrow{RUf} & RUc' \end{array}$$

From this we conclude that f is Cartesian and that $Uf = f'$ which finishes the proof. \square

Definition 5.9. Assume that $U : \mathcal{C} \rightarrow \mathcal{D}$ exhibits a recollement and $d \in \mathcal{D}$. We define the Tate object $T(d) \in \ker(U)$ to be the cofiber of the canonical map $L(d) \rightarrow R(d)$ where L is the left adjoint of U and R is the right adjoint of U .

We now give our criterion for a pullback. This is similar to the main result of [BG16b].

Proposition 5.10. *Assume that we have a commutative square*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ \downarrow U & & \downarrow U' \\ \mathcal{D} & \xrightarrow{G} & \mathcal{D}' \end{array}$$

of stable ∞ -categories and exact functors. If U and U' exhibit recollements then the square is a pullback if and only if the following two conditions are satisfied:

- (1) *the vertical kernels agree, i.e. the canonical map $\ker(U) \rightarrow \ker(U')$ is an equivalence;*
- (2) *under this equivalence the Tate object $T(d)$ is taken to the Tate object $T'(Gd)$, more precisely for every $d \in \mathcal{D}$ the canonical map $FT(d) \rightarrow T'(Gd)$ is an equivalence in $\ker(U')$.*

Proof. The functor $U : \mathcal{C} \rightarrow \mathcal{D}$ is a (co)Cartesian fibration. The functor $U' : \mathcal{C}' \rightarrow \mathcal{D}'$ also is a coCartesian fibration, thus so is the pullback $\mathcal{C}' \times_{\mathcal{D}'} \mathcal{D} \rightarrow \mathcal{D}$. Therefore, in order to show that the functor $\mathcal{C} \rightarrow \mathcal{C}' \times_{\mathcal{D}'} \mathcal{D}$ is an equivalence, it suffices to show that it is a fiberwise equivalence over \mathcal{D} , or equivalently that the functor $\mathcal{C} \rightarrow \mathcal{C}'$ induces an equivalence $\mathcal{C}_d \rightarrow \mathcal{C}'_{Gd}$, and that the functor preserves coCartesian lifts. An easy adjunction argument shows that the fiber over an object $d \in \mathcal{D}$ is equivalent to $\ker(U)_{/T(d)}$. Similarly for the fibration the fiber over Gd is given by $\ker(U')_{/T'(Gd)}$. Under these identifications the map on fibers is given by the evident map

$$\ker(U)_{/T(d)} \rightarrow \ker(U')_{/T'(Gd)}$$

coming from the functor $F : \ker(U) \rightarrow \ker(U')$ and the map of Tate objects $FT(d) \rightarrow T'(Gd)$. This immediately implies that the map is a fiberwise equivalence under the assumptions of the Proposition.

To see that the functor $\mathcal{C} \rightarrow \mathcal{C}'$ preserves coCartesian lifts we note that for a map $d \rightarrow d'$ in \mathcal{D} the induced map on fibers of $\mathcal{C} \rightarrow \mathcal{D}$ is given by the functor

$$\ker(U)_{/T(d)} \rightarrow \ker(U)_{/T(d')}$$

induced from the map $Td \rightarrow Td'$. This follows immediately from the way the fibers are identified and from the description of coCartesian lifts in the proof of Lemma 5.8. Similar the map on fibers $\mathcal{C}' \rightarrow \mathcal{D}'$ induced from the map $Gd \rightarrow Gd'$ is described by

$$\ker(U')_{/T'(Gd)} \rightarrow \ker(U')_{/T'(Gd')}$$

induced from the map $T(Gd) \rightarrow T(G'd')$. Then the claim follows from the fact that in the commutative square

$$\begin{array}{ccc} FT(d) & \longrightarrow & FT(d') \\ \downarrow & & \downarrow \\ T(Gd) & \longrightarrow & T(G'd') \end{array}$$

the vertical maps are equivalences by assumption. \square

Below, $\mathbb{TSp}_{C_{p^n}}^{\text{gen}}$ is the ∞ -category of genuine S^1 -spectra with respect to the family of subgroup of C_{p^n} .

Theorem 5.11. *The square*

$$\begin{array}{ccc} \mathbb{TSp}_{C_{p^{n+1}}}^{\text{gen}} & \xrightarrow{(-)^{C_p}} & \mathbb{TSp}_{C_{p^n}}^{\text{gen}} \\ U_{n+1} \downarrow & & \downarrow U_n \\ \mathbb{TSp}_{C_{p^n}}^{\text{gen}} & \xrightarrow{(-)^{C_p}} & \mathbb{TSp}_{C_{p^{n-1}}}^{\text{gen}} \end{array} \quad (23)$$

is a pullback of stable ∞ -categories.

Proof. By Example 5.7, the vertical maps U_{n+1} and U_n exhibit recollements. We want to apply the criterion given in Proposition 5.10. The vertical fibers are both equivalent to Sp^{BS^1} , where the equivalence $\ker(U_{n+1}) \rightarrow \text{Sp}^{BS^1}$ is induced by taking $C_{p^{n+1}}$ -fixed points for the left vertical map and similarly by taking C_{p^n} -fixed points for the right vertical map. Using the description of the adjoint given in Example 5.7, it follows immediately that the Tate objects are $T(X) \simeq (X^{C_{p^n}})^{tC_p} \in \text{Sp}^{BS^1}$ for X in $\mathbb{TSp}_{C_{p^n}}^{\text{gen}}$ and $T(X) \simeq (X^{C_{p^{n-1}}})^{tC_p} \in \text{Sp}^{BS^1}$ for X in $\mathbb{TSp}_{C_{p^{n-1}}}^{\text{gen}}$. This together with the description of the functors implies that criterion (2) of Proposition 5.10 is satisfied. \square

5.3 Consequences

Now we can deduce the following statement from the results of Section 5.1.

Theorem 5.12. *We have an adjunction*

$$L : \mathbf{TCart}_p \rightleftarrows \mathbf{CycSp}_p^{\text{gen}} : \text{TR}$$

with $(LM)^{C_{p^n}} = M/V^{n+1}$ and $\text{TR}(X) = \lim_{n,R} X^{C_{p^n}}$. The unit of the adjunction is given by the ‘naive’ V -completion $M \rightarrow \lim M/V^{n+1}$ and the counit is given on underlying spectra by the map $\text{TR}(X)/V \rightarrow X$.

Proof. By Proposition 5.2 we have an adjunction

$$L : \mathbf{TCart}_p^{\text{gen}} \rightleftarrows \mathbf{CycSp}_p^{\text{gen}} : \overline{\text{TR}}$$

which we compose with the equivalence $U : \mathbf{TCart}_p^{\text{gen}} \xrightarrow{\simeq} \mathbf{TCart}_p$ of Proposition 5.5 which extracts the ‘underlying’ p -typical topological Cartier module. The value of the composite $U \circ \overline{\text{TR}}$ at X has by definition of $\overline{\text{TR}}$ the underlying spectrum $\text{TR}(X) \simeq \lim_{n,R} X^{C_{p^n}}$ as classically defined (we have used that taking the underlying spectrum commutes with taking limits of genuine spectra).

Now we need to determine the fixed points of the spectrum associated with a p -typical topological Cartier module $M \in \mathbf{TCart}_p$. First, by the way the equivalence $U : \mathbf{TCart}_p^{\text{gen}} \xrightarrow{\simeq} \mathbf{TCart}_p$ we find that the map $M_{hC_p^{n+1}} \rightarrow M^{C_p^{n+1}}$ (where we have abusively identified $U^{-1}M$ with M) is given by the composite

$$V^{n+1} : M_{hC_p^{n+1}} \rightarrow \cdots \rightarrow M_{hC_p} \rightarrow M.$$

Thus the claim follows from Corollary 5.4. \square

So far, we have only defined what it means to be V -complete for bounded below p -typical topological Cartier modules. In general we shall consider the Bousfield localization of \mathbf{TCart}_p at the mod V -equivalences. For this definition it does not matter if one considers $(-)/V$ as taking values in p -typical cyclotomic spectra, genuine p -typical cyclotomic spectra, or spectra. The local objects for this Bousfield localization are in the bounded below case precisely the V -complete p -typical topological Cartier modules which follows from Proposition 3.17 and Theorem 3.21.

Corollary 5.13. *For every genuine p -typical cyclotomic spectrum X the induced p -typical topological Cartier module $\text{TR}(X)$ is complete with respect to the Bousfield localization at the mod V equivalences (see Remark 3.23). Moreover, there is an induced adjunction*

$$L : \widehat{\mathbf{TCart}}_p \rightleftarrows \mathbf{CycSp}_p^{\text{gen}} : \text{TR},$$

which is an equivalence on bounded below objects, where $\widehat{\mathbf{TCart}}_p$ denotes the Bousfield localization at the mod V -equivalences.

Proof. We have to verify that $\text{TR}(X)/V \rightarrow X$ is an equivalence for X bounded below. But this follows from the cofiber sequences $\text{TR}^n(X)_{hC_p} \rightarrow \text{TR}^{n+1}(X) \rightarrow X$ by passing to the limit and noting that homotopy orbits commutes with this limit since it is uniformly bounded below.

Moreover from the description in Theorem 5.12 it follows that both functors restrict to an adjunction between bounded below objects. Now, if M is derived V -complete, then by definition the unit of the adjunction is an equivalence. \square

Remark 5.14. In Proposition 3.17 we have proven that there is an adjunction

$$(-)/V : \mathbf{TCart}_p \rightleftarrows \mathbf{CycSp}_p : \text{TR}$$

for the ‘simplistic’ version of p -typical cyclotomic spectra. Clearly the functor $(-)/V$ factors over the forgetful functor $\mathbf{CycSp}_p^{\text{gen}} \rightarrow \mathbf{CycSp}_p$ which preserves colimits and is therefore left adjoint as well. Thus we get the adjunction of Proposition 3.17 as the composite of the two adjunctions

$$\mathbf{TCart}_p \rightleftarrows \mathbf{CycSp}_p^{\text{gen}} \rightleftarrows \mathbf{CycSp}_p.$$

Both of these adjunctions induce equivalences on subcategories of bounded below objects, where on the left we look at the full subcategory of V -complete bounded below objects. The left hand one as just shown in Corollary 5.13 and the right hand one as shown in [NS18]. The hardest part of the latter is in fact understanding the right adjoint to the forgetful functor $\mathbf{CycSp}_p^{\text{gen}} \rightarrow \mathbf{CycSp}_p$. Thus Corollary 5.13 also gives an independent proof of Theorem 3.21. Alternatively, Corollary 5.13 and Theorem 3.21 can be used to reprove the p -typical part of [NS18, Theorem II.3.8].

It is now natural to ask if the adjunction of Corollary 5.13 is an equivalence. For the simplistic adjunction $(-)/V : \mathbf{TCart}_p \rightleftarrows \mathbf{CycSp}_p : \mathrm{TR}$ this is obviously not the case since we have used the Tate orbit lemma in a crucial way. The genuine version has a much better chance of inducing an equivalence. The following result shows that the genuine adjunction is not an equivalence.

Proposition 5.15. *The adjunction of Corollary 5.13 does not form an equivalence of stable ∞ -categories.*

Proof. If the adjunction of Corollary 5.13 were an equivalence, then the counit of the adjunction $\mathrm{TR}(X)/V \rightarrow X$ would be an equivalence for every genuine p -typical cyclotomic spectrum X . This is equivalent to the assertion that the inverse limit defining $\mathrm{TR}(X)$ commutes with taking homotopy orbits (since it commutes with taking cofibers and X is the cofiber of the ‘commuted’ map). We will show that this is not the case for $X = \mathrm{KU}^{\mathrm{triv}}$ where $\mathrm{KU}^{\mathrm{triv}}$ refers to the genuine S^1 -spectrum given by the inflation of KU along the projection $\mathbb{T} \rightarrow e$. This carries a canonical p -typical cyclotomic structure, since the geometric fixed points are given by KU itself. More precisely there is a functor

$$(-)^{\mathrm{triv}} : \mathrm{Sp} \rightarrow \mathbf{CycSp}_p^{\mathrm{gen}}$$

which is uniquely determined by requiring that it is left adjoint (i.e. preserves all colimits) and sends the sphere to the genuine p -typical cyclotomic sphere. In other words we have that $\mathrm{KU}^{\mathrm{triv}} \simeq \mathbb{S}^{\mathrm{triv}} \otimes \mathrm{KU}$. The composite

$$\mathrm{Sp} \rightarrow \mathbf{CycSp}_p^{\mathrm{gen}} \rightarrow \mathbb{T}\mathrm{Sp}_p^{\mathrm{gen}} \xrightarrow{(-)^{C_{p^n}}} \mathrm{Sp}$$

then also preserves all colimits for each n . Thus, we find using the tom Dieck splitting that

$$\begin{aligned} \mathrm{TR}^{n+1}(\mathrm{KU}^{\mathrm{triv}}) &= (\mathrm{KU}^{\mathrm{triv}})^{C_{p^n}} \\ &\simeq \mathrm{KU} \otimes \mathbb{S}^{C_{p^n}} \\ &\simeq \mathrm{KU} \otimes \left(\bigoplus_{k=0}^n \mathbb{S} \otimes BC_{p^k} \right) \simeq \bigoplus_{k=0}^n \mathrm{KU} \otimes BC_{p^k}. \end{aligned}$$

The transition maps $R : \mathrm{TR}^{n+1}(\mathrm{KU}^{\mathrm{triv}}) \rightarrow \mathrm{TR}^n(\mathrm{KU}^{\mathrm{triv}})$ are under these identifications given by the projections away from the last factor. As a result we find that

$$\mathrm{TR}(\mathrm{KU}^{\mathrm{triv}}) \simeq \prod_{k=0}^{\infty} (\mathrm{KU} \otimes BC_{p^k}).$$

Now we will prove that the canonical map

$$\left(\prod_{k=0}^{\infty} \mathrm{KU} \otimes BC_{p^k} \right)_{hC_p} \rightarrow \prod_{k=0}^{\infty} (\mathrm{KU} \otimes BC_{p^k})_{hC_p} \quad (24)$$

is not surjective on π_1 . We write $\mathrm{KU} \otimes BC_{p^k}$ as the colimit of its negative Postnikov tower $\mathrm{colim}_{\tau_{\geq n}} (\mathrm{KU} \otimes BC_{p^k})$ for $n \rightarrow -\infty$. Then

$$\left(\prod_{k=0}^{\infty} \mathrm{KU} \otimes BC_{p^k} \right)_{hC_p} \simeq \mathrm{colim}_n \prod_{k=0}^{\infty} \tau_{\geq n} (\mathrm{KU} \otimes BC_{p^k})_{hC_p}$$

since for uniformly bounded below spectra we can commute the homotopy orbits with the infinite product. Similarly we get that the right hand side of (24) is given as

$$\prod_{k=0}^{\infty} (\mathrm{KU} \otimes BC_{p^k})_{hC_p} \simeq \prod_{k=0}^{\infty} \mathrm{colim}_n \tau_{\geq n} (\mathrm{KU} \otimes BC_{p^k})_{hC_p}$$

so that the map (24) is the canonical map that commutes the filtered colimit with the infinite product. On π_1 this map induces the map

$$\operatorname{colim}_n \prod_{k=0}^{\infty} M_{k,n} \rightarrow \prod_{k=0}^{\infty} \operatorname{colim}_n M_{k,n}$$

with $M_{k,n} := \pi_1 \left(\tau_{\geq n} \left(\mathrm{KU} \otimes BC_{p^k} \right)_{hC_p} \right)$. Algebraically such a map is a surjective precisely if there exists an n_0 such that for almost all k the map

$$M_{k,n_0} \rightarrow \operatorname{colim}_n M_{k,n}$$

is surjective. We will show that in our case none of the map $M_{k,n_0} \rightarrow \operatorname{colim}_n M_{k,n}$ is surjective.

We begin by determining the homotopy groups of $\mathrm{KU}_{hC_{p^k}}$. Greenlees [Gre93] has shown that they are isomorphic to the local cohomology groups of the representation ring (and explicitly the homotopy groups had been computed by Wilson [Wil73] and Knapp [Kna78] before). We get that

$$\pi_* \left(\mathrm{KU}_{hC_{p^k}} \right) = \begin{cases} \mathbb{Z} & * \text{ even} \\ I(C_{p^k}) \otimes \mathbb{Q}_p / \mathbb{Z}_p & * \text{ odd} \end{cases}$$

where $I(C_{p^k}) \subseteq R(C_{p^k})$ is the augmentation ideal¹⁷ in the representation ring of C_{p^k} which is isomorphic to the group ring $\mathbb{Z}[C_{p^k}^\vee]$. This can also be seen by a direct computation using the cofiber sequence

$$\mathrm{KU}_{hC_{p^k}} \rightarrow \mathrm{KU}^{hC_{p^k}} \rightarrow \mathrm{KU}^{tC_{p^k}}$$

(the second and third term are even and the last term is rational) or by using the universal coefficients theorem for KU and the Atiyah–Segal completion theorem. Now the filtration in question comes from the horizontal lines in the homotopy orbits spectral sequence

$$E_{i,j}^2 = H_i \left(BC_p, \pi_j \left(\mathrm{KU}_{hC_{p^k}} \right) \right) \Rightarrow \pi_{i+j} \left(\left(\mathrm{KU}_{hC_{p^k}} \right)_{hC_p} \right) \quad (25)$$

which is conditionally convergent. Recall that

$$H_i(BC_p, \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0, \\ \mathbb{F}_p & * > 0 \text{ odd}, \\ 0 & \text{else} \end{cases} \quad \text{and} \quad H_i(BC_p, \mathbb{Q}_p / \mathbb{Z}_p) = \begin{cases} \mathbb{Q}_p / \mathbb{Z}_p & * = 0, \\ \mathbb{F}_p & * > 0 \text{ even}, \\ 0 & * \text{ odd}. \end{cases}$$

Thus the homotopy orbits spectral sequence is in homological Serre grading a right half plan spectral sequence which is for $i > 0$ (i.e. right of the axis) concentrated in odd total degree where each entry is a sum of copies of \mathbb{F}_p 's. There cannot be any differentials in this spectral sequence since the only differentials that are possible for degree reasons would have to map a p -torsion group to \mathbb{Z} . Thus the whole spectral sequence collapses at E^2 and is strongly convergent.¹⁸ For every value of j there are elements in total degree 1 that are detected at the horizontal line through $-j$. Since this line corresponds to elements in the image of $M_{k,j} \rightarrow \operatorname{colim}_n M_{k,n}$, this finishes the proof. \square

¹⁷Really we should have the dual $I(C_{p^k})$ of the augmentation ideal in odd degrees but we will only need that $I(C_{p^k}) = \mathbb{Z}^{\oplus(p^k-1)}$ as abelian groups.

¹⁸Note that we know the homotopy groups of $\Sigma \left(\mathrm{KU}_{hC_{p^k}} \right)_{hC_p} = \Sigma \left(\mathrm{KU}_{hC_{p^{k+1}}} \right)$ by the above computation. It is interesting how the extensions work out to lead to this result.

6 THH of schemes and the cyclotomic t -structure

In this section, we give a sampling of applications of the cyclotomic t -structure to the study of THH of commutative rings and to schemes. We will continue an abuse of notation by where we view $\pi_n^{\text{cyc}}\text{THH}(X)$ simultaneously as an object of the abelian category $\widehat{\mathbf{Cart}}_p$ and as a cyclotomic spectrum (with homotopy groups given by Figure 1 on page 6).

6.1 THH of ring spectra and schemes

Our first result follows from a computations of Hesselholt–Madsen and Hesselholt.

Theorem 6.1. *If R is a connective \mathbb{E}_1 -ring spectrum, then $\pi_0^{\text{cyc}}\text{THH}(R) \cong \pi_0^{\text{cyc}}\text{THH}(\pi_0 R) \cong W(\pi_0 R)$ as a p -typical Cartier module, where $W(R)$ is equipped with the Witt vector Frobenius and Verschiebung maps.*

Proof. Since $\pi_0\text{THH}(R) \cong \pi_0\text{THH}(\pi_0 R) \cong \pi_0 R$, we see from the long exact sequence in cyclotomic homotopy groups that $\pi_0^{\text{cyc}}\text{THH}(R) \cong \pi_0^{\text{cyc}}\text{THH}(\pi_0 R)$. Now, for any associative ring A , $\pi_0\text{TR}(A) \cong W(A)$. In the commutative case, this is the content of [HM97, Theorem F]. In the noncommutative case, see [Hes97, Theorem 2.2.9] and [Hes05]. A recent exposition is given in [KN18, Corollary 10.2]. The theorem now follows from Theorem 9. \square

Remark 6.2. (a) As explained in Section 4.3, Theorem 6.1 gives a topological proof of Theorem 4.16, i.e., that $W(R)\widehat{\boxtimes}W(S) \cong W(R \otimes S)$ for rings R and S .

(b) It follows from Theorem 6.1 that for any cyclotomic spectrum X , the cyclotomic homotopy groups $\pi_i^{\text{cyc}}X$ are modules over $W(\mathbb{Z})$ and hence over the cyclotomic spectra \mathbb{Z}^{triv} and $\text{THH}(\mathbb{Z})$. Indeed, both maps $\mathbb{Z}^{\text{triv}} \leftarrow \mathbb{S}^{\text{triv}} \rightarrow \text{THH}(\mathbb{Z})$ induce equivalences on π_0^{cyc} .

Recall from Variant 2.4 that there is a t -structure on $\mathbf{CycSp}_{\text{THH}(R)} = \text{Mod}_{\text{THH}(R)}(\mathbf{CycSp}_p)$, the ∞ -category of $\text{THH}(R)$ -modules in cyclotomic spectra. We identify the heart in the next corollary.

Corollary 6.3. *If R is a connective \mathbb{E}_∞ -ring spectrum, then the heart $\mathbf{CycSp}_{\text{THH}(R)}^\heartsuit$ is equivalent to $\text{Mod}_{W(\pi_0 R)}(\mathbf{CycSp}_p^\heartsuit) \simeq \text{Mod}_{W(\pi_0 R)}(\widehat{\mathbf{Cart}}_p)$, the abelian category of $W(\pi_0 R)$ -modules in derived V -complete p -typical Cartier modules with respect to the $\widehat{\boxtimes}$ -symmetric monoidal structure.*

Proof. This follows from Theorem 4.15, Theorem 6.1, and Proposition A.15. \square

Remark 6.4. Lemma 4.17 gives a concrete description of the objects of this abelian category.

Now, we examine the cyclotomic homotopy groups of schemes in general. In Section 6.2, we will give more precise results in the case of regular \mathbb{F}_p -schemes.

It has been shown in [BMS19, Corollary 3.3] that $\text{THH}(-)$ is a Sp -valued fpqc sheaf on CAlg , the category of commutative rings. They also note in [BMS19, Remark 3.4] that the proof extends to show that $\text{THH}(-)^{tC_p}$ is a Sp -valued fpqc sheaf on CAlg . This is enough to prove that in fact $\text{THH}(-)$ is a \mathbf{CycSp}_p -valued fpqc sheaf on CAlg . We expand on this and establish pro-étale hyperdescent for THH in \mathbf{CycSp}_p . We claim no originality in our proof, which closely follows the argument of [BMS19].

For details on the pro-étale topology, see [BS15]. We give a brief summary here. A map $R \rightarrow S$ is **weakly étale** if $R \rightarrow S$ and $S \otimes_R S \rightarrow S$ are both flat. We let $\text{CAlg}_R^{\text{proét}}$ be the full subcategory of CAlg_R on the weakly étale R -algebras. Note that any map $S \rightarrow T$ of weakly étale R -algebras is itself weakly étale. The faithful weakly étale maps make $\text{CAlg}_R^{\text{proét}}$ into a site, the pro-étale site of R . For the next proof, we will need only to know that for every weakly étale map $R \rightarrow S$ one has $L_{S/R} \simeq 0$ (see [BS15, Proposition 2.3.3(2)], which follows from [GR03, Theorem 2.5.36]).

Proposition 6.5. *Write $\mathrm{THH}(\mathcal{O})$ for the presheaf which sends a commutative ring R to $\mathrm{THH}(R)$.*

- (a) *The presheaf $\mathrm{THH}(\mathcal{O})$ is a hypercomplete Sp -valued pro-étale sheaf.*
- (b) *The presheaf $\mathrm{THH}(\mathcal{O})$ is a hypercomplete \mathbf{CycSp}_p -valued pro-étale sheaf.*

Proof. One can use the main result of [Mat17], which says that $S \otimes_R \mathrm{THH}(R) \simeq \mathrm{THH}(S)$ for an étale map $R \rightarrow S$, to give a proof of part (a). We will give a different proof, which will serve to motivate our proof of (b). Let $\mathrm{CAlg}_R^{\mathrm{proét}}$ be the category of weakly étale R -algebras. As mentioned above, $\mathrm{THH}(\mathcal{O})$ is a Sp -valued pro-étale sheaf by [BMS19, Corollary 3.3]. To prove (a), consider the presheaf $\mathrm{THH}(\mathcal{O}) \otimes_{\mathrm{THH}(\mathbb{Z})} \tau_{\leq n} \mathrm{THH}(\mathbb{Z})$ (where $\tau_{\leq n} \mathrm{THH}(\mathbb{Z})$ is the ordinary (not cyclotomic) truncation), which sends a $S \in \mathrm{CAlg}_R^{\mathrm{proét}}$ to $\mathrm{THH}(S) \otimes_{\mathrm{THH}(\mathbb{Z})} \tau_{\leq n} \mathrm{THH}(\mathbb{Z})$. For each $S \in \mathrm{CAlg}_R^{\mathrm{proét}}$, we have that $\mathrm{THH}(S) \simeq \lim_n \mathrm{THH}(S) \otimes_{\mathrm{THH}(\mathbb{Z})} \tau_{\leq n} \mathrm{THH}(\mathbb{Z})$. In particular, $\mathrm{THH}(\mathcal{O}) \simeq \lim_n \mathrm{THH}(\mathcal{O}_X) \otimes_{\mathrm{THH}(\mathbb{Z})} \tau_{\leq n} \mathrm{THH}(\mathbb{Z})$ as Sp -valued presheaves. Since hypercomplete sheaves are closed under limits, it is enough to see that each $\mathrm{THH}(\mathcal{O}) \otimes_{\mathrm{THH}(\mathbb{Z})} \tau_{\leq n} \mathrm{THH}(\mathbb{Z})$ is hypercomplete. As in the proof of [BMS19, Corollary 3.3], this reduces to showing that each $\mathrm{THH}(\mathcal{O}) \otimes_{\mathrm{THH}(\mathbb{Z})} \pi_n \mathrm{THH}(\mathbb{Z})$ is hypercomplete and then to showing that $\mathrm{THH}(\mathcal{O}) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z}$ is hypercomplete, since

$$\mathrm{THH}(\mathcal{O}) \otimes_{\mathrm{THH}(\mathbb{Z})} \pi_n \mathrm{THH}(\mathbb{Z}) \simeq \mathrm{THH}(\mathcal{O}) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z} \otimes_{\mathbb{Z}} \pi_n \mathrm{THH}(\mathbb{Z}).$$

However, $\mathrm{THH}(\mathcal{O}) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z} \simeq \mathrm{HH}(\mathcal{O}/\mathbb{Z})$, the presheaf which sends $S \in \mathrm{CAlg}_R^{\mathrm{proét}}$ to $\mathrm{HH}(S/\mathbb{Z})$, the Hochschild homology of S . Thus, we are reduced to proving that Hochschild homology is hypercomplete as a Sp -valued pro-étale presheaf. Now, $\mathrm{HH}(\mathcal{O}/\mathbb{Z})$ admits a complete decreasing \mathbb{N} -indexed filtration $F_{\mathrm{HKR}}^* \mathrm{HH}(\mathcal{O}/\mathbb{Z})$ with graded pieces given by

$$\mathrm{gr}_{\mathrm{HKR}}^n \mathrm{HH}(\mathcal{O}/\mathbb{Z}) \simeq \Lambda^n \mathrm{L}_{\mathcal{O}/\mathbb{Z}}[n].$$

In particular, $\mathrm{gr}_{\mathrm{HKR}}^0 \mathrm{HH}(\mathcal{O}/\mathbb{Z}) \simeq \mathcal{O}$, which is hypercomplete. Since each $\Lambda^n \mathrm{L}_{\mathcal{O}/\mathbb{Z}}$ is quasi-coherent (we use here that $\mathrm{L}_{S/R} \simeq 0$ for $R \rightarrow S$ pro-étale) and since quasi-coherent sheaves are hypercomplete (see for example [Lur18, Proposition 2.2.6.1]), it follows by induction that each $F_{\mathrm{HKR}}^n \mathrm{HH}(\mathcal{O}/\mathbb{Z})$ is a hypercomplete pro-étale sheaf of spectra. Part (a) follows since the HKR filtration is complete.

Now, we would like to prove that $\mathrm{THH}(\mathcal{O})$ is a hypercomplete \mathbf{CycSp}_p -valued pro-étale sheaf. Since the forgetful functor $\mathbf{CycSp}_p \rightarrow \mathrm{Sp}$ does not commute with limits in general, this is not an immediate consequence of the previous paragraph. We invoke [NS18, Proposition II.1.5], which implies that it is enough for $\mathrm{THH}(R)^{tC_p} \rightarrow \lim_{\Delta} \mathrm{THH}(S^\bullet)^{tC_p}$ to be an equivalence for every pro-étale hypercover $R \rightarrow S^\bullet$. Since $(-)^{tC_p}$ commutes with sequential limits of increasingly connected maps (by Lemma 2.11(a)), we can reduce as in the previous paragraph to checking that $\mathrm{HH}(R)^{tC_p} \rightarrow \lim_{\Delta} \mathrm{HH}(S^\bullet)^{tC_p}$ is an equivalence. And, again, we can use the HKR filtration to reduce to checking that $(\Lambda^n \mathrm{L}_{R/\mathbb{Z}})^{tC_p} \simeq \lim_{\Delta} (\Lambda^n \mathrm{L}_{S^\bullet/\mathbb{Z}})^{tC_p}$ is an equivalence for each $n \in \mathbb{N}$. Since $(-)^{hC_p}$ commutes with limits, it is enough to prove that $(\Lambda^n \mathrm{L}_{R/\mathbb{Z}})_{hC_p} \rightarrow \lim_{\Delta} (\Lambda^n \mathrm{L}_{S^\bullet/\mathbb{Z}})_{hC_p}$ is an equivalence. This follows from faithfully flat descent since $(\mathrm{L}_{R/\mathbb{Z}})_{hC_p}$ is quasicohherent. \square

Corollary 6.6. *The presheaf $\mathrm{TR}(\mathcal{O})$ which sends a commutative ring R to $\mathrm{TR}(R)$ is a hypercomplete \mathbf{TCart}_p -valued pro-étale sheaf.*

Proof. This is immediate as $\mathrm{TR}: \mathbf{CycSp}_p \rightarrow \mathbf{TCart}_p$ preserves limits by Proposition 3.17. \square

It follows that $\mathrm{THH}(\mathcal{O})$ and $\mathrm{TR}(\mathcal{O})$ are hypercomplete for any topology coarser than the pro-étale topology as well. We will be mainly interested in the Zariski topology below.

Definition 6.7. Let X be a quasi-compact and quasi-separated scheme. Let $\tau_{\leq n} \mathrm{TR}(\mathcal{O}_X)$ denote the Zariski sheafification of $U \mapsto \tau_{\leq n} \mathrm{TR}(U)$ in the ∞ -category \mathbf{TCart}_p . Similarly, let $\tau_{\geq n} \mathrm{TR}(\mathcal{O}_X)$ denote the Zariski sheafification of $U \mapsto \tau_{\geq n} \mathrm{TR}(U)$.

Remark 6.8. (a) The forgetful functor $\mathbf{TCart}_p \rightarrow \mathbf{Sp}$ preserves limits and colimits, so in what follows we could as well work with presheaves of spectra.

(b) In particular, $\tau_{\leq n} \mathrm{TR}(\mathcal{O}_X)$ and $\tau_{\geq n} \mathrm{TR}(\mathcal{O}_X)$ are both hypercomplete since the inclusion of hypercomplete sheaves of spectra into all sheaves of spectra is t -exact.¹⁹

Lemma 6.9. *If X is a quasi-compact scheme of finite Krull dimension, then the natural map $\mathrm{TR}(\mathcal{O}_X) \rightarrow \lim_n \tau_{\leq n} \mathrm{TR}(\mathcal{O}_X)$ is an equivalence.*

Proof. In this case, the ∞ -topos of sheaves of spaces on the Zariski site of X has finite homotopy dimension by [CM19, Theorem 3.17]. This implies that every sheaf of spaces on the Zariski site is Postnikov complete by [Lur09, Proposition 7.2.1.10]. By taking infinite loop spaces, this implies that every connective sheaf of spectra on the Zariski site of X is Postnikov complete.²⁰ \square

Definition 6.10. Write $\pi_n^{\mathrm{cyc}} \mathrm{THH}(\mathcal{O}_X)$ for the Zariski sheafification of $U \mapsto \pi_n^{\mathrm{cyc}} \mathrm{THH}(U) \cong \pi_n \mathrm{TR}(U)$, which is naturally a sheaf in the abelian category \mathbf{Cart}_p .

Example 6.11. By Theorem 6.1, we see that $\pi_0^{\mathrm{cyc}} \mathrm{THH}(\mathcal{O}_X) \simeq W(\mathcal{O}_X)$, Serre’s Witt vector sheaf.

Proposition 6.12. *Let X be a quasi-separated scheme of finite Krull dimension. There is a conditionally convergent Zariski descent spectral sequence*

$$E_2^{s,t} \cong H^{-s}(X, \pi_t^{\mathrm{cyc}} \mathrm{THH}(\mathcal{O}_X)) \Rightarrow \pi_{s+t}^{\mathrm{cyc}} \mathrm{THH}(X)$$

in \mathbf{Cart}_p .

Proof. This is the spectral sequence associated to the complete decreasing \mathbb{N} -indexed Whitehead tower $\tau_{\geq t} \mathrm{TR}(\mathcal{O}_X)$ for $\mathrm{TR}(\mathcal{O}_X)$. We use that $\pi_{t-s}^{\mathrm{cyc}} \mathrm{THH}(X) \cong \pi_{t-s} \mathrm{TR}(X)$. \square

Example 6.13. Suppose that X has Krull dimension d . The differentials d_r in the descent spectral sequence have bidegree $(r, r-1)$, from which we see that the bottom cyclotomic homotopy group of $\mathrm{THH}(X)$ is given by

$$\pi_{-d}^{\mathrm{cyc}} \mathrm{THH}(X) \cong H^d(X, \pi_0^{\mathrm{cyc}} \mathrm{THH}(\mathcal{O}_X)) \cong H^d(X, W(\mathcal{O}_X)),$$

where $W(\mathcal{O}_X)$ is Serre’s sheaf of Witt vectors.

6.2 THH of smooth schemes over perfect rings in characteristic p

The following theorem follows easily from the description of the t -structure in terms of TR and the work of Hesselholt and Madsen.

¹⁹Let \mathcal{X} be an ∞ -topos, $\mathcal{X}^{\mathrm{hyp}}$ its hypercompletion, and consider the adjunction $f^*: \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \rightleftarrows \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}^{\mathrm{hyp}}): f_*$. As for general geometric morphisms of ∞ -topoi, f^* is t -exact and f_* is left t -exact (see [Lur18, 1.3.2.8]). We want to show that f_* is right t -exact as well in this case. Thus, let $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}^{\mathrm{hyp}})_{\geq 0}$ and consider the cofiber sequence

$$\tau_{\geq 0} f_* \mathcal{F} \rightarrow f_* \mathcal{F} \rightarrow \tau_{\leq -1} f_* \mathcal{F}.$$

Applying f^* again, we see that

$$f^* \tau_{\geq 0} f_* \mathcal{F} \simeq \tau_{\geq 0} f^* f_* \mathcal{F} \simeq \tau_{\geq 0} \mathcal{F} \simeq \mathcal{F}$$

using the t -exactness of f^* and the fully faithfulness of f_* (see [Lur18, 1.3.3.2]). Thus, $f^* \tau_{\leq -1} f_* \mathcal{F} \simeq 0$. In other words, $\tau_{\leq -1} f_* \mathcal{F}$ is both ∞ -connective and bounded above. Hence, since the t -structure on $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ is right complete by [Lur18, 1.3.2.7], $\tau_{\leq -1} f_* \mathcal{F} \simeq 0$ so $f_* \mathcal{F}$ is connective.

²⁰Note that the ∞ -category of connective sheaves of spectra is equivalent to the ∞ -category of sheaves of connective spectra.

Theorem 6.14. *Suppose that k is a perfect ring of characteristic p . Then, $\mathrm{THH}(k) \in \mathbf{CycSp}_p^\heartsuit$. The associated Cartier module $\pi_0^{\mathrm{cyc}}\mathrm{THH}(k)$ is isomorphic to $W(k)$ with the Witt vector Frobenius F and Verschiebung V .*

Proof. By Theorem 3.21, it suffices to show that $\mathrm{TR}(R)$ is concentrated in degree zero and that $\mathrm{TR}_0(k) \cong W(k)$. When k is a perfect field of characteristic p , this is the content of [HM97, Theorem 5.5]. When k is an arbitrary perfect ring of characteristic p , there is a map $\mathrm{THH}(k) \rightarrow \pi_0^{\mathrm{cyc}}\mathrm{THH}(k) \simeq W(k)/V$ of commutative algebra objects in \mathbf{CycSp}_p by Theorem 6.1. We already know that $\pi_*\mathrm{THH}(k) \cong k[b]$ where b has degree 2 (for instance by [BMS19, Theorem 6.1]). Since k is perfect, $\pi_*W(k)/V \cong k[b]$ as well. Thus, it is enough to see that b maps to b up to a unit. But, this follows by the commutative diagram

$$\begin{array}{ccc} \mathrm{THH}(\mathbb{F}_p) & \longrightarrow & \mathrm{THH}(k) \\ \downarrow & & \downarrow \\ W(\mathbb{F}_p)/V & \longrightarrow & W(k)/V \end{array}$$

and the fact that we know the result when $k = \mathbb{F}_p$. \square

Remark 6.15. (i) The fact that $\mathrm{THH}(\mathbb{F}_p) \in \mathbf{CycSp}_p^\heartsuit$ is equivalent to Bökstedt's original calculation via Theorem 6.1. We have already seen one direction as Bökstedt's calculation is used to prove [HM97, Theorem 5.5]. So, assume that $\mathrm{THH}(\mathbb{F}_p) \in \mathbf{CycSp}_p^\heartsuit$. By Theorem 6.1, we find that in fact $\mathrm{THH}(\mathbb{F}_p) \simeq W(\mathbb{F}_p)/V$. But, additively, $\pi_*W(\mathbb{F}_p)/V \cong \mathbb{F}_p[b]$ where $|b| = 2$ using Figure 1 on page 6. Thus, it is enough to determine the multiplicative structure. For this, we use that $\mathrm{THH}(\mathbb{F}_p)$ is a $\mathbb{Z}_p^{\mathrm{triv}}$ -module in cyclotomic spectra as the trace map $K(\mathbb{F}_p) \rightarrow \mathrm{THH}(\mathbb{F}_p)$ factors through $\tau_{\geq 0}\mathrm{TC}(\mathbb{F}_p) \simeq \mathbb{Z}_p$. Thus, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}_p^{\mathrm{triv}} & \longrightarrow & \mathrm{THH}(\mathbb{F}_p) & \longrightarrow & W(\mathbb{F}_p)/V \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}_p^{tC_p} & \longrightarrow & \mathrm{THH}(\mathbb{F}_p)^{tC_p} & \longrightarrow & (W(\mathbb{F}_p)/V)^{tC_p} \end{array}$$

of commutative algebra objects in Sp^{BS^1} . We know that $\mathbb{Z}_p^{tC_p} \simeq W(\mathbb{F}_p)^{tC_p} \rightarrow (W(\mathbb{F}_p)/V)^{tC_p}$ is an equivalence by the Tate orbit lemma. Thus, the composition of the bottom arrows is an equivalence. Since we are assuming that $\mathrm{THH}(\mathbb{F}_p) \simeq W(\mathbb{F}_p)/V$, this means that both bottom arrows are equivalences. On the other hand, this means that the middle and right vertical arrows are both 0-truncated by Proposition 2.25. Since $\pi_1\mathbb{Z}_p^{tC_p} = 0$ and because the map $\mathrm{THH}(\mathbb{F}_p) \rightarrow \mathrm{THH}(\mathbb{F}_p)^{tC_p}$ is a ring map, we see that in fact $\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geq 0}\mathrm{THH}(\mathbb{F}_p)^{tC_p} \simeq \tau_{\geq 0}\mathbb{Z}_p^{tC_p}$. This shows that $\pi_*\mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p[b]$ multiplicatively as well.

- (ii) Our original proof, prior to the discovery of p -typical topological Cartier modules, used only the fact that $\pi_*\mathrm{THH}(k) \cong k[b]$, where b has degree 2 when k is a perfect ring of characteristic p . Indeed, one can construct a tower of spectra with S^1 -action whose limit is $\tau_{\geq 1}^{\mathrm{cyc}}X$ for any p -typical cyclotomic spectrum X . Studying this tower for $X \simeq \mathrm{THH}(k)$ where k is a perfect ring of characteristic p gives an alternative proof of Theorem 6.14.

Remark 6.16. (a) Recall from Example 4.18 that if $R = \mathbb{F}_p$, then the objects of $\mathrm{Mod}_{W(\mathbb{F}_p)}(\mathbf{TCart}_p)$ are called Dieudonné modules. Thus, $\mathbf{CycSp}_{\mathrm{THH}(\mathbb{F}_p)}^\heartsuit$ is the abelian category of derived V -complete Dieudonné modules.

- (b) Since $\mathrm{THH}(\mathbb{F}_p)$ is a $\mathbb{Z}^{\mathrm{triv}}$ -module in \mathbf{CycSp}_p , it follows that $\eta = 0$ on $\pi_n^{\mathrm{cyc}} X$ for any $\mathrm{THH}(\mathbb{F}_p)$ -module X in p -typical cyclotomic spectra.

Definition 6.17. A **Dieudonné complex** is a p -typical Cartier complex (see Definition 3.34) with $VF = p$ and $\eta = 0$. Saturated Dieudonné complexes in the sense of [BLM18, Definition 2.2.1] naturally admit the structure of Dieudonné complexes in our sense by [BLM18, Proposition 2.2.4].

Combining the previous remarks, Theorem 6.14, and Lemma 3.33, we see that for any $X \in \mathbf{CycSp}_{\mathrm{THH}(\mathbb{F}_p)}$, the graded abelian group $\pi_*^{\mathrm{cyc}} X$ naturally admits the structure of a Dieudonné complex.

Theorem 6.18. *Let k be a perfect field of characteristic p . If R is a smooth k -algebra, then $\pi_*^{\mathrm{cyc}} \mathrm{THH}(R) \cong \mathrm{W}\Omega_R^*$, as Dieudonné complexes, where $\mathrm{W}\Omega_R^*$ is the de Rham–Witt complex. In particular, if R has relative dimension at most d over k , then $\mathrm{THH}(R) \in (\mathbf{CycSp}_p)_{[0,d]}$.*

Proof. At the level of $\mathrm{TR}(R)$, this is the content of [Hes96, Theorem C], which says that, with the F , V , and d operations, $\mathrm{TR}_*(R) \cong \mathrm{W}\Omega_R^*$. \square

Remark 6.19. In particular, we find that $\mathrm{W}\Omega_R^n$ is derived V -complete for all n . Since this is an unusual filtration to consider, we note that this is easy to deduce classically. Let $p^* \mathrm{W}\Omega_R^n$ be the p -adic filtration on $\mathrm{W}\Omega_R^n$ and let $V^* \mathrm{W}\Omega_R^n + dV^* \mathrm{W}\Omega_R^n \subseteq \mathrm{W}\Omega_R^n$ be the submodule generated by the images of V^* and dV^* . Then, $p^* \mathrm{W}\Omega_R^n \subseteq V^* \mathrm{W}\Omega_R^n \subseteq V^* \mathrm{W}\Omega_R^n + dV^* \mathrm{W}\Omega_R^{n-1}$. Since $\mathrm{W}\Omega_R^n$ is complete with respect to $p^* \mathrm{W}\Omega_R^n$ and to $V^* \mathrm{W}\Omega_R^n + dV^* \mathrm{W}\Omega_R^{n-1}$, it is complete with respect to $V^* \mathrm{W}\Omega_R^n$. Since V is injective, this is the same as derived V -completeness. By Proposition 3.38, $\mathrm{TR}_*(R)$ is also complete with respect to the derived version of the $(V + dV)$ -filtration. In fact, in this case, the derived and non-derived filtrations are pro-equivalent.

Corollary 6.20. *Let k be a perfect field of characteristic p and let S be an ind-smooth k -algebra. Then, $\mathrm{TR}_*(S) \cong \mathrm{W}\Omega_S^*$.*

Proof. Because TR does not commute with filtered colimits as a functor to \mathbf{TCart}_p , this theorem is not a formal consequence of Theorem 6.18. To correct this, we need to dig into the proof of [Hes96, Theorem C]. Hesselholt’s result follows from a finer result [Hes96, Theorem B], which says that if R is a smooth k -algebra, then $\pi_* \mathrm{THH}(R)^{C_{p^{n-1}}} \cong \mathrm{W}_n \Omega_R^*[b_n]$ and that moreover the map $R: \mathrm{THH}(R)^{C_{p^{n-1}}} \rightarrow \mathrm{THH}(R)^{C_{p^{n-2}}}$ sends b_n to pb_{n-1} (up to a unit). Passing up the tower, one obtains the computation of TR .

Now, each $\mathrm{THH}(-)^{C_{p^{n-1}}}$ commutes with sifted colimits of commutative rings and in particular filtered colimits. Suppose that $S \simeq \mathrm{colim}_{i \in I} S_i$ where I is a filtered category and each S_i is a smooth k -algebra. Then, we have

$$\pi_* \mathrm{THH}(S)^{C_{p^{n-1}}} \cong \mathrm{colim}_i \pi_* \mathrm{THH}(S_i)^{C_{p^{n-1}}} \cong \mathrm{colim}_i \mathrm{W}_n \Omega_{S_i}^*[b_n] \cong \mathrm{W}_n \Omega_S^*[b_n]$$

for each $n \geq 1$, as $\mathrm{W}_n \Omega_-^*$ commutes with filtered colimits (see [Ill79, I.1.10]). We still have that $R: \mathrm{THH}(S)^{C_{p^{n-1}}} \rightarrow \mathrm{THH}(S)^{C_{p^{n-2}}}$ sends b_n to pb_{n-1} up to a unit, so taking the limit gives the result. \square

Example 6.21. (1) By Popescu’s theorem, every regular noetherian \mathbb{F}_p -algebra is a filtered colimit of smooth \mathbb{F}_p -algebras. So, the theorem applies in particular to all regular noetherian \mathbb{F}_p -algebras.

- (2) We can say somewhat more in the special case when R is a filtered colimit of smooth k -algebras of a uniformly bounded dimension. For example, suppose that $K = k(x_1, \dots, x_d)$. Then, $\mathrm{THH}(K) \in (\mathbf{CycSp}_p)_{[0,d]}$. Indeed, in that case, each $\mathrm{W}_n \Omega_K^*$ is concentrated in degrees 0 to k and hence so is the limit.

- (3) If R is a smooth \mathbb{F}_p -algebra, then the theorem applies pro-étale locally on the pro-étale site of R . Indeed, for every weakly étale map $R \rightarrow S$ there is a faithfully flat ind-étale map $S \rightarrow T$ such that $R \rightarrow T$ is ind-étale. In particular, $k \rightarrow T$ is ind-smooth. See [BS15, Theorem 2.3.4]. We will use this below to compare the BMS filtration on TC with the filtration coming from the cyclotomic t -structure.

Corollary 6.22. *Let k be a perfect field of characteristic p and let X be a smooth quasi-compact k -scheme. There is a convergent spectral sequence*

$$E_2^{s,t} = H^s(X, W\Omega_X^t) \Rightarrow \pi_{t-s}^{\text{cyc}} \text{THH}(X)$$

in the abelian category of derived V -complete p -typical Cartier modules. If X has Krull dimension at most d , then $E_2^{s,t} = 0$ for $t > d$ and $s > d$.

Proof. Combine Proposition 6.12 and Theorem 6.18. In this case, the spectral sequence converges because it collapses at some finite stage for each connected component of X (above the dimension of the connected components). \square

Now, we can compare the BMS filtration and the cyclotomic t -structure filtrations on $\text{TC}(R)$ when $X = \text{Spec } R$ is a smooth affine scheme over a perfect field k . The BMS filtration is a decreasing multiplicative \mathbb{N} -indexed filtration $F_{\text{BMS}}^* \text{TC}(R)$ on $\text{TC}(R)$ with graded pieces given by $\text{gr}_{\text{BMS}}^n \text{TC}(R) \simeq \mathbb{Z}_p(n)(R)[2n] \simeq W\Omega_{X,\log}^n(R)[n]$ by [BMS19, Theorems 1.12 and 1.15]. It is defined in the following way. If S is a quasiregular semiperfect \mathbb{F}_p -algebra, then $\text{TC}^-(S)$ and $\text{TP}(S)$ are concentrated in even degrees. One sets $F_{\text{BMS}}^* \text{TC}^-(S) = \tau_{\geq 2*} \text{TC}^-(S)$ and $F_{\text{BMS}}^* \text{TP}(S) = \tau_{\geq 2*} \text{TP}(S)$. The canonical and Frobenius maps $\text{TC}^-(S) \rightarrow \text{TP}(S)$ preserve the BMS filtration on each side and hence define a filtration $F_{\text{BMS}}^* \text{TC}(S)$. One now constructs the BMS filtration on a general quasisyntomic \mathbb{F}_p -algebra by quasisyntomic descent.

Note that the filtrations $F_{\text{BMS}}^* \text{TC}^-(R)$ and $F_{\text{BMS}}^* \text{TP}(R)$ do not come from a cyclotomic filtration on $\text{THH}(R)$. Indeed, if $R = \mathbb{F}_p$ then $\text{gr}_{\text{BMS}}^n \text{TP}(\mathbb{F}_p) \simeq \mathbb{Z}_p[2n]$ whereas TP of any cyclotomic $\text{THH}(\mathbb{F}_p)$ -module will be 2-periodic. However, we will see that if R is smooth, then the filtration on $\text{TC}(R)$ does come from a cyclotomic filtration on $\text{THH}(R)$.

Consider the decreasing multiplicative \mathbb{N} -indexed filtration $\text{TC}(\tau_{\geq * }^{\text{cyc}} \text{THH}(R))$ on $\text{TC}(R)$. Since

$$\text{TC}(\tau_{\geq * }^{\text{cyc}} \text{THH}(R)) \simeq \text{fib} \left(\tau_{\geq * } \text{TR}(R) \xrightarrow{1-F} \tau_{\geq * } \text{TR}(R) \right),$$

we see from Theorem 6.18 that the graded pieces are given by

$$\text{fib} \left(W\Omega_R^n[n] \xrightarrow{1-F} W\Omega_R^n[n] \right) \simeq \text{fib} \left(W\Omega_R^n \xrightarrow{1-F} W\Omega_R^n \right) [n].$$

Theorem 6.23. *Let $X = \text{Spec } R$ be a smooth affine scheme over a perfect field k of characteristic p . There is a natural equivalence $\text{TC}(\tau_{\geq * }^{\text{cyc}} \text{THH}(R)) \rightarrow F_{\text{BMS}}^* \text{TC}(R)$ of filtered spectra.*

Proof. The BMS filtration satisfies pro-étale descent, by definition, since it satisfies quasisyntomic descent. In other words, the assignment which sends an pro-étale R -algebra S to $F_{\text{BMS}}^* \text{TC}(S)$ is a pro-étale sheaf. In fact, it is hypercomplete as it is bounded above on $\text{CAlg}_R^{\text{proét}}$. Indeed, if R has Krull dimension d , then for any pro-étale R -algebra S , $F_{\text{BMS}}^* \text{TC}(S)$ is d -truncated. Moreover, pro-étale-locally, the BMS filtration reduces to the Whitehead tower on TC by [BMS19, Theorem 1.15]. Thus, to map in to $F_{\text{BMS}}^* \text{TC}(R)$ it is enough to prove that pro-étale locally $\text{TC}(\tau_{\geq n}^{\text{cyc}} \text{THH}(R))$ is n -connective. For this, we need to know that $W\Omega_X^n \xrightarrow{1-F} W\Omega_X^n$ is pro-étale-locally surjective. As shown in the proof of [BMS19, Proposition 8.4], this follows from a result of Illusie [Ill79, Théorème I.5.7.2]. Moreover, they show that the kernel (in pro-étale sheaves) is $W\Omega_{X,\log}^n$. This

proves both that there is a map of filtered objects $\mathrm{TC}(\tau_{\geq \star}^{\mathrm{cyc}} \mathrm{THH}(R)) \rightarrow \mathrm{F}_{\mathrm{BMS}}^{\star} \mathrm{TC}(R)$ and that the map on graded pieces is the natural map

$$\mathrm{fib} \left(\mathrm{W}\Omega_R^n \xrightarrow{1-F} \mathrm{W}\Omega_R^n \right) \rightarrow \mathrm{W}\Omega_{X, \log}^n(R),$$

which is an equivalence (see for example [GH99, Lemma 4.1.3] and the following discussion). \square

6.3 Crystalline cohomology and TP

Let k be a perfect ring of characteristic p . Let R be a smooth commutative k -algebra. We show in this section how to use Theorem 6.18 to extract the de Rham–Witt complex from $\mathrm{TP}(R)$.

Recall that when R is smooth over a perfect field k , then the crystalline cohomology $\mathrm{R}\Gamma_{\mathrm{crys}}(R/W(k))$ has a canonical cochain complex model given by the de Rham–Witt complex

$$0 \rightarrow \mathrm{W}\Omega_R^0 \rightarrow \mathrm{W}\Omega_R^1 \rightarrow \cdots,$$

which is a Dieudonné complex in the sense of Definition 6.17.

Our next theorem complements a result of Bhatt, Morrow, and Scholze. They prove in [BMS19, Theorem 1.10] that if R is a smooth k -algebra for a perfect field k , then there is a filtration $\mathrm{F}_{\mathrm{BMS}}^{\star} \mathrm{TP}(R)$ with graded pieces

$$\mathrm{gr}_{\mathrm{BMS}}^n \mathrm{TP}(R) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(R/W(k))[2n].$$

Moreover, each $\mathrm{F}_{\mathrm{BMS}}^n \mathrm{TP}(R)$ is itself equipped with a filtration and the maps $\mathrm{F}_{\mathrm{BMS}}^{n+1} \mathrm{TP}(R) \rightarrow \mathrm{F}_{\mathrm{BMS}}^n \mathrm{TP}(R)$ are compatible with this secondary filtration. On graded pieces, one obtains the Nygaard filtration on $\mathrm{R}\Gamma_{\mathrm{crys}}(R/W(k))[2n]$.

We recover the BMS filtration on $\mathrm{TP}(R)$ in the next theorem by using the cyclotomic t -structure. Our filtration also comes equipped with a secondary filtration as well, but this time the induced filtration on the graded pieces is the Hodge, or Hodge–Witt, filtration, which lets us say that the graded pieces are given by shifts of the de Rham–Witt complex, as opposed to an object of the derived ∞ -category equivalent to the de Rham–Witt complex.

Theorem 6.24. *Let k be a perfect field of characteristic p and let R be an ind-smooth k -algebra and let $\mathrm{TP}(\tau_{\geq \star}^{\mathrm{cyc}} \mathrm{THH}(R))$ be the filtration on $\mathrm{TP}(R)$ induced by the cyclotomic Whitehead tower. The induced Whitehead tower with respect to the Beilinson t -structure on filtered spectra defines a natural complete exhaustive multiplicative decreasing \mathbb{Z} -indexed filtration $\mathrm{F}_{\mathrm{B}}^{\star} \mathrm{TP}(R)$ in filtered spectra with graded pieces given by*

$$\mathrm{gr}_{\mathrm{B}}^i \mathrm{TP}(R) \simeq \mathrm{W}\Omega_R^{\bullet}[2i]$$

for all i . Moreover, this filtration agrees with the BMS filtration on $\mathrm{TP}(R)$ after forgetting the secondary filtrations.

Proof. Applying the S^1 -Tate construction to the cyclotomic Whitehead tower of $\mathrm{THH}(R)$, we obtain a complete decreasing \mathbb{N} -indexed filtration $(\tau_{\geq \star}^{\mathrm{cyc}} \mathrm{THH}(R))^{tS^1}$ on $\mathrm{TP}(R)$. The graded pieces are

$$(\pi_{\star}^{\mathrm{cyc}} \mathrm{THH}(R)[\star])^{tS^1} \simeq (\mathrm{W}\Omega_R^{\star}[\star])^{tS^1}$$

by Corollaries 10 and 6.20. We let $\mathrm{F}_{\mathrm{B}}^{\star} \mathrm{TP}(R)$ denote the double-speed Whitehead tower with respect to the Beilinson t -structure (see [BMS19, Section 5]) on filtered spectra associated to the filtered spectrum $(\tau_{\geq \star}^{\mathrm{cyc}} \mathrm{THH}(R))^{tS^1}$. This is a complete exhaustive \mathbb{Z} -indexed filtration on $\mathrm{TP}(R)$ (see for example [Ant19, Lemma 3.2]). Moreover, $\mathrm{gr}_{\mathrm{B}}^n \mathrm{TP}(R)[-2n]$ is an object of the heart of $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{D}(W(\mathbb{F}_p)))$, the ∞ -category of

filtered objects of the derived ∞ -category of $W(\mathbb{F}_p)$. Thus, $\mathrm{gr}_B^n \mathrm{TP}(R)[-2n]$ is canonically a cochain complex; it is not hard to see that it is of the form

$$0 \rightarrow W\Omega_R^0 \rightarrow W\Omega_R^1 \rightarrow \cdots,$$

where the differential is Connes' B -operator. By Theorem 6.18, the B -operator is given by the differential in the de Rham–Witt complex. This completes the proof of the first part of the theorem. For more details, see [Ant19, Example 2.4].

To continue, we must produce a map $F_B^* \mathrm{TP}(R) \rightarrow F_{\mathrm{BMS}}^* \mathrm{TP}(R)$ of filtered spectra. To do so, we first Kan extend so that we can compare in the quasiregular semiperfect case, and then we descend back to the smooth case.

As in the first paragraph, if R is smooth over \mathbb{F}_p , we can produce a complete filtration $F_B^* \mathrm{TR}(R)_{hS^1}$ on $\mathrm{TR}(R)_{hS^1}$ with graded pieces $\mathrm{gr}_B^n \mathrm{TR}(R)_{hS^1} \simeq W\Omega_R^{\bullet \leq n}[2n]$. The natural map $\mathrm{TP}(R) \rightarrow \mathrm{TR}(R)_{hS^1}[2]$ induces a filtered map $F_B^* \mathrm{TP}(R) \rightarrow F_B^* \mathrm{TR}(R)_{hS^1}[2]$ which on graded pieces is the natural quotient $W\Omega_R^{\bullet \leq n}[2n] \rightarrow W\Omega_R^{\bullet \leq n-1}[2n]$ of chain complexes.

Fix a generator $v \in \pi_{-2} \mathbb{Z}^{hS^1} \cong \mathbb{Z}$. Recall that if $X \in \mathrm{Fun}(BS^1, D(\mathbb{Z}))$ is a chain complex with S^1 -action, then X_{hS^1} is a module over \mathbb{Z}^{hS^1} and the natural map $X^{tS^1} \rightarrow X_{hS^1}[2]$ induces an equivalence

$$X^{tS^1} \simeq \lim \left(\cdots \rightarrow X_{hS^1}[2n-2] \xrightarrow{v} X_{hS^1}[2n] \xrightarrow{v} X_{hS^1}[2n+2] \rightarrow \cdots \right).$$

In the case of $\mathrm{TR}(R)_{hS^1}$, v induces a filtered map $F_B^* \mathrm{TR}(R)_{hS^1} \rightarrow F_B^{*-1} \mathrm{TR}(R)_{hS^1}[2]$ which on graded pieces identifies with the quotient map $W\Omega^{\bullet \leq n}[2n] \rightarrow W\Omega^{\bullet \leq n-1}[2n]$.

For a general object $X \in \mathrm{Mod}_{\mathbb{Z}^{hS^1}}$, we let $T_v X = \lim \left(\cdots \rightarrow X[2n-2] \xrightarrow{v} X[2n] \xrightarrow{v} X[2n+2] \rightarrow \cdots \right)$. This construction defines a functor $\mathrm{Mod}_{\mathbb{Z}^{hS^1}} \rightarrow \mathrm{Mod}_{\mathbb{Z}^{tS^1}}$.

Let $\mathrm{LTP}(-)$ denote the left Kan extension of $\mathrm{TP}(-)$ from polynomial \mathbb{F}_p -algebras to all simplicial commutative \mathbb{F}_p -algebras as a functor with values in $\mathrm{Mod}_{\mathbb{Z}^{tS^1}}$. We let $\mathrm{LTR}(-)$ denote the left Kan extension of TR as a functor with values in $\mathrm{Fun}(BS^1, D(\mathbb{Z}))$. We let $\mathrm{L}(\mathrm{TR}(-)_{hS^1})$ be the left Kan extension of $\mathrm{TR}(-)_{hS^1}$ as a functor with values in $D(\mathbb{Z}^{hS^1})$. Since taking homotopy orbits commutes with colimits, there is an equivalence of functors $(\mathrm{LTR}(-))_{hS^1} \simeq \mathrm{L}(\mathrm{TR}(-)_{hS^1})$. We write $\mathrm{LTR}(-)_{hS^1}$ for this common functor.

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{LTP}(S) & \longrightarrow & T_v \mathrm{LTR}(S)_{hS^1} \\ \downarrow & & \downarrow \\ \mathrm{TP}(S) & \xlongequal{\quad} & T_v \mathrm{TR}(S)_{hS^1} \end{array}$$

in $\mathrm{Mod}_{\mathbb{Z}^{tS^1}}$. The spectrum $\mathrm{LTP}(S)$ carries the Kan extended Beilinson filtration $F_B^* \mathrm{LTP}(S)$ with graded pieces $\mathrm{LW}\Omega_S$. The spectrum $T_v \mathrm{TR}(S)_{hS^1}$ carries the inverse limit of the Kan extended Beilinson filtration on $\mathrm{TR}(S)_{hS^1}$. This is a filtration $F_B^* T_v \mathrm{LTR}(S)_{hS^1}$ with graded pieces given by $\widehat{\mathrm{LW}}\Omega_S$, the completion of $\mathrm{LW}\Omega_S$ with respect to the Hodge filtration $\mathrm{LW}\Omega_S^{\geq *}$.

The Kan extended filtration $F^* \mathrm{TR}(S)_{hS^1}$ is complete. Indeed, if S is smooth, $\mathrm{gr}_B^n \mathrm{TR}(S)_{hS^1} \simeq W\Omega^{\leq n}[2n]$ is in $D(\mathbb{Z})_{[n, 2n]}$. Thus, since the filtration is complete in this case, $F_B^n \mathrm{TR}(S)_{hS^1}$ is in $D(\mathbb{Z})_{\geq n}$. Hence, the Kan extension $F^n \mathrm{LTR}(S)_{hS^1}$ is in $D(\mathbb{Z})_{\geq n}$. In particular, the Kan extended Beilinson filtration is complete on $\mathrm{TR}(S)_{hS^1}$ for any S . Since $F_B^* T_v \mathrm{TR}(S)_{hS^1}$ is an inverse limit of complete filtrations, it is complete.

It follows that the map $F_B^* \mathrm{LTP}(S) \rightarrow F_B^* T_v \mathrm{LTR}(S)_{hS^1}$ factors through the completion $\widehat{\mathrm{LTP}}(S)$ of $\mathrm{LTP}(S)$ with respect to the Beilinson filtration. On $\widehat{\mathrm{LTP}}(S)$ we have the completed Beilinson filtration $F_B^* \widehat{\mathrm{LTP}}(S)$ with graded pieces $\mathrm{gr}_B^n \widehat{\mathrm{LTP}}(S) \simeq \mathrm{LW}\Omega_S[2n]$.

We thus have obtained a map $\widehat{\text{LTP}}(S) \rightarrow \text{TP}(S)$ for any commutative ring S . If S is quasiregular semiperfect, then $\text{LW}\Omega_S[2n]$ is p -adically concentrated in degree $2n$ (see [BMS19, Theorem 8.14]). The completeness of the filtration implies that $\widehat{\text{F}}_{\mathbb{B}}^n \widehat{\text{LTP}}(S)$ is p -adically $2n$ -connective. Thus, since $\text{TP}(S)$ is p -complete, the map $\widehat{\text{LTP}}(S) \rightarrow \text{TP}(S)$ automatically upgrades to a filtered map $\widehat{\text{F}}_{\mathbb{B}}^* \widehat{\text{LTP}}(S) \rightarrow \tau_{\geq 2*} \text{TP}(S) \simeq \text{F}_{\text{BMS}}^* \text{TP}(S)$.

Now, by quasisyntomic descent, it follows that there exists a natural map $\text{F}_{\mathbb{B}}^* \text{TP}(R) \rightarrow \text{F}_{\text{BMS}}^* \text{TP}(R)$ for any smooth \mathbb{F}_p -algebra R . Standard arguments using the Künneth isomorphism in crystalline cohomology and étale descent reduce us to checking that it is an equivalence for $R = \mathbb{F}_p[x]$, the ring of functions on $\mathbb{A}_{\mathbb{F}_p}^1$. But, one sees in this case that since the crystalline cohomology is concentrated in (homological) degrees 0 and -1 , both filtrations satisfy $\text{F}_{\mathbb{B}}^n \text{TP}(R) \simeq \tau_{\geq 2n-1} \text{TP}(R) \simeq \text{F}_{\text{BMS}}^n \text{TP}(R)$, so we are done. \square

Remark 6.25. We expect that there is a common refinement of the Beilinson and BMS filtrations on $\text{TP}(R)$ when R is smooth over a perfect field k of characteristic p . This would say that there is a filtration $\text{F}^* \text{TP}(R)$ on $\text{TP}(R)$ where each $\text{F}^n \text{TP}(R)$ is a bifiltered spectrum with one filtration reducing to the Nygaard filtration on the graded pieces and the other reducing to the de Rham–Witt filtration on the graded pieces.

In conclusion, we will briefly discuss three spectral sequences computing $\text{TP}(X)$ when X is smooth and proper over a perfect ring k of characteristic p : the Hesselholt spectral sequence, the Bhatt–Morrow–Scholze spectral sequence (or BMS spectral sequence for short), and a new spectral sequence arising from the cyclotomic Postnikov tower of $\text{THH}(X)$.

Hesselholt’s spectral sequence, which is given in [Hes18, Theorem 6.8] when k is a perfect field of characteristic p , is obtained Zariski locally from the inverse limit of the Postnikov filtrations of $\text{THH}(\mathcal{O}_X)^{tC_p^n}$. It has the form

$$E_{i,j}^2 = \bigoplus_{m \in \mathbb{Z}} \lim_{n,F} H^{-i}(X, W_n \Omega_X^{j+2m}) \Rightarrow \text{TP}_{i+j}(X). \quad (26)$$

The differentials d^r have bidegree $(-r, r-1)$. Note that, as explained in [Hes18, Section 5], the terms in this spectral sequence arise also in the conjugate spectral sequence

$$E_2^{s,t} = \lim_{n,F} H^s(X, W_n \Omega_X^t) \Rightarrow H_{\text{crys}}^{s+t}(X/W(k)) \quad (27)$$

computing the crystalline cohomology of X over $W(k)$. The point is that $\lim_{n,F} H^s(X, W_n \Omega_X^t)$ is isomorphic to $H^s(X, \mathcal{H}_{\text{crys}}^t)$ by the Cartier isomorphism, where $\mathcal{H}_{\text{crys}}^t$ is the Zariski sheafification of $U \mapsto H_{\text{crys}}^t(U/W(k))$.

The BMS spectral sequence arises because of a quasisyntomic-local filtration on TP . The filtration then has n th graded piece

$$\text{gr}_{\text{BMS}}^n \text{TP}(X) \simeq \text{R}\Gamma_{\text{crys}}(X/W(k))\{n\}[2n].$$

The twist $\{n\}$ is used in [BMS19] to keep track of the twist of the action of Frobenius. In this case, it means that the action of Frobenius is given by p^{-n} times the action of Frobenius on $\text{R}\Gamma_{\text{crys}}(X/W(k))$. The spectral sequence of the filtration then takes the form

$$E_2^{s,t} = H_{\text{crys}}^{s-t}(X/W(k))\{-t\} \Rightarrow \text{TP}_{-s-t}(X) \quad (28)$$

(see [BMS19, Theorem 1.12]).

Our spectral sequence for TP is induced from the cyclotomic Whitehead tower. Specifically, the cyclotomic Postnikov tower gives a complete exhaustive decreasing multiplicative \mathbb{N} -indexed filtration $\text{F}_{\text{cyc}}^t \text{THH}(X) = \tau_{\geq t}^{\text{cyc}} \text{THH}(X)$ of $\text{THH}(X)$ with associated graded pieces $\text{gr}_{\text{cyc}}^t \text{THH}(X) \simeq \text{R}\Gamma(X, W\Omega^t)[t]$.

Theorem 6.26. *Let X be a smooth and quasi-compact k -scheme where k is a perfect field of characteristic p . There is a complete decreasing multiplicative \mathbb{N} -indexed filtration $\text{F}_{\text{cyc}}^* \text{TP}(X)$ on $\text{TP}(X)$ with associated*

graded $\mathrm{gr}_{\mathrm{cyc}}^j \mathrm{TP}(X) \simeq (\pi_j^{\mathrm{cyc}} \mathrm{THH}(X)[j])^{tS^1}$. The associated multiplicative and conditionally convergent²¹ spectral sequence takes the form

$$E_{i,j}^1 = \pi_* \left(\mathrm{R}\Gamma(X, \mathrm{W}\Omega_X^j) \right) [t^\pm] \Rightarrow \mathrm{TP}_{i+j}(X). \quad (29)$$

where the bidegree of t is $(-2, 0)$ and the elements in $\pi_i(\mathrm{R}\Gamma(X, \mathrm{W}\Omega_X^j))$ sit in bidegree (i, j) .

Proof. It follows immediately from Theorem 6.18 and Corollary 10 that we get a spectral sequence of the form

$$E_{i,j}^1 = \pi_i \left(\mathrm{R}\Gamma(X, \mathrm{W}\Omega_X^j)^{tS^1} \right) \Rightarrow \mathrm{TP}_{i+j}(X).$$

Now the action of S^1 on $\mathrm{W}\Omega_X^j \simeq \pi_j \mathrm{TR}(\mathcal{O}_X)$ is trivial since there is just a single homotopy groups. Thus the action on the derived sections is trivial as well (in fact, canonically trivialized) and the Tate spectral sequence for $\mathrm{R}\Gamma(X, \mathrm{W}\Omega_X^j)^{tS^1}$ degenerates for all j . Moreover the trivialization is multiplicative which gives the result. \square

Remark 6.27. Theorem 6.24 implies that the spectral sequence (29) agrees with (28) when X is affine from the E_2 -page forward (up to reindexing). In general, they do not agree on the E_2 -page.

Remark 6.28. If X is smooth and proper over a perfect field k of characteristic p , then the Hesselholt, BMS, and cyclotomic spectral sequences for TP degenerate rationally. To prove degeneration of the spectral sequences for TP in the smooth proper case, it is enough to check it for one spectral sequence since rationally all three spectral sequences start with the same ranks contributing to $\mathrm{TP}_n(X)$ (using rational degeneration of the Hodge and conjugate spectral sequences for crystalline cohomology due to [IR83]). However, an argument of Scholze proves rational degeneration of the BMS spectral sequence for smooth *affine* schemes over perfect fields (see [Elm18]). This comes from a canonical splitting due to the action Adams operations and thus extends to smooth proper schemes, giving the desired degeneration.

A Background on t -structures

For further background on t -structures [Lur17, Chapter 1] or the original source [BBD82]. For prestable and especially Grothendieck prestable ∞ -categories, see [Lur18, Appendix C].

A.1 Right complete t -structures

The purpose of this section is to introduce some terminology on t -structures and especially right complete t -structures.

Definition A.1. Let \mathcal{C} be a stable ∞ -category. A **t -structure** on \mathcal{C} is a pair $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ of full subcategories of \mathcal{C} such that

- (1) $\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}[-1] \subseteq \mathcal{C}_{\leq 0}$;
- (2) if $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}$, then the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y[-1])$ is contractible;
- (3) for every $X \in \mathcal{C}$ there is a fiber sequence $\tau_{\geq 0} X \rightarrow X \rightarrow \tau_{\leq -1} X$ where $\tau_{\geq 0} X \in \mathcal{C}$ and $\tau_{\leq -1} X[1] \in \mathcal{C}_{\leq 0}$.

Remarks A.2. (a) It is useful to let $\mathcal{C}_{\geq n} = \mathcal{C}_{\geq 0}[n]$ and $\mathcal{C}_{\leq n} = \mathcal{C}_{\leq 0}[n]$ for integers n .

²¹The filtration might in general not be finite. But if the scheme is of finite dimension it is, so that the spectral sequence is strongly convergent.

- (b) A t -structure is determined by either $\mathcal{C}_{\geq 0}$ or $\mathcal{C}_{\leq 0}$. In other words, if $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is a t -structure on \mathcal{C} , then $\mathcal{C}_{\leq 0}$ is the full subcategory of objects $Y \in \mathcal{C}$ such that $\mathrm{Map}_{\mathcal{C}}(X, Y[-1]) \simeq 0$ for all $X \in \mathcal{C}_{\geq 0}$; similarly, $\mathcal{C}_{\geq 0}$ is the left orthogonal to $\mathcal{C}_{\leq -1}$. Therefore, we will often think of a t -structure as a pair consisting of a stable ∞ -category \mathcal{C} and a full subcategory $\mathcal{C}_{\geq 0}$ such that $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ defines a t -structure in the sense of Definition A.1, where $\mathcal{C}_{\leq 0}$ is the right orthogonal to $\mathcal{C}_{\geq 1}$.
- (c) If $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is a t -structure on \mathcal{C} , then $\mathcal{C}_{\geq 0}$ is often called the **aisle** and $\mathcal{C}_{\leq 0}$ the co-aisle. In this case, $\mathcal{C}_{\geq 0}$ is an example of a **prestable ∞ -category**, a notion studied in [Lur18, Appendix C] that abstracts the properties of the ∞ -category of connective spectra.
- (d) The truncations of condition (3) are functorial: the inclusion $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ admits a right adjoint $\tau_{\geq 0}$, and $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ admits a left adjoint $\tau_{\leq 0}$. For $X \in \mathcal{C}$, $\tau_{\geq 0}X \rightarrow X$ is the counit of the adjunction for $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ and $X \rightarrow \tau_{\leq 0}X$ is the unit for the adjunction for $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$.
- (e) The **heart** of a t -structure is the full subcategory $\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ of \mathcal{C} . It is an abelian category, by [BBD82]. It also coincides with the full subcategory of 0-truncated objects in $\mathcal{C}_{\geq 0}$, by [Lur17, 1.2.1.9].

Definition A.3. Let \mathcal{C}, \mathcal{D} be stable ∞ -categories equipped with t -structures $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$, respectively. An exact functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is **right t -exact** if $F(X) \in \mathcal{D}_{\geq 0}$ whenever $X \in \mathcal{C}_{\geq 0}$. Similarly, the exact functor F is **left t -exact** if $F(X) \in \mathcal{D}_{\leq 0}$ whenever $X \in \mathcal{C}_{\leq 0}$. An exact functor F is **t -exact** if it is left and right t -exact.

Now, we review some essentially well-known facts about right completions of t -structures.

Definition A.4. Let \mathcal{C} be a stable ∞ -category equipped with a t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$.

- We say that \mathcal{C} is **right separated** if the full subcategory of ∞ -coconnective objects

$$\bigcap_n \mathcal{C}_{\leq n}$$

is contractible.

- We say that \mathcal{C} is **right complete** if the natural map

$$\mathcal{C} \rightarrow \lim_n \mathcal{C}_{\geq n} \simeq \lim \left(\cdots \rightarrow \mathcal{C}_{\geq n} \xrightarrow{\tau_{\geq n+1}} \mathcal{C}_{\geq n+1} \rightarrow \cdots \right)$$

is an equivalence.

Remark A.5. **Left separated** and **left complete** t -structures are defined in the analogous way.

Lemma A.6. *Let \mathcal{C} be a stable ∞ -category with a t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Assume that \mathcal{C} admits countable coproducts and that $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ is closed under countable coproducts. Then, \mathcal{C} is right complete if and only if it is right separated.*

Proof. This is the right complete version of [Lur17, 1.2.1.19]. □

Definition A.7. If \mathcal{C} is a stable ∞ -category with a t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, then the ∞ -category

$$\lim \left(\cdots \rightarrow \mathcal{C}_{\geq n} \xrightarrow{\tau_{\geq n+1}} \mathcal{C}_{\geq n+1} \rightarrow \cdots \right)$$

is the **right completion** of \mathcal{C} .

Lemma A.8. *If \mathcal{C} is a stable ∞ -category with a t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, then the right completion is naturally equivalent to $\mathrm{Sp}(\mathcal{C}_{\geq 0})$, the ∞ -category of spectrum objects in $\mathcal{C}_{\geq 0}$.*

Proof. Write ι_{n+1} for the inclusion $\mathcal{C}_{\geq n+1} \subseteq \mathcal{C}_{\geq n}$. In other words, ι_{n+1} is the left adjoint to $\tau_{\geq n+1}$. We may rewrite the limit

$$\lim \left(\cdots \rightarrow \mathcal{C}_{\geq n} \xrightarrow{\tau_{\geq n+1}} \mathcal{C}_{\geq n+1} \rightarrow \cdots \right) \simeq \lim \left(\cdots \rightarrow \mathcal{C}_{\geq -2} \xrightarrow{\tau_{\geq -1}} \mathcal{C}_{\geq -1} \xrightarrow{\tau_{\geq 0}} \mathcal{C}_{\geq 0} \right)$$

as

$$\lim \left(\cdots \rightarrow \mathcal{C}_{\geq 0} \xrightarrow{\Omega} \mathcal{C}_{\geq 0} \xrightarrow{\Omega} \mathcal{C}_{\geq 0} \right). \quad (30)$$

If \mathcal{D} is an ∞ -category with finite limits, Lurie defines $\mathrm{Sp}(\mathcal{D})$ in [Lur17, 1.4.2.8] as the full subcategory of reduced, excisive functors $\mathcal{S}_*^{\mathrm{fin}} \rightarrow \mathcal{C}_{\geq 0}$, where $\mathcal{S}_*^{\mathrm{fin}}$ is the full subcategory of pointed spaces on those spaces that can be built out of finite colimits from a point.²² Lurie later shows, in [Lur17, 1.4.2.24], that $\mathrm{Sp}(\mathcal{D})$ is equivalent to the limit $\cdots \rightarrow \mathcal{D} \xrightarrow{\Omega} \mathcal{D} \xrightarrow{\Omega} \mathcal{D}$. Since $\mathcal{C}_{\geq 0}$ has finite limits, we see that $\mathrm{Sp}(\mathcal{C}_{\geq 0})$ is the right completion, as claimed. \square

The right completion $\mathrm{Sp}(\mathcal{C}_{\geq 0})$ can be identified with the full subcategory of $\mathrm{Fun}(\mathbb{Z}, \mathcal{C})$ consisting of those functors $X(\star): \mathbb{Z} \rightarrow \mathcal{C}$ such that $X(m) \in \mathcal{C}_{\geq m}$ for all m and $X(m) \rightarrow X(n)$ induces an equivalence $\tau_{\geq n} X(m) \simeq X(n)$ for $m \leq n$. Let $\mathrm{Sp}(\mathcal{C}_{\geq 0})_{\geq 0} \subseteq \mathrm{Sp}(\mathcal{C}_{\geq 0})$ denote the full subcategory of those objects $X(\star)$ where, additionally, $X(m) \in \mathcal{C}_{\geq 0}$ for all m . Let $\mathrm{Sp}(\mathcal{C}_{\geq 0})_{\leq 0}$ be the full subcategory of those sequences $X(\star)$ in $\mathrm{Sp}(\mathcal{C}_{\geq 0})$ where, additionally, $X(m) \in \mathcal{C}_{\leq 0}$ for all m .

Lemma A.9. *Let \mathcal{C} be a stable ∞ -category with a t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$.*

- (a) *The right completion $\mathrm{Sp}(\mathcal{C}_{\geq 0})$ is stable.*
- (b) *The full subcategories $(\mathrm{Sp}(\mathcal{C}_{\geq 0})_{\geq 0}, \mathrm{Sp}(\mathcal{C}_{\geq 0})_{\leq 0})$ define a right complete t -structure on $\mathrm{Sp}(\mathcal{C}_{\geq 0})$.*
- (c) *The natural functor $\mathrm{Sp}(\mathcal{C}_{\geq 0}) \rightarrow \mathcal{C}$ is right t -exact (and in particular exact).*
- (d) *The functor in (c) induces an equivalence $\mathcal{C}_{\geq 0} \simeq \mathrm{Sp}(\mathcal{C})_{\geq 0}$.*

Proof. Part (a) follows from [Lur17, 1.4.2.21] since $\mathcal{C}_{\geq 0}$ has finite limits. The remainder is a right complete version of [Lur17, 1.2.1.17]. \square

A.2 Compatibility with symmetric monoidal structures

In this section, we discuss the interaction of t -structures and tensor products.

Definition A.10. Let \mathcal{C} be a stable ∞ -category equipped with a t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and a symmetric monoidal structure \mathcal{C}^{\otimes} . We say that the t -structure is **compatible** with the symmetric monoidal structure if the following conditions hold:

- (i) the tensor product $\mathcal{C} \times \mathcal{C} \xrightarrow{-\otimes-} \mathcal{C}$ is exact in each variable;
- (ii) the unit $\mathbb{1}_{\mathcal{C}}$ is in $\mathcal{C}_{\geq 0}$;
- (iii) $X \otimes Y \in \mathcal{C}_{\geq 0}$ whenever $X, Y \in \mathcal{C}_{\geq 0}$.

Example A.11. The Postnikov t -structure on the ∞ -category of spectra is compatible with the smash product symmetric monoidal structure. Indeed, the tensor product commutes with all colimits in each variable, the sphere spectrum is connective, and the smash product of connective spectra is connective.

²²Thus, the idempotent completion of $\mathcal{S}_*^{\mathrm{fin}}$ is \mathcal{S}_*^{ω} .

In general, if $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is a t -structure compatible with \mathcal{C}^{\otimes} , then the ∞ -category $\mathcal{C}_{\geq 0}$ and the fully faithful inclusion functor $\mathcal{C}_{\geq 0} \rightarrow \mathcal{C}$ inherit unique t -structures. For example, $\mathcal{C}_{\geq 0}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is the full subcategory spanned by the objects $(X_1, \dots, X_n) \in \mathcal{C}_{\langle n \rangle}^{\otimes} \simeq \mathcal{C}^n$ such that $X_i \in \mathcal{C}_{\geq 0}$ for each i .

Lemma A.12. *Let \mathcal{C} be a stable ∞ -category equipped with a symmetric monoidal structure \mathcal{C}^{\otimes} and a t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ that is compatible with \mathcal{C}^{\otimes} . Then, there is a unique symmetric monoidal structure on \mathcal{C}^{\heartsuit} and a unique symmetric monoidal structure on the functor $\pi_0: \mathcal{C}_{\geq 0} \rightarrow \mathcal{C}^{\heartsuit}$.*

Proof. The proof is the same as that of [NS18, Theorem I.3.6] since $\mathcal{C}_{\geq 0} \rightarrow \mathcal{C}^{\heartsuit}$ is a localization at the class W of maps $X \rightarrow Y$ such that $\pi_0 X \rightarrow \pi_0 Y$ is an isomorphism in \mathcal{C}^{\heartsuit} . We just have to check that if $f: X \rightarrow Y$ is in the class W and if Z is an arbitrary object of $\mathcal{C}_{\geq 0}$, then $X \otimes Z \xrightarrow{f \otimes \text{id}_Z} Y \otimes Z$ is in W . This is a trivial consequence of the next lemma. \square

Lemma A.13. *In the situation of Lemma A.12, for any pair $X, Y \in \mathcal{C}_{\geq 0}$, the natural map $\pi_0(X \otimes Y) \rightarrow \pi_0(\pi_0 X \otimes \pi_0 Y)$ is an isomorphism in \mathcal{C}^{\heartsuit} .*

Proof. To see this, consider the 3×3 diagram obtained by tensoring $\tau_{\geq 1} X \rightarrow X \rightarrow \pi_0 X$ with the corresponding sequence for Y . We get a commutative diagram

$$\begin{array}{ccccc}
 \tau_{\geq 1} X \otimes \tau_{\geq 1} Y & \longrightarrow & X \otimes \tau_{\geq 1} Y & \longrightarrow & \pi_0 X \otimes \tau_{\geq 1} Y \\
 \downarrow & & \downarrow & & \downarrow \\
 \tau_{\geq 1} X \otimes Y & \longrightarrow & X \otimes Y & \longrightarrow & \pi_0 X \otimes Y \\
 \downarrow & & \downarrow & & \downarrow \\
 \tau_{\geq 1} X \otimes \pi_0 Y & \longrightarrow & X \otimes \pi_0 Y & \longrightarrow & \pi_0 X \otimes \pi_0 Y,
 \end{array}$$

where each row and column is a fiber sequence in \mathcal{C} . Any term with at least one $\tau_{\geq 1}$ is in $\mathcal{C}_{\geq 1}$. Hence, applying π_0 , we see using the bottom right four objects that

$$\pi_0(X \otimes Y) \simeq \pi_0(\pi_0 X \otimes \pi_0 Y),$$

as desired. \square

Definition A.14. We say that a t -structure on \mathcal{C} is **compatible with countable products** if \mathcal{C} admits countable products and $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ is closed under countable products. Similarly, we say that \mathcal{C} is **compatible with countable coproducts** if \mathcal{C} admits countable coproducts and $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ is closed under countable coproducts. We also say that a t -structure on \mathcal{C} is **compatible with products** if \mathcal{C} admits products and $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ is closed under products.

For the next proposition, let P be one of the following properties of a t -structure: left separated, right separated, compatible with countable products, or compatible with products.

Proposition A.15. *Let \mathcal{C} be a presentably symmetric monoidal stable ∞ -category equipped with a compatible accessible t -structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Let $A \in \text{Alg}_{\mathbb{E}_1}(\mathcal{C}_{\geq 0})$.*

- (1) *Then, $\text{Mod}_A(\mathcal{C})$ admits an accessible t -structure with $\text{Mod}_A(\mathcal{C})_{\geq 0} \simeq \text{Mod}_A(\mathcal{C}_{\geq 0})$.*
- (2) *If the t -structure on \mathcal{C} satisfies property P , then so does the t -structure on $\text{Mod}_A(\mathcal{C})$.*
- (3) *There is a natural equivalence $\text{Mod}_A(\mathcal{C})^{\heartsuit} \simeq \text{Mod}_{\pi_0 A}(\mathcal{C}^{\heartsuit})$.*

- (4) If A is in $\text{Alg}_{\mathbb{S}\mathbb{E}_\infty}(\mathcal{C}_{\geq 0})$, then $\text{Mod}_A(\mathcal{C})$ admits a presentably symmetric monoidal structure and the t -structure is compatible with the symmetric monoidal structure.

Proof. For (1) it is enough by [Lur17, 1.4.4.11] to note that $\text{Mod}_A(\mathcal{C}_{\geq 0}) \subseteq \text{Mod}_A(\mathcal{C})$ are presentable (see [Lur17, 4.2.3.7]), the inclusion preserves colimits, and that $\text{Mod}_A(\mathcal{C})$ is closed under extensions in $\text{Mod}_A(\mathcal{C})$. For (2), we will use that the forgetful functor $\mathcal{C} \leftarrow \text{Mod}_A(\mathcal{C})$ is conservative and t -exact. Indeed, right t -exactness follows by definition and left t -exactness follows from the fact that the forgetful functor is right adjoint to the right t -exact extension of scalars functor. Now, (2) for left or right separatedness follows from conservativity and t -exactness. For compatibility with (countable) products, we use conservativity, t -exactness, and preservation of limits. Part (3) is a general fact about symmetric monoidal localizations. Part (4) follows from [Lur17, 4.5.2.1]. \square

Remark A.16. By [Lur17, 1.2.1.19], compatibility with countable products and left separated together imply left complete. Similarly, compatibility with countable coproducts and right separated together imply right complete.

In one way or another, we will typically be starting with a t -structure on spectra with an S^1 -action. This is a special case of a t -structure on parametrized spectra, which we introduce below.

Let X be a space and let $\text{Sp}^X = \text{Fun}(X, \text{Sp})$ be the ∞ -category of spectra parametrized over X . There is a natural t -structure on Sp^X , the **Postnikov** t -structure, where $(\text{Sp}^X)_{\geq 0} = \text{Fun}(X, \text{Sp}_{\geq 0})$ and $(\text{Sp}^X)_{\leq 0} = \text{Fun}(X, \text{Sp}_{\leq 0})$. The heart $\text{Sp}^{X, \heartsuit}$ is naturally equivalent to $\prod_{x \in \pi_0 X} \text{Mod}_{\mathbb{Z}[\pi_1(X, x)]}^{\heartsuit}$, the product over the connected components of X of the abelian group of discrete $\mathbb{Z}[\pi_1(X, x)]$ -modules, where $\mathbb{Z}[\pi_1(X, x)]$ is the group algebra of $\pi_1(X, x)$.

Proposition A.17. *Let X be a space. The Postnikov t -structure on Sp^X is*

- (a) *accessible,*
- (b) *left and right complete,*
- (c) *compatible with products and filtered colimits, and*
- (d) *compatible with the pointwise symmetric monoidal structure*

$$(\text{Sp}^X)^{\otimes} = \text{Fun}(X, \text{Sp}^{\otimes}) \times_{\text{Fun}(X, \text{Fin}_*)} \text{Fin}_*$$

on Sp^X .

Proof. Statement (d) follows immediately from the definitions. For the rest, note that each property is stable under products of stable ∞ -categories. Thus, it is enough to check the case when X is path-connected. Let $x \in X$ be a point. Then, $\text{Sp}^X \xrightarrow{x^*} \text{Sp}$ admits a left adjoint $x_!$. Moreover, $x_! \mathbb{S}$ is a compact generator of Sp^X [NS18, Theorem I.4.1]. Thus, $\text{Sp}^X \simeq \text{Mod}_{\mathbf{End}_{\text{Sp}^X}(x_! \mathbb{S})}$ by Morita theory, where $\mathbf{End}_{\text{Sp}^X}(x_! \mathbb{S})$ denotes the endomorphism algebra spectrum of $x_! \mathbb{S}$. Now, $\mathbf{End}_{\text{Sp}^X}(x_! \mathbb{S}) \simeq \text{Map}_{\text{Sp}}(\mathbb{S}, x^* x_! \mathbb{S}) \simeq \mathbb{S}[\Omega_x X]$, the spherical group algebra of the grouplike \mathbb{E}_1 -space $\Omega_x X$. Since $\mathbb{S}[\Omega_x X]$ is a connective \mathbb{E}_1 -algebra, $\text{Mod}_{\mathbb{S}[\Omega_x X]}$ admits an accessible, left and right complete t -structure which is compatible with products and filtered colimits by [Lur17, Proposition 7.1.1.13]. Thus, it is enough to note that the two t -structures agree, which follows from the fact $M \in \text{Sp}_{\geq 0}^X$ if and only if $x^* M$ is in $\text{Sp}_{\geq 0}$. \square

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